

High precision black hole scattering: Tutti frutti vs worldline effective field theory

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We consider black hole scattering up to the fifth post-Minkowskian (G^5) order and compare the predictions of the tutti frutti formalism to the results obtained within two different versions of worldline effective field theory. At the G^4 order we highlight the complete agreement between tutti frutti results and the results of [C. Dlapa *et al.*, [Phys. Rev. Lett. **130**, 101401 (2023)]], and show how the tutti frutti approach allows one to extract the $O(G^3)$ angular momentum loss from the $O(G^4)$ impulse. We compare the sixth post-Newtonian (6PN) accurate tutti frutti predictions to the recent results of [M. Driesse *et al.*, [Nature **641**, 603 (2025)]], which are at the G^5 order, and at the leading order in the two mass ratios, finding complete agreement. We highlight that this agreement involves the presence at the 5.5PN level of a nonlocal tail-of-tail contribution to the scattering (first computed in [D. Bini *et al.*, [Phys. Rev. D **102**, 084047 (2020)]]), and involves, at the 6PN level, the presence of a $O(G^4)$ contribution to the angular momentum loss [C. Heissenberg, [Phys. Rev. D **111**, 126012 (2025)]]. At the second order in the mass ratios of the $O(G^5)$ order we predict two independent gauge-invariant observables to high-PN accuracy.

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I. INTRODUCTION

Gravitational scattering has been the focus of an intense activity in recent years, both for conservative and dissipative (i.e., gravitational radiation related) effects. A number of approximation methods have been applied to this problem: post-Newtonian (PN), see [1,2], multipolar post-Minkowskian (MPM) [3–5], nonrelativistic general relativity (NRGR) [6–10], gravitational self-force (GSF) [11], tutti frutti (TF) [12–16], and various avatars of post-Minkowskian (PM) perturbation theory, including amplitude-based effective field theory approaches and worldline effective field theory (WEFT), see, e.g., [17–42].

The aim of the present work is to relate several recent and ongoing WEFT developments in the radiation reacted scattering of binary systems [41,43] to state-of-the-art TF results [12–16]. Namely, we consider the radiatively corrected $O(G^4)$ scattering computations of Ref. [41] and the recent $O(G^5)$ radiation reacted scattering results of Ref. [43] and compare them to the results obtained in the TF approach [44,45]. The comparison between these results will be done at the level of the observables Δp_a^μ (where $a = 1, 2$ is a body label and $\mu = 0, 1, 2, 3$ a spacetime index)

$$\begin{aligned}\Delta p_a^\mu &= p_a^{(\text{out})\mu} - p_a^{(\text{in})\mu} \\ &= \Delta p_a^{G^1\mu} + \Delta p_a^{G^2\mu} + \dots + \Delta p_a^{G^5\mu} + O(G^6),\end{aligned}\quad (1.1)$$

measuring the change in four momentum of each body during scattering. Δp_a^μ is also known as the impulse of body a . In the second line of Eq. (1.1) we have indicated the PM expansion of Δp_a^μ , i.e., its expansion in powers of G at fixed impact parameter b . The initial four momentum $p_a^{(\text{in})\mu}$ of each body takes the form $p_a^{(\text{in})\mu} = m_a u_a^\mu$, where m_a is the mass of body a and u_a^μ (satisfying $u_a \cdot u_a = -1$) its initial four velocity. Each impulse admits a decomposition along the three 4-vectors: u_1^μ , u_2^μ and

$$\hat{b}^\mu \equiv \frac{b_{12}^\mu}{b} \equiv \frac{b_1^\mu - b_2^\mu}{b}. \quad (1.2)$$

The vectorial impact parameter b_{12}^μ is orthogonal to u_1^μ and u_2^μ and the scalar impact parameter b denotes its magnitude (so $\hat{b} \cdot \hat{b} = 1$ in our mostly plus signature). Denoting by $\gamma \equiv -u_1 \cdot u_2$ the relative Lorentz factor of the two incoming worldlines, the decomposition of the impulse Δp_a^μ reads

$$\begin{aligned}\Delta p_a^\mu &= \bar{C}_b^a \left[\frac{G}{b}, \gamma, m_1, m_2 \right] \hat{b}^\mu + \bar{C}_{u_1}^a \left[\frac{G}{b}, \gamma, m_1, m_2 \right] u_1^\mu \\ &\quad + \bar{C}_{u_2}^a \left[\frac{G}{b}, \gamma, m_1, m_2 \right] u_2^\mu,\end{aligned}\quad (1.3)$$

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or, equivalently,

$$\Delta p_a^\mu = \overset{a}{C}_b \left[\frac{G}{b}, \gamma, m_1, m_2 \right] \hat{b}^\mu + \overset{a}{C}_{\check{u}_1} \left[\frac{G}{b}, \gamma, m_1, m_2 \right] \check{u}_1^\mu + \overset{a}{C}_{\check{u}_2} \left[\frac{G}{b}, \gamma, m_1, m_2 \right] \check{u}_2^\mu, \quad (1.4)$$

where we introduced the dual four velocities $\check{u}_a^\mu$ defined as

$$\check{u}_1 = \frac{u_1 - \gamma u_2}{\gamma^2 - 1}, \quad \check{u}_2 = \frac{u_2 - \gamma u_1}{\gamma^2 - 1}, \quad (1.5)$$

and such that $u_a \cdot \check{u}_b = +\delta_{ab}$ ($a, b = 1, 2$). The various coefficients $\overset{a}{C}_b(\frac{G}{b}, \gamma, m_1, m_2), \dots, \overset{a}{C}_{\check{u}_2}(\frac{G}{b}, \gamma, m_1, m_2)$ admit an expansion in powers of $\frac{G}{b}$ (PM-expansion), and, at each order in $(\frac{G}{b})^n$ ($n \geq 1$), are polynomials of order $n+1$ in the two masses m_1 and m_2 of the form

$$\overset{a}{C}_X^{G^n}(\gamma, m_1, m_2) \sim \left(\frac{G}{b} \right)^n m_1 m_2 \times (m_1^{n-1} + m_1^{n-2} m_2 + \dots + m_2^{n-1}). \quad (1.6)$$

Each coefficient in the latter polynomial in masses is only a function of γ . The PN expansion of the impulse at any given PM order G^n is obtained by expanding the latter functions of γ in a small velocity expansion around $\gamma = 1$. When combining the structures indicated in Eqs. (1.4) and (1.6) the impulses read (where $\check{k}$ will be defined below)

$$\Delta p_a^\mu = m_1 m_2 \sum_{n=1}^{\infty} \sum_{k=0}^n G^n m_2^{n-k} m_1^k \Delta p_{a\check{k}\text{SF}}^{(n)\mu} = \overset{a}{C}_b \hat{b}^\mu + \overset{a}{C}_{\check{u}_1} \check{u}_1^\mu + \overset{a}{C}_{\check{u}_2} \check{u}_2^\mu. \quad (1.7)$$

In the following we will generally use as small velocity parameter the quantity

$$p_\infty = \sqrt{\gamma^2 - 1} = \frac{v}{\sqrt{1 - v^2}}, \quad (1.8)$$

where v is the relative velocity between the two incoming bodies.¹ By contrast to the WEFT computations which directly derive the total impulse in the above indicated polynomial form, the TF approach obtains the impulse in the form

$$\Delta p_a^\mu = \Delta p_a^{\text{cons}\mu} + \Delta p_a^{\text{rrtot}\mu}, \quad (1.9)$$

where $\Delta p_a^{\text{cons}\mu}$ is a conservative contribution (which starts at order G^1), while $\Delta p_a^{\text{rrtot}\mu}$ denotes a radiation-reaction contribution (which starts at order G^3 and contains squared

of radiation reaction effects starting at order G^4). In the TF formalism both the conservative contribution and the radiation reaction one are separately polynomial in masses at each order in G .

At the first two orders in G we have (when considering body 1)²

$$\begin{aligned} \Delta p_1^{(1)\mu} &= -\frac{G m_1 m_2}{b} 2\chi_1 \hat{b}^\mu \\ &= -\frac{G m_1 m_2}{b} \frac{2(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \hat{b}^\mu, \\ \Delta p_1^{(2)\mu} &= -\frac{G^2 m_1 m_2}{b^2} [2(m_1 + m_2) \tilde{\chi}_2 \hat{b}^\mu \\ &\quad - 2\chi_1^2 (m_1 \check{u}_2 - m_2 \check{u}_1)] \\ &= -\frac{G^2 m_1 m_2}{b^2} \left[(m_1 + m_2) \frac{3\pi(5\gamma^2 - 1)}{4\sqrt{\gamma^2 - 1}} \hat{b}^\mu \right. \\ &\quad \left. - \frac{2(2\gamma^2 - 1)^2}{\gamma^2 - 1} (m_1 \check{u}_2^\mu - m_2 \check{u}_1^\mu) \right]. \end{aligned} \quad (1.10)$$

Here

$$\chi_1 = \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}, \quad \tilde{\chi}_2 = \frac{3\pi(5\gamma^2 - 1)}{8\sqrt{\gamma^2 - 1}}, \quad (1.11)$$

are related to the PM expansion coefficients of the conservative scattering angles as

$$\chi = \sum_{n \geq 1} \frac{2\chi_n}{j^n} = h \sum_{n \geq 1} 2\tilde{\chi}_n \left(\frac{GM}{p_\infty b} \right)^n, \quad (1.12)$$

where $M = m_1 + m_2$,

$$j = \frac{J}{G m_1 m_2} = \frac{p_\infty b}{GMh}, \quad \tilde{\chi}_n = h^{n-1} \chi_n. \quad (1.13)$$

Here

$$h = \frac{E}{M} = \sqrt{1 + 2\nu(\gamma - 1)}, \quad \nu = \frac{m_1 m_2}{(m_1 + m_2)^2}, \quad (1.14)$$

where E denotes the (incoming) center-of-mass (c.m.) energy. Note that $\tilde{\chi}_1 \equiv \chi_1$. See Appendix A for a summary of our notation.

These impulses are conservative in the sense that they are symmetric (or better, equivariant) under time-reversal. Here, T -symmetry can be technically defined by considering the velocity-parity properties of the ratio $\Delta p_1^\mu / p_\infty$. Indeed, under velocity-reversal such that $u_a^\mu \rightarrow -u_a^\mu$, $p_\infty = \sqrt{\gamma^2 - 1} \rightarrow -p_\infty = -\sqrt{\gamma^2 - 1}$ [corresponding to $v \rightarrow -v$

¹We mainly use units such that $c = 1$.

²We denote the contribution of order G^n to some quantity Q either as Q^{G^n} or $Q^{(n)}$.

in Eq. (1.8)], the ratios $\Delta p_1^{(1)\mu}/p_\infty$ and $\Delta p_1^{(2)\mu}/p_\infty$ are invariant.

Starting at order G^3 , where (using the notation of Ref. [43])

$$\begin{aligned}\Delta p_1^{(3)\mu} = & m_1 m_2 (m_2^2 \Delta p_{\text{OSF}}^{(3)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] \\ & + m_1 m_2 \Delta p_{\text{ISF}}^{(3)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] \\ & + m_1^2 \Delta p_{\text{OSF}}^{(3)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu]),\end{aligned}\quad (1.15)$$

the dynamics is no longer T -symmetric, but can be uniquely decomposed in a T -even, conservative part (as measured by $\Delta p_1^{(3)\text{cons}\mu}/p_\infty$) and a T -odd radiation reaction part (as measured by $\Delta p_1^{(3)\text{rr}\mu}/p_\infty$) with opposite time reversal (and velocity reversal) properties. The TF approach decomposes $\Delta p_1^{(3)\mu}$ as follows,

$$\Delta p_1^{(3)\mu} = \Delta p_1^{(3)\text{cons}\mu} + \Delta p_1^{(3)\text{rr}\mu}, \quad (1.16)$$

where

$$\begin{aligned}\Delta p_1^{(3)\text{cons}\mu} = & -\frac{G^3 m_1 m_2 (m_1 + m_2)}{b^3} \\ & \times \left[\frac{(m_1 + m_2)}{\gamma^2 - 1} [2\check{\chi}_3^{\text{cons}} - \frac{4}{3} h^2 (\chi_1^{\text{cons}})^3] \hat{b}^\mu \right. \\ & \left. + \frac{4\chi_1^{\text{cons}} \check{\chi}_2^{\text{cons}}}{\sqrt{\gamma^2 - 1}} (m_2 \check{u}_1 - m_1 \check{u}_2) \right],\end{aligned}\quad (1.17)$$

and

$$\Delta p_1^{(3)\text{rr}\mu} = \frac{G^3 m_1^2 m_2^2}{b^3} \left[-\frac{\check{\chi}_1^{\text{cons}} \hat{J}_2(\gamma)}{(\gamma^2 - 1)} \hat{b}^\mu + \frac{\hat{E}_3(\gamma)}{(\gamma^2 - 1)^{3/2}} \check{u}_2^\mu \right]. \quad (1.18)$$

Here $\hat{J}_2$ measures the $O(G^2)$ angular momentum loss [46,47]

$$\hat{J}_2(\gamma) \equiv h^2 J_2(\gamma, \nu) \equiv 2(2\gamma^2 - 1) \sqrt{\gamma^2 - 1} \mathcal{I}(\nu), \quad (1.19)$$

where (see Eqs. (4.7) and (4.8) of Ref. [46])

$$\mathcal{I}(\nu) \equiv -\frac{16}{3} + \frac{2}{\nu^2} + \frac{2(3\nu^2 - 1)}{\nu^3} \text{arctanh}(\nu), \quad (1.20)$$

while $\hat{E}_3$ measures the $O(G^3)$ energy loss [39,48]

$$\hat{E}_3(\gamma) \equiv h^4 E_3(\gamma, \nu) \equiv (\gamma^2 - 1)^{3/2} \mathcal{E}(\gamma). \quad (1.21)$$

Here, $J_n(\gamma, \nu)$ and $E_n(\gamma, \nu)$ are defined so that

$$\begin{aligned}\frac{J_{\text{rad}}}{J_{\text{c.m.}}} &= +\nu^1 \sum_{n=2}^{\infty} \frac{J_n}{j^n}, \\ \frac{E_{\text{rad}}}{m_1 + m_2} &= +\nu^2 \sum_{n=3}^{\infty} \frac{E_n}{j^n}.\end{aligned}\quad (1.22)$$

At order G^4 , where the impulse has the general mass structure

$$\begin{aligned}\Delta p_1^{(4)\mu} = & m_1 m_2 (m_2^3 \Delta p_{\text{OSF}}^{(4)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] \\ & + m_1 m_2^2 \Delta p_{\text{ISF}}^{(4)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] \\ & + m_1^2 m_2 \Delta p_{\text{ISF}}^{(4)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] \\ & + m_1^3 \Delta p_{\text{OSF}}^{(4)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu]),\end{aligned}\quad (1.23)$$

the (TF-defined) conservative part of the dynamics (as measured by $\Delta p_1^{(4)\text{cons}\mu}/p_\infty$, see the first of Eq. (3.2) below) is still T -even (and velocity-even), while the radiation reaction part is no longer purely T -odd (nor velocity-odd) but can be uniquely decomposed in a velocity-odd part (linear in radiation reaction) and a velocity-even part (quadratic in radiation reaction). See detailed discussion in Sec. III.

As will be discussed in Sec. IV below, the situation at order G^5 becomes more involved because, though the (TF-defined) conservative part of the dynamics is physically time-symmetric, the conservative impulse $\Delta p_1^{(4)\text{cons}\mu}/p_\infty$, starting at order 5.5 PN, also contains odd powers of p_∞ . The latter odd powers of p_∞ (which are, in fact, odd powers of $|p_\infty| = +\sqrt{\gamma^2 - 1}$, and do not correspond to a time-reversal antisymmetry) come from the conservative part of the tail-of-tail nonlocal interaction (see below). This progressive loss of velocity-parity properties as the PM order increases is displayed in Table I where the words even and odd refer to the parity under $p_\infty \rightarrow -p_\infty$.

A companion supplemental material file gives explicit electronic forms of our main results (e.g., impulses at order G^5).

II. REMINDERS OF THE STRUCTURE OF THE IMPULSE IN THE TF APPROACH

In this section we summarize some results from the TF approach, and rewrite them in a new form which will clarify

TABLE I. Progressive loss of velocity-parity properties as the PM order increases.

G^n	cons	rr
G^1	even	...
G^2	even	...
G^3	even	odd
G^4	even	odd + even
G^5	even + odd	odd + even

both the comparison with recent EFT results, and the ability of the TF formalism to predict to high-PN-accuracy some observables that have not yet been computed in a PM-accurate way.

In the TF formalism the impulse is decomposed in a conservative part and a radiation reaction one (which includes terms linear and, starting at G^4 , also quadratic in radiation reaction)

$$\Delta p_a^\mu = \Delta p_a^{\text{cons}\mu} + \Delta p_a^{\text{rr tot}\mu}. \quad (2.1)$$

The conservative part of the impulse of body 1 reads (see [44])

$$\begin{aligned} \Delta p_1^{\text{cons}\mu} = & -P_{\text{cm}} \sin(\chi^{\text{cons}}) \hat{b}^\mu \\ & + (\cos(\chi^{\text{cons}}) - 1) \frac{m_1 m_2}{E^2} [(m_2 + m_1 \gamma) u_1^\mu \\ & - (m_1 + m_2 \gamma) u_2^\mu]. \end{aligned} \quad (2.2)$$

We recall the general expression of the PM expansion of the (TF-defined) *conservative* angle χ^{cons} , namely

$$\begin{aligned} \chi^{\text{cons}} &= \sum_{n \geq 1} \frac{2\tilde{\chi}_n^{\text{cons}}}{j^n} = \sum_{n \geq 1} \frac{2\tilde{\chi}_n^{\text{cons}}(\gamma, \nu)}{h^{n-1} j^n} \\ &= h \sum_{n \geq 1} 2\tilde{\chi}_n^{\text{cons}}(\gamma, \nu) \left(\frac{G(m_1 + m_2)}{p_\infty b} \right)^n, \end{aligned} \quad (2.3)$$

where $\tilde{\chi}_n^{\text{cons}}(\gamma, \nu) = h^{n-1} \chi_n^{\text{cons}}(\gamma, \nu)$ is a polynomial in ν of order $\lfloor \frac{n-1}{2} \rfloor$. Explicitly, $\tilde{\chi}_1^{\text{cons}} \equiv \chi_1$ and $\tilde{\chi}_2^{\text{cons}}$ are only functions of γ while

$$\begin{aligned} \tilde{\chi}_3^{\text{cons}}(\gamma, \nu) &= \tilde{\chi}_3^{0\text{cons}}(\gamma) + \nu \tilde{\chi}_3^{1\text{cons}}(\gamma), \\ \tilde{\chi}_4^{\text{cons}}(\gamma, \nu) &= \tilde{\chi}_4^{0\text{cons}}(\gamma) + \nu \tilde{\chi}_4^{1\text{cons}}(\gamma), \\ \tilde{\chi}_5^{\text{cons}}(\gamma, \nu) &= \tilde{\chi}_5^{0\text{cons}}(\gamma) + \nu \tilde{\chi}_5^{1\text{cons}}(\gamma) + \nu^2 \tilde{\chi}_5^{2\text{cons}}(\gamma). \end{aligned} \quad (2.4)$$

Here the $O(\nu^0)$ terms, namely $\tilde{\chi}_1^{\text{cons}}$, $\tilde{\chi}_2^{\text{cons}}$ and all the $\tilde{\chi}_n^{0\text{cons}}(\gamma)$ ($n \geq 3$) are known from the probe limit of a test mass in a Schwarzschild background. See, e.g., Eq. (8.1) of Ref. [16]. For convenience they are reproduced in Table IV below.

The ν -polynomiality structure of Eq. (2.4) ensures that the corresponding impulse defined by Eq. (2.2) is a polynomial in the two masses [as indicated in Eqs. (1.10), (1.15) and (1.23)]. More precisely, we can rewrite the conservative impulse as

$$\begin{aligned} \Delta p_1^{\text{cons}\mu} &= -2P_{\text{c.m.}} \sin\left(\frac{\chi^{\text{cons}}}{2}\right) e_x^\mu \\ &= -2m_1 m_2 p_\infty \frac{\sin(\frac{\chi^{\text{cons}}}{2})}{E} e_x^\mu, \end{aligned} \quad (2.5)$$

with the unit vector

$$e_x^\mu \equiv \hat{b}_{\text{eik}}^\mu \equiv \cos \frac{\chi^{\text{cons}}}{2} \hat{b}^\mu + \sin \frac{\chi^{\text{cons}}}{2} n^\mu, \quad (2.6)$$

where

$$n^\mu = \frac{\sqrt{\gamma^2 - 1}}{E} (m_2 \check{u}_1^\mu - m_1 \check{u}_2^\mu). \quad (2.7)$$

e_x^μ can be identified with the vectorial *eikonal* impact parameter used in Refs. [49,50].

In this decomposition both $\frac{\sin(\frac{\chi^{\text{cons}}}{2})}{E}$ and e_x^μ are polynomial in the masses (see Eq. (11.21) of Ref. [45]), as shown in Appendix C.

The total radiative contribution to the impulses is obtained in the TF approach as a sum of various contributions, namely (see Sec. XIII of [45])

$$\begin{aligned} \Delta p_{a\mu}^{\text{rr tot}} &= \chi^{\text{rr rel}} \frac{d}{d\chi^{\text{cons}}} \Delta p_{a\mu}^{\text{cons}} + \frac{\Delta P_{\text{c.m.}}}{P_{\text{c.m.}}} p_{a\mu}^{\text{out}} - \frac{m_a^2}{E_a} \frac{\Delta P_{\text{c.m.}}}{P_{\text{c.m.}}} U_\mu \\ &\quad - \frac{E_a}{E} P_\mu^{\text{rad}} - \frac{(P_{a\nu}^{\text{out}} P_{\text{rad}}^\nu)}{E} U_\mu + \Delta p_{a\mu}^{\text{rr remain}}, \end{aligned} \quad (2.8)$$

where the first contribution is the usual linear-response contribution with [51]

$$\chi^{\text{rr rel}} \equiv -\left(\frac{1}{2} \frac{\partial \chi^{\text{cons}}}{\partial E} E_{\text{rad}} + \frac{1}{2} \frac{\partial \chi^{\text{cons}}}{\partial J} J_{\text{rad}} \right), \quad (2.9)$$

where the next four contributions are linked to recoil effects [44,45],

$$\Delta P_{\text{c.m.}} = -\frac{E_1 E_2}{EP_{\text{c.m.}}} E_{\text{rad}}, \quad (2.10)$$

and where $\Delta p_a^{\text{rr remain}}$ (which starts at order G^4/c^{10}) was introduced in Eq. (12.21) of Ref. [45]. As shown there, the latter “remain” contribution is constrained by three different requirements: (i) the antisymmetric constraint (12.23) of Ref. [45]; (ii) its second-gravitational-self-force³ (2GSF) character, i.e., the fact that it contains a factor m_a^3 ; and (iii) the property of canceling the non polynomiality in masses of several terms entering Eq. (2.8). [Indeed, Eq. (2.8) contains denominators such that E_a , $P_{\text{c.m.}}$, and E which, considered by themselves, violate the mass polynomiality of Δp_a .]

It was shown in Sec. XII of Ref. [45] that the requirements (i), (ii), (iii) determine most of the structure of $\Delta p_1^{\text{rr remain}\mu}$ at orders G^4 and G^5 , modulo a supplementary $O(G^5)$ term denoted $\Delta p_1^{\text{rr remain sup}\mu}$ below. In other words, we have

³As shown below, there is a distinction (as well as a relation) between the usual concept of 2GSF (linked to the square of radiation-reaction on a particle of mass m_1) and the notation 1SF/1SF/2SF introduced by [43] and used already in Eq. (1.15) above, and Eq. (4.7) below.

$$\Delta p_1^{\text{rr remain } \mu} = \Delta p_1^{\text{rr remain } P_{\text{rad}} \mu} + \Delta p_1^{\text{rr remain sup } \mu}, \quad (2.11)$$

where $\Delta p_1^{\text{rr, remain } P_{\text{rad}}^{\mu}} \sim G^4 + G^5$ is fully determined by the TF approach, and is *linear* in P_{rad}^{μ} , and where

$$\Delta p_1^{\text{rr remain sup } \mu} = \frac{G^5}{b^5} m_1^3 m_2^3 f_b^{G^5 \text{ remain}} \hat{b}^{\mu} + O(G^6). \quad (2.12)$$

In view of the decomposition (2.11) it is convenient to gather all the contributions linear in P_{rad}^{μ} , and to define

$$\begin{aligned} \Delta p_a^{\text{rr lin } P_{\text{rad}} \mu} \equiv & -\frac{1}{2} \frac{\partial \chi^{\text{cons}}}{\partial E} E_{\text{rad}} \frac{d}{d\chi_{\text{cons}}} \Delta p_{a\mu}^{\text{cons}} \\ & + \frac{\Delta P_{\text{c.m.}}}{P_{\text{c.m.}}} p_{a\mu}^{\text{out}} - \frac{m_a^2}{E_a} \frac{\Delta P_{\text{c.m.}}}{P_{\text{c.m.}}} U_{\mu} \\ & - \frac{E_a}{E} p_{\mu}^{\text{rad}} - \frac{(p_{a\nu}^{\text{out}} P_{\text{rad}}^{\nu})}{E} U_{\mu} \\ & + \Delta p_{a\mu}^{\text{rr remain } P_{\text{rad}}}. \end{aligned} \quad (2.13)$$

The total impulse is then naturally decomposed in the TF formalism as follows:

$$\begin{aligned} \Delta p_a^{\mu} = & \Delta p_a^{\text{cons } \mu} + \Delta p_a^{\text{rr lin } J_{\text{rad}} \mu} \\ & + \Delta p_a^{\text{rr lin } P_{\text{rad}} \mu} + \Delta p_1^{\text{rr remain sup } \mu}. \end{aligned} \quad (2.14)$$

Here $\Delta p_a^{\text{rr lin } J_{\text{rad}} \mu}$ is the part of the linear response formula that is *linear* in J_{rad} , namely,⁴

$$\Delta p_a^{\text{rr lin } J_{\text{rad}}} \equiv -\frac{1}{2} \frac{\partial \chi^{\text{cons}}}{\partial J} J_{\text{rad}} \frac{d}{d\chi_{\text{cons}}} \Delta p_{a\mu}^{\text{cons}}, \quad (2.15)$$

while $\Delta p_a^{\text{rr lin } P_{\text{rad}} \mu}$, defined in Eq. (2.13), is *linear* in the radiated four momentum P_{rad}^{μ} . We recall that the last term $\Delta p_1^{\text{rr remain sup } \mu}$ in Eq. (2.14) starts at G^5 [see Eq. (2.12)]. The decomposition (2.14) generalizes the structure that was already present at G^3 , see Eq. (1.16), where the radiation reaction part given in Eq. (1.18) is linear in $\hat{J}_2$ and $\hat{E}_3$.

Each part in the decomposition (2.14) has the property of being polynomial in the two masses.

The mass polynomiality of $\Delta p_a^{\text{cons } \mu}$ follows from the ν -structure of the conservative scattering angle displayed in Eq. (2.4) above. The mass polynomiality of $\Delta p_a^{\text{rr lin } J_{\text{rad}}}$ follows from the ν -structure of J_{rad} enunciated in Refs. [44,45], see Eqs. (7.18) and (7.19) in Ref. [44] and Eq. (10.3) in Ref. [45] namely,

$$\begin{aligned} \hat{J}_n(\gamma, \nu) & \equiv h^n J_n + h^{n-1} \nu E_n \\ & = \hat{J}_n^0(\gamma) + \nu \hat{J}_n^1(\gamma) + \dots + \hat{J}_n^{d(n)}(\gamma) \nu^{d(n)}, \end{aligned} \quad (2.16)$$

where $d(n)$ is the integer part of $(n-2)/2$: $d(n) = \lfloor \frac{n-2}{2} \rfloor$, and where the coefficients E_n and J_n denote the PM expansion coefficients of J_{rad} and E_{rad} defined in Eq. (1.22). In particular, the latter polynomiality rule yields

$$\begin{aligned} h^2 J_2 & = \hat{J}_2(\gamma), \\ h^3 J_3 + h^2 \nu E_3 & = \hat{J}_3(\gamma), \\ h^4 J_4 + h^3 \nu E_4 & = \hat{J}_4^0(\gamma) + \nu \hat{J}_4^1(\gamma). \end{aligned} \quad (2.17)$$

Finally, the mass polynomiality of $\Delta p_a^{\text{rr lin } P_{\text{rad}}}$ follows from the structure of $\Delta p_a^{\text{rr remain } P_{\text{rad}}}$ together with the mass polynomiality structure of P_{rad}^{μ} , which also implies ν -polynomiality rules of the x and y components of P_{rad}^{μ} , as discussed in Secs. X and XI of Ref. [45]. Let us, in particular, recall the ν -polynomiality rule (which follows from Eq. (11.18) of Ref. [45])

$$\hat{E}_n = h^{n+1} E_n = P_{\lfloor \frac{n-2}{2} \rfloor}(\nu). \quad (2.18)$$

III. COMPARISON BETWEEN THE $O(G^4)$ TF IMPULSE AND WEFT

At order G^4 , the three contributions (conservative, linear in J_{rad} , linear in P_{rad}) of the TF-predicted impulse, namely

$$\Delta p_a^{G^4 \mu} = [\Delta p_a^{\text{cons } \mu} + \Delta p_a^{\text{rr lin } J_{\text{rad}}} + \Delta p_a^{\text{rr lin } P_{\text{rad}}}]_{G^4}, \quad (3.1)$$

read

$$\begin{aligned} \Delta p_1^{G^4 \text{ cons } \mu} & = -2P_{\text{c.m.}} \sin \frac{\chi^{\text{cons}}}{2} e_x^{\mu} |_{G^4}, \\ \Delta p_1^{G^4 \text{ rr lin } J_{\text{rad}} \mu} & = \frac{G^4 m_1^2 m_2^2}{b^4} \left[\frac{(m_1 + m_2) \chi_1^{\text{cons}}}{(\gamma^2 - 1)^{3/2}} \hat{J}_3 \hat{b}^{\mu} - 2 \frac{\hat{J}_2}{(\gamma^2 - 1)} \left[(\chi_1^{\text{cons}})^2 (m_2 \check{u}_1^{\mu} - m_1 \check{u}_2^{\mu}) + \frac{(m_1 + m_2) \tilde{\chi}_2^{\text{cons}}}{\sqrt{\gamma^2 - 1}} \hat{b}^{\mu} \right] \right], \\ \Delta p_1^{G^4 \text{ rr lin } P_{\text{rad}} \mu} & = \frac{G^4 m_1^2 m_2^2}{b^4} \left[m_1 \hat{P}_b^4 \hat{b}^{\mu} - (\gamma - 1)(m_1 - m_2) \hat{P}_{1-2}^4 \check{u}_2 + (\gamma + 1)(m_1 + m_2) \hat{P}_{1+2}^4 \check{u}_2 + \frac{(m_1 + m_2)}{\gamma - 1} \hat{P}_{1+2}^3 [2\gamma \chi_1 \right. \\ & \quad \left. + (\gamma^2 - 1) \chi_1'] \hat{b}^{\mu} \right]. \end{aligned} \quad (3.2)$$

⁴As discussed below, contributions nonlinear in J_{rad} start at order G^6 .

Here, Δp_1^μ at order G^4 , has been expressed in terms of $\hat{J}_2$, $\hat{J}_3$, and of the form factors $\hat{P}_{1+2}(\gamma)$, $\hat{P}_{1-2}(\gamma)$, and $\hat{P}_b(\gamma)$ of P_μ^{rad} defined as

$$P_{\text{rad}}^\mu = P_b \hat{b}^\mu + P_{1+2}(u_1^\mu + u_2^\mu) + P_{1-2}(u_1^\mu - u_2^\mu), \quad (3.3)$$

with mass-factorized expressions at orders G^3 and G^4 . At order G^3 only one function of γ enters. Indeed, $P_b^{G^3} = 0$, $P_{1-2}^{G^3} = 0$ and

$$P_{1+2}^{G^3} = \frac{G^3 m_1^2 m_2^2}{b^3} \hat{P}_{1+2}^3(\gamma). \quad (3.4)$$

At order G^4 the various components of P_{rad}^μ involve (after factoring the masses) three functions of γ , namely

$$\begin{aligned} P_b^{G^4} &= \frac{G^4 m_1^2 m_2^2}{b^4} (m_2 - m_1) \hat{P}_b^4(\gamma), \\ P_{1-2}^{G^4} &= \frac{G^4 m_1^2 m_2^2}{b^4} (m_2 - m_1) \hat{P}_{1-2}^4(\gamma), \\ P_{1+2}^{G^4} &= \frac{G^4 m_1^2 m_2^2}{b^4} (m_1 + m_2) \hat{P}_{1+2}^4(\gamma). \end{aligned} \quad (3.5)$$

The form factor $\hat{P}_{1+2}^3(\gamma)$ is related to $\hat{E}_3 = h^4 E_3 = (\gamma^2 - 1)^{3/2} \mathcal{E}(\gamma)$ by

$$\hat{P}_{1+2}^3(\gamma) = \frac{\mathcal{E}(\gamma)}{\gamma + 1}, \quad (3.6)$$

while at order G^4 we have $\hat{E}_4 = h^5 E_4 = \tilde{E}_4^0 + \nu \tilde{E}_4^1$, with the relations

$$\begin{aligned} \hat{P}_{1-2}^4(\gamma) &= -\frac{1}{4(\gamma - 1)} \frac{\tilde{E}_4^1}{(\gamma^2 - 1)^2}, \\ \hat{P}_{1+2}^4(\gamma) &= \frac{1}{4(\gamma + 1)} \frac{4\tilde{E}_4^0 + \tilde{E}_4^1}{(\gamma^2 - 1)^2}. \end{aligned} \quad (3.7)$$

It is convenient in the following to decompose the various contributions to Δp_a according to the mass polynomiality as in Eq. (1.23). Each $\Delta p_{\text{NSF}}^\mu[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu]$ will be decomposed along $\hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu$ according to the following notation

$$\Delta p_{\text{NSF}}^\mu[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] = \hat{b}^\mu f_b^{\text{NSF}}(\gamma) + \check{u}_1^\mu f_{\check{u}_1}^{\text{NSF}}(\gamma) + \check{u}_2^\mu f_{\check{u}_2}^{\text{NSF}}(\gamma), \quad (3.8)$$

where, the index n takes the values $n = 0, 1, \bar{1}, \bar{0}$ at G^4 , and the values $n = 0, 1, \bar{1}, 2, \bar{0}$ at G^5 . [Here, the notation $\bar{\text{ISF}}$ refers to $\bar{\text{ISF}}$.]

The G^4 contributions to the various $f_X^{\text{NSF}}(\gamma)$ (suppressing an overall factor G^n/b^n to ease notation) predicted by the TF formalism have the following structure (in which the

general notation K refers to contributions known from lower PM orders, e.g., in $f_b^{G^4\text{TF,ISF}}$ K stands for a function involving $\chi_1, \tilde{\chi}_2^{\text{cons}}, \tilde{\chi}_4^{0\text{cons}}$, and $\hat{J}_2$),

$$\begin{aligned} f_b^{G^4\text{TF,ISF}} &= -\frac{2}{(\gamma^2 - 1)^{3/2}} \tilde{\chi}_4^{1\text{cons}} - \frac{1}{(\gamma^2 - 1)^{3/2}} \tilde{\chi}_1^{1\text{cons}} \hat{J}_3 + K, \\ f_{\check{u}_1}^{G^4\text{TF,ISF}} &= K, \\ f_{\check{u}_2}^{G^4\text{TF,ISF}} &= (\gamma - 1) \hat{P}_{1-2}^4 + (\gamma + 1) \hat{P}_{1+2}^4 + K, \end{aligned} \quad (3.9)$$

while the $\bar{\text{ISF}}$ contributions read

$$\begin{aligned} f_b^{G^4\text{TF,}\bar{\text{ISF}}} &= \hat{P}_b^4 - \frac{2}{b^4(\gamma^2 - 1)^{3/2}} \tilde{\chi}_4^{1\text{cons}} \\ &\quad - \frac{1}{(\gamma^2 - 1)^{3/2}} \tilde{\chi}_1^{1\text{cons}} \hat{J}_3 + K, \\ f_{\check{u}_1}^{G^4\text{TF,}\bar{\text{ISF}}} &= K, \\ f_{\check{u}_2}^{G^4\text{TF,}\bar{\text{ISF}}} &= -(\gamma - 1) \hat{P}_{1-2}^4 + (\gamma + 1) \hat{P}_{1+2}^4 + K. \end{aligned} \quad (3.10)$$

One can recognize on the rhs of Eqs. (3.9) and (3.10) above the origin of the various contributions. For instance: (1) the terms $\propto \tilde{\chi}_4^{1\text{cons}}$ come from the conservative impulse Δp_1^{cons} ; (2) the terms $\hat{J}_3$ come from $\Delta p^{\text{rr lin } J_{\text{rad}}}$, while (3) the terms $\propto \hat{P}_{1+2}^4, \propto \hat{P}_{1-2}^4$ or $\propto \hat{P}_b^4$ all come from $\Delta p_a^{\text{rr lin } P_{\text{rad}}}$.

At the G^4 order, the conservative contribution to the impulse Δp_a^{cons} has the same meaning in TF and in WEFT. The exact value of $\Delta p_a^{G^4,\text{cons}}$ has been computed in Refs. [27,30,31,52]. The corresponding conservative TF impulse (which was computed earlier at the 6PN accuracy [44]) was found to be in full agreement with the WEFT results, up to the 6PN accuracy.

At this G^4 order, the radiation reaction contribution to the impulse $\Delta p_a^{G^4\text{rr}}$ can be defined both in TF and WEFT as being

$$\Delta p_a^{G^4\text{rr}\mu} \equiv \Delta p_a^{G^4\mu} - \Delta p_a^{G^4\text{cons}\mu}. \quad (3.11)$$

The exact value of $\Delta p_a^{G^4\text{rr}}$ has been computed by Refs. [41,53].

Reference [41] used a decomposition of $\Delta p_a^{G^4\text{rr}}$ in terms of dissipative-1 rad and dissipative-2 rad contributions, which are defined by their character under velocity-reversal: the 1 rad part being T -odd and the 2 rad part being T -even. A similar decomposition makes sense in the TF approach and was used in Ref. [45]. In particular, Eq. (12.34) of Ref. [45] at the G^4 level has shown that the radiation-reaction part of the b -projection of Δp_1 , namely $\Delta p_{1b}^{G^4\text{rr}} = \Delta p_{1bG^4} - \Delta p_{1bG^4}^{\text{cons}}$ was given in the TF approach by

$$\Delta p_{1b}^{G^4 \text{rr}} = \Delta p_{1bG^4}^{\text{rr, lin-odd}} + \frac{m_1}{m_2 - m_1} P_{xG^4}^{\text{rad}}, \quad (3.12)$$

where $\Delta p_{1bG^4}^{\text{rr, lin-odd}}$ is T -odd while $\frac{m_1}{m_2 - m_1} P_{xG^4}^{\text{rad}}$ is T -even. In terms of the decomposition in Eqs. (3.1) and (3.2), the TF T -odd part is equal to

$$\Delta p_{1bG^4}^{\text{rr, lin-odd}} = \hat{b} \cdot \Delta p_1^{\text{rr lin} J_{\text{rad}}} + \hat{b} \cdot \Delta p_1^{\text{rr lin} P_{\text{rad}}} \Big|_{T\text{-odd}}, \quad (3.13)$$

where the T -odd part of $\hat{b} \cdot \Delta p_1^{\text{rr lin} P_{\text{rad}}}$ is obtained from the third Eq. (3.2) in the following two-step way. First, one replaces $\hat{P}_b^4$ on the right-hand side of Eq. (3.2) by

$$\hat{P}_b^4 = -\frac{h^4}{(\gamma^2 - 1)^2} (-\check{P}_x^4 + \chi_1^{\text{cons}} \check{P}_y^3), \quad (3.14)$$

where $\check{P}_x^4$ and $\check{P}_y^3$ are related to the $1/j$ expansion of

$$P_x \equiv P_{\text{rad}\mu} e_x^\mu, \quad P_y \equiv P_{\text{rad}\mu} e_y^\mu, \quad (3.15)$$

in the following way

$$P_x = (m_2 - m_1) \nu^2 \sum_{n \geq 4} \frac{\check{P}_x^n}{j^n},$$

$$P_y = (m_2 - m_1) \nu^2 \sum_{n \geq 3} \frac{\check{P}_y^n}{j^n}. \quad (3.16)$$

We recall that the unit vector e_x^μ is defined as

$$e_x^\mu \equiv \hat{b}_{\text{cik}}^\mu \equiv \cos \frac{\chi_{\text{cons}}}{2} \hat{b}^\mu + \sin \frac{\chi_{\text{cons}}}{2} n^\mu, \quad (3.17)$$

see Eq. (3.49) in Ref. [44]. As a consequence e_x^μ has a nontrivial PM expansion so that

$$e_x^\mu = \hat{b}^\mu + G e_x^{G^1 \mu} + \dots, \quad (3.18)$$

which implies, in particular, that $[P_x]^{G^4}$ is the sum of a term proportional to $P_b^{G^4}$ and a term linear in $P_{\text{rad}}^{G^3}$ [as was already exhibited in Eq. (3.14)].

In addition, $\check{P}_y^3$ in Eq. (3.14) is given by (see Eq. (7.15) of Ref. [44])

$$\check{P}_y^3 = \sqrt{\frac{\gamma-1}{\gamma+1}} E_3, \quad h^4 E_3 = \hat{E}_3 = (\gamma^2 - 1)^{3/2} \mathcal{E}(\gamma). \quad (3.19)$$

In the second step of the definition of the T -odd part of $\hat{b} \cdot \Delta p_1^{\text{rr lin} P_{\text{rad}}}$ one sets to zero the term proportional to $\check{P}_x^4$ (which is T -even, contrary to the $\check{P}_y^3$ term which is T -odd).

This leads us to the following identifications between WEFT and TF quantities

$$c_{1b,1 \text{ rad}}^{4 \text{ diss}} = \Delta p_{1bG^4}^{\text{rr, lin-odd}},$$

$$c_{1b,2 \text{ rad}}^{4 \text{ diss}} = \frac{m_1}{m_2 - m_1} P_{xG^4}^{\text{rad}}. \quad (3.20)$$

In the latter equation [as well as in Eq. (3.12)] the notation $P_{xG^4}^{\text{rad}} \equiv [P_x]^{G^4}$ denotes the G^4 contribution to $P_x \equiv P_{\text{rad}\mu} e_x^\mu$ (without factoring masses or symmetric mass ratio), so that

$$P_{xG^4}^{\text{rad}} \equiv [P_x]^{G^4} = \frac{G^4 m_1^2 m_2^2}{b^4} (m_2 - m_1) \frac{h^4}{(\gamma^2 - 1)^2} \check{P}_x^4. \quad (3.21)$$

Using the explicit expression for $\Delta p_{1bG^4}^{\text{rr, lin-odd}}$ given in Eqs. (12.37) and (12.38) of Ref. [45] we get from the first equation above the following TF-predictions for $c_{1b,1 \text{ rad}}^{4 \text{ diss}}$

$$c_{1b,1 \text{ rad}}^{4 \text{ diss}} = \frac{G^4}{b^4} m_1^2 m_2^2 \left\{ (m_1 + m_2) \left[\mathcal{E}(\gamma) \frac{\gamma(6\gamma^2 - 5)}{(\gamma^2 - 1)^{3/2}} \right. \right.$$

$$\left. - \pi \frac{3}{4} \hat{J}_2(\gamma) \frac{(5\gamma^2 - 1)}{(\gamma^2 - 1)^{3/2}} - \hat{J}_3(\gamma) \frac{(2\gamma^2 - 1)}{(\gamma^2 - 1)^2} \right]$$

$$\left. - m_1 \mathcal{E}(\gamma) \frac{2\gamma^2 - 1}{(\gamma + 1) \sqrt{\gamma^2 - 1}} \right\}. \quad (3.22)$$

In this relation we see that the TF formalism points out the presence of angular momentum losses of G^2 and G^3 orders in $c_{1b,1 \text{ rad}}^{4 \text{ diss}}$. In particular, focusing on the 1GSF projection, i.e., the coefficient of $m_1^2 m_2^2$ of this equation, we can obtain an expression of $\hat{J}_3$ in terms of $c_{1b,1 \text{ rad}}^{4 \text{ diss}}$ together with the G^2 angular momentum loss and the G^3 energy loss ($\mathcal{E} = \frac{\hat{E}_3}{(\gamma^2 - 1)^{3/2}}$). This yields

$$\hat{J}_3 = C^{\text{ac}} \text{arccosh}(\gamma) + C^{\text{at}} \quad \text{at} \left(\frac{\sqrt{\gamma^2 - 1}}{\gamma} \right)$$

$$+ C^{\text{ln}} \ln \left(\frac{\gamma + 1}{2} \right), \quad (3.23)$$

with

$$C^{\text{ac}} = -\pi \frac{\gamma(2\gamma^2 - 3)}{8(\gamma^2 - 1)^{3/2}} (35\gamma^5 - 240\gamma^4$$

$$- 70\gamma^3 + 288\gamma^2 + 19\gamma - 48),$$

$$C^{\text{at}} = -\pi \frac{3\gamma(10\gamma^6 - 27\gamma^4 + 20\gamma^2 - 3)}{(\gamma^2 - 1)^{3/2}},$$

$$C^{\text{ln}} = \frac{1}{4} \pi (35\gamma^5 - 90\gamma^4 - 70\gamma^3 + 16\gamma^2 + 155\gamma - 62), \quad (3.24)$$

which indeed coincides with the result obtained in Ref. [40].

Finally, the second line in Eq. (3.20) points out the existence of a simple proportionality relation between $c_{1b,2\text{ rad}}^{4\text{ diss}}$ and $P_{xG^4}^{\text{rad}}$. This TF-predicted relation is indeed satisfied by the WEFT result for $c_{1b,2\text{ rad}}^{4\text{ diss}}$ of Ref. [41].

IV. TF-PREDICTED STRUCTURE OF THE $O(G^5)$ IMPULSE

At order G^5 the TF formalism predicts that the impulse contains again three main contributions: a conservative one, a term linear in J_{rad} and a term linear in P_{rad} . However, at order G^5 we have to take into account the presence of a *supplementary* contribution $\Delta p_1^{\text{rr remain sup } \mu}$, namely

$$\Delta p_a^\mu = \Delta p_a^{\text{cons } \mu} + \Delta p_a^{\text{rr lin } J_{\text{rad}} \mu} + \Delta p_a^{\text{rr lin } P_{\text{rad}} \mu} + \Delta p_a^{\text{rr remain sup } \mu}. \quad (4.1)$$

Separating out the supplementary contribution

$$\Delta p_1^{\text{rr remain sup } \mu} = \frac{G^5}{b^5} m_1^3 m_2^3 f_b^{G^5 \text{ remain}} \hat{b}^\mu + O(G^6), \quad (4.2)$$

the linear-in- P_{rad} contribution at order G^5 is given by the right-hand side (rhs) of Eq. (2.13) where the last term, $\Delta p_a^{\text{rr remain } P_{\text{rad}} \mu}$, was given in Ref. [45] as the sum of four contributions given in Eqs. (12.53), (12.62), (12.67) and (12.78) of Ref. [45], describing its components along the basis $\hat{b}^\mu$, $(u_1^\mu + u_2^\mu)$ and $(u_1^\mu - u_2^\mu)$. Explicitly (using Eq. (A8) below),

$$\Delta p_1^{G^5 \text{ rr remain } P_{\text{rad}} \mu} = c_{1b}^{G^5 \text{ rr remain } P_{\text{rad}}} \hat{b}^\mu + c_{1+2}^{G^5 \text{ rr remain}} (u_1^\mu + u_2^\mu) + c_{1-2}^{G^5 \text{ rr remain}} (u_1^\mu - u_2^\mu), \quad (4.3)$$

where

$$\begin{aligned} c_{1b}^{G^5 \text{ rr remain } P_{\text{rad}}} &= \frac{G^5}{b^5} m_1^3 m_2^3 (m_1 + m_2)^2 \frac{(\gamma + 1)}{E^2} p_{xG^5}(\gamma), \\ c_{1+2}^{G^5 \text{ rr remain}} &= \frac{G^5}{b^5} m_1^3 m_2^3 (m_1 - m_2)(m_1 + m_2) \\ &\quad \times \frac{(2\gamma^2 - 1)}{E^2 \sqrt{\gamma^2 - 1}} p_{xG^4}(\gamma), \\ c_{1-2}^{G^5 \text{ rr remain}} &= -2 \frac{G^5}{b^5} m_1^4 m_2^4 \frac{(2\gamma^2 - 1)(\gamma + 1)}{E^2 \sqrt{\gamma^2 - 1}} p_{xG^4}(\gamma) \\ &\quad + \frac{G^5}{b^5} m_1^3 m_2^3 \frac{(2\gamma^2 - 1)(\gamma + 1)}{(\gamma - 1) \sqrt{\gamma^2 - 1}} p_{xG^4}(\gamma). \end{aligned} \quad (4.4)$$

Here we introduced the notation

$$\begin{aligned} [P_x]^{G^4} &\equiv \frac{G^4 m_1^2 m_2^2}{b^4} (m_2 - m_1) p_{xG^4}(\gamma), \\ [P_x]^{G^5} &\equiv \frac{G^5 m_1^2 m_2^2 (m_1 + m_2)}{b^5} (m_2 - m_1) p_{xG^5}(\gamma). \end{aligned} \quad (4.5)$$

Let us note that, when gathering both the G^4 and G^5 contributions, the remain- P_{rad} contribution to the impulse lies in the direction of the e_x^μ vector and can be written in the simple form

$$\begin{aligned} \Delta p_{1x}^{\text{rr remain } P_{\text{rad}}} &= m_1^3 m_2^3 (m_1 + m_2) \frac{\gamma + 1}{E^2} \left(\frac{G^4}{b^4} p_{xG^4}(\gamma) \right. \\ &\quad \left. + \frac{G^5}{b^5} (m_1 + m_2) p_{xG^5}(\gamma) \right) \\ &= m_1 m_2 (m_1 + m_2) \frac{\gamma + 1}{E^2} \frac{P_x^{G^4} + P_x^{G^5}}{m_2 - m_1}. \end{aligned} \quad (4.6)$$

We have explicitly checked that the TF predicted impulse, which in its original definition given by Eq. (2.8), for Δp_1 at 5PM contains $1/E^2 = 1/(m_1^2 + m_2^2 + 2\gamma m_1 m_2)$ denominators [as well as $1/(m_1 + m_2)^{2n}$ coming from the powers of ν in $\tilde{\chi}^{\text{cons}}$, Eq. (2.4)] simplifies (because of the additional $\Delta p_1^{G^5 \text{ rr remain}}$ terms) to a purely polynomial-in-masses expression of the type

$$\begin{aligned} \Delta p_1^{(5)\mu} &= m_1 m_2 (m_2^4 \Delta p_{\text{0SF}}^{(5)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] + m_1 m_2^3 \Delta p_{\text{1SF}}^{(5)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] + m_1^2 m_2^2 \Delta p_{\text{2SF}}^{(5)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] \\ &\quad + m_1^3 m_2 \Delta p_{\text{3SF}}^{(5)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] + m_1^4 \Delta p_{\text{0SF}}^{(5)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu]), \\ \Delta p_2^{(5)\mu} &= m_1 m_2 (m_1^4 \Delta p_{\text{0SF}}^{(5)\mu}[\gamma, -\hat{b}^\mu, \check{u}_2^\mu, \check{u}_1^\mu] + m_2 m_1^3 \Delta p_{\text{1SF}}^{(5)\mu}[\gamma, -\hat{b}^\mu, \check{u}_2^\mu, \check{u}_1^\mu] + m_1^2 m_2^2 \Delta p_{\text{2SF}}^{(5)\mu}[\gamma, -\hat{b}^\mu, \check{u}_2^\mu, \check{u}_1^\mu] \\ &\quad + m_2^3 m_1 \Delta p_{\text{3SF}}^{(5)\mu}[\gamma, -\hat{b}^\mu, \check{u}_2^\mu, \check{u}_1^\mu] + m_2^4 \Delta p_{\text{0SF}}^{(5)\mu}[\gamma, -\hat{b}^\mu, \check{u}_2^\mu, \check{u}_1^\mu]). \end{aligned} \quad (4.7)$$

Each $\Delta p_{\text{nSF}}^{(5)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu]$ is then decomposed along $\hat{b}^\mu$, $\check{u}_1^\mu$, $\check{u}_2^\mu$ according to the following notation

$$\Delta p_{\text{nSF}}^{(5)\mu}[\gamma, \hat{b}^\mu, \check{u}_1^\mu, \check{u}_2^\mu] = \hat{b}^\mu f_b^{G^5, \text{nSF}}(\gamma) + \check{u}_1^\mu f_{\check{u}_1}^{G^5, \text{nSF}}(\gamma) + \check{u}_2^\mu f_{\check{u}_2}^{G^5, \text{nSF}}(\gamma), \quad (4.8)$$

where $n = 0, 1, 2, \bar{1}, \bar{0}$.

The 0SF and $\overline{0SF}$ contributions are defined by the probe limit and must satisfy the constraints

$$\begin{aligned} f_b^{G^5, 0SF} &= f_b^{G^5, \overline{0SF}}, & f_{\tilde{u}_1}^{0SF} &= -f_{\tilde{u}_2}^{0SF}, \\ f_{G^5, \tilde{u}_2}^{0SF} &= 0 = f_{G^5, \tilde{u}_1}^{\overline{0SF}}. \end{aligned} \quad (4.9)$$

They are explicitly given in Table V of Appendix C.

In other words, the TF approach predicts explicit forms for all the γ -dependent coefficients, say $f_b^{0SF}, \dots, f_b^{2SF}, \dots$ entering the 5PM impulse, Eq. (4.7), in terms of the basic TF ingredients χ^{cons} , P_μ^{rad} and the center-of-mass angular momentum loss J_{rad} . The new ingredients entering at the $O(G^5)$ level are the following nine functions of γ : $\tilde{\chi}_5^{1\text{cons}}(\gamma)$, $\tilde{\chi}_5^{2\text{cons}}(\gamma)$, $\hat{J}_4^0(\gamma)$, $\hat{J}_4^1(\gamma)$, $\hat{P}_{1+2}^{5,0}(\gamma)$, $\hat{P}_{1+2}^{5,1}(\gamma)$, $\hat{P}_b^5(\gamma)$, $\hat{P}_{1-2}^5(\gamma)$, and $f_b^{G^5, \text{remain}}(\gamma)$. As discussed later, the latter quantity [which was defined in Eq. (2.12)] is degenerate with an additional term in the $O(G^5)$ conservative scattering angle, $\tilde{\chi}_5^{\text{cons}}$. The four, P_{rad}^μ -related, functions of γ , namely $\hat{P}_b^5(\gamma)$, $\hat{P}_{1-2}^5(\gamma)$, $\hat{P}_{1+2}^{5,0}(\gamma)$, and $\hat{P}_{1+2}^{5,1}(\gamma)$ are defined such that (see, e.g., Eq. (11.8) in Ref. [45])

$$\begin{aligned} P_b^{G^5} &= \frac{G^5 m_1^2 m_2^2}{b^5} (m_2 - m_1)(m_1 + m_2) \hat{P}_b^5(\gamma), \\ P_{1-2}^{G^5} &= \frac{G^5 m_1^2 m_2^2}{b^5} (m_2 - m_1)(m_1 + m_2) \hat{P}_{1-2}^5(\gamma), \\ P_{1+2}^{G^5} &= \frac{G^5 m_1^2 m_2^2}{b^5} (m_1 + m_2)^2 [\hat{P}_{1+2}^{5,0}(\gamma) + \nu \hat{P}_{1+2}^{5,1}(\gamma)]. \end{aligned} \quad (4.10)$$

Denoting generically with K “known terms,” we find at 1SF, $\overline{1SF}$, and 2SF levels the following results:

$$\begin{aligned} f_b^{G^5, 1SF} &= -\frac{2\tilde{\chi}_5^{1\text{cons}}}{(\gamma^2 - 1)^2} - \frac{\chi_1 \hat{J}_4^0}{b^5 (\gamma^2 - 1)^2} + K, \\ f_{\tilde{u}_1}^{G^5, 1SF} &= K, \\ f_{\tilde{u}_2}^{G^5, 1SF} &= (\gamma - 1) \hat{P}_{1-2}^5 + (\gamma + 1) \hat{P}_{1+2}^{5,0} + K, \end{aligned} \quad (4.11)$$

$$\begin{aligned} f_b^{G^5, \overline{1SF}} &= \hat{P}_b^5 - \frac{2\tilde{\chi}_5^{1\text{cons}}}{(\gamma^2 - 1)^2} - \frac{\hat{J}_4^0 \chi_1}{(\gamma^2 - 1)^2} + K, \\ f_{\tilde{u}_1}^{G^5, \overline{1SF}} &= K, \\ f_{\tilde{u}_2}^{G^5, \overline{1SF}} &= -(\gamma - 1) \hat{P}_{1-2}^5 + (\gamma + 1) \hat{P}_{1+2}^{5,0} + K, \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} f_b^{G^5, 2SF} &= f_b^{G^5, \text{remain}} + P_b^5 - \frac{4}{(\gamma^2 - 1)^2} \tilde{\chi}_5^{1\text{cons}} \\ &\quad + \frac{2}{(\gamma^2 - 1)^2} \tilde{\chi}_5^{2\text{cons}} \\ &\quad - \frac{2\chi_1}{(\gamma^2 - 1)^2} \hat{J}_4^0 - \frac{\chi_1}{(\gamma^2 - 1)^2} \hat{J}_4^1 + K, \\ f_{\tilde{u}_1}^{G^5, 2SF} &= K, \\ f_{\tilde{u}_2}^{G^5, 2SF} &= (\gamma + 1)(2\hat{P}_{1+2}^{5,0} + \hat{P}_{1+2}^{5,1}) + K. \end{aligned} \quad (4.13)$$

These TF-predicted results satisfy the constraints coming from the linear momentum balance, namely

$$P_{\text{rad}}^\mu = -\Delta p_1^\mu - \Delta p_2^\mu. \quad (4.14)$$

This linear momentum balance implies the following constraints between the expansion coefficients of the impulse and those of the linear momentum loss

$$\begin{aligned} \hat{P}_b^5(\gamma) &= f_b^{G^5, 1SF} - f_b^{G^5, \overline{1SF}}, \\ \hat{P}_{1-2}^5(\gamma) &= \frac{1}{2(\gamma - 1)} [f_{\tilde{u}_1}^{G^5, 1SF} - f_{\tilde{u}_1}^{G^5, \overline{1SF}} - (f_{\tilde{u}_2}^{G^5, 1SF} - f_{\tilde{u}_2}^{G^5, \overline{1SF}})], \\ \hat{P}_{1+2}^{5,0}(\gamma) &= \frac{1}{2(\gamma + 1)} [f_{\tilde{u}_1}^{G^5, 1SF} + f_{\tilde{u}_1}^{G^5, \overline{1SF}} + f_{\tilde{u}_2}^{G^5, 1SF} + f_{\tilde{u}_2}^{G^5, \overline{1SF}}], \\ \hat{P}_{1+2}^{5,1}(\gamma) &= \frac{1}{\gamma + 1} [(f_{\tilde{u}_1}^{G^5, 2SF} + f_{\tilde{u}_2}^{G^5, 2SF}) \\ &\quad - (f_{\tilde{u}_1}^{G^5, 1SF} + f_{\tilde{u}_1}^{G^5, \overline{1SF}} + f_{\tilde{u}_2}^{G^5, 1SF} + f_{\tilde{u}_2}^{G^5, \overline{1SF}})]. \end{aligned} \quad (4.15)$$

Remembering that at any given PM order the mass-shell constraint

$$0 = (p_1^{\text{out}})^2 - (p_1^{\text{in}})^2 = 2p_1 \cdot \Delta p_1 + (\Delta p_1)^2, \quad (4.16)$$

determines the $\tilde{u}_1$ component of Δp_1 in terms of lower G contributions (i.e., $f_{\tilde{u}_1}^{\text{nSF}} = K$ at any PM order) we see that the relations (4.15) uniquely determine $f_{\tilde{u}_2}^{1SF}$ and $f_{\tilde{u}_2}^{\overline{1SF}}$ in terms of P_{rad} , namely at G^5 ,

$$\begin{aligned} f_{\tilde{u}_2}^{G^5, 1SF} &\sim \hat{P}_{1-2}^{5,0} + \hat{P}_{1+2}^{5,0} + K, \\ f_{\tilde{u}_2}^{G^5, \overline{1SF}} &\sim \hat{P}_{1-2}^5 + \hat{P}_{1+2}^{5,0} + K, \\ f_{\tilde{u}_2}^{G^5, 2SF} &\sim \hat{P}_{1-2}^{5,0} + \hat{P}_{1+2}^{5,0} + \hat{P}_{1+2}^{5,1} + K, \end{aligned} \quad (4.17)$$

while they only determine the difference between $f_b^{G^5, 1SF}$ and $f_b^{G^5, \overline{1SF}}$:

$$f_b^{G^5, 1SF} - f_b^{G^5, \overline{1SF}} \sim \hat{P}_b^5 + K. \quad (4.18)$$

On the other hand the balance equation (4.14) gives no constraints on $f_b^{G^5 2SF}$.

It is easily seen that the TF derived results (4.11), (4.12), and (4.13) are compatible with all those constraints. Note for instance how the same combination

$$-\frac{2\tilde{\chi}_5^{1\text{cons}}}{(\gamma^2 - 1)^2} - \frac{\chi_1 \hat{J}_4^0}{(\gamma^2 - 1)^2} \quad (4.19)$$

enters both $f_b^{G^5 1SF}$ and $f_b^{G^5 1SF}$.

V. TAIL-OF-TAIL EFFECTS AND THE APPEARANCE OF p_∞ -ODD CONTRIBUTIONS IN THE CONSERVATIVE ACTION AT G^5

As mentioned at the end of the introduction and summarized in Table I, there is a progressive loss of p_∞ -parity property of the impulse as the PM order increases. In the previous section we showed how at order G^4 it was still possible to identify the conservative (and p_∞ -even) part of the dynamics in various formalisms, TF and WEFT, and to uniquely separate the radiation reaction part in p_∞ -odd and p_∞ -even parts.

A new feature happens at order G^5 (see Fig. 1) because of the arising of nonlinear hereditary (tail) effects. We recall that the presence of tail-transported hereditary (nonlocal in time) effects in binary dynamics was first pointed out in Ref. [54]. Such effects are included in the MPM formalism when solving for the metric at higher PM orders [55]. Tail effects, related to the scattering of gravitational waves by the total energy E of the binary system, arise at various orders in GE . Linear tail effects include only one power of GE (and first arise in the binary dynamics at the G^4/c^8 level, i.e., 4PM and 4PN, see [54,56,57]). Quadratically nonlinear tail effects [55,58] (henceforth called tail-of-tail, tail-squared or quadratic-tail) involve two powers of (GE) . They first arise in the binary dynamics at the G^5/c^{11} level,

i.e., 5PM and 5.5 PN as was progressively understood in Refs. [59–62]. The conservative part of the binary dynamics mediated by quadratic-tail hereditary effects was described at 5.5PN order by an action involving “split” bilinear expressions in the (even parity) quadrupole moment of the binary system [14,63], see Eq. (9.19) in Ref. [63] and Eq. (12.2) in Ref. [14] and Eq. (12.2) in Ref. [14]. See also Refs. [64,65] for a derivation of the corresponding quadrupolar tail-of-tail, Schwinger-Keldysh (doubled-variable) nonconservative action.

Generalizing the derivation of the quadratic-tail, even-quadrupole conservative action of Refs. [14,63] to general- l (even-parity and odd-parity) multipoles, we get the following *conservative* quadratic-tail, or “tail-of-tail,” nonlocal action. This action can be written (modulo integration by parts) in two forms: a manifestly time-symmetric form involving the logarithmic kernel $\ln \frac{c|t-t'|}{2r_0}$ (where r_0 is an arbitrary scale which actually does not contribute to the nonlocal part of the action), namely

$$\begin{aligned} S_{\text{time-sym}}^{\text{tail-of-tail}} = & \frac{1}{2} \left(\frac{GM}{c^3} \right)^2 G \sum_{l \geq 2} \frac{a_l}{c^{2l+1}} \beta_l^{\text{even}} \int dt \\ & \times \int_{-\infty}^{\infty} dt' I_L^{(l+2)}(t) I_L^{(l+2)}(t') \ln \frac{c|t-t'|}{2r_0} \\ & + \frac{1}{2} \left(\frac{GM}{c^3} \right)^2 G \sum_{l \geq 2} \frac{b_l}{c^{2l+3}} \beta_l^{\text{odd}} \int dt \\ & \times \int_{-\infty}^{\infty} dt' J_L^{(l+2)}(t) J_L^{(l+2)}(t') \ln \frac{c|t-t'|}{2r_0}, \quad (5.1) \end{aligned}$$

or a nonmanifestly time-symmetric (“time-split”) form involving a scale-free principal-value integral, namely

$$S_{\text{time-split}}^{\text{tail-of-tail}} = \frac{1}{4} \left(\frac{GM}{c^3} \right)^2 G \int dt \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \mathcal{G}^{\text{split}}(t, \tau), \quad (5.2)$$

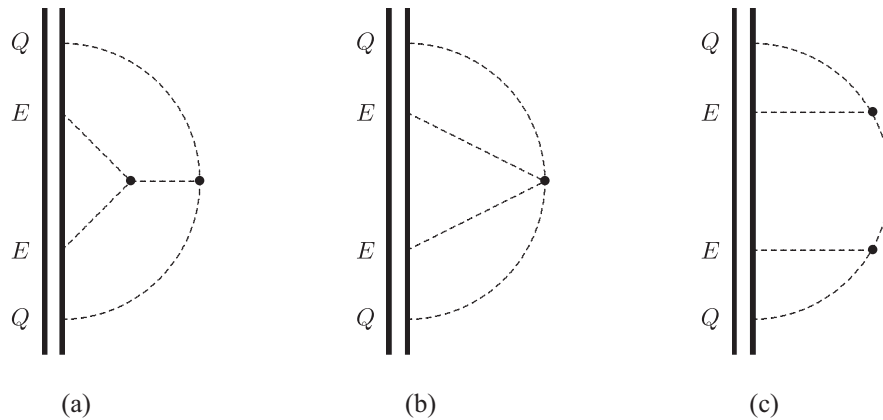


FIG. 1. Sketch of the various quadratic-tail [panels (a) and (b)] and tail-of-tail [panel (c)] diagrammatic contributions to the nonlocal (radiation-zone-mediated) effective action of a binary system. They start at order G^5 and get contributions from all multipole couplings (here indicated as Q).

with

$$\mathcal{G}^{\text{split}}(t, \tau) = \sum_{l \geq 2} \left(\frac{a_l}{c^{2l+1}} \beta_l^{(\text{even})} \mathcal{G}_l^l(t, \tau) + \frac{b_l}{c^{2l+3}} \beta_l^{(\text{odd})} \mathcal{G}_l^l(t, \tau) \right), \quad (5.3)$$

where

$$\begin{aligned} \mathcal{G}_l^l(t, \tau) &\equiv I_L^{(l+1)}(t) [I_L^{(l+2)}(t + \tau) - I_L^{(l+2)}(t - \tau)], \\ \mathcal{G}_l^l(t, \tau) &\equiv J_L^{(l+1)}(t) [J_L^{(l+2)}(t + \tau) - J_L^{(l+2)}(t - \tau)]. \end{aligned} \quad (5.4)$$

Here, $\mathcal{M} \equiv E/c^2$, I_L and J_L are the even- and odd-parity source multipole moments (as defined in the multipolar post-Minkowskian formalism), and a_l and b_l denote the coefficients entering the even and odd multipolar parts of the energy flux [1]

$$\begin{aligned} a_l &= \frac{(l+1)(l+2)}{l(l-1)l!(2l+1)!!}, \\ b_l &= \frac{4l(l+2)}{(l-1)(l+1)l!(2l+1)!!}. \end{aligned} \quad (5.5)$$

In addition $\beta_l^{(\text{even, odd})}$ are the beta function coefficients describing the scale dependence of the multipole moments. $\beta_l^{(\text{even})}$ was first computed in the Appendix of Ref. [54]; its meaning as a coefficient related to scale-dependence was then pointed out in Ref. [58] and understood as a renormalization-group β coefficient in Ref. [10]. According to recent works [66,67] the odd-parity $\beta_l^{(\text{odd})}$'s are equal to their even-parity counterparts.

We recall that linear-tail conservative (time-symmetric) nonlocal actions are logarithmically UV-divergent and must be completed by corresponding nonlocal, time-antisymmetric radiation-reaction terms which are UV-finite (see Sec. VI of [56]). By contrast, the quadratic-tail conservative nonlocal actions, Eqs. (5.1) and (5.2), are UV-finite at $t' - t = \tau = 0$, and must be completed by corresponding (in-out) multipolar radiation-reaction terms which are logarithmically divergent.⁵

The leading PN order value of the action (5.3) in Refs. [14,63] is obtained by using $\beta_2 = -\frac{214}{105}$ and $a_2 = \frac{1}{5}$ and reads, with $B = -\frac{107}{105} = \frac{\beta_2}{2}$,

$$\begin{aligned} S_{\text{quadrupolar}}^{\text{tail-of-tail}} &= -\frac{B}{10} \frac{G}{c^5} \left(\frac{G\mathcal{M}}{c^3} \right)^2 \int dt \\ &\quad \times I_{ij}^{(3)}(t) \int_{-\infty}^{\infty} \frac{d\tau}{\tau} [I_{ij}^{(4)}(t + \tau) - I_{ij}^{(4)}(t - \tau)] \\ &= -\frac{B}{10} \frac{G}{c^5} \left(\frac{G\mathcal{M}}{c^3} \right)^2 \int dt \int_{-\infty}^{\infty} \frac{d\tau}{\tau} \mathcal{G}_I^2(t, \tau). \end{aligned} \quad (5.6)$$

The 5.5PN and $O(G^5)$ level contribution to the conservative scattering angle induced by Eq. (5.6) was computed in Ref. [13], see Eq. (5.10) there, and found to be

$$\chi_{5.5} = -\nu \frac{95872}{675} \frac{p_\infty^6}{j^5} + O(G^6). \quad (5.7)$$

Compared to the leading, Newtonian-order scattering angle $\chi_N \approx \frac{2}{p_{\infty J}} \approx \frac{2GM}{p_\infty b}$, the latter, leading tail-of-tail contribution is $\sim |p_\infty| (\frac{GM}{b})^5$, i.e., it is smaller by a factor $(\frac{|p_\infty|}{c})^3 (\frac{GM}{c^2 b})^4$. As emphasized by our notation, it is the absolute value of p_∞ which enters when expressing $\chi_{5.5}$ in terms of the impact parameter. This can be clearly seen when starting from the manifestly time-symmetric version, Eq. (5.1), of the leading-order tail-of-tail action, $\propto \int \int dt dt' I_{ij}^{(4)}(t) I_{ij}^{(4)}(t') \ln \frac{c|t-t'|}{2r_0}$. We will prove below that the term $\chi_{5.5}$, Eq. (5.7), together with several other TF-computed terms, is included in the recent G^5 1SF result of Ref. [43].

VI. COMPARISON BETWEEN THE $O(G^5)$ TF IMPULSE AND THE 1SF WEFT RESULTS

We have presented in Sec. IV above the TF ingredients entering the 1SF $O(G^5)$ impulse. Namely, from Eqs. (4.11) and (4.12),

- (1) The $O(\nu^1)$ contribution to the 5PM conservative scattering angle: $\tilde{\chi}_5^{1\text{cons}}(\gamma)$;
- (2) The $O(\nu^0)$ contribution to the 4PM angular momentum loss: $\hat{J}_4^0(\gamma)$;
- (3) The $O(G^5)$ (mass-factorized) component of P_{rad}^μ along $u_1^\mu - u_2^\mu$: $\hat{P}_{1-2}^5(\gamma)$;
- (4) The $O(G^5)$ (mass-factorized) component of P_{rad}^μ along $\hat{b}^\mu$: $\hat{P}_b^5(\gamma)$;
- (5) The $O(\nu^0)$, $O(G^5)$ (mass-factorized) component of P_{rad}^μ along $u_1^\mu + u_2^\mu$: $\hat{P}_{1+2}^{5,0}(\gamma)$.

In other words, a PN-exact comparison between TF and WEFT at 1SF would require the knowledge of five independent functions of γ . However, the TF formalism gives only access to the PN expansion of those five functions. The current PN knowledge of these functions in the TF formalism is the following:

- (1) $\tilde{\chi}_5^{1\text{cons}}(\gamma)$ has been computed in Refs. [13–16] up to the absolute 6PN accuracy (including the tail-of-tail 5.5PN contribution highlighted in the previous section),

⁵The latter, external-zone, logarithmic UV-divergences will be compensated by corresponding IR-divergences in the matched source multipole moments.

$$\begin{aligned}\tilde{\chi}_{5\text{TF}}^{1\text{cons}} = & \frac{2}{5p_\infty^3} - \frac{121}{10p_\infty} + \left(-\frac{19457}{60} + \frac{41}{8}\pi^2\right)p_\infty + \left(-\frac{5211479}{4320} + \frac{5069}{144}\pi^2 - \frac{6272}{45}\ln(2p_\infty)\right)p_\infty^3 \\ & + \left(-\frac{782142451}{504000} + \frac{111049}{960}\pi^2 - \frac{74432}{525}\ln(2p_\infty)\right)p_\infty^5 - \frac{1}{2}\frac{95872}{675}p_\infty^6 \\ & + \left(\frac{57623004613}{98784000} - \frac{184881}{4480}\pi^2 - \frac{881392}{11025}\ln(2p_\infty)\right)p_\infty^7 + O(p_\infty^8),\end{aligned}\quad (6.1)$$

where the extra factor 1/2 in the 5.5PN tail-of-tail contribution comes from the definition of $\tilde{\chi}_n$, Eq. (2.3).

(2) $\hat{J}_4^0(\gamma)$ was computed in Ref. [45] up to the 5.5PN absolute accuracy (fractional 3PN), namely

$$\hat{J}_{4\text{TF}}^0(\gamma) = \frac{176}{5}p_\infty + \frac{8144}{105}p_\infty^3 + \frac{448}{5}p_\infty^4 - \frac{93664}{1575}p_\infty^5 + \frac{1184}{21}p_\infty^6 - \frac{4955072}{121275}p_\infty^7 + O(p_\infty^8). \quad (6.2)$$

In addition, Ref. [68] recently computed the even-in- p_∞ part of $\hat{J}_4^0$ to all PN orders, with a result whose PN expansion reads

$$\begin{aligned}\hat{J}_{4\text{even}}^0 = & \frac{448}{5}p_\infty^4 + \left(\frac{1184}{21} - \frac{431936}{1575}\nu\right)p_\infty^6 + \left(-\frac{13736}{315} - \frac{126736\nu}{1575} + \frac{857152}{1575}\nu^2\right)p_\infty^8 \\ & + \left(\frac{724868}{17325} + \frac{3100472}{121275}\nu + \frac{4544}{315}\nu^2 - \frac{472256}{525}\nu^3\right)p_\infty^{10} + O(p_\infty^{12}).\end{aligned}\quad (6.3)$$

Therefore, we can combine the two results to reach the (absolute) 6PN accuracy in the knowledge of $\hat{J}_{4\text{TF}}^0(\gamma)$, namely

$$\hat{J}_{4\text{combined}}^0(\gamma) = \frac{176}{5}p_\infty + \frac{8144}{105}p_\infty^3 + \frac{448}{5}p_\infty^4 - \frac{93664}{1575}p_\infty^5 + \frac{1184}{21}p_\infty^6 - \frac{4955072}{121275}p_\infty^7 - \frac{13736}{315}p_\infty^8 + O(p_\infty^9). \quad (6.4)$$

(3) $\hat{P}_{1-2}^5(\gamma)$ was computed in Ref. [45] up to the 5.5PN absolute accuracy (fractional 3PN), namely

$$\begin{aligned}\hat{P}_{1-2}^5(\gamma) = & \frac{82\pi}{15p_\infty^3} - \frac{5207\pi}{630p_\infty} + \frac{939\pi^3}{280} - \frac{1491\pi}{400} - \frac{963239\pi}{40320}p_\infty - \left(\frac{13603\pi^3}{4480} - \frac{902743\pi}{33600}\right)p_\infty^2 \\ & - \left(\frac{1591\pi}{980}\log\left(\frac{p_\infty}{2}\right) - \frac{313\pi^3}{140} + \frac{4809573323\pi}{434649600}\right)p_\infty^3 + O(p_\infty^4).\end{aligned}\quad (6.5)$$

(4) $\hat{P}_b^5(\gamma)$ was computed in [45] up to the 5.5PN absolute accuracy (fractional 3PN), namely

$$\begin{aligned}\hat{P}_b^5(\gamma) = & -\frac{64}{3p_\infty^2} - \frac{37\pi^2}{20} - \frac{27392}{525} - \frac{30208}{225}p_\infty + \left(-\frac{856768}{33075} - \frac{3429\pi^2}{1120}\right)p_\infty^2 + \frac{462592}{7875}p_\infty^3 \\ & + \left(-\frac{74417152}{363825} - \frac{7915\pi^2}{2688}\right)p_\infty^4 + O(p_\infty^5).\end{aligned}\quad (6.6)$$

(5) $\hat{P}_{1+2}^{5,0}(\gamma)$ was computed in [45] up to the 5.5PN absolute accuracy (fractional 3PN), namely

$$\begin{aligned}\hat{P}_{1+2}^{5,0}(\gamma) = & \frac{61\pi}{5p_\infty^3} + \frac{34073\pi}{1680}\frac{1}{p_\infty} + \frac{297\pi^3}{40} - \frac{23923\pi}{2880}p_\infty + \left(-\frac{31029\pi^3}{2240} + \frac{1484997\pi}{11200}\right)p_\infty^2 \\ & + \left(-\frac{10593\pi}{700}\log\left(\frac{p_\infty}{2}\right) + \frac{99\pi^3}{20} + \frac{34695068413\pi}{620928000}\right)p_\infty^3 + O(p_\infty^4).\end{aligned}\quad (6.7)$$

Injecting the 6PN accurate results for $\tilde{\chi}_5^{\text{TF}}$ and for $\hat{J}_{4\text{combined}}^0(\gamma)$ in Eq. (4.11) we found perfect agreement for $f_b^{G^5\text{1SF}}$ with the (1SF, WEFT) results of Ref. [43].

This agreement is remarkable in view of the presence of both T -even and T -odd contributions in $\chi_5^{1\text{cons}}$ and $\hat{J}_{4\text{combined}}^0(\gamma)$. The T -odd contribution in $\chi_5^{1\text{cons}}$ is linked to the tail-of-tail effects discussed above, while the T -odd terms in $\hat{J}_4^0$ come from time-asymmetric contributions to the angular momentum loss.

Reference [43] has computed the all PN order values of $\hat{P}_{1-2}^5, \hat{P}_b^5, \hat{P}_{1+2}^{5,0}$. The latter three observables are respectively denoted in [43] r_1, r_2 and r_3 . The TF-computed high-PN expansion values of $\hat{P}_{1-2}^5, \hat{P}_b^5, \hat{P}_{1+2}^{5,0}$ displayed above nicely agree, within their accuracy, with the results of [43]. Evidently, one can use the latter results to complete the TF-computed results for those three observables to any needed accuracy. For instance, Ref. [43] allows us to get the next 6PN level terms in

$$\begin{aligned}\hat{P}_b^5|_{6\text{PN}} &= -\frac{99136}{2625} p_\infty^5, \\ \hat{P}_{1-2}^5|_{6\text{PN}} &= \left(\frac{4118497\pi}{44800} - \frac{40459\pi^3}{3584}\right) p_\infty^4, \\ \hat{P}_{1+2}^{5,0}|_{6\text{PN}} &= \left(\frac{100349\pi^3}{17920} - \frac{1815089\pi}{26880}\right) p_\infty^4.\end{aligned}\quad (6.8)$$

Let us finally emphasize that the TF formalism predicts the hidden presence of the angular momentum loss $\hat{J}_4^0$ and $\hat{J}_4^1$ in the b -component of the impulse, see Eqs. (4.11)–(4.13). However, it comes together with the G^5 conservative scattering angle $\tilde{\chi}_5^{1\text{cons}}$. At order G^4 we could

explicitly check the hidden presence of J_{rad} because $\tilde{\chi}_4^{\text{cons}}$ was separately known, see Sec. III above.

As soon as the WEFT formalism will compute the all-PN-order value of $\hat{J}_4^0(\gamma)$ then one will be able to read off the exact value $\tilde{\chi}_5^{1\text{cons}}$ from the current 1SF G^5 results of Ref. [43].

VII. PREDICTIONS FOR THE 2SF-LEVEL IMPULSE

We have presented in Sec. IV above the TF ingredients entering the 2SF $O(G^5)$ impulse. Namely, from Eq. (4.13)

- (1) The $O(\nu^1)$ and $O(\nu^2)$ contributions to the 5PM conservative scattering angle: $\tilde{\chi}_5^{1\text{cons}}(\gamma), \tilde{\chi}_5^{2\text{cons}}(\gamma)$;
- (2) The $O(\nu^0)$ and $O(\nu^1)$ contributions to the 4PM angular momentum loss: $\hat{J}_4^0(\gamma)$ and $\hat{J}_4^1(\gamma)$;
- (3) The $O(G^5)$ component of P_{rad}^μ along $\hat{b}^\mu$: $\hat{P}_b^5(\gamma)$;
- (4) The $O(\nu^0)$ and $O(\nu^1)$, $O(G^5)$ component of P_{rad}^μ along $u_1^\mu + u_2^\mu$: $\hat{P}_{1+2}^{5,0}(\gamma)$ and $\hat{P}_{1+2}^{5,1}(\gamma)$;
- (5) The supplementary contribution to $\Delta p_1^{\text{remain}}$: $f_b^{G^5\text{remain}}(\gamma)$.

In other words, a PN-exact comparison between TF and WEFT at 2SF would require (beyond what was needed at 1SF) the knowledge of three independent functions of γ , namely $\hat{J}_4^1(\gamma), \hat{P}_{1+2}^{5,1}(\gamma), \tilde{\chi}_5^{2\text{cons}}(\gamma)$. However, the TF formalism gives only access to the PN expansion of those three functions. Namely, the current PN knowledge of these functions in the TF formalism is the following:

- (1) $\tilde{\chi}_5^{2\text{cons}}(\gamma)$ has been computed in Refs. [13–16] up to the absolute 6PN accuracy and reads

$$\begin{aligned}\tilde{\chi}_5^{2\text{cons}}(\gamma) &= \frac{1}{5} \frac{1}{p_\infty} + \frac{59}{10} p_\infty + \left(\frac{10681}{144} - \frac{41}{24} \pi^2\right) p_\infty^3 + \left(\frac{1408}{45} \ln(2p_\infty) - \frac{4}{15} \bar{d}_5^2 - \frac{365555}{6048} + \frac{23407}{5760} \pi^2\right) p_\infty^5 \\ &\quad + \left(-\frac{4}{35} q_{4.5}^2 - \frac{2273170241}{2352000} + \frac{1219303}{20160} \pi^2 - \frac{6944}{225} \ln(2p_\infty)\right) p_\infty^7 + O(p_\infty^8).\end{aligned}\quad (7.1)$$

- (2) $\hat{J}_4^1(\gamma)$ was computed in Ref. [45] up to the 5.5PN absolute accuracy $O(p_\infty^7)$ (fractional 3PN), and its knowledge can be raised to the 6PN ($O(p_\infty^8)$) level by using the results of Ref. [68]. This yields the combined result

$$\begin{aligned}\hat{J}_4^1(\gamma) &= -\frac{208}{15} p_\infty^3 + \frac{988}{63} p_\infty^5 - \frac{13312}{525} p_\infty^6 \\ &\quad + \frac{5458}{1575} p_\infty^7 - \frac{208}{225} p_\infty^8 + O(p_\infty^9).\end{aligned}\quad (7.2)$$

- (3) $\hat{P}_{1+2}^{5,1}(\gamma)$ was computed in Ref. [45] up to the 5.5PN absolute accuracy (fractional 3PN), see below.

Concerning the supplementary remain contribution, $f_b^{G^5\text{remain}}(\gamma)$, it was pointed out in Ref. [45] that it could

start at the 5PN level, i.e., its PN expansion would *a priori* read

$$\begin{aligned}f_b^{G^5\text{remain}}(\gamma) &= f_1^{G^5,\text{sup}} p_\infty + f_2^{G^5,\text{sup}} p_\infty^2 \\ &\quad + f_3^{G^5,\text{sup}} p_\infty^3 + O(p_\infty^4).\end{aligned}\quad (7.3)$$

We see from Eq. (4.13) that the supplementary $f_b^{G^5\text{remain}}(\gamma)$ term is degenerate with $\tilde{\chi}_5^{2\text{cons}}(\gamma)$ and can formally be absorbed in a redefinition of the $\tilde{\chi}_5^{2\text{cons}}(\gamma)$ as follows

$$\tilde{\chi}_5^{2\text{cons}'}(\gamma) = \tilde{\chi}_5^{2\text{cons}}(\gamma) + \frac{1}{2} p_\infty^4 f_b^{G^5}(\gamma).\quad (7.4)$$

However, from the TF point of view, the p_∞ structure of $\tilde{\chi}_5^{2\text{cons}}(\gamma)$ is not arbitrary but is predicted by the structure of

the local-in-time and nonlocal-in-time contributions to the dynamics. On the one hand, one expects the local-in-time dynamics to be T -even (and leading to a p_∞ -even conservative scattering angle). On the other hand, the nonlocal-in-time contribution to the conservative dynamics has a p_∞ -character dictated by the structure of the tail-mediated interactions. We saw above that tail-squared effects, connected with the action (5.6), lead to p_∞ -odd contributions to the scattering angle. However, at the 5.5PN level [corresponding to the term $f_2^{G^5, \text{sup}} p_\infty^2$ in Eq. (7.3)] we

have seen above that the conservative scattering angle is linear in ν and therefore does not contribute to the presently considered 2SF level in $f_b^{G^5}$.

Finally, at the 2SF-level, the TF approach makes predictions for the beginning of the PN expansion of two independent functions of γ : $f_b^{G^5, 2\text{SF}}$ and $\hat{P}_{1+2}^{5,1}(\gamma)$ (the latter corresponding to the quantity denoted as r_4 in [43]). The value of $\hat{P}_{1+2}^{5,1}(\gamma)$ has been computed to the absolute 5.5PN level in Ref. [45] and reads

$$\begin{aligned} \hat{P}_{1+2}^{5,1}(\gamma) = & -\frac{55\pi}{12} \frac{1}{p_\infty} + \frac{6427\pi}{10080} p_\infty + \left(-\frac{939\pi^3}{560} + \frac{877\pi}{400} \right) p_\infty^2 + \left(-\frac{4059\pi^3}{1280} + \frac{255491\pi}{10080} \right) p_\infty^3 \\ & + \left(\frac{\pi E_{5,9}^1}{2} - \frac{190387\pi^3}{4480} + \frac{14186677\pi}{33600} \right) p_\infty^4 + O(p_\infty^5). \end{aligned} \quad (7.5)$$

The presently known 5.5PN level knowledge of $\hat{P}_{1+2}^{5,1}(\gamma)$ corresponds to the first four terms in the latter equation, up to the term $\propto p_\infty^3$. For information, we parametrized the next, 6PN-level term, $\propto p_\infty^4$ in the latter equation, by the (currently unknown) coefficient $E_{5,9}^1$ entering the $O(\nu)$ term in $E_5 = E_5(\gamma, \nu)$ defined from

$$E_{\text{rad}} = M\nu^2 \left(\frac{G^3 E_3}{j^3} + \frac{G^4 E_4}{j^4} + \frac{G^5 E_5}{j^5} + \dots \right), \quad (7.6)$$

with (denoting below the fractional PN accuracy such that $\text{nPNabs} = \text{nPNfrac} + 2.5\text{PN}$)

$$E_5 = E_5^{\leq 3\text{PNfrac}} + (E_{5,9}^0 + E_{5,9}^1 \nu + E_{5,9}^2 \nu^2) p_\infty^9 + O(p_\infty^{10}), \quad (7.7)$$

where

$$\begin{aligned} E_5^{\leq 3\text{PNfrac}} = & \pi \left\{ \frac{122}{5} p_\infty^2 + \left(\frac{13831}{280} - \frac{933\nu}{10} \right) p_\infty^4 + \frac{297\pi^2}{20} p_\infty^5 + \left(\frac{2067\nu^2}{10} - \frac{187559\nu}{1680} - \frac{64579}{5040} \right) p_\infty^6 \right. \\ & + \left[\left(-\frac{15291\pi^2}{280} + \frac{296}{25} \right) \nu - \frac{24993\pi^2}{1120} + \frac{9216}{35} \right] p_\infty^7 + \left[-\frac{1823\nu^3}{5} + \frac{12269\nu^2}{80} + \left(-\frac{4059\pi^2}{640} + \frac{76897}{480} \right) \nu \right. \\ & \left. \left. - \frac{10593}{350} \log\left(\frac{p_\infty}{2}\right) + \frac{99\pi^2}{10} + \frac{29573617463}{310464000} \right] p_\infty^8 \right\}, \end{aligned} \quad (7.8)$$

as can be read from Eqs. (C10)–(C13) of Ref. [44] up to the 2PNfrac level, and from Eqs. (6.5) and (6.6) of Ref. [45] for the contributions to 2.5PNfrac and 3PNfrac, respectively. The first (ν^0) and the last (ν^2) contributions to the next, 3.5PNfrac, $O(p_\infty^9)$ term [as written in parametric form in Eq. (7.7)] are given by

$$\begin{aligned} E_{5,9}^0 &= \left(\frac{1577\pi^2}{1120} - \frac{384}{7} \right), \\ E_{5,9}^2 &= \left(\frac{33399\pi^2}{280} - \frac{888}{25} \right). \end{aligned} \quad (7.9)$$

Here $E_{5,9}^0$ is known from Ref. [43], while $E_{5,9}^2$ follows by using the ν -polynomiality property (2.18)

$$E_5 = \frac{\hat{E}_5}{h^6} = \frac{\hat{E}_5^0 + \nu \hat{E}_5^1}{(1 + 2\nu(\gamma - 1))^3}, \quad (7.10)$$

which implies that all the powers of ν starting at ν^2 in E_5 come from PN-expanding the denominator in Eq. (7.10). See also the Appendices B and D below.

The value of $f_b^{G^5, 2\text{SF}}$ is predicted up to the absolute 6PN level and reads (using the corresponding PN expressions for $\hat{J}_2, \hat{J}_3, \hat{J}_4^0, \hat{J}_4^1, \hat{P}_{1+2}^3, \hat{P}_{1+2}^4, \hat{P}_{1+2}^5, \hat{P}_{1+2}^6, \hat{P}_b^5, \chi_1, \tilde{\chi}_2^{\text{cons}}, \tilde{\chi}_3^{\text{cons}}, \tilde{\chi}_3^{1\text{cons}}, \tilde{\chi}_5^{0\text{cons}}, \tilde{\chi}_5^{1\text{cons}}, \tilde{\chi}_5^{2\text{cons}}$)

$$f_{b\text{TF}}^{G^5,2\text{SF}} = f_b^{G^5,2\text{SF},4.5\text{PNab}} + C_{b1}(d_5^{\nu^2}, f_1^{G^5,\text{sup}})p_\infty + C_{b2}(f_2^{G^5,\text{sup}})p_\infty^2 + C_{b3}(q_{4,5}^{\nu^2}, f_3^{G^5,\text{sup}})p_\infty^3 + O(p_\infty^4). \quad (7.11)$$

The 4.5PNabs-accurate TF contribution to $f_b^{G^5,2\text{SF}}$ is fully predicted to have the following value

$$\begin{aligned} f_{b\text{TF}}^{G^5,2\text{SF},4.5\text{PNabs}} = & -\frac{12}{p_\infty^9} + \frac{64}{p_\infty^7} + \frac{496 + 54\pi^2}{p_\infty^5} + \frac{832}{45p_\infty^4} + \left(\frac{1552}{3} + \frac{481\pi^2}{2}\right)\frac{1}{p_\infty^3} + \left(\frac{409888}{1575} - \frac{94\pi^2}{5}\right)\frac{1}{p_\infty^2} \\ & + \left(\frac{25088}{45}\log(2p_\infty) + \frac{21131\pi^2}{72} + \frac{190576}{135}\right)\frac{1}{p_\infty} - \frac{1489\pi^2}{35} + \frac{2283544}{11025}. \end{aligned} \quad (7.12)$$

Beyond the $O(p_\infty^0)$ level there enter several dimensionless parameters that are left undetermined by the TF formalism. On the one hand, the parameters $f_1^{G^5,\text{sup}}$, $f_2^{G^5,\text{sup}}$, $f_3^{G^5,\text{sup}}$ parametrizing $f_b^{G^5,\text{remain}}(\gamma)$, Eq. (7.3), enter at the absolute 5PN, 5.5PN and 6PN levels, respectively. On the other hand, the parameters $\bar{d}_5^{\nu^2}$ and $q_{4,5}^{\nu^2}$, which respectively belong to the absolute 5PN and 6PN levels, enter the above expression in the form

$$\sim(\bar{d}_5^{\nu^2} + \dots)p_\infty + (q_{4,5}^{\nu^2} + \dots)p_\infty^3. \quad (7.13)$$

Though the full value of $\bar{d}_5^{\nu^2}$ has not been yet determined, the work of Refs. [69,70] (see also Eq. (9.42) in [44]) has determined its transcendental contribution (which comes from the exchange of potential gravitons), namely

$$\bar{d}_5^{\nu^2} = \frac{306545}{512}\pi^2 + d_5^{\text{rat}}, \quad (7.14)$$

where d_5^{rat} is a rational number. Inserting the latter result in Eq. (7.11) above leads to the following TF-based prediction for the $O(p_\infty^1)$ (5PNabs) coefficient C_{b1} entering the $O(p_\infty^1)$ contribution to $f_b^{G^5,2\text{SF}}(\gamma)$

$$\begin{aligned} C_{b1} = & \frac{8}{15}d_5^{\text{rat}} + f_1^{G^5,\text{sup}} + \frac{112221208}{23625} \\ & + \frac{4225\pi^2}{36} + \frac{794624}{1575}\log(2p_\infty). \end{aligned} \quad (7.15)$$

We note in passing that the coefficient of π^2 in the combined result (7.15) is significantly simpler than the one appearing in the original (not combined) expression. Though only the $p_\infty \ln(2p_\infty)$ term is uniquely predicted by this result, it will be interesting to see whether the π^2 term in the latter result agrees with future 2SF-level G^5 scattering results. If it is the case, we are entitled to absorb $f_1^{G^5,\text{sup}}$ in a redefinition of the rational part, d_5^{rat} of $\bar{d}_5^{\nu^2}$, say

$$\frac{8}{15}d_5^{\text{rat}'} = \frac{8}{15}d_5^{\text{rat}} + f_1^{G^5,\text{sup}}. \quad (7.16)$$

At the $O(p_\infty^2)$ level (5.5PNabs), the corresponding term in $f_b^{G^5,2\text{SF}}(\gamma)$ reads

$$C_{b2} = \frac{67876972}{363825} + f_2^{G^5,\text{sup}} + \frac{23783}{3360}\pi^2. \quad (7.17)$$

As in the discussion of the $O(p_\infty^1)$ term, we expect our undetermined parameter $f_2^{G^5,\text{sup}}$ to be, similarly to the presently undetermined rational (nonlocal-related) contribution to $\bar{d}_5^{\nu^2}$, to be rational. In view of the fact that $f_2^{G^5,\text{sup}}$ was an addition to the algebraically simplest solution to the constraints (i), (ii), (iii) above (see Eq. (12.51) in Ref. [45]) that have to be satisfied by $\Delta p_1^{\text{remain}}$, it is even reasonable to expect that $f_2^{G^5,\text{sup}}$ may simply vanish. An additional argument suggesting that $f_2^{G^5,\text{sup}}$ might vanish is the fact that the direct impact of the G^2 radiation-reaction force on scattering proceeds via the $O(G^3)\chi'^{\text{rad}}$ term whose square is $O(G^6)$ and not $O(G^5)$, see discussion in the Concluding Remarks below. Indeed, the latter fact indicates that there is no ambiguity at $O(G^5)$ linked to radiation-reaction squared effects. In this case both the π^2 contribution and the rational contribution in Eq. (7.17) should agree with future 2SF-level G^5 scattering results.

At $O(p_\infty^3)$ the contribution to $f_b^{G^5,2\text{SF}}(\gamma)$ reads

$$\begin{aligned} C_{b3} = & \frac{8}{35}q_{4,5}^{\nu^2} + f_3^{G^5,\text{sup}} + \frac{64192021}{308700} \\ & + \frac{28637}{360}\pi^2 + \frac{841216}{2205}\ln(2p_\infty). \end{aligned} \quad (7.18)$$

Again it will be interesting to compare this TF-based result with future 2SF G^5 results (contrary to the 5PNabs level parameter $\bar{d}_5^{\nu^2}$ the transcendental part of 6PN-level one $q_{4,5}^{\nu^2}$ has not been determined).

Let us finally emphasize that one of the defining characteristics of the TF approach is to point out the presence of the angular momentum loss in the radiation reacted impulse. It would be interesting to understand this occurrence within the WEFT formalism. Furthermore, when the $O(G^4)$ angular momentum loss is computed it

will be interesting to compare the 2SF-level TF prediction to the corresponding 2SF EFT impulse result.

VIII. CONCLUDING REMARKS

In the present work we summarized the results of the TF formalism concerning black hole scattering that were disseminated in the series of papers that developed the tutti frutti formalism (starting with Ref. [12] and including [13–16]). We found a way to simplify the structure of the TF-obtained radiation-reacted impulse by rewriting it in the following form [see Eq. (2.14)]:

$$\Delta p_a^\mu = \Delta p_a^{\text{cons}\mu} + \Delta p_a^{\text{rr lin } J_{\text{rad}}\mu} + \Delta p_a^{\text{rr lin } P_{\text{rad}}\mu} + \Delta p_1^{\text{rr remain sup } \mu}. \quad (8.1)$$

Here $\Delta p_a^{\text{cons}\mu}$ was defined in Eq. (2.2) above, $\Delta p_a^{\text{rr lin } J_{\text{rad}}\mu}$ was defined in Eq. (2.15), $\Delta p_a^{\text{rr lin } P_{\text{rad}}\mu}$ was defined in Eq. (2.13) and $\Delta p_1^{\text{rr remain sup } \mu}$ was defined in Eq. (2.12). This form is a 4-vectorial generalization of the original linear-response formula of Ref. [51] in that it decomposes the total impulse Δp_a^μ in: (1) a conservative contribution; (2) a radiative contribution linear in the c.m.-radiated angular momentum, J_{rad} ; (3) a radiative contribution linear in the radiated four momentum P_{rad}^μ ; and (4) a supplementary contribution which starts only at the G^5 order. Both $\Delta p_a^{\text{rr lin } J_{\text{rad}}\mu}$ and $\Delta p_a^{\text{rr lin } P_{\text{rad}}\mu}$ start at order G^3 ,

$$\begin{aligned} \Delta p_a^{\text{rr lin } J_{\text{rad}}\mu} &\sim G^3 + G^4 + G^5 + \dots \\ \Delta p_a^{\text{rr lin } P_{\text{rad}}\mu} &\sim G^3 + G^4 + G^5 + \dots \end{aligned} \quad (8.2)$$

By definition of the TF formalism, each term in the decomposition (8.1) is polynomial in the two masses. In addition, defining

$$0 = p_{a\text{out}}^{\text{cons}\mu} = m_a u_a^\mu + \Delta p_a^{\text{cons}\mu}, \quad (8.3)$$

taking into account that $p_{a\text{out}}^{\text{cons}\mu}$ separately satisfies the mass-shell condition $(p_{a\text{out}}^{\text{cons}})^2 = (p_{a\text{in}}^{\text{cons}})^2 = -m_a^2$ and remembering that $\Delta p_a^{\text{rr lin } J_{\text{rad}}\mu}$ and $\Delta p_a^{\text{rr lin } P_{\text{rad}}\mu}$ both start at order G^3 , Eq. (8.2), the mass-shell constraint on the total, radiation-reacted impulse reads

$$\begin{aligned} 0 &= (p_{a\text{out}}^{\text{cons}} + \Delta p_a^{\text{rr lin } J_{\text{rad}}} + \Delta p_a^{\text{rr lin } P_{\text{rad}}}) \cdot p_{a\text{in}}^2 \\ &= 2p_{a\text{out}}^{\text{cons}} \cdot \Delta p_a^{\text{rr lin } J_{\text{rad}}} + 2p_{a\text{out}}^{\text{cons}} \cdot \Delta p_a^{\text{rr lin } P_{\text{rad}}} \\ &\quad + O(G^6). \end{aligned} \quad (8.4)$$

Each term on right-hand-side of the latter equation (which starts at order G^3) vanishes separately. We note that if one extends the definition of the J_{rad} part of the impulse by defining it as being the all-order G -expansion of, for example, $\Delta p_1^{\text{cons}\mu}(\chi^{\text{cons}}(J - \frac{1}{2}J_{\text{rad}}))$, the so defined

radiation-reacted nonlinear-in- J_{rad} impulse, $\Delta p_1^{\text{nonlin}, J_{\text{rad}}}$, would *exactly* satisfy the mass-shell condition. Any such definition involving nonlinear-in- J_{rad} extensions of J differs anyway from $\Delta p_1^{\text{nonlin}, J_{\text{rad}}}$ at $O(G^6)$.

At order G^4 we compared the TF predicted result (8.1) to the state-of-the-art post-Minkowskian results. We highlighted the agreement between the results of Ref. [41] and two structural features predicted by the TF formalism:

- (i) the simple proportionality relation between the “2 rad” dissipative component along b computed in [41] and the projection of the $O(G^4)$ radiated four momentum along the vector e_x^μ

$$c_{1b,2\text{ rad}}^{\text{4diss}} = \frac{m_1}{m_2 - m_2} P_{xG^4}^{\text{rad}}; \quad (8.5)$$

- (ii) the possibility to extract the exact value of the $O(G^3)$ radiated angular momentum from the “1 rad” dissipative component along b computed in [41]. As exhibited in Eqs. (3.23) and (3.24), the so extracted value of $\hat{J}_3$ coincides with the results obtained in Ref. [40]. In addition we recalled that the PN-accurate expansion of the radiation reacted impulse which was computed in TF at the absolute 6PN accuracy agreed with the PN expansion of the exact G^4 result of [41].

At order G^5 , we compared the TF predicted result (4.1) to the state-of-the-art post-Minkowskian results [43], which are currently limited to the 1SF level in the sense of the mass decomposition (4.7) and (4.8) (i.e., terms with first-order in either m_1/m_2 or m_2/m_1). The TF predictions for the structure of the 1SF G^5 impulse were given in Eqs. (4.11) and (4.12), where we recall that the notation “K” denotes terms known from previous orders. The full expressions of the TF-predicted impulse coefficients are given in the Appendix. We highlighted the presence of the ν^2 -rescaled $O(G^4)$ angular momentum loss $\hat{J}_4^0(\gamma)$, as defined in the third of Eq. (2.17) (see Eq. (6.4) for its PN evaluation). When the WEFT formalism will compute the exact value of $\hat{J}_4^0(\gamma)$ one will be able to read-off the exact value of $\chi_5^{\text{cons}1}$ from [43]. At the time being we know from the TF formalism high-PN accuracy values for all the ingredients entering the TF-predicted impulse, see Eqs. (6.4)–(6.8) displayed in Sec. VI. In particular, we highlight again that a subtle nonlocal 5.5PN tail-squared contribution to $\tilde{\chi}_5^{\text{cons}}$ enters $f_b^{G^5\text{1SF}}$ and agrees with the corresponding recent results of [43]. We presented in Eqs. (5.1) and (5.2) the all-multipole generalization of the quadrupolar-level action of Refs. [14,63] describing the $O(G^5)$ nonlocal-in-time conservative dynamics, mediated by time-symmetrized tail-squared interactions. We leave to future works a corresponding PN-extension of the tail-of-tail contributions to scattering, Eq. (5.10) in Ref. [14] where it was first derived (see Eq. (5.7) here). It would be

interesting to have an independent derivation of the nonlocal tail-of-tail action within the NRGR approach.

We also found agreement between TF and WEFT at the next, 6PN level. This agreement was obtained by combining TF results with a recent, partial, computation of $\hat{J}_4^0$ [68].

The TF-predicted PN expansion of three more independent functions of γ related to radiated linear momentum, $\hat{P}_{1-2}^{G^5}$, $\hat{P}_b$, $\hat{P}_{102}^{5,0}$ have been found to be in perfect agreement with corresponding results in Ref. [43].

We highlighted in Eq. (4.15) the extent to which the linear momentum balance, $\Delta p_1^\mu + \Delta p_2^\mu + P_{\text{rad}}^\mu = 0$ determines the expansion coefficients of the individual impulses Δp_1 and Δp_2 . When combining these constraints with the mass-shell constraint, Eq. (4.16), the main conclusion is that, at order G^5 , all the separate impulse components are determined except for the average $\frac{1}{2}(f_b^{G^5,1\text{SF}} + f_b^{G^5,\overline{1\text{SF}}})$ and $f_b^{G^5,2\text{SF}}$. Concerning the 1SF level, only the difference $f_b^{G^5,1\text{SF}} - f_b^{G^5,\overline{1\text{SF}}}$ is determined from the b component of P_{rad}^μ (together with known lower-order results). At the 1SF level, the resulting ambiguity in the separate determination of $f_b^{G^5,1\text{SF}}$ and $f_b^{G^5,\overline{1\text{SF}}}$ is completely lifted in the TF formalism which expresses both $f_b^{G^5,1\text{SF}}$ and $f_b^{G^5,\overline{1\text{SF}}}$ in terms of the three TF-meaningful observables: $\tilde{\chi}_5^{1\text{cons}}$, $\hat{J}_4^0$ and P_b^5 . We note in particular that the fact that $\hat{P}_b^5$ contributes only to $f_b^{G^5,\overline{1\text{SF}}}$ and not to $f_b^{G^5,1\text{SF}}$ is related to the constraint stated in [45] that terms in Δp_1 which are nonlinear in the radiation reaction force acting on body 1, should contain a factor m_1^3 . [At G^4 the same argument allowed the TF formalism to predict the full structure of Δp_1^π in terms of P_{rad}^μ and J_{rad}^μ .]

However, at the 2SF level there exists a remaining ambiguity in the TF determination of $\Delta p_{1b}^{2\text{SF}}$, as displayed in the first Eq. (4.13). The TF formalism expresses $f_b^{G^5,2\text{SF}}$ in terms of six ingredients, namely $f_b^{G^5}$, $\hat{P}_b^5$, $\tilde{\chi}_5^{1\text{cons}}$, $\tilde{\chi}_5^{2\text{cons}}$, $\hat{J}_4^0$ and $\hat{J}_4^1$. Among these ingredients, the last five of them are TF-predicted at the current 6PN accuracy as displayed in Sec. VII.

On the other hand, the function $f_b^{G^5\text{remain}}(\gamma)$ entering the supplementary impulse contribution $\Delta p_1^{\text{rr remain sup}}$, Eq. (2.12), is only constrained by the TF formalism to start at the 5PN order. As a consequence, the TF predictions for the value of $f_b^{G^5\text{remain}}(\gamma)$ is only fully predicted at the 4.5PNabs accuracy, as displayed in Eq. (7.12). Beyond this order, as discussed in Sec. VII, the TF formalism can however predict the value of the logarithmic coefficient at 5PN order. Moreover, if one assumes that the supplementary contribution has rational PN-expansion coefficients, Eq. (7.15) predicts the π^2 term in the 5PN contribution $f_{b,\text{TF}}^{G^5,2\text{SF}}(\gamma)$.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [71].

APPENDIX A: NOTATION AND USEFUL RELATIONS

We collect here some definitions and relations which are often used. We use the mostly positive signature.

The masses of the two bodies are denoted as m_1 and m_2 , with the convention $m_1 \leq m_2$. The symmetric mass ratio $\nu \equiv (m_1 m_2)/(m_1 + m_2)^2$ is the ratio of the reduced mass $\mu \equiv (m_1 m_2)/(m_1 + m_2)$ to the total mass $M = m_1 + m_2$.

The 4-momenta in the asymptotic future ($t \rightarrow +\infty$, outgoing) are denoted by $p_a^{\text{out}} = m_a u_a^{\text{out}}$, with $a = 1, 2$, and the asymptotic energies by E_a^{out} . To simplify notation, in the asymptotic past ($t \rightarrow -\infty$, incoming) we will often omit the corresponding “in” label for all these quantities. We often work within the incoming c.m. frame of the system, with time axis

$$U \equiv \frac{p_1 + p_2}{|p_1 + p_2|} = \frac{m_1 u_1 + m_2 u_2}{E}, \quad (\text{A1})$$

where $E = E_1 + E_2$ is the incoming c.m. energy, which is related to the incoming value of the momentum by

$$P_{\text{c.m.}} = \frac{m_1 m_2}{E} \sqrt{\gamma^2 - 1} = \frac{m_1 m_2}{E} p_\infty, \quad (\text{A2})$$

with

$$\gamma \equiv -u_1 \cdot u_2 = -\frac{p_1 \cdot p_2}{m_1 m_2}, \quad p_\infty \equiv \sqrt{\gamma^2 - 1}, \quad (\text{A3})$$

so that $EP_{\text{c.m.}} = m_1 m_2 p_\infty$. The total incoming energy can also be written as

$$\frac{E}{Mc^2} \equiv h(\gamma, \nu) = \sqrt{1 + 2\nu(\gamma - 1)}, \quad (\text{A4})$$

implying that $P_{\text{c.m.}} = \mu p_\infty / h$ and

$$\frac{GE}{b} \equiv \frac{GMh}{b} = \frac{p_\infty}{j}, \quad (\text{A5})$$

where b is the impact parameter and

$$j \equiv \frac{cJ_{\text{c.m.}}}{Gm_1m_2} = \frac{cJ_{\text{c.m.}}}{GM\mu}, \quad (\text{A6})$$

is a dimensionless rescaled version of the total center-of-mass angular momentum $J_{\text{c.m.}}$ at $t \rightarrow -\infty$. The individual incoming energies (see Eq. (A9) of Ref. [44]) read

$$\frac{E_1}{m_1} = \frac{m_1 + \gamma m_2}{E}, \quad \frac{E_2}{m_2} = \frac{m_2 + \gamma m_1}{E}, \quad (\text{A7})$$

so that, for example,

$$\frac{m_1 E_2 + m_2 E_1}{E} = m_1 m_2 (m_1 + m_2) \frac{(\gamma + 1)}{E^2}. \quad (\text{A8})$$

The vectorial impact parameter (orthogonal to u_1 and u_2) $\mathbf{b}_{12} = \mathbf{b}_1 - \mathbf{b}_2 = b\hat{\mathbf{b}}_{12}$ together with the conservative scattering angle χ_{cons} enters the definition of the Cartesianlike basis vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ as follows (see Eq. (3.49) of Ref. [44]):

$$\begin{aligned} \mathbf{e}_x &= \cos \frac{\chi_{\text{cons}}}{2} \hat{\mathbf{b}} + \sin \frac{\chi_{\text{cons}}}{2} \mathbf{n}, \\ \mathbf{e}_y &= -\sin \frac{\chi_{\text{cons}}}{2} \hat{\mathbf{b}} + \cos \frac{\chi_{\text{cons}}}{2} \mathbf{n}, \end{aligned} \quad (\text{A9})$$

where $\mathbf{n}$ is the direction of the incoming momenta (orthogonal to the c.m. four velocity, $\mathbf{n} \cdot U = 0$),

$$\mathbf{n} = \frac{m_1 m_2}{P_{\text{c.m.}} E} \left(\frac{E_2}{m_2} u_1 - \frac{E_1}{m_1} u_2 \right) = \frac{(u_2 \wedge u_1) \cdot U}{\sqrt{\gamma^2 - 1}}. \quad (\text{A10})$$

Here the wedge product of two vectors A and B is standardly defined as $A \wedge B = A \otimes B - B \otimes A$.

Boldface vectors denote spatial vectors in the c.m. frame with time axis U : $p_a = m_a u_a = E_a U + \mathbf{p}_a$ (where $\mathbf{p}_a$ is orthogonal to U , and $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{P}_{\text{c.m.}}$). When describing the conservative scattering it is useful to introduce the c.m. direction of the (conservative) outgoing momenta, $\mathbf{n}_{\text{out}}^{\text{cons}}$, as well as its associated orthogonal direction $\hat{\mathbf{B}}$, namely

$$\begin{aligned} \hat{\mathbf{B}} &= \cos(\chi_{\text{cons}}) \hat{\mathbf{b}} + \sin(\chi_{\text{cons}}) \mathbf{n}, \\ \mathbf{n}_{\text{out}}^{\text{cons}} &= -\sin(\chi_{\text{cons}}) \hat{\mathbf{b}} + \cos(\chi_{\text{cons}}) \mathbf{n}. \end{aligned} \quad (\text{A11})$$

Note the relation

$$\hat{\mathbf{B}} = -\frac{d}{d\chi_{\text{cons}}} \mathbf{n}_{\text{out}}^{\text{cons}}. \quad (\text{A12})$$

The dyad $(\hat{\mathbf{B}}, \mathbf{n}_{\text{out}}^{\text{cons}})$ differs from the incoming dyad $(\hat{\mathbf{b}}, \mathbf{n})$ by a rotation of angle χ_{cons} . The dyad $(\mathbf{e}_x, \mathbf{e}_y)$ is midway between the latter two dyads, being obtained from the incoming dyad by a rotation of angle $\frac{1}{2}\chi_{\text{cons}}$.

APPENDIX B: PN-EXPANDED FORMS OF VARIOUS QUANTITIES

We list in Table II, and Table IV the PN-expanded forms of various quantities at $O(G^3)$, $O(G^4)$, and $O(G^5)$. Table III instead contains the exact expressions for the scattering angle in the Schwarzschild spacetime at PM orders 1 to 7.

1. P_b and P_n vs $P_{1\pm 2}$ at all orders in G

Starting from the relations (valid at all orders in G)

$$\begin{aligned} 2P_{1+2} &= \frac{E^{\text{rad}}}{h} + \frac{P_n}{\sqrt{\gamma^2 - 1}} \left(\frac{E_2}{m_2} - \frac{E_1}{m_1} \right), \\ 2P_{1-2} &= \frac{(m_1 - m_2)}{Mh} E^{\text{rad}} + \frac{P_n}{\sqrt{\gamma^2 - 1}} \left(\frac{E_2}{m_2} + \frac{E_1}{m_1} \right), \end{aligned} \quad (\text{B1})$$

TABLE II. PN-expanded forms of the various quantities at $O(G^3)$, $O(G^4)$ and $O(G^5)$.

$\hat{P}_{1+2}^3$	$\pi \left[\frac{37}{30} p_\infty^3 + \frac{839}{1680} p_\infty^3 + \frac{2699}{2016} p_\infty^5 - \frac{1531643}{1182720} p_\infty^7 + \frac{70348799}{61501440} p_\infty^9 - \frac{30685679}{30750720} p_\infty^{11} + O(p_\infty^{13}) \right]$
$\hat{J}_2$	$\frac{16}{5} p_\infty^3 + \frac{176}{35} p_\infty^5 - \frac{608}{315} p_\infty^7 + \frac{3712}{3465} p_\infty^9 - \frac{10496}{15015} p_\infty^{11} + O(p_\infty^{13})$
$\hat{J}_3$	$\pi \left[\frac{28}{5} p_\infty^2 + \frac{739}{84} p_\infty^4 - \frac{5777}{2520} p_\infty^6 + \frac{115769}{126720} p_\infty^8 - \frac{16548173}{23063040} p_\infty^{10} + \frac{3912731}{6589440} p_\infty^{12} + O(p_\infty^{13}) \right]$
$\hat{J}_4^0$	$\frac{176}{5} p_\infty^3 + \frac{8144}{105} p_\infty^5 + \frac{448}{5} p_\infty^7 - \frac{93664}{1575} p_\infty^9 + \frac{1184}{21} p_\infty^{11} - \frac{4955072}{121275} p_\infty^{13} - \frac{13736}{315} p_\infty^8 + O(p_\infty^9)$
$\hat{J}_4^1$	$-\frac{208}{15} p_\infty^3 + \frac{988}{63} p_\infty^5 - \frac{13312}{525} p_\infty^7 + \frac{5458}{1575} p_\infty^9 - \frac{208}{225} p_\infty^{11} + O(p_\infty^9)$
$\hat{E}_4^0(\gamma)$	$\frac{1568}{45} p_\infty^3 + \frac{18608}{525} p_\infty^5 + \frac{3136}{45} p_\infty^7 + \frac{220348}{11025} p_\infty^9 + \frac{1216}{105} p_\infty^{11} - \frac{151854}{13475} p_\infty^{13} + \frac{117248}{1575} p_\infty^{10} - \frac{405087523}{9909900} p_\infty^{11} - \frac{224512}{3465} p_\infty^{12} + O(p_\infty^{13})$
$\hat{E}_4^1(\gamma)$	$-\frac{352}{45} p_\infty^5 + \frac{1736}{225} p_\infty^7 - \frac{704}{45} p_\infty^9 + \frac{8068}{4725} p_\infty^{11} - \frac{1808}{225} p_\infty^{13} + \frac{1967239}{103950} p_\infty^{11} - \frac{47176}{4725} p_\infty^{12} + O(p_\infty^{13})$
$\hat{P}_b^4$	$-\pi \left[\frac{37}{30} p_\infty^2 + \frac{1661}{560} p_\infty^4 + \frac{1491}{400} p_\infty^6 + \frac{23563}{10080} p_\infty^8 - \frac{26757}{5600} p_\infty^{10} + \frac{700793}{506880} p_\infty^{12} + \frac{4046561}{451584} p_\infty^{14} - \frac{8085643}{5591040} p_\infty^{16} - \frac{628523873}{41395200} p_\infty^{18} \right]$
$\hat{P}_{1+2}^4$	$\frac{784}{45} p_\infty^3 + \frac{2168}{175} p_\infty^5 + \frac{1568}{45} p_\infty^7 + \frac{98666}{11025} p_\infty^9 - \frac{512}{105} p_\infty^{11} - \frac{2702747}{363825} p_\infty^{13} + \frac{62392}{1575} p_\infty^{15} - \frac{2178650927}{138738600} p_\infty^{17} - \frac{2335876}{51975} p_\infty^{19}$
$\hat{P}_{1-2}^4$	$\frac{176}{45} p_\infty^3 + \frac{72}{25} p_\infty^5 + \frac{352}{45} p_\infty^7 - \frac{9746}{4725} p_\infty^9 + \frac{448}{75} p_\infty^{11} - \frac{484019}{51975} p_\infty^{13} + \frac{26024}{4725} p_\infty^{15} + \frac{157355621}{46246200} p_\infty^{17} - \frac{79612}{14553} p_\infty^{19}$

TABLE III. Exact expressions for the scattering angle in the Schwarzschild spacetime.

$\chi_1^{\text{Schw}}(p_\infty)$	$\frac{1}{p_\infty} + 2p_\infty$
$\chi_2^{\text{Schw}}(p_\infty)$	$\pi(\frac{3}{2} + \frac{15}{8}p_\infty^2)$
$\chi_3^{\text{Schw}}(p_\infty)$	$-\frac{1}{3p_\infty^3} + \frac{4}{p_\infty} + 24p_\infty + \frac{64}{3}p_\infty^3$
$\chi_4^{\text{Schw}}(p_\infty)$	$\pi(\frac{105}{8} + \frac{315}{8}p_\infty^2 + \frac{3465}{128}p_\infty^4)$
$\chi_5^{\text{Schw}}(p_\infty)$	$\frac{1}{5p_\infty^5} - \frac{2}{p_\infty} + \frac{32}{p_\infty} + 320p_\infty + 640p_\infty^3 + \frac{1792}{5}p_\infty^5$
$\chi_6^{\text{Schw}}(p_\infty)$	$\pi(\frac{1155}{8} + \frac{45045}{64}p_\infty^2 + \frac{135135}{128}p_\infty^4 + \frac{255255}{512}p_\infty^6)$
$\chi_7^{\text{Schw}}(p_\infty)$	$-\frac{1}{7p_\infty^7} + \frac{8}{5p_\infty^5} - \frac{16}{p_\infty} + \frac{320}{p_\infty} + 4480p_\infty + 14336p_\infty^3 + \frac{86016}{5}p_\infty^5 + \frac{49152}{7}p_\infty^7$

TABLE IV. First terms in the PN expansion of $\hat{P}_b^{4,(b)}$, $\hat{P}_b^{5,(b)}$, $\hat{P}_n^{3,(b)}$, $\hat{P}_n^{4,(b)}$, and $\hat{P}_n^{5,(b)}$.

$\hat{P}_b^{4,(b)}$	$\pi[-\frac{37}{30} - \frac{1661}{560}p_\infty^2 - \frac{1491}{400}p_\infty^3 - \frac{23563}{10080}p_\infty^4 + \frac{26757}{5600}p_\infty^5 - \frac{700793}{506880}p_\infty^6 - \frac{238033}{26564}p_\infty^7 + O(p_\infty^8)]$
$\hat{P}_b^{5,(b)}$	$-\frac{64}{3p_\infty^7} - \frac{27392}{525} - \frac{37}{20}\pi^2 - \frac{30208}{225}p_\infty + (-\frac{856768}{33075} - \frac{3429}{1120}\pi^2)p_\infty^2 + \frac{462592}{7875}p_\infty^3 - \frac{7915}{2688}\pi^2 - \frac{74417152}{363825}p_\infty^4 - \frac{99136}{2625}p_\infty^5 + O(p_\infty^6)$
$\hat{P}_n^{3,(b)}$	$\pi[\frac{37}{30}p_\infty^2 + (-\frac{37}{60}\nu + \frac{839}{1680})p_\infty^4 + (-\frac{107}{1120}\nu + \frac{37}{80}\nu^2 + \frac{2699}{2016})p_\infty^6 + (-\frac{27581}{40320}\nu - \frac{197}{4480}\nu^2 - \frac{37}{96}\nu^3 - \frac{1531643}{1182720})p_\infty^8 + O(p_\infty^{10})]$
$\hat{P}_n^{4,(b)}$	$\frac{64}{3} + (-\frac{32}{3}\nu + \frac{1664}{175})p_\infty^2 + \frac{128}{3}p_\infty^3 + (8\nu^2 - \frac{1096}{525}\nu + \frac{227776}{33075})p_\infty^4 + (-\frac{64}{3}\nu + \frac{192}{175})p_\infty^5$ $+ (-\frac{118676}{33075}\nu - \frac{76}{175}\nu^2 - \frac{20}{3}\nu^3 - \frac{1218176}{72765})p_\infty^6 + (\frac{8528}{189} + \frac{2512}{525}\nu + 16\nu^2)p_\infty^7 + O(p_\infty^8)$
$\hat{P}_n^{5,(b)}$	$\pi[\frac{53}{3p_\infty^5} + \frac{60563}{5040} - \frac{161}{12}\nu + (-\frac{1491}{400} + \frac{1509}{140}\pi^2)p_\infty + (-\frac{1298161}{40320} - \frac{7969}{2520}\nu + \frac{107}{12}\nu^2)p_\infty^2$ $+ (\frac{2678867}{16800} - \frac{218867}{67200}\nu - \frac{75661}{4480}\pi^2 + \frac{491447}{67200}\nu - \frac{3957}{560}\pi^2\nu)p_\infty^3$ $+ (\frac{2435}{8064}\nu^2 - \frac{695}{96}\nu^3 - \frac{41053}{2450}\ln(p_\infty) + \frac{32135}{768}\nu + \frac{194769745661}{4346496000} - \frac{4059}{1280}\pi^2\nu + \frac{503}{70}\pi^2 + \frac{41053}{2450}\ln(2))p_\infty^4$ $+ (-\frac{2678867}{16800} + \frac{22984309}{67200}\nu - \frac{7981}{3200}\nu^2 + \frac{75661}{4480}\pi^2 - \frac{41863}{1280}\nu\pi^2 + \frac{2733}{560}\nu^2\pi^2 + \frac{1}{2}\nu E_{5,9}^1)p_\infty^5 + O(p_\infty^6)]$

where

$$\frac{E_2}{m_2} - \frac{E_1}{m_1} = \frac{(\gamma - 1)(m_1 - m_2)}{Mh},$$

$$\frac{E_2}{m_2} + \frac{E_1}{m_1} = \frac{(\gamma + 1)(m_1 + m_2)}{Mh}, \quad (\text{B2})$$

we find

$$P_{1+2} = \frac{1}{2h} \left[E^{\text{rad}} + \frac{(m_1 - m_2)}{M} \sqrt{\frac{\gamma - 1}{\gamma + 1}} P_n \right],$$

$$P_{1-2} = \frac{1}{2h} \left[\frac{(m_1 - m_2)}{M} E^{\text{rad}} + \sqrt{\frac{\gamma + 1}{\gamma - 1}} P_n \right]. \quad (\text{B3})$$

2. P_b , P_n vs P_x , P_y and their expressions in powers of $1/j$ and $1/b$

The PM expansions of P_b and P_n in powers of GM/b read

$$P_b = (m_2 - m_1)\nu^2 \hat{P}_b^{(b)}, \quad P_n = (m_2 - m_1)\nu^2 \hat{P}_n^{(b)}, \quad (\text{B4})$$

where

$$\hat{P}_b^{(b)} = \hat{P}_b^{3,(b)} \left(\frac{GM}{b} \right)^3 + \hat{P}_b^{4,(b)} \left(\frac{GM}{b} \right)^4$$

$$+ \hat{P}_b^{5,(b)} \left(\frac{GM}{b} \right)^5 + O(G^6),$$

$$\hat{P}_n^{(b)} = \hat{P}_n^{3,(b)} \left(\frac{GM}{b} \right)^3 + \hat{P}_n^{4,(b)} \left(\frac{GM}{b} \right)^4$$

$$+ \hat{P}_n^{5,(b)} \left(\frac{GM}{b} \right)^5 + O(G^6). \quad (\text{B5})$$

Let us also recall the relations between j -expanded quantities [referred to the frame $(\mathbf{e}_x, \mathbf{e}_y)$] and b -expanded quantities [referred to the frame $(\hat{\mathbf{b}}, \mathbf{n})$], namely

$$\hat{P}_b^{3,(b)} = 0,$$

$$\hat{P}_b^{4,(b)} = -\frac{h^4}{(\gamma^2 - 1)^2} (-\check{P}_x^4 + \chi_1^{\text{cons}} \check{P}_y^3),$$

$$\hat{P}_b^{5,(b)} = -\frac{h^5}{(\gamma^2 - 1)^{5/2}} (\chi_1^{\text{cons}} \check{P}_y^4 + \chi_2^{\text{cons}} \check{P}_y^3 - \check{P}_x^5),$$

$$\hat{P}_n^{3,(b)} = \check{P}_y^3,$$

$$\hat{P}_n^{4,(b)} = \check{P}_y^4,$$

$$\hat{P}_n^{5,(b)} = \check{P}_y^5 - \frac{1}{2} (\chi_1^{\text{cons}})^2 \check{P}_y^3 + \chi_1^{\text{cons}} \check{P}_x^4, \quad (\text{B6})$$

where $\check{P}_x^n$ and $\check{P}_y^n$ and $\hat{P}_b^{n,(b)}$ are defined such that

$$P_i = (m_2 - m_1)\nu^2 \sum_{n \geq 3} \frac{\check{P}_i^n}{j^n}, \quad i = x, y$$

$$P_b = (m_2 - m_1)\nu^2 \sum_{n \geq 4} \hat{P}_b^{n,(b)} \left(\frac{GM}{b}\right)^n. \quad (\text{B7})$$

APPENDIX C: POLYNOMIALITY STRUCTURE OF VARIOUS QUANTITIES

From the expansions

$$\sin\left(\frac{\chi^{\text{cons}}}{2}\right) = \frac{GMh}{b} \left[\frac{\chi_1^{\text{cons}}}{p_\infty} + \frac{\tilde{\chi}_2^{\text{cons}}}{p_\infty^2} \left(\frac{GM}{b}\right)^1 + \left(\frac{\tilde{\chi}_3^{\text{cons}}}{p_\infty^3} - \frac{(\chi_1^{\text{cons}})^3 h^2}{6p_\infty^3} \right) \left(\frac{GM}{b}\right)^2 + \dots \right],$$

$$\cos\left(\frac{\chi^{\text{cons}}}{2}\right) = 1 - \frac{(\chi_1^{\text{cons}})^2 h^2}{2p_\infty^2} \left(\frac{GM}{b}\right)^2 - \frac{\tilde{\chi}_2^{\text{cons}} h^2 \chi_1^{\text{cons}}}{p_\infty^3} \left(\frac{GM}{b}\right)^3 + \dots, \quad (\text{C1})$$

we recognize that $E^{-1} \sin(\frac{\chi^{\text{cons}}}{2}) = \frac{1}{Mh} \sin(\frac{\chi^{\text{cons}}}{2})$ and $\cos(\frac{\chi^{\text{cons}}}{2})$ are two polynomial function of m_1 and m_2 . For the vector

$$n = \frac{1}{\sqrt{\gamma^2 - 1}} \left(\frac{E_2}{m_2} u_1 - \frac{E_1}{m_1} u_2 \right)$$

$$= \frac{\sqrt{\gamma^2 - 1}}{E} (m_2 \check{u}_1 - m_1 \check{u}_2), \quad (\text{C2})$$

we see that the product En is linear in the masses m_1 and m_2 . Consequently, recalling

$$e_x = \cos \frac{\chi^{\text{cons}}}{2} \hat{b} + \sin \frac{\chi^{\text{cons}}}{2} n$$

$$= \cos \frac{\chi^{\text{cons}}}{2} \hat{b} + E^{-1} \sin \frac{\chi^{\text{cons}}}{2} (En)$$

$$Ee_y = -E \sin \frac{\chi^{\text{cons}}}{2} \hat{b} + \cos \frac{\chi^{\text{cons}}}{2} En$$

$$= -E^2 \frac{\sin \frac{\chi^{\text{cons}}}{2}}{E} \hat{b} + \cos \frac{\chi^{\text{cons}}}{2} En, \quad (\text{C3})$$

where $E^2 = m_1^2 + m_2^2 + 2\gamma m_1 m_2$, we have that e_x^μ and Ee_y^μ are both polynomial in masses.

APPENDIX D: RELATIONS AMONG

$$\hat{E}_5 \text{ AND } \hat{P}_{1\pm 2}^{5,0}, \hat{P}_{1\pm 2}^{5,1}$$

We have

$$\hat{P}_{1+2}^{5,0} = \frac{\hat{E}_5^{0\text{SF}}}{(\gamma + 1)(\gamma^2 - 1)^{5/2}} - \frac{\gamma - 1}{\gamma + 1} \hat{P}_{1-2}^5,$$

$$\hat{P}_{1+2}^{5,1} = \frac{\hat{E}_5^{1\text{SF}}}{(\gamma + 1)(\gamma^2 - 1)^{5/2}} + 4 \frac{\gamma - 1}{\gamma + 1} \hat{P}_{1-2}^5, \quad (\text{D1})$$

so that, for example,

$$4\hat{P}_{1+2}^{5,0} + \hat{P}_{1+2}^{5,1} = \frac{4\hat{E}_5^{0\text{SF}} + \hat{E}_5^{1\text{SF}}}{(\gamma + 1)(\gamma^2 - 1)^{5/2}}. \quad (\text{D2})$$

TABLE V. PN expansion of all the structure functions at $O(G^5)$.

$f_{\check{u}_2}^{G^5 \text{ 2SF}}$	$\pi \left[-\frac{18}{p_\infty^6} + \frac{114}{p_\infty^4} + \frac{60}{p_\infty^3} + \left(\frac{3409}{4} + \frac{123\pi^2}{64} \right) \frac{1}{p_\infty^2} + \frac{21787}{140p_\infty} - \frac{148}{5} \log\left(\frac{p_\infty}{2}\right) + \frac{854981\pi^2}{20480} + \frac{115753}{96} + \frac{122533}{720} p_\infty \right. \\ \left. + \left(-\frac{5501}{70} \log\left(\frac{p_\infty}{2}\right) - \frac{965609\pi^2}{40960} + \frac{24412151}{33600} \right) p_\infty^2 + \left(-\frac{10593}{175} \log\left(\frac{p_\infty}{2}\right) + \frac{8613\pi^2}{640} + \frac{69397802563}{155232000} \right) p_\infty^3 \right. \\ \left. + \left(E_{5,9} - \frac{60521}{840} \log\left(\frac{p_\infty}{2}\right) - \frac{3109120219\pi^2}{58720256} + \frac{14402621}{39200} \right) p_\infty^4 + O(p_\infty^5) \right]$
$f_{\check{u}_1}^{G^5 \text{ 2SF}}$	$\pi \left[\frac{18}{p_\infty^6} - \frac{114}{p_\infty^4} - \frac{56}{5p_\infty^3} + \left(-\frac{3409}{4} - \frac{123\pi^2}{64} \right) \frac{1}{p_\infty^2} - \frac{15007}{210p_\infty} - \frac{49345\pi^2}{4096} - \frac{115753}{96} + \frac{148}{5} \log\left(\frac{p_\infty}{2}\right) - \frac{235867}{1260} p_\infty \right. \\ \left. + \left(\frac{5501}{70} \log\left(\frac{p_\infty}{2}\right) - \frac{227435\pi^2}{8192} - \frac{920693}{4800} \right) p_\infty^2 - \frac{11646937}{63360} p_\infty^3 + \left(\frac{60521}{840} \log\left(\frac{p_\infty}{2}\right) - \frac{219729011\pi^2}{8388608} + \frac{20028103}{58800} \right) p_\infty^4 + O(p_\infty^5) \right]$
$f_b^{G^5 \text{ 1SF}}$	$-\frac{8}{p_\infty^6} + \frac{44}{p_\infty^4} + \frac{329+36\pi^2}{p_\infty^3} + \frac{416}{45p_\infty^2} + \left(\frac{889}{6} + \frac{643\pi^2}{4} \right) \frac{1}{p_\infty} + \left(\frac{203264}{1575} - \frac{47\pi^2}{5} \right) \frac{1}{p_\infty^2} \\ + \left(\frac{12544}{45} \log(2p_\infty) + \frac{14695\pi^2}{72} + \frac{575123}{2160} \right) \frac{1}{p_\infty} + \frac{116992}{1225} - \frac{5697\pi^2}{280} + \left(\frac{148864}{525} \log(2p_\infty) - \frac{33019\pi^2}{480} + \frac{534248951}{252000} \right) p_\infty \\ + \left(\frac{3407\pi^2}{672} - \frac{9728}{275} \right) p_\infty^2 + \left(\frac{1762784}{11025} \log(2p_\infty) + \frac{224571\pi^2}{2240} - \frac{12181731011}{16464000} \right) p_\infty^3 + O(p_\infty^4)$
$f_{\check{u}_2}^{G^5 \text{ 1SF}}$	$\pi \left[-\frac{6}{p_\infty^6} + \frac{45}{p_\infty^4} + \frac{122}{5p_\infty^3} + \frac{717}{2p_\infty^2} + \frac{13831}{280p_\infty} + \frac{297\pi^2}{20} + \frac{20457}{32} - \frac{64579}{5040} p_\infty + \left(\frac{335931}{560} - \frac{24993\pi^2}{1120} \right) p_\infty^2 \right. \\ \left. + \left(-\frac{10593}{350} \log\left(\frac{p_\infty}{2}\right) + \frac{99\pi^2}{10} + \frac{29573617463}{310464000} \right) p_\infty^3 + \left(+\frac{1577\pi^2}{1120} - \frac{384}{7} \right) p_\infty^4 + O(p_\infty^5) \right]$

(Table continued)

TABLE V. (Continued)

$f_{\tilde{u}_1}^{G^5 \text{ ISF}}$	$\pi \left[\frac{18}{p_\infty^6} - \frac{114}{p_\infty^4} - \frac{56}{5p_\infty^3} + \left(-\frac{3409}{4} - \frac{123\pi^2}{64} \right) \frac{1}{p_\infty^2} - \frac{15007}{210} p_\infty + \frac{148}{5} \log\left(\frac{p_\infty}{2}\right) - \frac{49345\pi^2}{4096} - \frac{115753}{96} - \frac{235867p_\infty}{1260} \right.$
$f_b^{G^5 \text{ ISF}}$	$\left. + \left(\frac{5501}{70} \log\left(\frac{p_\infty}{2}\right) - \frac{227435\pi^2}{8192} - \frac{884909}{4800} \right) p_\infty^2 - \frac{11646937}{63360} p_\infty^3 + \left(\frac{60521}{840} \log\left(\frac{p_\infty}{2}\right) - \frac{219729011\pi^2}{8388608} + \frac{10171457}{29400} \right) p_\infty^4 + O(p_\infty^5) \right]$
$f_{\tilde{u}_2}^{G^5 \text{ ISF}}$	$-\frac{8}{p_\infty^6} + \frac{44}{p_\infty^4} + \frac{329+36\pi^2}{p_\infty^5} + \frac{416}{45p_\infty^4} + \left(\frac{889}{6} + \frac{643\pi^2}{4} \right) \frac{1}{p_\infty^3} + \left(\frac{169664}{1575} - \frac{47\pi^2}{5} \right) \frac{1}{p_\infty^2}$
$f_{\tilde{u}_1}^{G^5 \text{ OSF}}$	$+ \left(\frac{12544 \log(2p_\infty)}{45} + \frac{14695\pi^2}{72} + \frac{575123}{2160} \right) \frac{1}{p_\infty} - \frac{1243\pi^2}{56} + \frac{159232}{3675} + \left(\frac{148864 \log(2p_\infty)}{525} - \frac{33019\pi^2}{480} + \frac{500415991}{252000} \right) p_\infty$
$f_b^{G^5 \text{ OSF}}$	$+ \left(\frac{241\pi^2}{120} - \frac{22294592}{363825} \right) p_\infty^2 + \left(\frac{1762784 \log(2p_\infty)}{11025} + \frac{224571\pi^2}{2240} - \frac{33643816009}{49392000} \right) p_\infty^3$
$f_{\tilde{u}_2}^{G^5 \text{ OSF}}$	$\pi \left[-\frac{18}{p_\infty^6} + \frac{114}{p_\infty^4} + \frac{178}{5p_\infty^3} + \frac{3409+123\pi^2}{4p_\infty^2} + \frac{13847}{120p_\infty} - \frac{148}{5} \log\left(\frac{p_\infty}{2}\right) + \frac{550853\pi^2}{20480} + \frac{115753}{96} + \frac{927433}{5040} p_\infty \right.$
$f_{\tilde{u}_1}^{G^5 \text{ OSF}}$	$\left. + \left(-\frac{5501}{70} \log\left(\frac{p_\infty}{2}\right) + \frac{85783\pi^2}{40960} + \frac{15166967}{33600} \right) p_\infty^2 + \left(-\frac{10593}{350} \log\left(\frac{p_\infty}{2}\right) + \frac{99\pi^2}{10} + \frac{10356321807}{34496000} \right) p_\infty^3 \right.$
$f_b^{G^5 \text{ OSF}}$	$\left. + \left(-\frac{60521}{840} \log\left(\frac{p_\infty}{2}\right) + \frac{1320222271\pi^2}{41943040} - \frac{50406217}{117600} \right) p_\infty^4 + O(p_\infty^5) \right]$
$f_{\tilde{u}_2}^{G^5 \text{ OSF}}$	$\pi \left[\frac{6}{p_\infty^6} - \frac{45}{p_\infty^4} - \frac{717}{2p_\infty^3} - \frac{20457}{32} - \frac{5385}{16} p_\infty^2 \right]$
$f_{\tilde{u}_1}^{G^5 \text{ OSF}}$	$-\frac{2}{p_\infty^6} + \frac{12}{p_\infty^4} + \frac{80+9\pi^2}{p_\infty^5} + \frac{81\pi^2-128}{p_\infty^3} + \left(\frac{225\pi^2}{8} - 384 \right) p_\infty + \frac{945\pi^2-576}{p_\infty}$
$f_b^{G^5 \text{ OSF}}$	0
$f_{\tilde{u}_2}^{G^5 \text{ OSF}}$	$\frac{6\pi}{p_\infty^6} - \frac{45\pi}{p_\infty^4} - \frac{5385\pi p_\infty^2}{16} - \frac{717\pi}{2p_\infty^3} - \frac{20457\pi}{32}$
$f_{\tilde{u}_1}^{G^5 \text{ OSF}}$	$-\frac{2}{p_\infty^6} + \frac{12}{p_\infty^4} + \frac{80+9\pi^2}{p_\infty^5} + \frac{81\pi^2-128}{p_\infty^3} + \left(\frac{225\pi^2}{8} - 384 \right) p_\infty + \frac{945\pi^2-576}{p_\infty}$
$f_b^{G^5 \text{ OSF}}$	$-\frac{6\pi}{p_\infty^6} + \frac{45\pi}{p_\infty^4} + \frac{5385\pi p_\infty^2}{16} + \frac{717\pi}{2p_\infty^3} + \frac{20457\pi}{32}$
$f_{\tilde{u}_2}^{G^5 \text{ OSF}}$	0

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