

Improved perturbative QCD study of the decay $B_c^+ \rightarrow \eta_c L^+$ Wen-Jing Zhang and Xin Liu *Department of Physics, Jiangsu Normal University, Xuzhou 221116, China*

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We perform an improved perturbative QCD study of the decays $B_c^+ \rightarrow \eta_c L^+$, where L denotes the light ground-state pseudoscalar, vector mesons and the corresponding p -wave scalar, axial-vector, and tensor ones, and predict their branching ratios (BRs) associated with relative ratios at leading order in the strong coupling α_s . Our results $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+) = (2.03_{-0.41}^{+0.53}) \times 10^{-3}$ and $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)/\text{BR}(B_c^+ \rightarrow J/\psi \pi^+) = 1.74_{-0.50}^{+0.66}$ are consistent with several available predictions in different approaches within uncertainties. Inputting the measured $\eta_c \rightarrow p\bar{p}$ and $\eta_c \rightarrow \pi^+\pi^-(\pi^+\pi^-, K^+K^-, p\bar{p})$ BRs with p here being a proton, we derive the multibody $B_c^+ \rightarrow \eta_c(\pi, \rho)^+$ BRs through secondary decay chains via resonance η_c under the narrow-width approximation, which might facilitate the (near-)future tests of $B_c \rightarrow \eta_c$ decays. Under the $q\bar{q}$ assignment for light scalars, different to B_c decaying into J/ψ plus a scalar meson and other $B_c^+ \rightarrow \eta_c L^+$ modes, surprisingly small $\Delta S = 0$ BRs around $\mathcal{O}(10^{-7} - 10^{-9})$ and highly large ratios near $\mathcal{O}(10^2)$ between the $\Delta S = 1$ and $\Delta S = 0$ BRs are found in the B_c decays to η_c plus light scalars, with S being strange number. Many large BRs and interesting ratios presented in this work could be tested by the Large Hadron Collider experiments, which would help us to examine the reliability of this improved perturbative QCD formalism for B_c -meson decays and further understand the QCD dynamics in the considered decay modes as well as in the related hadrons.

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I. INTRODUCTION

It is well known that, disparate from the $b\bar{b}$ and $c\bar{c}$ states, the B_c meson is flavor antisymmetric while unique since it is the only ground state containing two different heavy quarks b and c simultaneously [1,2]. Peculiar to the more extensively studied B_u , B_d , and B_s mesons, both constituents in a B_c meson can decay separately, which offers a precious opportunity to expand our understanding of heavy B -meson physics through thoroughly exploring the more rich and complicated QCD in the perturbative and non-perturbative regimes. Thus, the weak decays of the B_c meson have prompted significant theoretical and experimental interest since its first discovery at the Tevatron in 1998 [3,4].

On the experimental side, the LHC experiments that started running in 2009 have observed many B_c -meson decay channels of interest, such as $B_c^+ \rightarrow J/\psi(\pi, 2\pi, 3\pi)^+$, even $B_c^+ \rightarrow \chi_{cJ}\pi^+(J = 0, 1, 2)$, etc. [5,6]. The η_c meson, the lowest-lying $c\bar{c}$ pseudoscalar state, has attracted considerable theoretical and experimental attention since its

discovery [7]. It decays primarily via $c\bar{c}$ annihilation into two gluons and is expected to have numerous hadronic decay modes into two- or three-body hadronic charged and/or neutral final states, which, unfortunately, seems not friendly for experimental studies at LHC experiments. Hence, though struggling against the large background and the small efficiency at LHC for these problems, the $B_c^+ \rightarrow \eta_c \pi^+$ decay is not yet observed presently. However, along with the successful upgrade and resumed running of Large Hadron Collider-beauty (LHCb) detector since 2022, the ensuingly exciting results are forthcoming [8]. The continuously collected data from the upgraded LHCb detector will offer a precious opportunity to advance B_c -meson physics into a precision era. It is therefore expected that a huge amount of data with superior quality can facilitate a promising measurement of $B_c^+ \rightarrow \eta_c \pi^+$, for example, via $\eta_c \rightarrow p\bar{p}$ decay chain. Here, p denotes a proton.

On the theoretical side, the $B_c^+ \rightarrow \eta_c \pi^+$ decay, as well as $B_c^+ \rightarrow \eta_c \rho^+$, has been investigated in different approaches [9–24], while the branching ratios (BRs) differ with a wide range of magnitude, notably, $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+) \in [0.25, 4.22] \times 10^{-3}$ and $\text{BR}(B_c^+ \rightarrow \eta_c \rho^+) \in [0.67, 13.16] \times 10^{-3}$, respectively. These numerical results imply different understanding of QCD dynamics in the $B_c \rightarrow \eta_c$ decays. The remarkable discrepancies suggest that our understanding of the involved dynamics is far from complete and more investigations need to be carried out in alternative

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approaches necessarily. Motivated by the general consistency between the LHC measurements [25–28] and the improved perturbative QCD (iPQCD) predictions [29,30] about the ratios among the BRs of $B_c^+ \rightarrow J/\psi\pi^+$, $B_c^+ \rightarrow J/\psi\rho^+(\rightarrow\pi^+\pi^0)$, $B_c^+ \rightarrow J/\psi a_1(1260)(\rightarrow\pi^+\pi^-\pi^+)$, $B_c^+ \rightarrow \chi_{c1}(1P)\pi^+$, and $B_c^+ \rightarrow \chi_{c2}(1P)\pi^+$, the present work will concentrate on the decays $B_c^+ \rightarrow \eta_c L^+$, where L denotes the light mesons such as pseudoscalars (P)— π and K ; vectors (V)— ρ and K^* ; axial vectors (A)— $a_1(1260)$, $b_1(1235)$ and $K_1(1270, 1400)$; scalars (S)— $a_0(980, 1450)$ and $K_0^*(700, 1430)$ [$K_0^*(700)$ also known as κ]; and tensors (T)— $a_2(1320)$ and $K_2^*(1430)$, respectively.

To the best of our knowledge, the nature of light scalars, especially those under or near 1 GeV, remains a long-standing puzzle in hadron physics. Recently, the CMS Collaboration found strong evidence of $f_0(980)$ being a normal quark-antiquark state [31], even though the ALICE Collaboration supported the $K_0^*(700)$ being a four-quark state [32]. Undoubtedly, more endeavors need to be devoted to this field. The useful clues about the nature of light scalars could be collected indirectly through probing their productions in the heavy hadron decays, just like the $B \rightarrow f_0(980)K$ decays observed in the B -factory experiments [33,34]. So far, there are two different scenarios to describe these scalar mesons $a_0(980, 1450)$ and $K_0^*(700, 1430)$ under the $q\bar{q}$ assignment [35]. In scenario 1 ($S1$), the scalar mesons $a_0(980)$ and $K_0^*(700)$ are treated as the lowest-lying states, and those $a_0(1450)$ and $K_0^*(1430)$ are the first excited states correspondingly. And, in scenario 2 ($S2$), the scalar mesons $a_0(1450)$ and $K_0^*(1430)$ are viewed as the ground states, while those $a_0(980)$ and $K_0^*(700)$ might be the four-quark states. However, as stressed in Ref. [35], it is difficult in practice to make quantitative predictions based on the four-quark or tetra-quark picture for light scalars because the calculations of decay constant and form factors of light scalars are beyond the conventional quark model and the involved nonfactorizable contributions cannot be calculated in the available QCD-based factorization framework. Moreover, the productions of light scalars from the vacuum in the $B_c^+ \rightarrow \eta_c S^+$ decays are expected to be highly suppressed originating from the nearly zero vector decay constants f_S [actually, the vector decay constants $f_S = 0$ in the $SU(3)$ limit] [35]. It means that investigations on these decays must go beyond naive factorization. Therefore, the predictions in this work are made on the basis of two-quark model for light scalars within the iPQCD framework.

The p -wave light axial vectors have been investigated at both experimental and theoretical aspects. However, our understanding about their nature is still far from complete [36]. In the spectroscopy study [5], $a_1(1260)$ and $b_1(1235)$ are the 1^3P_1 and 1^1P_1 axial-vector states, respectively, carrying quantum numbers $J^{PC} = 1^{++}$ and $J^{PC} = 1^{+-}$ correspondingly. Although it is very interesting that the strange $K_1(1270)$ and $K_1(1400)$ mesons are generally

regarded as the mixtures of 1^3P_1 state K_{1A} and 1^1P_1 state K_{1B} due to the $SU(3)$ flavor broken symmetry [37],

$$\begin{pmatrix} |K_1(1270)\rangle \\ |K_1(1400)\rangle \end{pmatrix} = \begin{pmatrix} \sin\theta_K & \cos\theta_K \\ \cos\theta_K & -\sin\theta_K \end{pmatrix} \begin{pmatrix} |K_{1A}\rangle \\ |K_{1B}\rangle \end{pmatrix}, \quad (1)$$

with mixing angle θ_K . The value of θ_K can be related to the masses of the $K_1(1270)$ and $K_1(1400)$, to the strong decays of the $K_1(1270)$ and $K_1(1400)$, and to rates of weak decays to final states involving the $K_1(1270)$ and $K_1(1400)$ [38,39]. Thus, the decays such as $B_c^+ \rightarrow (c\bar{c})K_1(1270, 1400)^+$ would be of great interest for exploring the information of θ_K . However, presently, there is no consensus on the value of the mixing angle θ_K , and the results from various approaches are still quite controversial; e.g., see a short overview in Ref. [40] (and references therein). We therefore take both referenced values, i.e., $\theta_{K_1} \approx 33^\circ$ and 58° [37,41] into account in the related numerical calculations of this work.

By incorporating the finite charm quark mass effects into Sudakov resummation of the large logarithmic corrections to wave functions through k_T resummation at the next-to-leading-logarithm accuracy [42], besides including them in the hard kernel, the iPQCD formalism is now self-consistent for systematically studying the B_c -meson decays and B -meson decaying into charmonia. So far, facilitated by the newly proposed transverse-momentum-dependent B_c -meson wave function [43], we have studied the $B_c^+ \rightarrow J/\psi M^+$ [29] and $B_c^+ \rightarrow \chi_{cJ}(P, V)^+$ [30] decays, in which, the BRs predicted in the iPQCD formalism are generally consistent with several available predictions in other approaches. Furthermore, the resultant relative ratios such as $\text{BR}(B_c^+ \rightarrow J/\psi a_1(1260)^+(\rightarrow\pi^+\pi^-\pi^+))/\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)$, $\text{BR}(B_c^+ \rightarrow J/\psi\rho^+(\rightarrow\pi^+\pi^0))/\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)$, $\text{BR}(B_c^+ \rightarrow \chi_{c2}\pi^+)/\text{BR}(B_c^+ \rightarrow J/\psi\pi^+)$, etc. agree well with the current data reported by the LHC experiments within theoretical uncertainties. In light of these successful outputs, we shall analyze the decays $B_c^+ \rightarrow \eta_c L^+$ in the iPQCD formalism at leading order in the strong coupling α_s [42,43]. Inputting the BRs [5] of strong decays $\eta_c \rightarrow p\bar{p}$ and $\eta_c \rightarrow \pi^+\pi^-(\pi^+\pi^-, K^+K^-, p\bar{p})$, we additionally present the BRs of multibody modes arising from $B_c^+ \rightarrow \eta_c(\pi, \rho)^+$ via resonance η_c under the narrow-width approximation. The (near)-future tests of our iPQCD predictions at experimental facilities are expected to help us to further understand the QCD dynamics involved in these B_c -meson decays, and even explore the inner structure of η_c and related light hadrons.

The rest of this paper is organized as follows. In Sec. II, the formalism and the perturbative calculations in association with factorization formulas of $B_c^+ \rightarrow \eta_c L^+$ are presented. The numerical results and phenomenological analyses are given in Sec. III. Section IV summarizes our main conclusions.

II. FORMALISM AND PERTURBATIVE CALCULATIONS

For the $B_c^+ \rightarrow \eta_c L^+$ decays, the related weak effective Hamiltonian H_{eff} at the quark level can be written as [44]

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \{V_{cb}^* V_{uq} [C_1(\mu) O_1(\mu) + C_2(\mu) O_2(\mu)]\} + \text{H.c.}, \quad (2)$$

with the Fermi constant $G_F = 1.16639 \times 10^{-5} \text{ GeV}^{-2}$, the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements V , and the Wilson coefficients $C_i(\mu)$ at the renormalization scale μ . The local four-quark tree operators O_1 and O_2 are read as

$$\begin{aligned} O_1 &= \bar{q}_\alpha \gamma_\mu (1 - \gamma_5) u_\beta \bar{c}_\beta \gamma_\mu (1 - \gamma_5) b_\alpha, \\ O_2 &= \bar{q}_\alpha \gamma_\mu (1 - \gamma_5) u_\alpha \bar{c}_\beta \gamma_\mu (1 - \gamma_5) b_\beta, \end{aligned} \quad (3)$$

where q denotes the light down quark $d(s)$ for the CKM-favored (-suppressed) $\Delta S = 0(\Delta S = 1)$ processes with S being strange number.

Similar to $B_c^+ \rightarrow J/\psi M^+$ decays [29], the kinematics of $B_c^+ \rightarrow \eta_c L^+$ could be defined in the light-cone coordinates as

$$\begin{aligned} P_1 &= \frac{m_{B_c}}{\sqrt{2}} (1, 1, \mathbf{0}_T), & P_2 &= \frac{m_{B_c}}{\sqrt{2}} (1 - r_3^2, r_2^2, \mathbf{0}_T), \\ P_3 &= \frac{m_{B_c}}{\sqrt{2}} (r_3^2, 1 - r_2^2, \mathbf{0}_T), \end{aligned} \quad (4)$$

where the ratios $r_2 = m_L/m_{B_c}$ and $r_3 = m_{\eta_c}/m_{B_c}$ with $m_L(m_{\eta_c})$ being the light (η_c) meson mass, P_1 denotes the momentum carried by B_c meson in its rest frame, and P_2 and P_3 denotes the momenta carried by L and η_c mesons moving along the plus and minus z directions, respectively. Thanks to conservation of the angular momentum, the possible polarization vectors ϵ_{2L} (here, the subscript L stands for the longitudinal polarization, not to be confused with the abbreviation L of light mesons) could be easily derived through the constraints $P_2 \cdot \epsilon_{2L} = 0$ and $\epsilon_{2L}^2 = -1$. Notice that, if L is a tensor meson, then a new longitudinal polarization vector could be constructed similarly relative to those of vectors and axial vectors but with an additional factor $\sqrt{2/3}$ [29,45–47]. Then, specifically, $\epsilon_{2L} = \frac{1}{\sqrt{2(1-r_3^2)r_2}} (1 - r_3^2, -r_2^2, \mathbf{0}_T)$ for vectors and axial vectors, while $\epsilon_{2L} = \frac{1}{\sqrt{3(1-r_3^2)r_2}} (1 - r_3^2, -r_2^2, \mathbf{0}_T)$ for tensors in this work. The momenta of valence quarks in the initial- and final-state mesons are parametrized as

$$\begin{aligned} k_1 &= (x_1 P_1^+, x_1 P_1^-, \mathbf{k}_{1T}), & k_2 &= (x_2 P_2^+, x_2 P_2^-, \mathbf{k}_{2T}), \\ k_3 &= (x_3 P_3^+, x_3 P_3^-, \mathbf{k}_{3T}), \end{aligned} \quad (5)$$

where x_i ($i = 1, 2, 3$) is the corresponding momentum fraction.

The $B_c^+ \rightarrow \eta_c L^+$ decay amplitude in the iPQCD formalism can therefore be conceptually written as

$$\begin{aligned} A(B_c^+ \rightarrow \eta_c L^+) &\sim \int dx_1 dx_2 dx_3 b_1 db_1 b_2 db_2 b_3 db_3 \\ &\cdot \text{Tr}[C(t) \Phi_{B_c}(x_1, b_1) \Phi_L(x_2, b_2) \\ &\times \Phi_{\eta_c}(x_3, b_3) H(x_i, b_i, t) e^{-S(t)}], \end{aligned} \quad (6)$$

where b_i is the conjugate space coordinate of transverse momentum k_{iT} , t is the largest running energy scale in hard kernel $H(x_i, b_i, t)$, Tr denotes the trace over Dirac and $SU(3)$ color indices, $C(t)$ stands for the Wilson coefficients including the large logarithms $\ln(m_W/t)$ [48], and Φ is the wave function describing the hadronization of quark and antiquark to the meson. The Sudakov factor $e^{-S(t)}$ arises from k_T resummation, which provides a strong suppression on the long distance contributions in the small k_T (or large b) region [49]. The detailed discussions for $e^{-S(t)}$ can be easily found in the original Refs. [42,43,49]. Thus, with Eq. (6), we can give the convoluted amplitudes of the decays $B_c^+ \rightarrow \eta_c L^+$ explicitly through the evaluations of hard kernel $H(x_i, b_i, t)$ at leading order in the α_s expansion within the iPQCD formalism.

The wave function for B_c meson with a “heavy-light” structure can generally be defined as [43,48,50]

$$\Phi_{B_c}(x, \mathbf{k}_T) = \frac{i}{\sqrt{2N_c}} \{(\not{P} + m_{B_c}) \gamma_5 \phi_{B_c}(x, \mathbf{k}_T)\}, \quad (7)$$

where P is the momentum of B_c meson, $N_c = 3$ is the color factor, and x and $\mathbf{k}_T$ are the momentum fraction and intrinsic transverse momentum of charm quark in the B_c meson. Here, $\phi_{B_c}(x, \mathbf{k}_T)$ is the B_c -meson leading-twist distribution amplitude, whose explicit form in the impact $\mathbf{b}$ space is [43]

$$\begin{aligned} \phi_{B_c}(x, \mathbf{b}) &= \frac{f_{B_c}}{2\sqrt{2N_c}} N_{B_c} x(1-x) \exp \left[-\frac{(1-x)m_c^2 + x m_b^2}{8\beta_{B_c}^2 x(1-x)} \right] \\ &\times \exp[-2\beta_{B_c}^2 x(1-x)\mathbf{b}^2], \end{aligned} \quad (8)$$

with the decay constant $f_{B_c} = 0.489 \pm 0.005 \text{ GeV}$ [51], the shape parameter $\beta_{B_c} = 1.0 \pm 0.1 \text{ GeV}$ [43], and m_c and m_b being the charm and bottom quark masses. Moreover, the normalization constant N_{B_c} is fixed by the following relation:

$$\int_0^1 \phi_{B_c}(x, \mathbf{b} = 0) dx \equiv \int_0^1 \phi_{B_c}(x) dx = \frac{f_{B_c}}{2\sqrt{2N_c}}. \quad (9)$$

For the η_c meson, its wave function has been studied within the nonrelativistic QCD approach [52] and

derived as

$$\Phi_{\eta_c}(x) = \frac{i}{\sqrt{2N_c}} \gamma_5 \{ \not{P} \phi_{\eta_c}^v(x) + m_{\eta_c} \phi_{\eta_c}^s(x) \}, \quad (10)$$

with P and m being the momentum and mass of η_c and x describing the charm-quark momentum fraction in $\phi_{\eta_c}^v(x)$ and $\phi_{\eta_c}^s(x)$ the twist-2 and twist-3 distribution amplitudes,

$$\begin{aligned} \phi_{\eta_c}^v(x) &= 9.58 \frac{f_{\eta_c}}{2\sqrt{2N_c}} x(1-x) \mathcal{C}(x), \\ \phi_{\eta_c}^s(x) &= 1.97 \frac{f_{\eta_c}}{2\sqrt{2N_c}} \mathcal{C}(x), \end{aligned} \quad (11)$$

where $f_{\eta_c} = 0.387 \pm 0.007$ [53] is the decay constant and the function $\mathcal{C}(x)$ reads

$$\mathcal{C}(x) = [(x(1-x))/(1-4x(1-x)(1-v^2))]^{1-v^2}, \quad (12)$$

with $v^2 = 0.3$ standing for small relativistic corrections to the Coulomb wave functions.

The light-cone wave functions including distribution amplitudes for light pseudoscalars, scalars, vectors, axial vectors, and tensors calculated in the QCD sum rules up to twist 3 have been collected in Ref. [29] (and references therein). For simplicity, their explicit expressions would no longer be presented in this work. The readers could refer to Ref. [29] for detail.

The leading-order Feynman diagrams for the decays $B_c^+ \rightarrow \eta_c L^+$ in the iPQCD formalism are shown in Fig. 1. Similar to those in Ref. [29], we use F_e and M_e to describe the factorizable emission and the nonfactorizable emission amplitudes induced by the $(V-A)(V-A)$ operators. The $B_c^+ \rightarrow \eta_c L^+$ decay amplitude can thus be decomposed into

$$A(B_c^+ \rightarrow \eta_c L^+) = V_{cb}^* V_{uq} (F_e \cdot f_L + M_e), \quad (13)$$

in which the related factorization formulas are given explicitly as follows:

(i) For $B_c^+ \rightarrow \eta_c(P, S)^+$ decays,

$$\begin{aligned} F_e(P) &= -8\pi C_F m_{B_c}^4 \int_0^1 dx_1 dx_3 \int_0^\infty b_1 db_1 b_3 db_3 \phi_{B_c}(x_1, b_1) (r_3^2 - 1) \\ &\quad \times \{ [r_3(r_b + 2x_3 - 2) \phi_{\eta_c}^s(x_3) - (2r_b + x_3 - 1) \phi_{\eta_c}^v(x_3)] h_a(x_1, x_3, b_1, b_3) E_f(t_a) \\ &\quad - [2r_3(1 + r_c - x_1) \phi_{\eta_c}^s(x_3) + (r_3^2(x_1 - 1) - r_c) \phi_{\eta_c}^v(x_3)] h_b(x_1, x_3, b_1, b_3) E_f(t_b) \}, \end{aligned} \quad (14)$$

where the ratios $r_b = m_b/m_{B_c}$ and $r_c = m_c/m_{B_c}$. The hard function $h_i(x_i, b_i)$ and the evolution function $E_f(t_i)$ could refer to those expressions in Ref. [29],

$$\begin{aligned} M_e(P) &= -\frac{32}{\sqrt{6}} \pi C_F m_{B_c}^4 \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_1 db_1 b_2 db_2 \phi_{B_c}(x_1, b_1) \phi_P^A(x_2) (r_3^2 - 1) \\ &\quad \times \{ [r_3(x_3 - x_1) \phi_{\eta_c}^s(x_3) + (x_1 + x_2 - 1) \phi_{\eta_c}^v(x_3) + r_3^2(x_1 - x_2 - 2x_3 + 1) \phi_{\eta_c}^v(x_3)] \\ &\quad \times E_f(t_c) h_c(x_1, x_2, x_3, b_1, b_2) + [(x_2 + x_3 - 2x_1 + r_3^2(x_3 - x_2)) \phi_{\eta_c}^v(x_3) \\ &\quad + r_3(x_1 - x_3) \phi_{\eta_c}^s(x_3)] h_d(x_1, x_2, x_3, b_1, b_2) E_f(t_d) \}, \end{aligned} \quad (15)$$

and

$$F_e(S) = -F_e(P), \quad M_e(S) = -M_e(P), \quad (16)$$

but with the corresponding replacement of $\phi_P^A(x) \rightarrow \phi_S(x)$ in Eq. (16).

(ii) For $B_c^+ \rightarrow \eta_c(V, A, T)^+$ decays,

$$\begin{aligned} F_e(V) &= 8\pi C_F m_{B_c}^4 \int_0^1 dx_1 dx_3 \int_0^\infty b_1 db_1 b_3 db_3 \phi_{B_c}(x_1, b_1) \sqrt{1 - r_3^2} \\ &\quad \times \{ [r_3(r_b + 2x_3 - 2) \phi_{\eta_c}^s(x_3) - (2r_b + x_3 - 1) \phi_{\eta_c}^v(x_3)] h_a(x_1, x_3, b_1, b_3) E_f(t_a) \\ &\quad - [2r_3(1 + r_c - x_1) \phi_{\eta_c}^s(x_3) + (r_3^2(x_1 - 1) - r_c) \phi_{\eta_c}^v(x_3)] h_b(x_1, x_3, b_1, b_3) E_f(t_b) \}, \end{aligned} \quad (17)$$

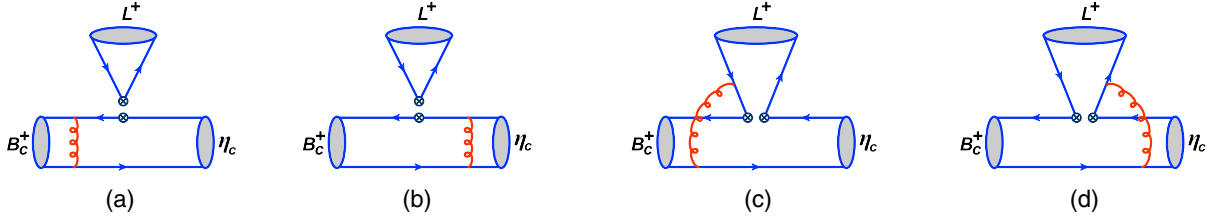


FIG. 1. Leading-order Feynman diagrams for the decays $B_c^+ \rightarrow \eta_c L^+$ in the iPQCD formalism: (a) and (b) [(c) and (d)] contributing to the (non)factorizable-emission amplitudes.

$$\begin{aligned}
 M_e(V) = & \frac{32}{\sqrt{6}} \pi C_F m_{B_c}^4 \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty db_1 db_2 db_3 \phi_{B_c}(x_1, b_1) \phi_V(x_2) \sqrt{1 - r_3^2} \\
 & \times \{ [(x_1 + x_2 - 1) \phi_{\eta_c}^v(x_3) + r_3^2(x_1 - x_2 - 2x_3 + 1) \phi_{\eta_c}^v(x_3) + r_3(x_3 - x_1) \phi_{\eta_c}^s(x_3)] \\
 & \times E_f(t_c) h_c(x_1, x_2, x_3, b_1, b_2) + [(x_2 + x_3 - 2x_1 + r_3^2(x_3 - x_2)) \phi_{\eta_c}^v(x_3) \\
 & + r_3(x_1 - x_3) \phi_{\eta_c}^s(x_3)] h_d(x_1, x_2, x_3, b_1, b_2) E_f(t_d) \}, \quad (18)
 \end{aligned}$$

and

$$\begin{aligned}
 F_e(A) = -F_e(V), \quad F_e(T) = 0, \\
 M_e(A) = -M_e(V), \quad M_e(T) = \sqrt{\frac{2}{3}} M_e(V). \quad (19)
 \end{aligned}$$

associated with the replacements of $\phi_V(x) \rightarrow \phi_{A,T}(x)$ in (19) correspondingly.

The corresponding BR is then given, in the rest frame of a heavy B_c meson, by

$$\begin{aligned}
 \text{BR}(B_c^+ \rightarrow \eta_c L^+) & \equiv \tau_{B_c} \cdot \Gamma(B_c^+ \rightarrow \eta_c L^+) \\
 & = \tau_{B_c} \cdot \frac{G_F^2 |\mathbf{P}_c|^2}{16\pi m_{B_c}^2} |A(B_c^+ \rightarrow \eta_c L^+)|^2, \quad (20)
 \end{aligned}$$

with the B_c -meson lifetime τ_{B_c} and the decay width Γ . Note that $|\mathbf{P}_c| \equiv |\mathbf{P}_L| = |\mathbf{P}_{\eta_c}| = \sqrt{\lambda(m_{B_c}^2, m_L^2, m_{\eta_c}^2)/(2m_{B_c})}$ is the momentum of either L or η_c meson in the final states, with the Källén function $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$ [5].

III. NUMERICAL RESULTS AND DISCUSSIONS

In numerical calculations, central values of the input parameters will be used implicitly unless otherwise stated. We adopt the relevant QCD scale (GeV), masses (GeV), and B_c -meson lifetime (ps) [5,48],

$$\begin{aligned}
 \Lambda_{\overline{\text{MS}}}^{(f=4)} &= 0.250, & m_W &= 80.41, \\
 m_{B_c} &= 6.275, & m_{\eta_c} &= 2.98, \\
 \tau_{B_c} &= 0.507, & m_b &= 4.8, & m_c &= 1.5. \quad (21)
 \end{aligned}$$

and the CKM matrix elements [5],

$$\begin{aligned}
 |V_{cb}| &= 0.04182_{-0.00074}^{+0.00085}, & |V_{ud}| &= 0.97435 \pm 0.00016, \\
 |V_{us}| &= 0.22500 \pm 0.00067. \quad (22)
 \end{aligned}$$

In the following context, we will classify the considered decays into two different groups conventionally, that is, the decays $B_c^+ \rightarrow \eta_c(P, V)^+$ and the decays $B_c^+ \rightarrow \eta_c(A, S, T)^+$, respectively, to present the related iPQCD predictions and phenomenological insights.

A. $B_c^+ \rightarrow \eta_c(P, V)^+$

The CP -averaged $B_c^+ \rightarrow \eta_c(\pi, K)^+$ BRs in the iPQCD formalism are presented as

$$\begin{aligned}
 \text{BR}(B_c^+ \rightarrow \eta_c \pi^+) &= 2.03_{-0.40}^{+0.51} (\beta_{B_c})_{-0.08}^{+0.09} (f_M)_{-0.00}^{+0.00} \\
 &\times (a_\pi)_{-0.07}^{+0.09} (V_{cb}) \times 10^{-3}, \quad (23)
 \end{aligned}$$

$$\begin{aligned}
 \text{BR}(B_c^+ \rightarrow \eta_c K^+) &= 1.52_{-0.29}^{+0.39} (\beta_{B_c})_{-0.06}^{+0.07} (f_M)_{-0.10}^{+0.11} \\
 &\times (a_K)_{-0.05}^{+0.07} (V_{cb}) \times 10^{-4}, \quad (24)
 \end{aligned}$$

where the uncertainties are dominated by the shape parameter β_{B_c} from the B_c -meson distribution amplitude. Note that, though having a smaller decay constant $f_{\eta_c} \sim 0.387$ than $f_{J/\psi} \sim 0.405$ and the same leading-twist distribution amplitudes in both η_c and J/ψ , the $B_c^+ \rightarrow \eta_c \pi^+$ BR is still larger than the $B_c^+ \rightarrow J/\psi \pi^+$ one in the iPQCD formalism. In fact, from the numerical results of decay amplitudes shown in Table I, it is found that $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)$ and $\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)$ are governed by highly different QCD dynamics, that is, the former (latter) determined by the contributions arising from the distribution amplitude of twist-3 $\phi_{\eta_c}^s(x)$ [twist-2 $\phi_{J/\psi}^L(x)$] of $\eta_c(J/\psi)$, even with enhancement (reduction) by that of twist-2 $\phi_{\eta_c}^v(x)$ [twist-3 $\phi_{J/\psi}^t(x)$]. Any precise measurements in various experiments to help understand the QCD behavior of these $c\bar{c}$ -mesons in depth are urgently demanded.

TABLE I. Decay amplitudes (in units of 10^{-3} GeV^{-3}) of $B_c^+ \rightarrow \eta_c \pi^+$ and $B_c^+ \rightarrow J/\psi \pi^+$ from different twists in the iPQCD formalism. For simplicity, only the central values are quoted for clarifications.

Modes	Decay amplitudes (F_e)		Decay amplitudes (M_e)	
$B_c^+ \rightarrow \eta_c \pi^+$	$\phi_{\eta_c}^v(x)$ 8.43 - i8.56	$\phi_{\eta_c}^s(x)$ -26.42 - i113.08	$\phi_{\eta_c}^v(x)$ 1.00 - i2.61	$\phi_{\eta_c}^s(x)$ 0.01 - i0.01
$B_c^+ \rightarrow J/\psi \pi^+$	$\phi_{J/\psi}^L(x)$ 30.76 + i101.07	$\phi_{J/\psi}^t(x)$ -12.83 - i9.09	$\phi_{J/\psi}^L(x)$ -1.85 + i4.21	$\phi_{J/\psi}^t(x)$ -0.24 - i0.92

The ratio between $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)$ and $\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)$ in the iPQCD framework could help measure the mode $B_c^+ \rightarrow \eta_c \pi^+$ experimentally,

$$R_{\eta_c/J/\psi}^\pi \equiv \frac{\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 1.74_{-0.50}^{+0.66}, \quad (25)$$

According to the LHCb data [54], that is, $\frac{\sigma(B_c^+)}{\sigma(B^+)} \times \text{BR}(B_c^+ \rightarrow J/\psi \pi^+) = (6.97 \pm 0.13) \times 10^{-6}$ [30] in the fiducial region corresponding to the transverse momentum $0 < p_T < 20 \text{ GeV}$ and the rapidity $2.0 < y < 4.5$, and the assumption $\text{BR}(B_c^+ \rightarrow J/\psi \pi^+) \sim \mathcal{O}(10^{-3})$, then the forthcoming measurements of $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)$ in the LHCb experiment might be

$$\frac{\sigma(B_c^+)}{\sigma(B^+)} \times \text{BR}(B_c^+ \rightarrow \eta_c \pi^+) \sim (1.41_{-0.29}^{+0.37}) \times 10^{-5}. \quad (26)$$

Based on the data for η_c strong decays to stable hadrons [5], one can derive the multibody B_c -meson decays via η_c resonance under the narrow-width approximation:

- (1) $\eta_c \rightarrow p\bar{p}$ and $p\bar{p}\pi^+\pi^-$. The inputs of $\eta_c \rightarrow p\bar{p}$ and $p\bar{p}\pi^+\pi^-$ decays with

$$\begin{aligned} \mathcal{B}(\eta_c \rightarrow p\bar{p}) &= (1.33 \pm 0.11) \times 10^{-3}, \\ \mathcal{B}(\eta_c \rightarrow p\bar{p}\pi^+\pi^-) &= (3.7 \pm 0.5) \times 10^{-3} \end{aligned} \quad (27)$$

lead to the three-body $B_c^+ \rightarrow \pi^+ \eta_c (\rightarrow p\bar{p})$ and five-body $B_c^+ \rightarrow \pi^+ \eta_c (\rightarrow p\bar{p}\pi^+\pi^-)$ BRs as

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \pi^+ \eta_c (\rightarrow p\bar{p})) & \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \pi^+) \cdot \mathcal{B}(\eta_c \rightarrow p\bar{p}) & \\ = (2.70_{-0.59}^{+0.74}) \times 10^{-6}, & \end{aligned} \quad (28)$$

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \pi^+ \eta_c (\rightarrow p\bar{p}\pi^+\pi^-)) & \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \pi^+) \cdot \mathcal{B}(\eta_c \rightarrow p\bar{p}\pi^+\pi^-) & \\ = (7.51_{-1.83}^{+2.21}) \times 10^{-6}. & \end{aligned} \quad (29)$$

- (2) $\eta_c \rightarrow 2(\pi^+\pi^-)$, $2(K^+K^-)$ and $\pi^+\pi^-K^+K^-$. The inputs of $\eta_c \rightarrow 2(\pi^+\pi^-)$, $\pi^+\pi^-K^+K^-$ and $2(K^+K^-)$ decays with

$$\begin{aligned} \mathcal{B}(\eta_c \rightarrow 2(\pi^+\pi^-)) &= (9.6 \pm 1.5) \times 10^{-3}, \\ \mathcal{B}(\eta_c \rightarrow 2(K^+K^-)) &= (1.4 \pm 0.4) \times 10^{-3}, \\ \mathcal{B}(\eta_c \rightarrow \pi^+\pi^-K^+K^-) &= (8.3 \pm 1.8) \times 10^{-3} \end{aligned} \quad (30)$$

result in the five-body $B_c^+ \rightarrow \pi^+ \eta_c (\rightarrow 2(\pi^+\pi^-), (\pi^+\pi^-K^+K^-), 2(K^+K^-))$ BRs as

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \pi^+ \eta_c (\rightarrow \pi^+\pi^-\pi^+\pi^-)) & \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \pi^+) \cdot \mathcal{B}(\eta_c \rightarrow \pi^+\pi^-\pi^+\pi^-) & \\ = (1.95_{-0.50}^{+0.59}) \times 10^{-5}, & \end{aligned} \quad (31)$$

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \pi^+ \eta_c (\rightarrow \pi^+\pi^-K^+K^-)) & \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \pi^+) \cdot \mathcal{B}(\eta_c \rightarrow \pi^+\pi^-K^+K^-) & \\ = (1.68_{-0.50}^{+0.57}) \times 10^{-5}, & \end{aligned} \quad (32)$$

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \pi^+ \eta_c (\rightarrow K^+K^-K^+K^-)) & \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \pi^+) \cdot \mathcal{B}(\eta_c \rightarrow K^+K^-K^+K^-) & \\ = (2.84_{-0.99}^{+1.10}) \times 10^{-6}. & \end{aligned} \quad (33)$$

In principle, the related measurements are previously difficult due to the large background and the small efficiency at LHC. However, these BRs around $\mathcal{O}(10^{-6})$ and above with fully charged final states are expected to be probed in the near future, since the LHCb detector has accomplished a successful upgrade in 2022.

The ratio between the BRs of $B_c^+ \rightarrow \eta_c K^+$ and $B_c^+ \rightarrow \eta_c \pi^+$ is given theoretically in the iPQCD formalism as

$$R_{K/\pi}^{\eta_c} \equiv \frac{\text{BR}(B_c^+ \rightarrow \eta_c K^+)}{\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)} = (7.49_{-0.49}^{+0.54}) \times 10^{-2}, \quad (34)$$

which agrees well with the naive anticipation $(7.95 \pm 0.04) \times 10^{-2}$ within uncertainties. That is, the decays $B_c^+ \rightarrow \eta_c(\pi, K)^+$ are indeed predominated by the factorizable decay amplitudes. Meanwhile, the a_1^K -term induced $SU(3)$ flavor symmetry-breaking effects arising from leading-twist kaon distribution amplitude in the nonfactorizable decay amplitudes lead to a slight deviation to the naive expectation, as stated in Refs. [29,30].

The $B_c^+ \rightarrow \eta_c V^+$ BRs in the iPQCD formalism can be read as

$$\text{BR}(B_c^+ \rightarrow \eta_c \rho^+) = 5.37_{-1.08}^{+1.42} (\beta_{B_c})_{-0.24}^{+0.24} (f_M)_{-0.00}^{+0.00} \times (a_\rho)_{-0.19}^{+0.22} (V_{cb}) \times 10^{-3}, \quad (35)$$

$$\text{BR}(B_c^+ \rightarrow \eta_c K^{*+}) = 3.05_{-0.62}^{+0.81} (\beta_{B_c})_{-0.19}^{+0.19} (f_M)_{-0.02}^{+0.02} \times (a_{K^*})_{-0.11}^{+0.12} (V_{cb}) \times 10^{-4}, \quad (36)$$

where the theoretical errors are also dominated mainly by the B_c -meson shape parameter β_{B_c} . By employing Eqs. (27) and (30), the measurable BRs of multibody $B_c^+ \rightarrow \eta_c \rho^+$ decays through resonance state η_c could be derived under the narrow-width approximation as follows:

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \eta_c(\rightarrow p\bar{p})\rho^+) \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \rho^+) \cdot \mathcal{B}(\eta_c \rightarrow p\bar{p}) \\ = (7.14_{-1.60}^{+2.03}) \times 10^{-6}, \end{aligned} \quad (37)$$

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \eta_c(\rightarrow p\bar{p}\pi^+\pi^-)\rho^+) \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \rho^+) \cdot \mathcal{B}(\eta_c \rightarrow p\bar{p}\pi^+\pi^-) \\ = (1.99_{-0.49}^{+0.60}) \times 10^{-5}, \end{aligned} \quad (38)$$

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \eta_c(\rightarrow \pi^+\pi^-\pi^+\pi^-)\rho^+) \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \rho^+) \cdot \mathcal{B}(\eta_c \rightarrow \pi^+\pi^-\pi^+\pi^-) \\ = (5.16_{-1.34}^{+1.62}) \times 10^{-5}, \end{aligned} \quad (39)$$

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \eta_c(\rightarrow \pi^+\pi^-K^+K^-)\rho^+) \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \rho^+) \cdot \mathcal{B}(\eta_c \rightarrow \pi^+\pi^-K^+K^-) \\ = (4.46_{-1.34}^{+1.55}) \times 10^{-5}, \end{aligned} \quad (40)$$

$$\begin{aligned} \text{BR}(B_c^+ \rightarrow \eta_c(\rightarrow K^+K^-K^+K^-)\rho^+) \\ \equiv \text{BR}(B_c^+ \rightarrow \eta_c \rho^+) \cdot \mathcal{B}(\eta_c \rightarrow K^+K^-K^+K^-) \\ = (7.52_{-2.66}^{+2.97}) \times 10^{-6}. \end{aligned} \quad (41)$$

The ratio between the $B_c^+ \rightarrow \eta_c \rho^+$ and $B_c^+ \rightarrow (J/\psi, \eta_c)\pi^+$ BRs could be deduced as

and

$$\frac{\text{BR}(B_c^+ \rightarrow \eta_c \rho^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 4.59_{-1.34}^{+1.77}, \quad (42)$$

$$R_{\rho/\pi}^{\eta_c} \equiv \frac{\text{BR}(B_c^+ \rightarrow \eta_c \rho^+)}{\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)} = 2.65_{-0.03}^{+0.02}. \quad (43)$$

Here, the latter ratio $R_{\rho/\pi}^{\eta_c}$ is close to that obtained from the $B_c^+ \rightarrow J/\psi(\pi, \rho)^+$ decays [29]. Moreover, the ratio between the $B_c^+ \rightarrow \eta_c K^{*+}$ and $B_c^+ \rightarrow \eta_c \rho^+$ BRs is

$$R_{K^*/\rho}^{\eta_c} \equiv \frac{\text{BR}(B_c^+ \rightarrow \eta_c K^{*+})}{\text{BR}(B_c^+ \rightarrow \eta_c \rho^+)} = (5.68_{-0.11}^{+0.11}) \times 10^{-2}, \quad (44)$$

which matches well with the value $(5.75 \pm 0.03) \times 10^{-2}$ anticipated by naive factorization within errors.

Finally, as aforementioned in the Introduction, the decays $B_c^+ \rightarrow \eta_c(P, V)^+$ have been studied extensively with various $B_c \rightarrow \eta_c$ form factors in different approaches [9–24]. However, the obtained values of BRs and relative ratios have a wide spread, as explicitly presented in Tables II and III. Based on the numerical results in the literature and this work, some remarks are in order: (a) For the absolute BRs of $B_c^+ \rightarrow \eta_c(\pi, \rho)^+$, as seen in the first line of Tables II and III, it is clear that, within 1σ errors, our iPQCD predictions are consistent with those in Refs. [9,15,20,21,23,24] for $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)$ and with those in Refs. [9,15,21,24] for $\text{BR}(B_c^+ \rightarrow \eta_c \rho^+)$, respectively. We can also easily find that the results of BRs for both $B_c^+ \rightarrow \eta_c \pi^+$ and $B_c^+ \rightarrow \eta_c \rho^+$ decays given in Refs. [11,13,14,16,18,22] are inconsistent with ours, even within 2σ errors. Needless to say, there are significant discrepancies in the understanding of the transition $B_c \rightarrow \eta_c$ across different frameworks. (b) For the relative ratios $\text{BR}(B_c^+ \rightarrow \eta_c(\pi, \rho)^+)/\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)$, as collected in the second line of Tables II and III, it is found that, within 1σ errors, our iPQCD predictions are consistent with those in Refs. [10,12–14,16,19–21,24] for $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)/\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)$ and with those in Refs. [10,14,19–21,24] for $\text{BR}(B_c^+ \rightarrow \eta_c \rho^+)/\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)$, respectively. (c) As shown in the third line of

TABLE II. Various predictions for $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)$, $\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)/\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)$ and $\text{BR}(B_c^+ \rightarrow \eta_c K^+)/\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)$ in this work and in the literature, respectively.

Observables	This work	[9]	[10]	[11]	[12]	[13]	[14]	[15]	[16]	[17] ^a	[18]	[19] ^b	[20]	[21]	[22]	[23]	[24] ^c
$10^3 \cdot \text{BR}(B_c^+ \rightarrow \eta_c \pi^+)$	$2.03_{-0.41}^{+0.53}$	1.80	1.30	0.26	1.40	9.30	0.85	1.90	0.94	0.91(1.16)	0.34	2.95	$2.98_{-1.05}^{+1.24}$	$1.89_{-0.37}^{+0.37}$	0.397	$1.40_{-0.40}^{+0.40}$	$0.79_{-0.55}^{+0.86}$
$\frac{\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)}$	$1.74_{-0.50}^{+0.66}$	1.00	1.78	0.19	1.27	2.07	1.39	1.12	1.24	...	1.00	1.33	$1.28_{-0.56}^{+0.69}$	$1.87_{-0.52}^{+0.52}$	1.02	$0.56_{-0.21}^{+0.21}$	$0.95_{-0.68}^{+1.04}$
$10^2 \cdot R_{K/\pi}^{\eta_c}$	$7.49_{-0.49}^{+0.54}$	7.78	10.00	8.00	7.86	5.05	8.24	7.89	7.98	8.13(8.10)	8.82	7.12	$8.05_{-0.99}^{+0.68}$	$7.94_{-0.05}^{+0.02}$	7.81	$7.86_{-0.08}^{+0.14}$	$7.51_{-0.00}^{+0.02}$

^aThe results are calculated on the basis of Coulomb plus linear confining and harmonic oscillator (in the parentheses) potentials.

^bThese results are calculated with the non-relativistic QCD approach at leading order in the strong coupling α_s . And the corresponding next-to-leading-order BR in α_s^2 is $(5.19_{-1.07}^{+0.70}) \times 10^{-3}$.

^cThese results are calculated with the QCD factorization approach in α_s . And the corresponding next-to-next-to-leading-order BR in α_s^2 is $(0.81_{-0.56}^{+0.88}) \times 10^{-3}$.

TABLE III. Same as Table II but for $B_c^+ \rightarrow \eta_c(\rho, K^*)^+$.

Observables	This work	[9]	[10]	[11]	[12]	[13]	[14]	[15]	[16]	[17]	[18]	[19]	[20]	[21]	[22]	[23]	[24]
$10^3 \cdot \text{BR}(B_c^+ \rightarrow \eta_c \rho^+)$	$5.37_{-1.12}^{+1.46}$	4.90	3.00	0.67	3.30	370.0	2.10	4.50	2.40	2.57(3.24)	1.06	7.89	$9.83_{-2.59}^{+3.33}$	$5.18_{-1.04}^{+1.04}$	1.24	$3.80_{-0.10}^{+0.10}$	$2.15_{-1.46}^{+2.23}$
$\frac{\text{BR}(B_c^+ \rightarrow \eta_c \rho^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)}$	$4.59_{-1.34}^{+1.77}$	2.72	4.11	0.52	3.0	82.2	3.44	2.65	3.16	...	3.12	3.55	$4.22_{-1.57}^{+2.05}$	$5.13_{-1.45}^{+1.45}$	3.18	$1.52_{-0.37}^{+0.37}$	$2.60_{-1.80}^{+2.71}$
$10^2 \cdot R_{K^*/\rho}^{\eta_c}$	$5.68_{-0.11}^{+0.11}$	5.10	7.00	5.97	5.45	3.70	5.24	5.56	5.42	5.06(5.25)	5.66	5.20	$5.80_{-0.45}^{+0.52}$	$5.60_{-0.04}^{+0.03}$	5.24	$5.53_{-1.48}^{+1.14}$	$5.12_{-0.06}^{+0.15}$

Tables II and III, within 1σ errors, our predictions for $R_{K/\pi}^{\eta_c}$ are consistent with most literature results except those in Refs. [10,13,14,18], while for $R_{K^*/\rho}^{\eta_c}$ they agree only with those in Refs. [12,15,18,20,21,23].

It is worth noting that all our numerical results, such as individual BRs and various relative ratios, for the decays $B_c^+ \rightarrow \eta_c(P, V)^+$ in the iPQCD formalism agree well with those calculated in covariant confined quark model [21] within 1σ uncertainties. To evaluate the reliability of the available theoretical calculations across different models, future precise measurements are highly essential.

B. $B_c^+ \rightarrow \eta_c(A, S, T)^+$

In this subsection, we present the predictions of the decays $B_c^+ \rightarrow \eta_c(A, S, T)^+$ for the first time within the QCD-based factorization framework, which could provide references for understanding the structure of these considered light hadrons in the future.

First of all, for the $B_c^+ \rightarrow \eta_c a_1(1260)^+$ channel, its BR in the iPQCD formalism is read as

$$\text{BR}(B_c^+ \rightarrow \eta_c a_1(1260)^+) = 6.91_{-1.41}^{+1.84} (\beta_{B_c})_{-0.64}^{+0.66} (f_M)_{-0.00}^{+0.00} \times (a_{a_1})_{-0.24}^{+0.29} (V_{cb}) \times 10^{-3}, \quad (45)$$

associated with the relevant ratios,

$$\frac{\text{BR}(B_c^+ \rightarrow \eta_c a_1(1260)^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 5.91_{-1.81}^{+2.33}, \quad (46)$$

and

$$R_{a_1/\rho}^{\eta_c} \equiv \frac{\text{BR}(B_c^+ \rightarrow \eta_c a_1(1260)^+)}{\text{BR}(B_c^+ \rightarrow \eta_c \rho^+)} = 1.29_{-0.40}^{+0.51},$$

$$R_{a_1/\pi}^{\eta_c} \equiv \frac{\text{BR}(B_c^+ \rightarrow \eta_c a_1(1260)^+)}{\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)} = 3.40_{-1.03}^{+1.32}. \quad (47)$$

In the above results, the uncertainties are mainly dominated by the variations of shape parameter β_{B_c} in the B_c -meson distribution amplitude. Future precise measurements will provide valuable information to aid in the study of QCD dynamics among the states pion, ρ , and $a_1(1260)$, which involve the same quark components.

For the decay $B_c^+ \rightarrow \eta_c b_1(1235)^+$, in principle, different from the $B_c^+ \rightarrow \eta_c(P, V)^+$ modes, its factorizable-emission contributions are generally suppressed due to the tiny zeroth Gegenbauer moment $a_{0,b_1}^{\parallel}$ in the longitudinal leading-twist $b_1(1235)^+$ distribution amplitude [55], while its iPQCD BR is read as

$$\text{BR}(B_c^+ \rightarrow \eta_c b_1(1235)^+) = 7.88_{-1.86}^{+2.52} (\beta_{B_c})_{-0.75}^{+0.80} (f_M)_{-2.57}^{+3.07} \times (a_{b_1})_{-0.28}^{+0.32} (V_{cb}) \times 10^{-4}, \quad (48)$$

accompanied by the two ratios

$$\frac{\text{BR}(B_c^+ \rightarrow \eta_c b_1(1235)^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 0.67_{-0.31}^{+0.39},$$

$$R_{b_1/\pi}^{\eta_c} \equiv \frac{\text{BR}(B_c^+ \rightarrow \eta_c b_1(1235)^+)}{\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)} = 0.39_{-0.18}^{+0.22}, \quad (49)$$

in which theoretical uncertainties of these numerical results are dominantly induced by the variations of shape parameter β_{B_c} and Gegenbauer moment a_{b_1} in the leading-twist distribution amplitudes of B_c and $b_1(1235)$, respectively. Notice that, the BRs of $B_c^+ \rightarrow \eta_c b_1(1235)^+$ and $B_c^+ \rightarrow J/\psi b_1(1235)^+$ in the iPQCD formalism indicate an interesting relation, namely, $\text{BR}(B_c^+ \rightarrow \eta_c b_1(1235)^+) \simeq \text{BR}(B_c^+ \rightarrow J/\psi b_1(1235)^+) \sim \mathcal{O}(10^{-3})$ within uncertainties, even though the latter BR includes three kinds of polarization contributions [29]. These values could be tested in the near-future experiments to help understand the QCD dynamics in the related decays, as well as in $b_1(1235)$.

The $B_c^+ \rightarrow \eta_c K_1(1270, 1400)^+$ BRs predicted in the iPQCD formalism with different θ_K are

$$\text{BR}(B_c^+ \rightarrow \eta_c K_1(1270)^+) = \begin{cases} 2.57_{-0.50}^{+0.63} (\beta_{B_c})_{-0.22}^{+0.23} (f_M)_{-0.63}^{+0.71} (B_{K_1})_{-0.09}^{+0.11} (V_{cb}) \times 10^{-4} \\ 4.00_{-0.78}^{+1.01} (\beta_{B_c})_{-0.38}^{+0.40} (f_M)_{-0.54}^{+0.58} (B_{K_1})_{-0.14}^{+0.16} (V_{cb}) \times 10^{-4} \end{cases}, \quad (50)$$

$$\text{BR}(B_c^+ \rightarrow \eta_c K_1(1400)^+) = \begin{cases} 2.10_{-0.45}^{+0.63} (\beta_{B_c})_{-0.26}^{+0.27} (f_M)_{-0.35}^{+0.37} (B_{K_1})_{-0.08}^{+0.08} (V_{cb}) \times 10^{-4} \\ 6.70_{-1.68}^{+2.48} (\beta_{B_c})_{-0.73}^{+0.78} (f_M)_{-2.12}^{+2.92} (B_{K_1})_{-0.24}^{+0.27} (V_{cb}) \times 10^{-5} \end{cases}, \quad (51)$$

where the first (second) entry in Eqs. (50) and (51) corresponds to $\theta_K = 33^\circ(58^\circ)$. The similar patterns to the BRs also appear in the following observables for related modes trivially:

$$\frac{\text{BR}(B_c^+ \rightarrow \eta_c K_1(1270)^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = \begin{cases} 0.22_{-0.08}^{+0.10} \\ 0.34_{-0.11}^{+0.14} \end{cases},$$

$$\frac{\text{BR}(B_c^+ \rightarrow \eta_c K_1(1400)^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = \begin{cases} 0.18_{-0.07}^{+0.08} \\ 0.06_{-0.03}^{+0.04} \end{cases}. \quad (52)$$

In the above numerical results, the dominant errors arise from the uncertainties of shape parameter β_{B_c} and Gegenbauer moments in the distribution amplitudes of B_c , and K_{1A} and K_{1B} states, respectively. The $B_c^+ \rightarrow \eta_c K_1(1270, 1400)^+$ BRs indicate that $\text{BR}(B_c^+ \rightarrow \eta_c K_1(1270)^+)$ is very consistent with $\text{BR}(B_c^+ \rightarrow \eta_c K_1(1400)^+)$ at $\theta_K \sim 33^\circ$ within uncertainties, while $\text{BR}(B_c^+ \rightarrow \eta_c K_1(1270)^+)$ is significantly larger than $\text{BR}(B_c^+ \rightarrow \eta_c K_1(1400)^+)$ at $\theta_K \sim 58^\circ$ with a factor around 6. It means that a destructive interference between $B_c^+ \rightarrow \eta_c K_{1A}^+$ and $B_c^+ \rightarrow \eta_c K_{1B}^+$ occurs significantly in $B_c^+ \rightarrow \eta_c K_1(1400)^+$ at $\theta_K \sim 58^\circ$. In other words, future measurements testing these $B_c^+ \rightarrow \eta_c K_1(1270, 1400)^+$ BRs with clearly different results would lead to a better determination of θ_K . A verified θ_K value is key to further guide the theoretical predictions with good precision.

It is worth emphasizing that the determination of θ_K with definite value is highly important because, if θ_K could be determined unambiguously, it could further help constrain the mixing between $f_1(1285)(h_1(1170))$ and $f_1(1420)(h_1(1450))$ promisingly [37]. The ratios between the $B_c^+ \rightarrow \eta_c K_1(1270)^+$ and $B_c^+ \rightarrow \eta_c K_1(1400)^+$ BRs are then derived to provide necessary reference for constraining the magnitude of θ_K ,

$$\frac{\text{BR}(B_c^+ \rightarrow \eta_c K_1(1400)^+)}{\text{BR}(B_c^+ \rightarrow \eta_c K_1(1270)^+)} = \begin{cases} 0.82_{-0.08}^{+0.09} \\ 0.17_{-0.04}^{+0.04} \end{cases}. \quad (53)$$

The above BRs and the associated ratios predicted in the iPQCD formalism would be helpful to explore the QCD dynamics in the considered axial vectors, especially in $K_1(1270, 1400)$.

Next, we turn to the $B_c^+ \rightarrow \eta_c S^+$ decays. As stressed in the Introduction, relative to the above $B_c^+ \rightarrow \eta_c(P, V)^+$ decays, the nearly zero vector decay constant f_S for light scalars will lead to the significantly suppressed factorizable-emission contributions in the $B_c^+ \rightarrow \eta_c S^+$ channels. According to the categorization of light scalar mesons in two scenarios, the CP -averaged $B_c^+ \rightarrow \eta_c S^+$ BRs in the iPQCD formalism are given as

$$\text{BR}(B_c^+ \rightarrow \eta_c a_0(980)^+) = 1.02_{-0.30}^{+0.44}(\beta_{B_c})_{-0.21}^{+0.24}(f_M)_{-0.05}^{+0.04} \times (B_i)_{-0.04}^{+0.04}(V_{cb}) \times 10^{-7}, \quad (54)$$

$$\text{BR}(B_c^+ \rightarrow \eta_c \kappa^+) = 7.20_{-2.07}^{+3.16}(\beta_{B_c})_{-1.52}^{+1.82}(f_M)_{-0.46}^{+0.49} \times (B_i)_{-0.25}^{+0.30}(V_{cb}) \times 10^{-6}, \quad (55)$$

and

$$\text{BR}(B_c^+ \rightarrow \eta_c a_0(1450)^+) = \begin{cases} 6.79_{-1.39}^{+1.61}(\beta_{B_c})_{-1.40}^{+4.53}(f_M)_{-0.65}^{+1.13}(B_i)_{-0.24}^{+0.27}(V_{cb}) \times 10^{-9} \\ 2.15_{-0.57}^{+0.83}(\beta_{B_c})_{-1.00}^{+1.00}(f_M)_{-0.04}^{+0.05}(B_i)_{-0.07}^{+0.09}(V_{cb}) \times 10^{-7} \end{cases}, \quad (56)$$

$$\text{BR}(B_c^+ \rightarrow \eta_c K_0^*(1430)^+) = \begin{cases} 6.68_{-1.38}^{+1.70}(\beta_{B_c})_{-2.06}^{+3.39}(f_M)_{-0.16}^{+0.21}(B_i)_{-0.23}^{+0.28}(V_{cb}) \times 10^{-7} \\ 1.29_{-0.34}^{+0.50}(\beta_{B_c})_{-0.47}^{+0.75}(f_M)_{-0.04}^{+0.04}(B_i)_{-0.04}^{+0.06}(V_{cb}) \times 10^{-5} \end{cases}, \quad (57)$$

where the first (second) entry in Eqs. (56) and (57) corresponds to $S1(S2)$ and the dominant errors come mainly from the B_c -meson shape parameter β_{B_c} and from the scalar decay constant $\bar{f}_S$ and the Gegenbauer moments B_i ($i = 1, 3$) of scalar mesons, respectively. The future precise constraints from experimental measurements and Lattice QCD calculations on these hadronic parameters are urgently demanded for theoretical predictions with good precision. Nevertheless, since the LHC experiments can measure the B_c decays with a branching ratio at 10^{-6} level [56], then the BRs $(7.20_{-2.62}^{+3.69}) \times 10^{-6}$ for $B_c^+ \rightarrow \eta_c \kappa^+$ and $(6.68_{-2.50}^{+3.81}) \times 10^{-7} [(1.29_{-0.58}^{+0.91}) \times 10^{-5}]$ for $B_c^+ \rightarrow \eta_c K_0^*(1430)^+$ are expected to be probed in the newly upgraded LHCb experiment. Furthermore, the experimental confirmations on the evident order difference of magnitudes in the predicted BRs of $B_c^+ \rightarrow \eta_c K_0^*(1430)^+$ could potentially provide important clues to ascertain the preferred scenario for understanding the scalar $K_0^*(1430)$.

In principle, the above-mentioned large uncertainties from nonperturbative inputs tend to be greatly cancelled by relative ratios between the relevant BRs. Then, the ratios between the $\Delta S = 1$ and $\Delta S = 0$ BRs in $B_c^+ \rightarrow \eta_c S^+$ are easily given as

$$\frac{\text{BR}(B_c^+ \rightarrow \eta_c \kappa^+)}{\text{BR}(B_c^+ \rightarrow \eta_c a_0(980)^+)} = (0.71_{-0.01}^{+0.02}) \times 10^2,$$

$$\frac{\text{BR}(B_c^+ \rightarrow \eta_c K_0^*(1430)^+)}{\text{BR}(B_c^+ \rightarrow \eta_c a_0(1450)^+)} = \begin{cases} (0.98_{-0.16}^{+0.08}) \times 10^2 \\ (0.60_{-0.01}^{+0.11}) \times 10^2 \end{cases}. \quad (58)$$

It seems that the errors in these results induced by the hadronic parameters are indeed canceled to a great extent. However, it is surprisingly noted that, drastically different from the values presented in Eqs. (34) and (44), the above two ratios are significantly large near $\mathcal{O}(10^2)$ within uncertainties, even though already with a known factor

TABLE IV. Decay amplitudes (in units of 10^{-3} GeV^{-3}) of $B_c^+ \rightarrow \eta_c(J/\psi)S^+$ in the iPQCD formalism. The upper (lower) entry corresponds to the scalars $a_0(1450)^+$ and $K_0^*(1430)^+$ in scenario 1 (2) at every line. For simplicity, only the central values are quoted for clarifications.

Modes	Decay amplitudes (F_e)	Decay amplitudes (M_e)
$B_c^+ \rightarrow \eta_c(J/\psi)a_0(980)^+$	$-0.32 - i1.88(-0.11 - i0.73)$	$0.59 + i2.76(-48.13 + i52.70)$
$B_c^+ \rightarrow \eta_c(J/\psi)\kappa^+$	$0.46 + i10.09(-1.17 - i7.60)$	$1.51 - i2.63(-7.87 + i12.70)$
$B_c^+ \rightarrow \eta_c(J/\psi)a_0(1450)^+$	$0.02 + i0.13(0.09 + i0.63)0.09$ $+i1.41(-0.15 - i0.97)$	$-0.14 + i0.08(-68.32 + i33.43)0.17$ $-i0.05(-44.16 + i7.08)$
$B_c^+ \rightarrow \eta_c(J/\psi)K_0^*(1430)^+$	$0.21 + i1.60(1.07 + i7.06)0.71$ $+i11.01(-1.60 - i10.60)$	$-1.04 + i0.68(-11.39 + i6.47)1.41$ $-i0.53(-10.30 + i2.90)$

of $|V_{us}/V_{ud}|^2 \sim 0.05$. Moreover, they are also remarkably larger than those in the $B_c^+ \rightarrow J/\psi S^+$ decays correspondingly within the iPQCD framework; for detail, see Eq. (61) in Ref. [29].

To understand this peculiar feature in the $B_c^+ \rightarrow \eta_c S^+$ decays, the related decay amplitudes calculated in the iPQCD formalism are presented explicitly in Table IV. For effective and clear clarifications, we also quote the available $B_c^+ \rightarrow J/\psi S^+$ decay amplitudes [29] and present them in Table IV simultaneously. Evidently, the originally constructive interferences to the $B_c^+ \rightarrow J/\psi S^+$ BRs between the nonfactorizable-emission diagrams in Figs. 1(c) and 1(d) with antisymmetric leading-twist distribution amplitude of light scalars do not appear in the $B_c^+ \rightarrow \eta_c S^+$ modes. In contrast, the sharply destructive interferences result in much smaller decay amplitudes as exhibited in the third column of Table IV. Furthermore, the destructions in S1 are heavier than those in S2 for the $B_c^+ \rightarrow \eta_c(a_0(1450), K_0^*(1430))^+$ decays. The situation is very contrary to that in the $B_c^+ \rightarrow J/\psi S^+$ decays [29]. Due to allowed $SU(3)$ symmetry-breaking effects, the comparably large factorizable-emission contributions, although evidently suppressed relative to those of $B_c^+ \rightarrow \eta_c(P, V)^+$, lead to further significant destructions between the decay amplitudes of factorizable-emission and nonfactorizable-emission topologies in these $B_c^+ \rightarrow \eta_c S^+$ decays. However, it should be stressed that, due to the light quark masses objectively satisfying the relation, that is, $m_s \gg m_d \sim m_u$, the apparently larger factorizable-emission contributions induced by the vector decay constants $f_{\kappa^+}, f_{K_0^*(1430)^+} \propto (m_s - m_u)$ [35] are therefore produced in $B_c^+ \rightarrow \eta_c(\kappa, K_0^*(1430))^+$ channels, relative to the factorizable-emission amplitudes in $B_c^+ \rightarrow \eta_c a_0(980, 1450)^+$ modes proportional to the value of $(m_d - m_u)$, namely, tiny broken isospin symmetry. The above iPQCD results predicted for the $B_c^+ \rightarrow \eta_c S^+$ decays await the future experimental tests, which could help us to further explore the complicated QCD dynamics potentially.

Lastly, we come to the discussions of B_c decays to η_c plus a light tensor state. For the $B_c^+ \rightarrow \eta_c T^+$ modes, to our best understanding, tensor states cannot be produced via the vector current and the conventionally factorizable-emission

contributions at leading order are thus vanished naturally. Therefore, their studies must go beyond naive factorization. Then, the $B_c^+ \rightarrow \eta_c T^+$ BRs calculated in the iPQCD formalism are

$$\text{BR}(B_c^+ \rightarrow \eta_c a_2(1320)^+) = 1.33_{-0.32}^{+0.41}(\beta_{B_c})_{-0.16}^{+0.16}(f_M)_{-0.05}^{+0.05} \times (V_{cb}) \times 10^{-4}, \quad (59)$$

$$\text{BR}(B_c^+ \rightarrow \eta_c K_2^*(1430)^+) = 8.21_{-2.00}^{+2.56}(\beta_{B_c})_{-0.76}^{+0.79}(f_M)_{-0.29}^{+0.33} \times (V_{cb}) \times 10^{-6}, \quad (60)$$

associated with the ratios

$$\frac{\text{BR}(B_c^+ \rightarrow \eta_c a_2(1320)^+)}{\text{BR}(B_c^+ \rightarrow J/\psi \pi^+)} = 0.11_{-0.04}^{+0.05},$$

$$R_{a_2/\pi}^{\eta_c} \equiv \frac{\text{BR}(B_c^+ \rightarrow \eta_c a_2(1320)^+)}{\text{BR}(B_c^+ \rightarrow \eta_c \pi^+)} = 0.07_{-0.01}^{+0.00}, \quad (61)$$

where the dominant errors come from the B_c -meson distribution amplitude shape parameter β_{B_c} . Future tests of these values, which are solely from nonfactorizable-emission contributions, would help us examine the reliability of this iPQCD formalism.

Similar to $R_{K^*/\rho}^{\eta_c}$ in the $B_c^+ \rightarrow \eta_c V^+$ sector, another ratio $R_{K_2/a_2}^{\eta_c}$ in the $B_c^+ \rightarrow \eta_c T^+$ decays is also defined by utilizing the $B_c^+ \rightarrow \eta_c a_2(1320)^+$ and $B_c^+ \rightarrow \eta_c K_2^*(1430)^+$ BRs in the iPQCD framework, and its value is then read as

$$R_{K_2/a_2}^{\eta_c} \equiv \frac{\text{BR}(B_c^+ \rightarrow \eta_c K_2^*(1430)^+)}{\text{BR}(B_c^+ \rightarrow \eta_c a_2(1320)^+)} = (6.17_{-0.13}^{+0.20}) \times 10^{-2}. \quad (62)$$

This result is induced by only nonfactorizable-emission contributions and is very close to the value presented in Eq. (44), as well as that naively anticipated in factorization ansatz for factorizable-emission predominated $B_c^+ \rightarrow \eta_c(\rho, K^*)^+$ modes within errors.

Analogous to $B_c^+ \rightarrow \eta_c b_1(1235)^+$ but different from $B_c^+ \rightarrow \eta_c a_0(980, 1450)^+$, the considerably constructive interferences due to the antisymmetric leading-twist

distribution amplitude between the two nonfactorizable-emission diagrams in Figs. 1(c) and 1(d) work in the $B_c^+ \rightarrow \eta_c a_2(1320)^+$ mode indeed. The future measurements on the iPQCD predictions of $B_c^+ \rightarrow \eta_c L^+$ in this work could help test, even differentiate, the reliability of the adopted approaches in the literature, which might help us to further understand the rich QCD dynamics in B_c -meson decays through pulling together disparate ideas.

IV. CONCLUSIONS

In summary, we have studied the B_c -meson decays into η_c plus a light charged meson in the self-consistent iPQCD formalism at leading order of strong coupling α_s . The numerical results and phenomenological insights on CP -averaged BRs in association with relative ratios are presented explicitly. The $B_c^+ \rightarrow \eta_c(\pi, \rho)^+$ and $B_c^+ \rightarrow \eta_c a_1(1260)^+$ BRs are generally larger than those $B_c \rightarrow J/\psi$ decay ones correspondingly in the iPQCD formalism, which reveal the remarkably different QCD dynamics between these two kinds of B_c -meson decays. For (near-)future tests at experiments, the BRs of multibody modes of $B_c^+ \rightarrow \eta_c(\pi, \rho)^+$ via η_c resonance through $\eta_c \rightarrow p\bar{p}$ and $\eta_c \rightarrow \pi^+\pi^-(\pi^+\pi^-, K^+K^-, p\bar{p})$ are also predicted under the narrow-width approximation. The experimental search performed with the successfully upgraded LHCb detector for the BRs around $\mathcal{O}(10^{-6})$ and above, for example, $\text{BR}(B_c^+ \rightarrow \eta_c(\rightarrow p\bar{p})\pi^+) = (2.70^{+0.74}_{-0.59}) \times 10^{-6}$, $\text{BR}(B_c^+ \rightarrow \eta_c(\rightarrow p\bar{p})\rho^+) = (7.14^{+2.03}_{-1.60}) \times 10^{-6}$, $\text{BR}(B_c^+ \rightarrow \eta_c[\rightarrow 2(\pi^+\pi^-)]\pi^+) = (1.95^{+0.59}_{-0.50}) \times 10^{-5}$, $\text{BR}(B_c^+ \rightarrow \eta_c[\rightarrow 2(\pi^+\pi^-)]\rho^+) = (5.16^{+1.62}_{-1.34}) \times 10^{-5}$, $\dots$, are expected to help probe the relevant decay channels and explore the

nature of charmonium η_c . The almost equal BRs of $B_c^+ \rightarrow (J/\psi, \eta_c)b_1(1235)^+$, while the surprisingly smaller ones of $B_c^+ \rightarrow \eta_c a_0(980, 1450)^+$ compared to those of $B_c^+ \rightarrow J/\psi a_0(980, 1450)^+$ need further explorations in both aspects of theory and experiment. Moreover, the predicted ratios $R_{K/\pi}^{\eta_c}$ and $R_{K^*/\rho}^{\eta_c}$ matching the anticipations based on the naive ansatz means that the $B_c^+ \rightarrow \eta_c(P, V)^+$ decays are predominated by factorizable-emission contributions. The experimental tests on the iPQCD ratios $R_{\eta_c/J/\psi}^\pi = 1.74^{+0.66}_{-0.50}$ and $\text{BR}(B_c^+ \rightarrow \eta_c S^+)_{\Delta S=0}/\text{BR}(B_c^+ \rightarrow \eta_c S^+)_{\Delta S=1} \sim \mathcal{O}(0.01)$ might be urgently demanded, which could help decipher the QCD dynamics involved in the considered B_c -meson decays as well as in the related charmonia η_c and J/ψ greatly.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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