

Phase diagram and finite temperature properties of negative coupling scalar field theory

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In this work, I consider scalar field theory with negative quartic self-interaction, corresponding to an upside-down classical potential. Despite not possessing a classically stable ground state, such potentials are known to behave properly when treated quantum mechanically, leading to stable and unitary time evolution. Using two different saddle-point expansions for the same theory, I discuss the phase diagram in terms of bare parameters in Euclidean dimensions one to four, as well as the generalization to finite temperature. Comparing to other methods where available, I find that negative coupling field theory is a promising candidate for an interacting scalar field theory in the continuum. In particular, in four dimensions it exploits a loophole in mathematical proofs of quantum triviality, suggesting that negative coupling scalar field theory could offer a UV-complete and interacting description of the Higgs.

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I. INTRODUCTION

In classical physics, upside-down potentials do not have stable ground states, as those who choose to play golf on a mountain will readily acknowledge. In quantum field theory, the notion that all “good” potentials must have stable classical ground states is enshrined as an “obvious” condition in almost all textbooks, and early ideas to build quantum field theories with negative coupling [1] were derided as “nonsense” by some of the most distinguished members of the community [2].

The same obvious condition could be applied to the electron in the hydrogen atom, because a charged classical particle on a circular orbit continually emits electromagnetic radiation, leading to orbital decay, thus preventing a stable configuration. Yet the electronic ground state of the hydrogen atom is quantum mechanically stable, despite possessing no classically stable ground state.

Further information surfaced in the twilight of the last millennium [3], where it was found that even non-Hermitian potentials can lead to real and bounded eigenspectra of quantum mechanical Hamiltonians if they possess certain characteristics [4–7]. Using the similarities between Schrödinger’s wave equation and wave optics and open quantum systems [8], experimental verification of these “exotic” theory predictions was possible (cf. Ref. [9]),

creating a new subfield of material physics; see, for instance, Ref. [10].

Unlike negative coupling quantum mechanics, very little is known about negative coupling quantum field theory. Studies of nonclassical quantum field theory were first conducted for $i\phi^3$ potentials in Refs. [11–14], and more recently for a broader class of complex potentials in Refs. [15–17]. Subsequently, non-Hermitian quantum field theory has been considered for fermionic theories (cf. Refs. [18–22]), quantum gravity (cf. Refs. [23,24]) and lattice systems (cf. Refs. [25,26]). Studies of negative coupling quantum field theory are even more recent, (see, e.g., Refs. [27–33]), but potentially important as descriptions of the Higgs field; cf. Refs. [34–36].

One of the issues that studies of negative coupling field theory have faced is that established techniques such as weak coupling perturbation theory and lattice field theory using Monte Carlo importance sampling fail, simply because the classical probabilistic interpretation underlying these established techniques is absent. By contrast, nonperturbative methods such as large N expansions pioneered in the 1970s [37,38] were found to work well [28,30,39–41], whereas contour-deformation methods for lattice studies [25,29,42] work in principle, but are numerically too expensive for extracting continuum physics in spacetime dimensions $d > 2$.

In this work, the focus is on uncovering the broad qualitative features of the phase diagram of negative coupling scalar field theory, both at zero and finite temperature. The main tool for this survey is the use of two distinct saddle-point expansions of the theory, which were presented and cross-checked against other methods for the case of positive coupling field theory in Ref. [43]. It should

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be emphasized that while the expansions are systematically improvable, there is no small expansion parameter, and therefore truncations correspond to uncontrolled approximations of the theory. However, the truncations made in Ref. [43] have the advantage that results are analytically tractable, and can be compared to other nonperturbative methods whenever available (to date basically only quantum mechanics, cf. Ref. [44]) to check on the quality of the approximation.

In this sense, the present work is intended as a survey of the phase diagram of negative coupling field theory both at zero and finite temperature, with the aim of identifying the approximate location of features such as transition lines that may subsequently be probed by numerical methods, such as lattice field theory [25].

II. CALCULATION

I consider a scalar field ϕ with Euclidean action

$$S = \int d^d x \left[\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{2} m_B^2 \phi^2 - g_B \phi^4 \right], \quad (1)$$

where the timelike Euclidean direction is a circle with radius of inverse temperature, $\beta = \frac{1}{T}$. For further simplicity, it will be useful to state the results for the propagator and pressure of a free massive boson in dimension d at finite temperature:

$$\Delta_{\text{free}}(M, T) = T \sum_n \int \frac{d^{d-1} k}{(2\pi)^d} \frac{1}{\omega_n^2 + k^2 + M^2}, \quad p_{\text{free}}(M, T) = -\frac{T}{2} \sum_n \int \frac{d^{d-1} k}{(2\pi)^d} \ln[\omega_n^2 + k^2 + M^2], \quad (2)$$

where $\omega_n = 2\pi nT$ are the bosonic Matsubara frequencies. Some integrations may be performed to give the results

$$\Delta_{\text{free}}(M, T) = \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} M^{d-2} + \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \frac{n_B(\sqrt{k^2 + M^2})}{\sqrt{k^2 + M^2}}, \quad (3)$$

$$p_{\text{free}}(M, T) = \frac{\Gamma\left(-\frac{d}{2}\right)}{2(4\pi)^{\frac{d}{2}}} M^d - T \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \ln\left(1 - e^{-\beta\sqrt{k^2 + M^2}}\right), \quad (4)$$

where $n_B(x) = \frac{1}{e^{\beta x} - 1}$ is the Bose-Einstein distribution function. Yet another form can be given in terms of the high-temperature expansion [Ref. [45], Eq. (2.90)]:

$$\begin{aligned} \Delta_{\text{free}}(M, T) &= \frac{TM^{d-3}\Gamma\left(\frac{3-d}{2}\right)}{(4\pi)^{\frac{d-1}{2}}} + \frac{2T}{(4\pi)^{\frac{d-1}{2}}(2\pi T)^{3-d}} \sum_{l=0}^{\infty} \left[\frac{-M^2}{(2\pi T)^2} \right]^l \frac{\Gamma\left(l + \frac{3-d}{2}\right)}{\Gamma(l+1)} \zeta(2l+3-d), \\ p_{\text{free}}(M, T) &= \frac{T^d \zeta(d) \Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}} - \frac{TM^{d-1}\Gamma\left(\frac{3-d}{2}\right)}{(d-1)(4\pi)^{\frac{d-1}{2}}} + \frac{T}{(4\pi)^{\frac{d-1}{2}}(2\pi T)^{1-d}} \sum_{l=0}^{\infty} \left[\frac{-M^2}{(2\pi T)^2} \right]^{l+1} \frac{\Gamma\left(l + \frac{3-d}{2}\right)}{\Gamma(l+2)} \zeta(2l+3-d), \end{aligned} \quad (5)$$

where $p_{\text{SB}}(T) = \frac{T^d \zeta(d) \Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}}$ can be recognized as the Stefan-Boltzmann pressure for a free massless boson in d dimension.

For the interacting theory, I use an analytically tractable resummation scheme called R1 originally introduced in Ref. [46]. The R1 resummation scheme is very versatile and can be implemented in all dimensions, both at zero and finite temperature, and both in continuum and for spacetime lattice discretizations. For the present manuscript, the relevant derivations of the R1 formulas can be found in the appendix of Ref. [43] with the replacement of $\lambda \rightarrow -g$.

A. Symmetric saddle expansion

Using the mathematical identity

$$e^{g\phi^4} = \int_{-\infty}^{\infty} \frac{d\zeta}{\sqrt{16g\pi}} e^{-\frac{\zeta^2}{16g} + \frac{1}{2}\zeta\phi^2}, \quad (6)$$

I rewrite the action of the theory into a form that only contains ϕ quadratically. Using the R1 resummation [46], I find for the partition function of the theory

$$Z = \int_{-\infty}^{\infty} d\zeta_0 e^{\beta V \left(p_{\text{free}}(M, T) - \frac{\zeta_0^2}{16g_B} - 2g_B \Delta_{\text{free}}^2(M, T) \right)}, \quad (7)$$

where βV is the volume of spacetime, and

$$M^2 \equiv m_B^2 + \zeta_0 + \nu^2, \quad \nu^2 = -8g_B \Delta_{\text{free}}(M, T), \quad (8)$$

is the pole mass of the field ϕ . In the large volume limit, the partition function may be calculated using the saddle-point method, finding

$$\frac{\ln Z}{\beta V} \equiv p(M, T) = p_{\text{free}}(M, T) - \frac{\zeta_0^2}{16g_B} - 2g_B \Delta_{\text{free}}^2(M, T), \quad (9)$$

with the saddle-point condition

$$0 = \frac{\partial p(M, T)}{\partial M^2} = -\frac{1}{2} \Delta_{\text{free}}(M, T) - \frac{\zeta_0}{8g_B}. \quad (10)$$

This implies that $\zeta_0 = -4g_B \Delta_{\text{free}}(M, T)$, such that $\nu^2 = 2\zeta_0$ and the saddle-point condition can be expressed in terms of the pole mass [46]. One finds for the symmetric phase that

$$M^2 = m_B^2 - 12g_B \Delta_{\text{free}}(M, T),$$

$$p(M, T) = p_{\text{free}}(M, T) - \frac{(M^2 - m_B^2)^2}{48g_B}. \quad (11)$$

B. Broken phase saddle expansion

In contrast to the symmetric phase expansion, one can expand the action using

$$\phi(x) = \phi_0 + \xi(x), \quad (12)$$

corresponding to an expansion around a phase with $\langle \phi \rangle \neq 0$. Using the R1 resummation [43], one obtains the pressure in the broken phase as

$$\tilde{p}(\tilde{M}, T) = -\frac{m_B^2 \phi_0^2}{2} + g_B \phi_0^4 + p_{\text{free}}(\tilde{M}, T) - 3g_B \Delta_{\text{free}}^2(\tilde{M}, T), \quad (13)$$

where the pole mass $\tilde{M}$ of the fluctuation field ξ fulfills

$$\tilde{M}^2 = m_B^2 - 12g_B \phi_0^2 + \tilde{\nu}^2, \quad \tilde{\nu}^2 = -12g_B \Delta_{\text{free}}(\tilde{M}, T). \quad (14)$$

In the large volume limit, the value of ϕ_0 is fixed through the saddle-point condition

$$0 = \frac{\partial \tilde{p}(\tilde{M}, T)}{\partial \phi_0^2} = -\frac{m_B^2}{2} + 2g_B \phi_0^2 + 6g_B \Delta_{\text{free}}(\tilde{M}, T). \quad (15)$$

Using this condition, the pole mass can be written as

$$\tilde{M}^2 = -2m_B^2 + 24g_B \Delta_{\text{free}}(\tilde{M}, T), \quad (16)$$

so that the saddle-point condition and broken phase pressure simplify to

$$\phi_0^2 = -\frac{\tilde{M}^2}{8g_B},$$

$$\tilde{p}(\tilde{M}, T) = p_{\text{free}}(\tilde{M}, T) + \frac{\tilde{M}^4}{96g_B} + \frac{\tilde{M}^2 m_B^2}{24g_B} - \frac{m_B^4}{48g_B}. \quad (17)$$

C. Transitions between saddles

Both the symmetric and broken phase saddle are descriptions of the same theory. However, when evaluating physical observables such as the free energy for the same theory parameters, results will in general differ between the two saddle-point expansions.

As outlined in Ref. [43], I interpret this as the presence of distinct phases for the same theory, describing a stable phase (with lowest free energy), and others describing a metastable or unstable phase (with higher or complex free energy). Which saddle corresponds to which phase can only be determined after evaluating physical quantities such as the free energy, but I will use the criterion from Ref. [43] to define the physical ground state as the phase with the lowest real part of the free energy. Note that in principle, the calculation could lead to complex (meta-stable) solutions as potential ground states, but in practice, it will be found that no such problematic case arises.

III. $d = 1$: $\mathcal{PT}$ -SYMMETRIC QUANTUM MECHANICS

For the case of quantum mechanics $d = 1$, the symmetric phase and broken phase results simplify to

$$p(M, T) = -\frac{M}{2} - T \ln(1 - e^{-\beta M}) - \frac{(M^2 - m_B^2)^2}{48g_B},$$

$$M^2 = m_B^2 - \frac{6g_B}{M} - \frac{12g_B n_B(M)}{M},$$

$$\tilde{p}(\tilde{M}, T) = -\frac{\tilde{M}}{2} - T \ln(1 - e^{-\beta \tilde{M}}) + \frac{\tilde{M}^4}{96g_B} + \frac{\tilde{M}^2 m_B^2}{24g_B} - \frac{m_B^4}{48g_B},$$

$$\tilde{M}^2 = -2m_B^2 + \frac{12g_B}{\tilde{M}} + \frac{24g_B n_B(\tilde{M})}{\tilde{M}}. \quad (18)$$

Let me first discuss the case of zero temperature $T = 0$, where one finds solutions $M \in \mathbb{R}^+$ for $m_B^2 \geq 243^{\frac{1}{3}} g_B^{\frac{2}{3}} \simeq 6.24 g_B^{\frac{2}{3}}$ for the symmetric phase, whereas solutions $\tilde{M} \in \mathbb{R}^+$ exist for all m_B^2 . Using the criterion outlined in Ref. [43], these solutions correspond to different phases of the theory, with the phase with the highest pressure (lowest

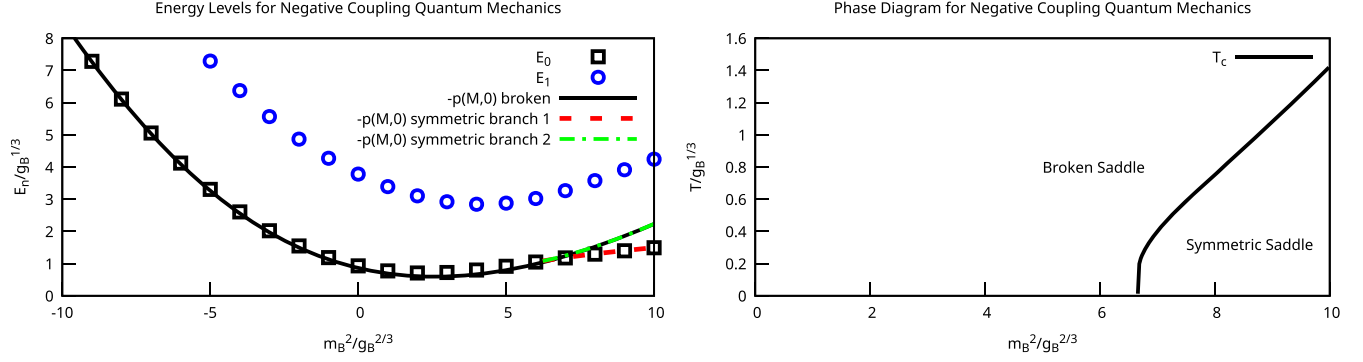


FIG. 1. Left: comparison of lowest lying eigenvalues E_0, E_1 of the Hamiltonian (20) to minus the pressure $p(M, T=0), \tilde{p}(\tilde{M}, T=0)$ from the symmetric and broken saddles obtained in the R1-level resummation. Right: phase diagram at finite temperature, indicating which saddle is thermodynamically preferred. Note that there is no actual phase transition here, just a transition from one saddle to another saddle. See text for details.

free energy) being thermodynamically favored. Comparing $p(M, 0) - \tilde{p}(\tilde{M}, 0)$ for the solutions outlined above, one finds that there is a transition from a broken phase for small $\frac{m_B^2}{g_B}$ to a symmetric phase at

$$m_B^2 \simeq 6.66 g_B^{\frac{2}{3}}. \quad (19)$$

As discussed in the Supplemental Material of Ref. [43], a transition from broken to symmetric phase is not taking place in the quantum mechanical theory, even though the pressure in the symmetric phase does give an accurate numerical description of the ground-state energy of the system. This is verified in Fig. 1 through the numerical diagonalization of the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} + 4g_B x^4 + \sqrt{2g_B} x - x^2 m_B^2 + \frac{m_B^4}{16g_B}, \quad (20)$$

which is isospectral to the negative coupling ($\mathcal{PT}$ -symmetric) Hamiltonian $\mathcal{H} = \frac{p^2}{2} + \frac{1}{2} m_B^2 x^2 - g_B x^4$ [6,44].

At finite temperature, real-valued solutions for the symmetric pole mass M exist only below $T < T_c(m_B^2)$, whereas real-valued solutions exist for $\tilde{M}$ for all m_B^2, T . It is possible to track the solution with the highest pressure for all m_B^2, T , and one obtains the phase diagram for the preferred thermodynamic saddle shown in Fig. 1. In the high-temperature limit $T \gg m_B$, results (18) simplify to

$$\begin{aligned} p(M, T) &= -\frac{M}{2} - T \ln(\beta M) - \frac{M^4}{48g_B}, & M^2 &= -\frac{12g_B T}{M^2}, \\ \tilde{p}(\tilde{M}, T) &= -\frac{\tilde{M}}{2} - T \ln(\beta \tilde{M}) + \frac{\tilde{M}^4}{96g_B}, & \tilde{M}^2 &= \frac{24g_B T}{\tilde{M}^2}. \end{aligned} \quad (21)$$

In the case of the symmetric saddle, there are no real-valued solutions M on the principal Riemann sheet, but instead one has

$$M^2 = \pm i(12g_B T)^{\frac{1}{2}}, \quad (22)$$

and complex-valued pressure at high temperature. This matches the well-documented cases of complex-valued pressure at high temperature for symmetric saddle expansions in the literature in various dimensions, cf. the discussions in Refs. [27–29,32,47,48]. For the case of $d=2$, Ref. [32] proposed a resolution of the problem by using solutions to the saddle-point condition on nonprincipal Riemann sheets. For $d=1$, there are solutions for the symmetric saddle-point condition (22) on nonprincipal Riemann sheets, but they are not real valued.

In this work, I point out a different way from Ref. [32] that also resolves the issue of complex valued pressures at high temperature. In particular, the broken saddle-point condition has real-valued solutions

$$\tilde{M} = (24g_B T)^{\frac{1}{4}}, \quad (23)$$

on the principal Riemann sheet, and the resulting pressure in the high-temperature limit is

$$\tilde{p}(\tilde{M} \ll T) = -\frac{T}{4} \ln\left(\frac{24g_B}{e^1 T^3}\right) + \dots, \quad (24)$$

which is real valued and positive for high temperatures, as advertised.

For the case of $d=1$, it is possible to test if the broken saddle solution (24) correctly captures the high-temperature limit of the theory. This can be done by numerically obtaining the eigenvalues E_n of the isospectral Hamiltonian (20), and from this the finite temperature pressure of the theory as

$$p(T) = T \ln \left(\sum_n e^{-\beta E_n} \right). \quad (25)$$

One finds that the numerically calculated $p(T)$ indeed approaches Eq. (24) in the high-temperature limit. This suggests that the real-valued broken phase saddle-point solution (23) on the principal Riemann sheet rather than

some complex-valued solution on nonprincipal Riemann sheets captures the high-temperature behavior of the theory.

IV. $d = 2$: NEGATIVE COUPLING FIELD THEORY

For the case of $d = 2 - 2\varepsilon$, the symmetric phase and broken phase results lead to

$$p(M, T) = -\frac{M^2}{8\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2 e^1}{M^2} \right) - \frac{(M^2 - m_B^2)^2}{48g_B} + \frac{TM}{\pi} \sum_{n=1}^{\infty} \frac{K_1(n\beta M)}{n},$$

$$M^2 = m_B^2 - \frac{3g_B}{\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2}{M^2} \right) - \frac{12g_B}{\pi} \sum_{n=1}^{\infty} K_0(n\beta M), \quad (26)$$

$$\tilde{p}(\tilde{M}, T) = -\frac{\tilde{M}^2}{8\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2 e^1}{\tilde{M}^2} \right) + \frac{\tilde{M}^4}{96g_B} + \frac{\tilde{M}^2 m_B^2}{24g_B} - \frac{m_B^4}{48g_B} + \frac{T\tilde{M}}{\pi} \sum_{n=1}^{\infty} \frac{K_1(n\beta \tilde{M})}{n},$$

$$\tilde{M}^2 = -2m_B^2 + \frac{6g_B}{\pi} \left(\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2}{\tilde{M}^2} \right) + \frac{24g_B}{\pi} \sum_{n=1}^{\infty} K_0(n\beta \tilde{M}), \quad (27)$$

where $\bar{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E}$ is the $\overline{\text{MS}}$ renormalization scale and Γ used

$$\int \frac{dk}{2\pi} \ln(1 - e^{-\beta\sqrt{k^2 + M^2}}) = -\frac{M}{\pi} \sum_{n=1}^{\infty} \frac{K_1(n\beta M)}{n}. \quad (28)$$

The divergencies can be nonperturbatively renormalized using

$$m_B^2 = m_R^2(\bar{\mu}) + \frac{3g_B}{\pi\varepsilon}, \quad (29)$$

which leads to the running renormalized mass in the $\overline{\text{MS}}$ scheme,

$$m_R^2(\bar{\mu}) = \frac{3g_B}{\pi} \ln \frac{\bar{\mu}^2}{\Lambda_{\overline{\text{MS}}}^2}, \quad (30)$$

where $\Lambda_{\overline{\text{MS}}}$ is the $\overline{\text{MS}}$ parameter. Plugging this result for m_B^2 into the above formulas leads to

$$p(M, T) = -\frac{M^2}{8\pi} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^1}{M^2} - \frac{M^4 + m_B^4}{48g_B} + \sum_{n=1}^{\infty} \frac{TMK_1(n\beta M)}{\pi n}, \quad -\frac{\pi M^2}{3g_B} = \ln \frac{\Lambda_{\overline{\text{MS}}}^2}{M^2} + 4 \sum_{n=1}^{\infty} K_0(n\beta M),$$

$$\tilde{p}(\tilde{M}, T) = -\frac{\tilde{M}^2}{8\pi} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^1}{\tilde{M}^2} + \frac{\tilde{M}^4 - 2m_B^4}{96g_B} + \sum_{n=1}^{\infty} \frac{T\tilde{M}K_1(n\beta \tilde{M})}{\pi n}, \quad \frac{\pi \tilde{M}^2}{6g_B} = \ln \frac{\Lambda_{\overline{\text{MS}}}^2}{\tilde{M}^2} + 4 \sum_{n=1}^{\infty} K_0(n\beta \tilde{M}).$$

At zero temperature $T = 0$, a pair of real-valued solutions for the symmetric phase exist for $g_B \geq \frac{\pi e^1}{3} \Lambda_{\overline{\text{MS}}}^2$ and are given by

$$M^2 = -\frac{3g_B}{\pi} W_{-1,0} \left(-\frac{\pi \Lambda_{\overline{\text{MS}}}^2}{3g_B} \right), \quad (31)$$

where W_k denotes the Lambert W -function of branch k . For the broken phase, real-valued solutions exist for all values of $g_B \geq 0$ and are given by

$$\tilde{M}^2 = \frac{6g_B}{\pi} W_0 \left(\frac{\pi \Lambda_{\overline{\text{MS}}}^2}{6g_B} \right). \quad (32)$$

While $p, \tilde{p}$ are divergent, the difference

$$\Delta p(M, \tilde{M}, T) \equiv p(M, T) - \tilde{p}(\tilde{M}, T), \quad (33)$$

is finite, and can be calculated to decide which saddle offers the thermodynamically preferred (higher pressure) phase. At zero temperature, one finds that starting at small g_B , the

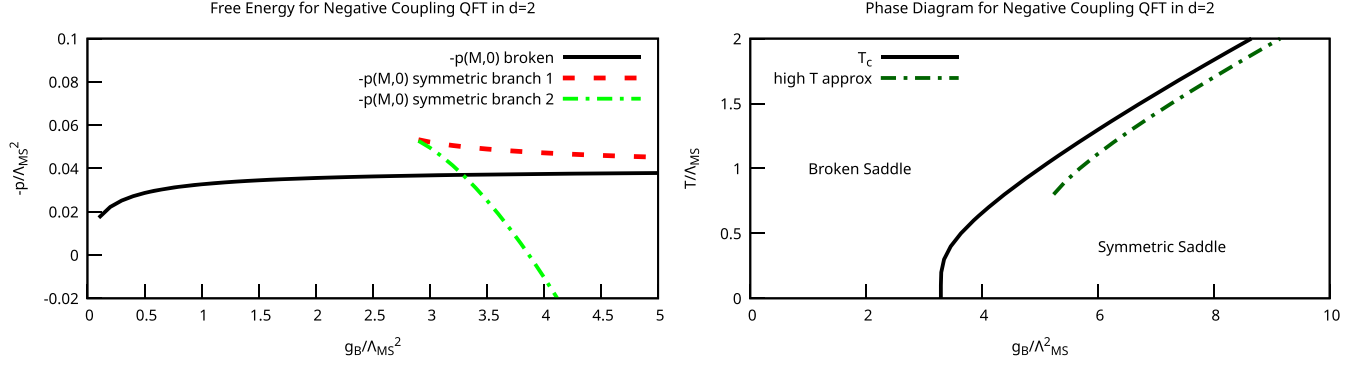


FIG. 2. Left: free energy (negative pressure) from the symmetric and broken phase saddle expansions as a function of the coupling. Right: phase diagram at finite temperature, indicating which saddle is thermodynamically preferred. Line labeled “high T approx” is Eq. (43). See text for details.

dominant phase is given by the broken phase until a critical value of

$$g_c \equiv \frac{g_B}{\Lambda_{\overline{MS}}^2} \simeq 3.2894, \quad (34)$$

above which the dominant phase is given by the symmetric phase with solution branch $k = -1$ in Eq. (31). I note that at the critical coupling, the symmetric pole mass fulfills

$$\frac{g_B}{M^2} = -\frac{\pi}{3W_{-1}\left(-\frac{\pi}{3g_c}\right)} \simeq 0.64, \quad (35)$$

which lends itself to comparison to lattice-based approaches (see, e.g., Ref. [49] for the positive coupling situation). A plot of the free energies at $T = 0$ for the

different phases is shown in Fig. 2. This figure shows that at low values of g_B , the lowest free-energy solution is provided by the broken phase saddle, whereas for sufficiently high values of g_B , the symmetric phase saddle-point solution (31) with index -1 has the lowest free energy. The symmetric phase saddle-point solution (31) with index 0 never has the lowest free energy, and therefore always corresponds to a metastable phase that is thermodynamically disfavored.

A. High temperature and dimensional reduction

As temperature T is increased, the critical coupling value where $\Delta p(M, \tilde{M}, T) = 0$ increases as well (see Fig. 2). In the high-temperature limit $T \gg \Lambda_{\overline{MS}}$, one can use the expansions (5) to express Eq. (31) after renormalization as

$$p(M, T) = \frac{T^2 \pi}{6} - \frac{TM}{2} - \frac{M^2}{8\pi} \ln \frac{\Lambda_{\overline{MS}}^2 e^{2\gamma_E}}{(4\pi T)^2} - \frac{M^4 + m_B^4}{48g_B}, \quad -\frac{\pi M^2}{3g_B} = \frac{2\pi T}{M} + \ln \frac{\Lambda_{\overline{MS}}^2 e^{2\gamma_E}}{(4\pi T)^2}, \quad (36)$$

$$\tilde{p}(\tilde{M}, T) = \frac{T^2 \pi}{6} - \frac{T\tilde{M}}{2} - \frac{\tilde{M}^2}{8\pi} \ln \frac{\Lambda_{\overline{MS}}^2 e^{2\gamma_E}}{(4\pi T)^2} + \frac{\tilde{M}^4 - 2m_B^4}{96g_B}, \quad \frac{\pi \tilde{M}^2}{6g_B} = \frac{2\pi T}{\tilde{M}} + \ln \frac{\Lambda_{\overline{MS}}^2 e^{2\gamma_E}}{(4\pi T)^2}. \quad (37)$$

For high temperatures, the action (1) dimensionally reduces to

$$S_{\text{red}} = \int d^{d-1}x \left[\frac{1}{2} \partial_i \phi \partial_i \phi + \frac{1}{2} m_{B,(d-1)}^2 \phi^2 - T g_{B,(d)} \phi^4 \right], \quad (38)$$

where $i = 1, \dots, d-1$ and I have rescaled $\phi \rightarrow \phi \sqrt{T}$ and I have indicated the original dimension for the coupling g_B by a subscript. Note that the mass term in Eq. (38) need not coincide with m_B^2 of the original theory because additional contributions are typically generated when integrating out

the “static” modes (cf. the discussion in Ref. [45]). In this form, S_{red} corresponds to Eq. (1) at zero temperature in one dimension lower, and one can immediately relate the coupling constants

$$g_{B,(d-1)} = T g_{B,(d)}. \quad (39)$$

In order to relate the mass parameter, one matches observables in the original d -dimensional theory at high temperature and the dimensionally reduced theory. For instance, let us consider the pole mass as an observable, which is determined by the solution to the saddle-point condition. Matching the symmetric saddle-point condition

at high temperature (36) in $d = 2$ to the saddle-point condition at zero temperature (18) in $d = 1$ with mass $m_{B,(1)}$ and coupling $g_{B,(1)}$, one finds

$$g_{B,(1)} = T g_{B,(2)}, \quad m_{B,(1)}^2 = -\frac{3g_{B,(2)}}{\pi} \ln \frac{\Lambda_{\overline{\text{MS}},(2)}^2 e^{2\gamma_E}}{(4\pi T)^2}, \quad (40)$$

where I stress that T here denotes the temperature in the $d = 2$ dimensional theory, whereas the effective $d = 1$ description is at zero temperature. Also, it is worth pointing out that the $d = 2$ renormalization scale $\Lambda_{\overline{\text{MS}},(2)}$ in this matching fulfills

$$\Lambda_{\overline{\text{MS}},(2)} = 4\pi T e^{-\gamma_E - \frac{\pi m_{B,(1)}^2}{6g_{B,(2)}}}; \quad (41)$$

e.g., is temperature dependent for fixed $m_{B,(1)}^2$.

A completely analogous construction can be performed for the broken phase pole mass $\tilde{M}$, where one finds exactly the same matching conditions (40) as for the symmetric saddle.

With the parameters for the effective $d = 1$ action fixed, one can directly reuse the results found in the previous section. One finds that for quantum mechanics ($d = 1$) at zero temperature, the system is in the broken phase as long as

$$\frac{m_{B,(1)}^2}{g_{B,(1)}^{\frac{2}{3}}} = -\frac{3g_{B,(2)}^{\frac{1}{3}}}{\pi T^{\frac{2}{3}}} \ln \frac{\Lambda_{\overline{\text{MS}},(2)}^2 e^{2\gamma_E}}{(4\pi T)^2} \lesssim 6.66, \quad (42)$$

which is always true in the high-temperature limit. From this requirement, one can immediately obtain an estimate for the critical temperature T_c in the $d = 2$ theory as

$$\frac{g_{B,(2)}}{\Lambda_{\overline{\text{MS}},(2)}^2} \simeq \frac{6.66^3 \pi^3 T_c^2}{27 \ln \frac{(4\pi T_c)^2}{e^{2\gamma_E} \Lambda_{\overline{\text{MS}},(2)}^2}}, \quad (43)$$

which is also shown in Fig. 2.

The result at high temperature is also interesting because the solution to the symmetric phase saddle (36) becomes

$$M = (-3g_{B,(2)} T)^{\frac{1}{3}}, \quad (44)$$

which is not real valued for the principal branch of the root. This leads to the well-documented case of complex-valued pressure functions at high temperature [27–29,32,47]. In a previous article [32], I argued that the nonprincipal branch of the root function leads to a well-behaved high-temperature limit. In this work, the existence of a real-valued pole mass and pressure from

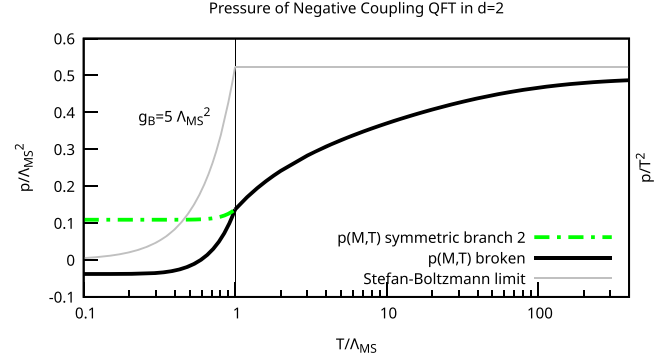


FIG. 3. Pressure for the dominant phase as a function of temperature for $g_B = 5\Lambda_{\overline{\text{MS}}}^2$. Left part of the plot shows $\frac{p(M,T)}{\Lambda_{\overline{\text{MS}}}^2}$; right part shows $\frac{p(M,T)}{T^2}$ to indicate approach to the Stefan-Boltzmann limit $p_{\text{SB}}(T) = \frac{\pi T^2}{6}$ (gray line).

the principal root of the broken saddle expansion, specifically

$$\tilde{M} = (6g_{B,(2)} T)^{\frac{1}{3}}, \quad (45)$$

make appeals to higher Riemann sheets unnecessary, and in my opinion provide a better resolution of the problems discussed in Refs. [27–29,32,47]. As discussed in the previous section, for the case of $d = 1$, one can show numerically that the real-valued broken phase saddle solution on the principal Riemann sheet correctly captures the high-temperature limit of the theory. For the case of $d = 2$, it will be possible to test the proposed broken phase saddle result as soon as numerical data become available [50].

A representative plot for the pressure for the thermodynamically dominant phase as a function of temperature is shown in Fig. 3 for $g_B = 5\Lambda_{\overline{\text{MS}}}^2$. As can be seen from this figure, the symmetric branch is preferred at low temperatures, but there is a phase transition at $\frac{T_c}{\Lambda_{\overline{\text{MS}}}} \simeq 1$. For $T > T_c$, the broken saddle expansion provides a real-valued pressure that approaches the Stefan-Boltzmann limit in the high-temperature limit.

V. $d = 3$: NEGATIVE COUPLING FIELD THEORY

For the case of $d = 3$, the symmetric phase and broken phase results lead to

$$p(M, T) = \frac{M^3}{12\pi} - \frac{(M^2 - m_B^2)^2}{48g_B} + \frac{T^3}{2\pi} (\text{Li}_3(e^{-\beta M}) + \beta M \text{Li}_2(e^{-\beta M})),$$

$$M^2 = m_B^2 + \frac{3g_B M}{\pi} + \frac{6g_B T}{\pi} \ln(1 - e^{-\beta M}), \quad (46)$$

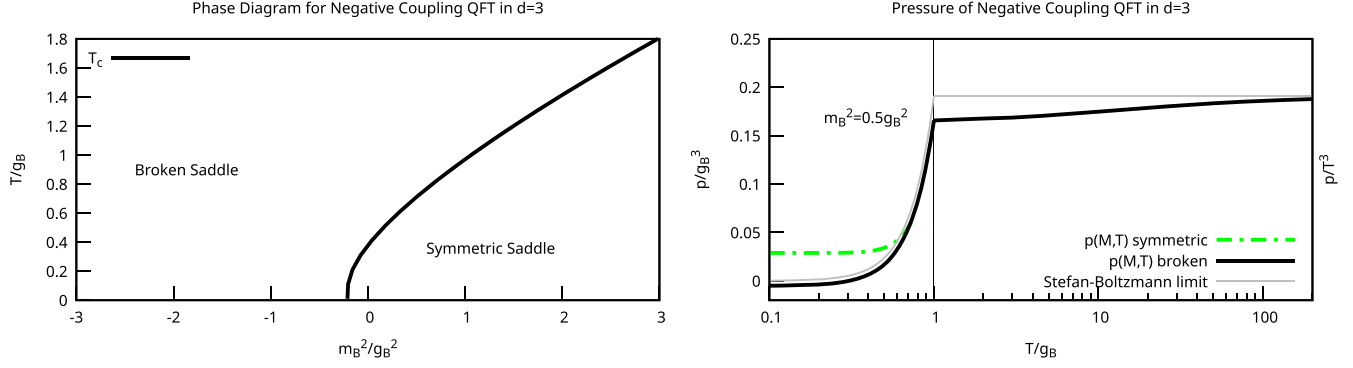


FIG. 4. Left: phase diagram at finite temperature, indicating which saddle is thermodynamically preferred. Right: pressure as a function of temperature for $m_B^2 = 0.5g_B^2$. Left part of the plot shows $\frac{p(M,T)}{g_B^3}$; right part shows $\frac{p(M,T)}{T^3}$ to indicate approach to the Stefan-Boltzmann limit $p_{SB}(T) = \frac{\zeta(3)T^3}{2\pi}$ (gray line). Note that this different normalization leads to a fictitious cusp in the figure at $T = \Lambda_{MS}$. For $m_B^2 = 0.5g_B^2$, the broken saddle pressure exceeds the symmetric saddle branch above $T_c \simeq 0.71g_B$. See text for details.

$$\begin{aligned} \tilde{p}(\tilde{M}, T) &= \frac{\tilde{M}^3}{12\pi} + \frac{\tilde{M}^4}{96g_B} + \frac{\tilde{M}^2 m_B^2}{24g_B} - \frac{m_B^4}{48g_B} + \frac{T^3}{2\pi} (\text{Li}_3(e^{-\beta\tilde{M}}) + \beta\tilde{M}\text{Li}_2(e^{-\beta\tilde{M}})), \\ &\quad - \frac{\tilde{M}^2}{2} = m_B^2 + \frac{3g_B\tilde{M}}{\pi} + \frac{6g_B T}{\pi} \ln(1 - e^{-\beta\tilde{M}}), \end{aligned} \quad (47)$$

where I note that because of the absence of logarithmic divergencies, no nontrivial renormalization is necessary, and I used

$$\int \frac{d^2k}{(2\pi)^2} \ln(1 - e^{-\beta\sqrt{k^2+M^2}}) = -\frac{T^3}{(2\pi)} \text{Li}_3(e^{-\beta M}) - \frac{MT^2}{(2\pi)} \text{Li}_2(e^{-\beta M}), \quad (48)$$

in terms of the polylogarithm functions Li_n .

In the zero temperature limit $T = 0$, the saddle-point equations admit solutions

$$M = \frac{3g_B}{2\pi} \left(1 \pm \sqrt{1 + \frac{4\pi^2 m_B^2}{9g_B^2}} \right), \quad \tilde{M} = -\frac{3g_B}{\pi} \left(1 \pm \sqrt{1 - \frac{2m_B^2\pi^2}{9g_B^2}} \right), \quad (49)$$

where $M \in \mathbb{R}^+$ for at least one branch for $m_B^2 \geq -\frac{9g_B^2}{4\pi^2}$ and $\tilde{M} \in \mathbb{R}^+$ for the minus branch and $m_B^2 < 0$. One finds that $\Delta p(M, \tilde{M}, 0) = 0$ for a critical value of

$$\frac{m_B^2}{g_B^2} \simeq -0.21, \quad \frac{g_B^2}{M^2} \simeq 2.68, \quad (50)$$

which may lend itself to comparison to lattice-based approaches.

A. Nonzero temperature and high-temperature limit

One can track the transition between the symmetric and broken saddles by following the line defined by $\Delta p(M, \tilde{M}, T) = 0$, which is shown in Fig. 4. In the high-temperature limit, one can use the expansions (5) or more straightforwardly directly expand Eq. (46) to find

$$\begin{aligned} p(M, T) &= \frac{\zeta(3)T^3}{2\pi} - \frac{M^2 T}{8\pi} (1 - 2 \ln(\beta M)), & M^2 &= \frac{6g_B T}{\pi} \ln(\beta M), \\ \tilde{p}(\tilde{M}, T) &= \frac{\zeta(3)T^3}{2\pi} - \frac{\tilde{M}^2 T}{8\pi} (1 - 2 \ln(\beta \tilde{M})), & -\frac{\tilde{M}^2}{2} &= \frac{6g_B T}{\pi} \ln(\beta \tilde{M}), \end{aligned}$$

which have solutions

$$M^2 = -\frac{3g_B T}{\pi} W_{0,-1}\left(-\frac{\pi T}{3g_B}\right), \quad \tilde{M}^2 = \frac{6g_B T}{\pi} W_0\left(\frac{\pi T}{6g_B}\right), \quad (51)$$

for the pole masses of the symmetric and broken saddle expansions. Using the matching condition (39) and comparing the pole masses to the zero-temperature expressions (31) and (32), one finds

$$g_{B,(2)} = T g_{B,(3)}, \quad \Lambda_{\overline{\text{MS}},(2)} = T. \quad (52)$$

Note that in particular, the renormalization scale $\Lambda_{\overline{\text{MS}}}$ for the two-dimensionally reduced theory takes the value of the temperature of the three-dimensional theory. As was found before, the matching conditions are independent of matching the symmetric or broken saddle expansion in the dimensionally reduced theory.

However, in the high-temperature limit, the symmetric saddle solution M as well as the corresponding pressure becomes complex valued; see the discussion for $d = 1$ around Eq. (22). The solution is indicated in Fig. 4: rather than tracking the symmetric saddle, at high temperatures a real-valued solution is provided by the broken saddle expansion.

A representative example of the pressure as a function of temperature is shown in Fig. 4 for the case of $m_B^2 = 0.5g_B^2$. For low temperatures, the dominant phase is given by the symmetric saddle expansion until a critical temperature of $T_c \simeq 0.71g_B$, above which the broken phase saddle provides higher pressure. At high temperatures, the broken phase saddle pressure approaches the Stefan-Boltzmann pressure asymptotically from below.

VI. $d = 4$: NONTRIVIAL INTERACTING SCALAR FIELD THEORY

Let me now consider the case $d = 4$. Before presenting the results, let me preface the discussion by pointing out that there is a mathematical proof that scalar field theory in four dimensions is trivial in the continuum, ruling out interacting field theory in four dimensions [51]. The mathematical proof is very specific, and in particular is only valid for Euclidean actions that are within the Griffiths-Simon class. Notably, the case of negative coupling $\lambda = -g$ with $g \in \mathbb{R}^+$ is *not* in the Griffiths-Simon class, providing a loophole for which the mathematical proof in Ref. [51] does not apply [52], which has been pointed out before in Ref. [53]. Similarly, lattice-based arguments against an interacting continuum theory from the 1980s [54,55] suffer from the same loophole in that positive coupling was implicitly assumed as a *sine qua non* condition. Lattice studies for the negative coupling theory using contour deformations and brute force numerical integration have only been performed recently [25], but only for tiny lattices, preventing the study of the continuum theory using this method.

Since no known result in the literature prohibits an interacting continuum scalar field theory in $d = 4 - 2\varepsilon$ with $\varepsilon \rightarrow 0$ and negative coupling [53], I now proceed to evaluate precisely this case using the saddle-point expansions from above. I find

$$p(M, T) = \frac{M^4}{64\pi^2} \left(\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{M^2} \right) - \frac{(M^2 - m_B^2)^2}{48g_B} + \frac{M^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta M)}{n^2},$$

$$M^2 = m_B^2 + \frac{3M^2 g_B}{4\pi^2} \left(\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2 e^1}{M^2} \right) - \frac{6g_B M T}{\pi^2} \sum_{n=1}^{\infty} \frac{K_1(\beta M n)}{n}, \quad (53)$$

$$\tilde{p}(\tilde{M}, T) = \frac{\tilde{M}^4}{64\pi^2} \left(\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{\tilde{M}^2} \right) + \frac{\tilde{M}^4}{96g_B} + \frac{\tilde{M}^2 m_B^2}{24g_B} - \frac{m_B^4}{48g_B} + \frac{\tilde{M}^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta \tilde{M})}{n^2},$$

$$-\frac{\tilde{M}^2}{2} = m_B^2 + \frac{3\tilde{M}^2 g_B}{4\pi^2} \left(\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2 e^1}{\tilde{M}^2} \right) - \frac{6g_B \tilde{M} T}{\pi^2} \sum_{n=1}^{\infty} \frac{K_1(\beta \tilde{M} n)}{n}, \quad (54)$$

where I used

$$\int \frac{dk}{2\pi^2} k^2 \ln \left(1 - e^{-\beta \sqrt{k^2 + M^2}} \right) = -\frac{M^2 T}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta M)}{n^2}. \quad (55)$$

The symmetric saddle can be nonperturbatively renormalized as

$$\frac{1}{g_B} = \frac{1}{g_R(\bar{\mu})} + \frac{3}{4\pi^2 \varepsilon}, \quad \frac{1}{g_R(\bar{\mu})} = \frac{3}{4\pi^2} \ln \frac{\bar{\mu}^2}{\Lambda_{\overline{\text{MS}}}^2}, \quad (56)$$

whereas the broken saddle can be nonperturbatively renormalized as

$$\frac{1}{g_B} = \frac{1}{g_R(\bar{\mu})} - \frac{3}{2\pi^2 \varepsilon}, \quad \frac{1}{\tilde{g}_R(\bar{\mu})} = \frac{3}{2\pi^2} \ln \frac{\tilde{\Lambda}_{\overline{\text{MS}}}^2}{\bar{\mu}^2}; \quad (57)$$

see the discussions in Ref. [43]. Note that unlike in lower dimensions, the symmetric and broken saddles cannot be simultaneously renormalized using the same renormalization condition, which introduces the need for two scales: $\Lambda_{\overline{\text{MS}}}^2, \tilde{\Lambda}_{\overline{\text{MS}}}^2$. However, after renormalization, Eq. (53) becomes

$$p(M, T) = \frac{M^4}{64\pi^2} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{\frac{3}{2}}}{M^2} + \frac{M^2 m_B^2}{24g_B} + \frac{M^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta M)}{n^2},$$

$$0 = \frac{m_B^2}{g_B} + \frac{3M^2}{4\pi^2} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^1}{M^2} - \frac{6MT}{\pi^2} \sum_{n=1}^{\infty} \frac{K_1(\beta Mn)}{n}, \quad (58)$$

$$\tilde{p}(\tilde{M}, T) = \frac{\tilde{M}^4}{64\pi^2} \ln \frac{\tilde{\Lambda}_{\overline{\text{MS}}}^2 e^{\frac{3}{2}}}{\tilde{M}^2} + \frac{\tilde{M}^2 m_B^2}{24g_B} + \frac{\tilde{M}^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{K_2(n\beta \tilde{M})}{n^2},$$

$$0 = \frac{m_B^2}{g_B} + \frac{3\tilde{M}^2}{4\pi^2} \ln \frac{\tilde{\Lambda}_{\overline{\text{MS}}}^2 e^1}{\tilde{M}^2} - \frac{6\tilde{M}T}{\pi^2} \sum_{n=1}^{\infty} \frac{K_1(\beta \tilde{M}n)}{n}, \quad (59)$$

implying that the free energy and pole masses in the symmetric and broken saddle expansion would be exactly identical if $\Lambda_{\overline{\text{MS}}} = \tilde{\Lambda}_{\overline{\text{MS}}}$.

However, for $\Lambda_{\overline{\text{MS}}} = \tilde{\Lambda}_{\overline{\text{MS}}}$, one has different values of g_B in the broken and symmetric phase, contradicting the derivation from the same action (1).

A partial resolution is indicated by considering the high-temperature limit of the theory provided by Eq. (5), which for the case at hand, becomes after renormalization

$$p(M, T) = \frac{\pi^2 T^4}{90} - \frac{M^2 T^2}{24} + \frac{M^3 T}{12\pi} + \frac{M^4}{64\pi^2} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2},$$

$$0 = \frac{3M^2}{4\pi^2} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2} - T^2 + \frac{3MT}{\pi}, \quad (60)$$

$$\tilde{p}(\tilde{M}, T) = \frac{\pi^2 T^4}{90} - \frac{\tilde{M}^2 T^2}{24} + \frac{\tilde{M}^3 T}{12\pi} + \frac{\tilde{M}^4}{64\pi^2} \ln \frac{\tilde{\Lambda}_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2},$$

$$0 = \frac{3\tilde{M}^2}{4\pi^2} \ln \frac{\tilde{\Lambda}_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2} - T^2 + \frac{3\tilde{M}T}{\pi}. \quad (61)$$

Comparing the saddle-point conditions for $M, \tilde{M}$ with the effective dimensionally reduced theory (46) leads to

$$g_{B,(3)} = -\frac{4\pi^2 T}{3 \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2}}, \quad m_{B,(3)}^2 = -T g_{B,(3)}, \quad (62)$$

$$\tilde{g}_{B,(3)} = \frac{2\pi^2 T}{3 \ln \frac{\tilde{\Lambda}_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2}}, \quad m_{B,(3)}^2 = -T \tilde{g}_{B,(3)}. \quad (63)$$

The effective three-dimensional theories for the symmetric and broken saddle expansion become identical if $g_{B,(3)} = \tilde{g}_{B,(3)}$ or

$$\tilde{\Lambda}_{\overline{\text{MS}}}^2 = \frac{(4\pi T)^3}{\Lambda_{\overline{\text{MS}}} e^{3\gamma_E}}. \quad (64)$$

Note the similarity to Eq. (41).

With the effective parameters for the three-dimensional theory fixed, one can use Eq. (50) to determine the approximate transition temperature T_c from broken to symmetric saddle as

$$\frac{m_{B,(3)}^2}{g_{B,(3)}^2} = \frac{3 \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T_c)^2}}{4\pi^2} \simeq -0.21, \quad (65)$$

implying $T_c \simeq 0.564 \Lambda_{\overline{\text{MS}}}$. The saddle-point conditions (60) in the high-temperature limit have the solutions

$$M = -\frac{2\pi T}{\ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2}} \left(1 \pm \sqrt{1 + \frac{1}{3} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2}} \right),$$

$$\tilde{M} = \frac{4\pi T}{\ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2}} \left(1 \pm \sqrt{1 - \frac{1}{6} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2}} \right), \quad (66)$$

where the only solution $\in \mathbb{R}^+$ is

$$\tilde{M} = \frac{4\pi T}{\ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2}} \left(1 - \sqrt{1 - \frac{1}{6} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{2\gamma_E}}{(4\pi T)^2}} \right). \quad (67)$$

In particular, the solution for the symmetric saddle mass M is complex valued, and the resulting pressure is complex valued as well, which has created much discussion in the recent literature [27–29, 32, 47, 48].

The existence of the real-valued broken saddle solution (67) implies a real-valued pressure for the high-temperature limit of negative coupling field theory, thereby resolving the issue of complex-valued pressure for the four-dimensional scalar field theory.

At zero temperature, the solution to the symmetric phase saddle-point condition (58) is

$$M^2 = \frac{4\pi^2 m_B^2}{3g_B W_0 \left(\frac{4m_B^2 \pi^2}{3e^1 g_B \Lambda_{\overline{\text{MS}}}^2} \right)}, \quad (68)$$

which implies $M \in \mathbb{R}^+$ for $\frac{m_B^2}{g_B \Lambda_{\overline{\text{MS}}}^2} \geq -\frac{3}{4\pi^2}$. For the broken phase, using the renormalization prescription (57) implies

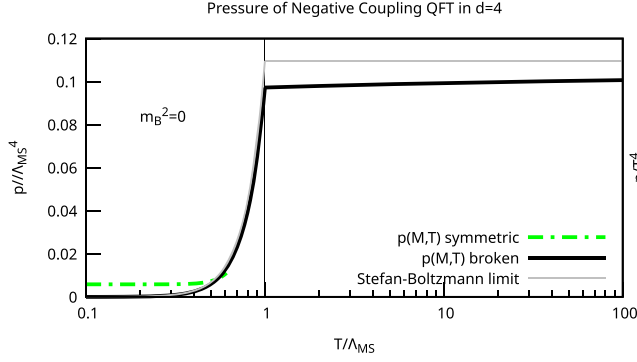


FIG. 5. Pressure as a function of temperature for $m_B^2 = 0$. Left part of the plot shows $\frac{p(M,T)}{\Lambda_{\text{MS}}^4}$; right part shows $\frac{p(M,T)}{T^4}$ to indicate approach to the Stefan-Boltzmann limit $p_{\text{SB}}(T) = \frac{\pi^2 T^4}{90}$ (gray line). Note that this different normalization leads to a fictitious cusp in the figure at $T = \Lambda_{\text{MS}}$. See text for details.

$$\tilde{M} = 0, \quad \forall \quad \frac{m_B^2}{g_B \Lambda_{\text{MS}}^2} = \text{finite}, \quad (69)$$

and also $\tilde{p}(\tilde{M}, T = 0) = 0$. This suggests that at low values of $\frac{m_B^2}{g_B \Lambda_{\text{MS}}^2}$, the broken phase is dominant, whereas for sufficiently high $\frac{m_B^2}{g_B}$, the symmetric phase is dominant. Increasing m_B^2 from $\frac{m_B^2}{g_B \Lambda_{\text{MS}}^2} = -\frac{3}{4\pi^2}$, one finds indeed that $\Delta p(M, \tilde{M}, T = 0) = 0$ at

$$\frac{m_B^2}{g_B \Lambda_{\text{MS}}^2} \simeq -0.06264, \quad (70)$$

suggesting a phase transition from the broken to symmetric saddle for this value of parameters.

At finite temperature and $m_B^2 = 0$, using Eq. (64) directly in Eq. (53) leads to two solutions for $\tilde{M}$, the smaller one of which reproduces Eq. (67) in the high-temperature limit. A plot for the pressure in the symmetric phase and broken phase tracking the smaller solution for $\tilde{M}$ is shown in Fig. 5.

As can be seen in Fig. 5, the pressure is well defined for all temperatures and approaches the Stefan-Boltzmann limit asymptotically from below. One finds $\Delta p(M, \tilde{M}, T_c) = 0$ for $T_c \simeq 0.543 \Lambda_{\text{MS}}$, which is well below the temperature $T \simeq 0.616 \Lambda_{\text{MS}}$ where the symmetric pole mass M becomes complex valued [28].

VII. SHORTCOMINGS AND POTENTIAL FUTURE DIRECTIONS

Since this work is intended as a survey of negative coupling field theory, it invariably falls short of providing a fully mature understanding of the subject. In particular,

interesting questions that are not addressed here include (but are not limited to):

- (i) Are the saddle-point approximations used in this survey sufficient, especially near phase transitions and in the high-temperature limit?
- (ii) Are the complex-valued saddle points an artifact of the approximation; i.e., will more sophisticated approximations result in real-valued saddle points for the same parameters?
- (iii) Which Riemann sheet is preferred for negative couplings, and on what basis should complex-valued saddle points be disregarded?
- (iv) For nonrelativistic bosons, cold-atom experiments find unstable behavior for negative quartic couplings [56]. Would the present analysis correctly describe the unstable behavior if adapted to the nonrelativistic case?

These questions go much beyond the survey character of the present study, and provide excellent starting points for follow-up work in future studies. However, incomplete answers to some of the above questions can already be gleaned from existing and ongoing work.

For instance, positive coupling field theory provides a good testing case for the reliability of the saddle-point approximations used in the present survey. A dedicated study on saddle-point approximations in $d = 2$ positive coupling field theory [57] found that the saddle-point approximations are quantitatively reliable for quantities such as the ground-state energy of the system, but become inaccurate at the 25% level on the location of the phase transition. Because of this study, it is reasonable to assume a similar performance of the saddle-point approximations for negative coupling field theory, which is indeed the case [50].

Perhaps surprisingly, compared to other approaches, the saddle-point approximations used in this work perform surprisingly well in the high-temperature limit, as is—for instance—evidenced by direct comparison with correlators from lattice Monte Carlo [58] obtained for positive coupling scalar field theory in $d = 4$; cf. Fig. 2 in Ref. [43].

Systematic improvements over the saddle-point approximations are possible within the framework of N-component scalar field theory. For positive coupling in $d = 4$ dimensions, the renormalizability and stability of the saddle-point approximation for the symmetric saddle has been established to next-to-next-to leading order in the large N expansion [41]. For $d = 3$, higher-order corrections in the large N expansion have been compared to critical exponents obtained through bootstrap methods [59]. For $d = 1$ (quantum mechanics), higher-order corrections have been compared to the ground-state energy obtained from Hamiltonian diagonalization [40]. These comparisons are in line with the expectation that the large N series is asymptotic, in complete analogy to the weak coupling expansions.

Carrying this program forward to include finite temperature properties and the broken phase saddle, as well as the

case of negative coupling field theory, is an exciting future research direction which warrants individual in-depth study.

There is no *a priori* reason why complex-valued saddle points should be discarded. However, in the present study, it was found in all cases that whenever complex-valued saddle points were present, there was a different phase saddle point that was real valued and had a lower free energy than the real part of the complex-valued saddle. This suggests the interpretation that complex-valued saddle points indicate the presence of a metastable phase in the system, whereas the true ground state is characterized by a real-valued lower free energy saddle.

The answer to which saddle points on which Riemann sheet are preferred can in principle be answered by constructing the respective Lefschetz thimbles and their intersection numbers. For the case of low-dimensional negative coupling field theory, this has been carried through in Ref. [29], where it was found that only the real-valued saddle points on the principal Riemann sheet gave a contribution. For the case of higher-dimensional quantum field theory with negative coupling, the corresponding construction of the thimbles and study of their intersection numbers is an interesting future research project. However, it should be pointed out that an upcoming study finds that the real-valued saddle points on the principal Riemann sheet used in this survey correctly capture the ground-state energy of $d = 2$ negative coupling field theory calculated via direct Hamiltonian diagonalization [50].

The reason for the burst of atoms observed in experiments conducted for nonrelativistic bosonic atoms in Ref. [56], in particular the role of three-body losses, has been debated extensively in the past; see, e.g., Refs. [60,61]. These experiments are very interesting, because they allow changing the sign of the coupling (in this case scattering length) by varying the magnetic field across a Feshbach resonance; see, e.g., Ref. [62]. For nonrelativistic fermionic atoms, it has been known for some time that saddle-point expansions very similar to those for the relativistic case can be used to describe the experimental findings; see, e.g., Refs. [40,63]. Whether or not saddle-point expansions for nonrelativistic bosonic atoms are able to correctly describe the burst of atoms found in the experimental setup of Ref. [56] is a question that requires a dedicated study in the future.

VIII. SUMMARY AND CONCLUSIONS

In this work, I have considered scalar field theory with negative quartic coupling using saddle-point expansions in various dimensions. It was found that the analytic saddle-point expansions predict an interesting phase diagram structure for the continuum theory, which persists at finite temperature. For the case of quantum mechanics at negative coupling, which is alternatively accessible through its isospectral Hermitian equivalent [44], the saddle-point expansions were found to be quantitatively reliable, agreeing with the numerically diagonalized Hamiltonian on the level of 15%. In two dimensions, the saddle-point expansions point to a phase transition at zero temperature that may be probed by numerical methods in the near future, such as brute-force numerical integration on the lattice [25]. At finite temperature, the issue of complex-valued pressure reported previously for negative coupling field theory [32] is resolved through a phase transition to the broken phase saddle, leading to a well-defined phase structure of the theory for all temperatures.

For three dimensions, the saddle-point expansions likewise suggest a phase transition at zero temperature located near bare parameter values $\frac{m_B^2}{g_B} \simeq -0.21$, and similarly a well-defined phase structure for the theory at finite temperature.

The case of four dimensions is considerably more complex, but the dimensionally reduced theory at high temperatures points to a well-defined real-valued pressure function described by the broken phase saddle, while at zero temperature and positive m_B^2 , the dominant thermodynamic configuration is described by the symmetric phase saddle. Nevertheless, the fact that negative coupling exploits a known loophole in quantum triviality proofs [51–53] and suggests asymptotic freedom for scalars [1,39] makes further study of this case particularly interesting for the future.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

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