

Gauge symmetry breaking with S^2 extra dimensions

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We consider symmetry breaking of arbitrary gauge groups on a six-dimensional space-time which consists of a four-dimensional Minkowski space-time M^4 and a two-dimensional sphere S^2 . We expand the gauge fields in the presence of a nontrivial background unique to S^2 . We analyze Kaluza-Klein (KK) modes of the gauge fields and derive the mass spectrum of the KK modes. We found that the gauge fields (not) commuting with the background fields (do not) remain symmetry operators in four dimensions. We also discuss the mass spectrum of the extra-dimensional components of the gauge fields and identify a physical scalar ϕ and a Nambu-Goldstone mode χ . As a result, we obtain a method to break gauge symmetry due to the nontrivial solution for gauge fields which is a unique feature of S^2 .

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I. INTRODUCTION

The Standard Model (SM) has achieved remarkable success as a fundamental theory in particle physics and established by fitting the last piece of the puzzle, that is, discovery of the Higgs boson in 2012 [1,2]. However, several unresolved issues remain, such as the complexity of the gauge charge of the SM fields and the theoretical origin of the Higgs particle.

The existence of multiple coupling constants arises from the fact that the SM gauge group is given by a direct product of simple groups: $G_{\text{SM}} = \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$. Grand unified theory (GUT) [3–6] provides a clear solution to this puzzle. In GUT, a single simple gauge group G_{GUT} is assumed and broken into G_{SM} through a spontaneous breaking of the gauge symmetry, and as a result, the unification of the three gauge interactions and quantization of hypercharge are realized.

In the SM, the Higgs field plays a crucial role in the mechanism of spontaneous symmetry breaking and in

giving masses to elementary particles. Nevertheless, the origin of the Higgs field remains unknown. One of the promising frameworks is the gauge-Higgs unification (GHU) theory [7–21]. In GHU, the SM gauge theory in the four-dimensional Minkowski spacetime is embedded in a higher-dimensional gauge theory. The extra-dimensional components of the gauge fields behave as scalar fields in four-dimensional spacetime and can be interpreted as a Higgs field. Moreover, GUT has been discussed in the context of the GHU, and this kind of frameworks is called grand gauge-Higgs unification [22–39].

In this paper, we introduce a two-dimensional sphere S^2 as an extra dimensional space in addition to the four-dimensional Minkowski spacetime M^4 and study a gauge theory on the six-dimensional spacetime. Since S^2 is defined as a quotient of compact Lie groups, $\text{SO}(3)/\text{SO}(2)$, it is suitable for gauge groups and ideal for studying gauge symmetry breaking. Indeed, the models with S^2 have been considered in various contexts [7,40–51]. Since the gauge fields on S^2 have a nontrivial classical solution proportional to $\cos\theta$ as a result of the curvature, we introduce this solution as a background field [7,44]. We extend the previous method and apply it to general gauge groups. We analyze Kaluza-Klein (KK) modes expansion of the gauge fields and derive the mass spectrum of the KK modes. As a result, only the gauge fields that commute with the background field remain in four dimensions. Furthermore, KK modes of the extra-dimensional components of the gauge fields, which behave as scalar fields in four dimensions, are also affected by the curvature and background fields.

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This paper is organized as follows. In Sec. II, we show the pure Yang-Mills theory on the six-dimensional $M^4 \times S^2$ spacetime, and introduce a nontrivial background of the gauge fields. In Sec. III, we analyze KK mode expansion of the gauge fields in the general framework, and derive the mass spectrum of the KK modes. In Sec. IV, we show the applications for the breaking of the SU(3) and SU(5) gauge symmetries. Finally, we summarize our work in Sec. V.

II. PURE YANG-MILLS THEORY ON $M^4 \times S^2$ SPACETIME

In this section, we review a pure Yang-Mills theory on the six-dimensional spacetime $M^4 \times S^2$ which consists of a four-dimensional Minkowski spacetime M^4 and a compact two-dimensional sphere S^2 . We denote the indices of the coordinates of $M^4 \times S^2$ by uppercase Greek letter as X^M ($M = 0, 1, \dots, 5$), and those of M^4 and S^2 by lowercase Greek letters as $X^\mu = x^\mu$ ($\mu = 0, 1, 2, 3$), and $X^{\hat{\alpha}} = \theta, \varphi$ ($\hat{\alpha} = 4, 5$), respectively. The metric of $M^4 \times S^2$ is given by

$$G_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{\hat{\alpha}\hat{\beta}} \end{pmatrix}, \quad (1)$$

where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and $g_{\hat{\alpha}\hat{\beta}} = \text{diag}(-R^2, -R^2 \sin^2 \theta)$ with R being the radius of S^2 are the metrics of M^4 and S^2 , respectively.

The action of the gauge theory under consideration can be written as

$$S = \int d^4x d\theta d\varphi \mathcal{L} = \int d^4x d\theta d\varphi R^2 \sin \theta \left(-\frac{1}{2} \text{Tr} F_{MN} F^{MN} \right), \quad (2)$$

where $F_{MN} = \partial_M A_N - \partial_N A_M - ig[A_M, A_N]$ with A_M being the gauge field and g the gauge coupling constant stands for the field strength of A_M . Then, we obtain the equation of motion

$$\nabla^N F_{MN} - ig[A^N, F_{MN}] = 0, \quad (3)$$

where ∇^M represents the covariant derivative of $M^4 \times S^2$. In terms of the gauge components of the Lie algebra,

$$\nabla^N F_{MNa} + gf_{ab}^c A^{Nb} F_{MNc} = 0. \quad (4)$$

Here, indices $a, b, \dots$ run over all components of the Lie algebra, and f_{ab}^c is a structure constant of the gauge group. By setting $M = \mu, \theta, \varphi$ in Eq. (4), the equations of motion for the vacuum values of $A_{\mu a}, A_\theta, A_{\varphi a}$ take the following form:

$$\begin{aligned} & \left(-\square + \frac{1}{R^2} L^2 \right) A_{\mu a} + \nabla_\mu \nabla^N A_{Na} + gf_{abc} A_\mu^b \nabla^N A_N^c \\ & = -gf_{ab}^c A^{Nb} \nabla_\mu A_{Nc} + 2gf_{ab}^c A^{Nb} \nabla_N A_{\mu c} - g^2 f_{abe} f_{cd}^e A^{Nb} A_\mu^c A_N^d, \end{aligned} \quad (5)$$

$$\begin{aligned} & \left\{ -\square + \frac{1}{R^2} \left(L^2 - \frac{1}{\sin^2 \theta} \right) \right\} A_{\theta a} - \frac{2 \cos \theta}{R^2 \sin^3 \theta} \partial_\varphi A_{\varphi a} + \nabla_\theta \nabla^N A_{Na} + gf_{abc} A_\theta^b \nabla^N A_N^c \\ & = -gf_{ab}^c A^{Nb} \nabla_\theta A_{Nc} + 2gf_{ab}^c A^{Nb} \nabla_N A_{\theta c} - g^2 f_{abe} f_{cd}^e A^{Nb} A_\theta^c A_N^d, \end{aligned} \quad (6)$$

$$\begin{aligned} & \frac{2 \cos \theta}{R^2 \sin \theta} \partial_\varphi A_{\theta a} + \left\{ -\square + \frac{1}{R^2} \left(L^2 - 2 \frac{\cos \theta}{\sin \theta} \partial_\theta \right) \right\} A_{\varphi a} + \nabla_\varphi \nabla^N A_{Na} + gf_{abc} A_\varphi^b \nabla^N A_N^c \\ & = -gf_{ab}^c A^{Nb} \nabla_\varphi A_{Nc} + 2gf_{ab}^c A^{Nb} \nabla_N A_{\varphi c} - g^2 f_{abe} f_{cd}^e A^{Nb} A_\varphi^c A_N^d, \end{aligned} \quad (7)$$

where $\square \equiv \partial_\mu \partial^\mu$ is the d'Alembertian, and $L^2 \equiv -(1/\sin \theta) \partial_\theta (\sin \theta \partial_\theta) - (1/\sin^2 \theta) \partial_\varphi^2$ is the square of angular momentum operator. As a very remarkable feature of S^2 , we have a nontrivial solution for Eq. (3),

$$A_M = \begin{cases} \mu \cdot H \cos \theta & (M = \varphi) \\ 0 & (\text{otherwise}), \end{cases} \quad (8)$$

where H are the Cartan generators. Therefore, $\langle A_\varphi \rangle = \mu \cdot H \cos \theta$ can be introduced as the background field [7,44]. This background fields excite the gauge field energy and

hence at first glance they must not be taken. However, as shown in the U(1) gauge case, these kinds of background fields yield massless fermions and stabilize the entire model [7,41,44,52], and accordingly, it is reasonable to assume that a similar discussion holds in general.

III. MASS SPECTRUM OF KALUZA-KLEIN MODES

In this section, we perform a KK expansion of the gauge fields in the presence of the background fields, substitute these modes into the action Eq. (2), and derive the mass

spectrum of the KK modes of the gauge field. Hereafter, we denote the quantum fluctuations as A_M which are the deviation from the background field $\langle A_M \rangle$.

A. Quadratic terms for gauge fields

In this subsection, we focus on the quadratic terms of gauge field A_M in the action to evaluate the mass spectrum of the gauge fields. We use the Cartan-Weyl basis

$$A_M = \sum_{i: \text{all Cartans}} A_M^i H_i + \sum_{\alpha: \text{all roots}} A_M^\alpha E_\alpha, \quad (9)$$

where $i, j, \dots$ run over the Cartan subalgebra and $E_{\pm\alpha}$ are the raising and lowering operators associated with the root α . In the following section, we omit the summation symbols. Whenever an index variable appears twice (or three times in some cases) in a single term, we automatically sum over all values.

Under this basis, the Lagrangian is as follows:

$$\begin{aligned} \mathcal{L}_{\text{gauge}}^{\text{quadratic}} = & -\frac{1}{4}R^2 \sin \theta \left\{ (\partial_\mu A_{\nu i} - \partial_\nu A_{\mu i})(\partial^\mu A^{\nu i} - \partial^\nu A^{\mu i}) + (\partial_\mu A_{\nu, -\alpha} - \partial_\nu A_{\mu, -\alpha})(\partial^\mu A^{\nu \alpha} - \partial^\nu A^{\mu \alpha}) \right. \\ & - \frac{2}{R^2}(\partial_\theta A_{\mu i})(\partial_\theta A^{\mu i}) - \frac{2}{R^2 \sin^2 \theta}(\partial_\varphi A_{\mu i})(\partial_\varphi A^{\mu i}) - \frac{2}{R^2}(\partial_\theta A_\mu^{-\alpha})(\partial_\theta A^{\mu \alpha}) \\ & - \frac{2}{R^2 \sin^2 \theta}[(\partial_\varphi A_\mu^{-\alpha})(\partial_\varphi A^{\mu \alpha}) + 2ik_\alpha \cos \theta A_\mu^{-\alpha} \partial_\varphi A^{\mu \alpha} + k_\alpha^2 \cos^2 \theta A_\mu^{-\alpha} A^{\mu \alpha}] \\ & - \frac{2}{R^2}[(\partial_\mu A_{\theta i})(\partial^\mu A_\theta^i) + (\partial_\mu A_\theta^{-\alpha})(\partial^\mu A_\theta^\alpha)] \\ & - \frac{2}{R^2 \sin^2 \theta}[(\partial_\mu A_{\varphi i})(\partial^\mu A_\varphi^i) + (\partial_\mu A_\varphi^{-\alpha})(\partial^\mu A_\varphi^\alpha)] \\ & + \frac{2}{R^4 \sin^2 \theta}(\partial_\theta A_\varphi^i - \partial_\varphi A_\theta^i)(\partial_\theta A_\varphi^i - \partial_\varphi A_\theta^i) + \frac{2}{R^4 \sin^2 \theta}(2ik_\alpha \sin \theta A_\theta^\alpha A_\varphi^{-\alpha}) \\ & + \frac{2}{R^4 \sin^2 \theta}(\partial_\theta A_\varphi^{-\alpha} - \partial_\varphi A_\theta^{-\alpha} - ik_\alpha \cos \theta A_\theta^{-\alpha})(\partial_\theta A_\varphi^\alpha - \partial_\varphi A_\theta^\alpha + ik_\alpha \cos \theta A_\theta^\alpha) \Big\} \\ & - R^2 \sin \theta \left\{ -\frac{1}{R^2 \sin \theta}(\partial_\mu A^\mu i)(\partial_\theta \sin \theta A_\theta^i) - \frac{1}{R^2 \sin^2 \theta}(\partial_\mu A^\mu i)(\partial_\varphi A_\varphi^i) \right. \\ & - \frac{1}{R^2 \sin \theta}(\partial_\mu A^{\mu -\alpha})(\partial_\theta \sin \theta A_\theta^\alpha) - \frac{1}{R^2 \sin^2 \theta}(\partial_\mu A^{\mu -\alpha})(\partial_\varphi A_\varphi^\alpha) \\ & \left. + ik_\alpha \frac{\cos \theta}{R^2 \sin^2 \theta}(\partial_\mu A^{\mu -\alpha})A_\varphi^\alpha \right\}. \end{aligned} \quad (10)$$

In a four-dimensional theory, the four-dimensional components A_μ behave as gauge fields, while the extra-dimensional components A_θ and A_φ behave as scalar fields. Therefore, in the discussion of gauge symmetry breaking, when some of the A_μ components acquire a mass, the corresponding components of A_θ and A_φ are expected to play the role of Nambu-Goldstone bosons. To clarify this picture, we perform the following transformation on A_θ and A_φ

$$A_\theta^i = -\frac{1}{\sin \theta} \partial_\varphi \phi^i + \partial_\theta \chi^i, \quad (11)$$

$$A_\varphi^i = \sin \theta \partial_\theta \phi^i + \partial_\varphi \chi^i, \quad (12)$$

$$A_\theta^\alpha = -\frac{1}{\sin \theta} \partial_\varphi \phi^\alpha + \partial_\theta \chi^\alpha + ik_\alpha \frac{\cos \theta}{\sin \theta} \phi^\alpha \quad (\text{no sum}), \quad (13)$$

$$A_\varphi^\alpha = \sin \theta \partial_\theta \phi^\alpha + \partial_\varphi \chi^\alpha - ik_\alpha \cos \theta \chi^\alpha \quad (\text{no sum}). \quad (14)$$

Here we define $k_\alpha \equiv g\alpha \cdot \mu$ as follows.

$$\begin{aligned} g[\mu \cdot H, A_M] &= gA_M^\alpha \mu \cdot [H, E_\alpha] \\ &= gA_M^\alpha \mu \cdot \alpha E_\alpha \\ &= k_\alpha A_M^\alpha E_\alpha. \end{aligned} \quad (15)$$

Therefore, k_α is interpreted as a charge of the α -direction gauge field with respect to the background field.

As shown in the later part of this section, the most components of ϕ denotes the physical scalar field, and χ denotes the Nambu-Goldstone boson. In fact, applying Eqs. (11)–(14) to linear parts of $F_{\theta\varphi}$ in Eq. (10), we find

that it is mainly occupied by the physical scalar field ϕ

$$\begin{aligned} F_{\theta\varphi}|_{\text{linear}} &= (\partial_\theta A_\varphi^i - \partial_\varphi A_\theta^i) H_i \\ &\quad + (\partial_\theta A_\varphi^\alpha - \partial_\varphi A_\theta^\alpha + ik_\alpha \cos \theta A_\theta^\alpha) E_\alpha \\ &= -\sin \theta (\mathbf{L}^2 \phi^i) H_i - \sin \theta (\tilde{\mathbf{J}}^{(\alpha)2} \phi^\alpha) E_\alpha \\ &\quad + ik_\alpha \sin \theta \chi^\alpha E_\alpha, \end{aligned} \quad (16)$$

where $\tilde{\mathbf{J}}^{(\alpha)2}$ is an operator that has the same algebra as $\mathbf{L}^2$, but a different representation. We define $\tilde{\mathbf{J}}^{(\alpha)2}$ for each root α as

$$\begin{aligned} \tilde{\mathbf{J}}^{(\alpha)2} &\equiv -\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) - \frac{1}{\sin^2 \theta} \partial_\varphi^2 \\ &\quad + 2 \frac{\cos \theta}{\sin^2 \theta} k_\alpha i \partial_\varphi + \frac{\cos^2 \theta}{\sin^2 \theta} k_\alpha^2 \\ &= \mathbf{J}^{(\alpha)2} - k_\alpha^2. \end{aligned} \quad (17)$$

The operators $\mathbf{J}^{(\alpha)2}$ are the square of the operators $J_1^{(\alpha)}$, $J_2^{(\alpha)}$, and $J_3^{(\alpha)}$, which are defined as follows:

$$\begin{aligned} J_1^{(\alpha)} &= i \left(\sin \varphi \partial_\theta + \frac{\cos \theta}{\sin \theta} \cos \varphi \partial_\varphi \right) - k_\alpha \frac{\cos \varphi}{\sin \theta}, \\ J_2^{(\alpha)} &= -i \left(\cos \varphi \partial_\theta - \frac{\cos \theta}{\sin \theta} \sin \varphi \partial_\varphi \right) - k_\alpha \frac{\sin \varphi}{\sin \theta}, \\ J_3^{(\alpha)} &= -i \partial_\varphi. \end{aligned} \quad (18)$$

Then, $\mathbf{J}^{(\alpha)2}$ and $J_i^{(\alpha)}$ satisfy the SU(2) algebra

$$\mathbf{J}^{(\alpha)2} = \sum_{i=1}^3 J_i^{(\alpha)} J_i^{(\alpha)}, \quad (19)$$

$$[J_i^{(\alpha)}, J_j^{(\alpha)}] = i \varepsilon_{ijk} J_k^{(\alpha)}, \quad (20)$$

with ε_{ijk} being the Levi-Civita symbol. The eigenvalue j of $\mathbf{J}^{(\alpha)2}$ takes non-negative integer and the eigenvalue m of $J_3^{(\alpha)}$ takes $-j, -j+1, \dots, j$. As a different point from $\mathbf{L}^2$, j is restricted to

$$j = |k_\alpha|, \quad |k_\alpha| + 1, \quad |k_\alpha| + 2, \dots, \quad (21)$$

which is proved in the Appendix. Therefore, $|k_\alpha|$ takes non-negative integer values. When k_α is a not integer, there are no solutions. We interpret that the corresponding gauge field does not exist. Incidentally, when $k_\alpha = 0$, $\mathbf{J}^{(\alpha)2}$ becomes $\mathbf{L}^2$.

We define the eigenfunctions of this representation $\mathcal{Y}_{k_\alpha j m}(\theta, \varphi)$

$$\mathbf{J}^{(\alpha)2} \mathcal{Y}_{k_\alpha j m}(\theta, \varphi) = j(j+1) \mathcal{Y}_{k_\alpha j m}(\theta, \varphi), \quad (22)$$

$$J_3^{(\alpha)} \mathcal{Y}_{k_\alpha j m}(\theta, \varphi) = m \mathcal{Y}_{k_\alpha j m}(\theta, \varphi). \quad (23)$$

The concrete form of $\mathcal{Y}_{k_\alpha j m}(\theta, \varphi)$ and the quantization of the eigenvalues are given in the Appendix, where it is shown that Eq. (21) is ensured by the condition that the surface terms vanish. Therefore, we omit all the surface terms by partial integration hereafter. Note that χ appears as a massless scalar with careful calculations of all terms of Eq. (10) although Eq. (16) still has χ in root components.

We perform the following calculation to consider gauge-fixing function. First of all, we define the covariant derivative in the M -direction as

$$\begin{aligned} D_M A_N &\equiv \nabla_M A_N - ig[\langle A_M \rangle, A_N] \\ &\equiv \partial_M A_N - \Gamma_{MN}^P A_P - ig[\langle A_M \rangle, A_N], \end{aligned} \quad (24)$$

where Γ_{MN}^P are Christoffel symbols. Using Eqs. (11)–(14), we obtain

$$\begin{aligned} D_{\hat{A}} A^{\hat{A}} &= \left\{ -\frac{1}{R^2 \sin \theta} (\partial_\theta \sin \theta A_\theta^i) - \frac{1}{R^2 \sin^2 \theta} (\partial_\varphi A_\theta^i) \right\} H_i \\ &\quad + \left\{ -\frac{1}{R^2 \sin \theta} (\partial_\theta \sin \theta A_\theta^\alpha) \right. \\ &\quad \left. - \frac{1}{R^2 \sin^2 \theta} (\partial_\varphi A_\theta^\alpha) + ik_\alpha \frac{\cos \theta}{R^2 \sin^2 \theta} A_\theta^\alpha \right\} E_\alpha \\ &= \frac{1}{R^2} (\mathbf{L}^2 \chi^i) H_i + \frac{1}{R^2} (\tilde{\mathbf{J}}^{(\alpha)2} \chi^\alpha) E_\alpha + \frac{1}{R^2} ik_\alpha \phi^\alpha E_\alpha. \end{aligned} \quad (25)$$

Then, we define gauge-fixing function $G(A)$ and the gauge-fixing Lagrangian $\mathcal{L}_{\text{gf}}$ as follows:

$$G(A) = D_\mu A^\mu + \xi (D_{\hat{A}} A^{\hat{A}}), \quad (26)$$

$$\begin{aligned} \mathcal{L}_{\text{gf}} &= R^2 \sin \theta \left(-\frac{1}{\xi} \text{tr}[G(A_M)^2] \right) \\ &= -\frac{1}{\xi} R^2 \sin \theta \{ \text{tr}[(\partial_\mu A^\mu)^2] + 2\xi \text{tr}[(\partial_\mu A^\mu)(D_{\hat{A}} A^{\hat{A}})] \\ &\quad + \xi^2 \text{tr}[(D_{\hat{A}} A^{\hat{A}})^2] \}, \end{aligned} \quad (27)$$

where ξ is a gauge-fixing parameter. Combining Eqs. (10) and (27) and performing partial integration, we obtain

$$\begin{aligned}
\mathcal{L}_{\text{gauge}}^{\text{quadratic}} + \mathcal{L}_{\text{gf}} = & -\frac{1}{4}R^2 \sin \theta \left\{ (\partial_\mu A_{\nu i} - \partial_\nu A_{\mu i})(\partial^\mu A^{\nu i} - \partial^\nu A^{\mu i}) + (\partial_\mu A_{\nu, -\alpha} - \partial_\nu A_{\mu, -\alpha})(\partial^\mu A^{\nu \alpha} - \partial^\nu A^{\mu \alpha}) \right. \\
& - \frac{2}{R^2}(\partial_\theta A_{\mu i})(\partial_\theta A^{\mu i}) - \frac{2}{R^2 \sin^2 \theta}(\partial_\varphi A_{\mu i})(\partial_\varphi A^{\mu i}) - \frac{2}{R^2}(\partial_\theta A_\mu^{-\alpha})(\partial_\theta A^{\mu \alpha}) \\
& - \frac{2}{R^2 \sin^2 \theta}[(\partial_\varphi A_\mu^{-\alpha})(\partial_\varphi A^{\mu \alpha}) + 2ik_\alpha \cos \theta A_\mu^{-\alpha} \partial_\varphi A^{\mu \alpha} + k_\alpha^2 \cos^2 \theta A_\mu^{-\alpha} A^{\mu \alpha}] \\
& - \frac{2}{R^2}[(\partial_\mu A_{\theta i})(\partial^\mu A_\theta^i) + (\partial_\mu A_\theta^{-\alpha})(\partial^\mu A_\theta^\alpha)] \\
& - \frac{2}{R^2 \sin^2 \theta}[(\partial_\mu A_{\varphi i})(\partial^\mu A_\varphi^i) + (\partial_\mu A_\varphi^{-\alpha})(\partial^\mu A_\varphi^\alpha)] \\
& + \frac{2}{R^4 \sin^2 \theta}(\partial_\theta A_\varphi^i - \partial_\varphi A_\theta^i)(\partial_\theta A_\varphi^i - \partial_\varphi A_\theta^i) + \frac{2}{R^4 \sin^2 \theta}(2ik_\alpha \sin \theta A_\theta^\alpha A_\varphi^{-\alpha}) \\
& + \frac{2}{R^4 \sin^2 \theta}(\partial_\theta A_\varphi^{-\alpha} - \partial_\varphi A_\theta^{-\alpha} - ik_\alpha \cos \theta A_\theta^{-\alpha})(\partial_\theta A_\varphi^\alpha - \partial_\varphi A_\theta^\alpha + ik_\alpha \cos \theta A_\theta^\alpha) \Big\} \\
& - \frac{1}{2\xi}R^2 \sin \theta \{ (\partial_\mu A_{\nu i})(\partial_\nu A^{\mu i}) + (\partial_\mu A^{\mu -\alpha})(\partial_\nu A^{\nu \alpha}) \} \\
& - \frac{\xi}{2R^2} \sin \theta \{ (L^2 \chi_i)(L^2 \chi^i) + (\tilde{J}^{(-\alpha)2} \chi^{-\alpha})(\tilde{J}^{(\alpha)2} \chi^\alpha) \\
& + 2ik_\alpha (\tilde{J}^{(-\alpha)2} \chi^{-\alpha}) \phi^\alpha + k_\alpha^2 \phi^{-\alpha} \phi^\alpha \}. \tag{28}
\end{aligned}$$

The cross terms between A_μ and $A_{\hat{\alpha}}$ in Eq. (10) are canceled by the gauge-fixing terms. The scalar fields, χ^i , χ^α , and some components of ϕ^α , are interpreted as Nambu-Goldstone bosons and convert into the degree of freedom of A_μ^i and A_μ^α , as shown in the later part of this section.

B. KK mass of A_μ

Hereafter, we consider KK expansion of the A_μ and $A_{\hat{\alpha}}$ in Eq. (28) and analyze mass spectrum of KK modes.

First, we consider four-dimensional gauge field A_μ . Gauge fields that do not possess massless modes correspond to the directions in which the symmetry is broken, whereas gauge fields that do possess massless modes correspond to the directions in which the symmetry remains in four dimensions.

By partial integration, we obtain

$$\begin{aligned}
\mathcal{L}_{\text{gauge}}^{\text{quadratic}} = & \frac{1}{2}R^2 \sin \theta \left[A_\mu^i \left(\square + \frac{1}{R^2} L^2 \right) A^{\mu i} \right. \\
& \left. + A_\mu^{-\alpha} \left\{ \square + \frac{1}{R^2} \tilde{J}^{(\alpha)2} \right\} A^{\mu \alpha} \right], \tag{29}
\end{aligned}$$

where we omit the gauge-fixing terms since these terms proportional to $\frac{1}{\xi}$ are the same as usual gauge fixing. Since the eigenfunctions of L^2 are the ordinary spherical harmonics Y_{lm} , the KK expansion of the Cartan components A_μ^i is given by

$$A_\mu^i(x, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{R} A_{\mu, lm}^i(x) Y_{lm}(\theta, \varphi), \tag{30}$$

where l takes non-negative integer values $0, 1, 2, \dots$ and m takes $-l, -l+1, \dots, l$. The corresponding KK masses are equal to

$$\frac{l(l+1)}{R^2}, \tag{31}$$

and the state with $l=0$ corresponds to the massless mode.

Since the eigenfunctions of $\tilde{J}^{(\alpha)2}$ are $\mathcal{Y}_{k_\alpha j m}$ which we defined in the last section, the KK expansion of the root components A_μ^α is given by

$$A_\mu^\alpha(x, \theta, \varphi) = \sum_{j=|k_\alpha|}^{\infty} \sum_{m=-j}^j \frac{1}{R} A_{\mu, jm}^\alpha(x) \mathcal{Y}_{k_\alpha j m}(z, \varphi), \tag{32}$$

where j takes non-negative integer values $|k_\alpha|, |k_\alpha|+1, |k_\alpha|+2, \dots$, and m takes $-j, -j+1, \dots, j$. The KK masses are given by

$$\frac{j(j+1) - k_\alpha^2}{R^2}. \tag{33}$$

It is important to mention that, compared to Eq. (31), the values of $j(j+1)$ are shifted to $j(j+1) - k_\alpha^2$.

By substituting Eqs. (30) and (32) into Eq. (29) and integrating over θ and φ , we obtain the kinetic and mass terms of the four-dimensional gauge fields

$$S_{\text{gauge}}^{\text{quadratic}} = \int d^4x \left[\sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2} A_{\mu,l,-m}^i \left\{ \square + \frac{l(l+1)}{R^2} \right\} A_{lm}^{\mu i} + \sum_{\alpha: \text{all roots}} \sum_{j=|k_\alpha|}^{\infty} \sum_{m=-j}^j \frac{1}{2} A_{\mu,j,-m}^{-\alpha} \left\{ \square + \frac{j(j+1) - k_\alpha^2}{R^2} \right\} A_{jm}^{\mu\alpha} \right], \quad (34)$$

where we use the convention that $(Y_{lm})^* = Y_{l,-m}$ and $(\mathcal{Y}_{k_\alpha j m})^* = \mathcal{Y}_{-k_\alpha j, -m}$.

Since the gauge fields A_μ are real fields, we can define the real fields $A_{\mu,1}^{(ilm)}, A_{\mu,2}^{(ilm)}$ the following way.

$$A_{\mu,lm}^i = \frac{A_{\mu,1}^{(ilm)} - iA_{\mu,2}^{(ilm)}}{\sqrt{2}} = (A_{\mu,l,-m}^i)^*, \quad A_{\mu,l0}^i = A_\mu^{(il0)}, \quad (35)$$

where $m > 0$. Also, we define the real fields $A_1^{(\alpha jm)}, A_2^{(\alpha jm)}$ as follows:

$$A_{\mu,jm}^\alpha = \frac{A_{\mu,1}^{(\alpha jm)} - iA_{\mu,2}^{(\alpha jm)}}{\sqrt{2}} = (A_{\mu,j,-m}^{-\alpha})^* \quad (36)$$

with positive roots α and $m = -j, -j+1, \dots, 0, \dots, j$. Correspondingly, we choose the Hermitian basis,

$$E_\alpha = \frac{E_1^{(\alpha)} + iE_2^{(\alpha)}}{\sqrt{2}} \quad (37)$$

where $\alpha > 0$. And we also chose the real functions

$$Y_{lm} = \frac{Y_1^{(lm)} + iY_2^{(lm)}}{\sqrt{2}}, \quad Y_{l0} = Y^{(l0)}, \quad (38)$$

where $m > 0$, and

$$\mathcal{Y}_{k_\alpha j m} = \frac{\mathcal{Y}_1^{(\alpha jm)} + i\mathcal{Y}_2^{(\alpha jm)}}{\sqrt{2}}, \quad (39)$$

where $\alpha > 0, m = -j, -j+1, \dots, 0, \dots, j$. Then, Eq. (34) is realized in the following way:

$$S_{\text{gauge}}^{\text{quadratic}} = \int d^4x \left[\sum_{l=1}^{\infty} \sum_{m=1}^l \sum_{r=1,2} \frac{1}{2} A_{\mu,r}^{(ilm)} \left\{ \square + \frac{l(l+1)}{R^2} \right\} A_r^{(ilm)\mu} + \sum_{l=0}^{\infty} \frac{1}{2} A_\mu^{(il0)} \left\{ \square + \frac{l(l+1)}{R^2} \right\} A_{(il0)}^\mu \right. \\ \left. + \sum_{\alpha>0} \sum_{j=|k_\alpha|}^{\infty} \sum_{m=-j}^j \sum_{r=1,2} \frac{1}{2} A_{\mu,r}^{(\alpha jm)} \left\{ \square + \frac{j(j+1) - k_\alpha^2}{R^2} \right\} A_r^{(\alpha jm)\mu} \right]. \quad (40)$$

The gauge fields of Cartan components always have a massless mode corresponding to $l=0$. The gauge fields of root components have a massless mode when $j=k_\alpha=0$. Therefore, when satisfying $k_\alpha=0$, the gauge fields in the α -direction are unbroken in four dimensions. In contrast, when $k_\alpha \neq 0$, the gauge symmetry in the α -direction is broken. If k_α is an integer, the corresponding gauge field becomes a massive vector boson; if k_α is a noninteger, the gauge field is projected out entirely.

C. KK mass of $A_{\hat{a}}$

Let us analyze the mass spectrum of the extra-dimensional components A_θ and A_ϕ . Using the transformation from A_θ and A_ϕ to ϕ and χ in Eqs. (11)–(14), we obtain the Lagrangian as follows:

$$\mathcal{L}_{\text{extra gauge}}^{\text{quadratic}} = -\frac{1}{2} \sin \theta \left\{ \phi_i \square (L^2 \phi^i) + \chi_i \square (L^2 \chi^i) + \frac{1}{R^2} (L^2 \phi_i) (L^2 \phi^i) + \frac{\xi}{R^2} (L^2 \chi_i) (L^2 \chi^i) \right. \\ + \phi^{-\alpha} \square (\tilde{J}^{(\alpha)2} \phi^\alpha) + \chi^{-\alpha} \square (\tilde{J}^{(\alpha)2} \chi^\alpha) - 2ik_\alpha \phi^{-\alpha} \square \chi^\alpha \\ + \frac{1}{R^2} (\tilde{J}^{(-\alpha)2} \phi^{-\alpha}) (\tilde{J}^{(\alpha)2} \phi^\alpha) - \frac{1}{R^2} k_\alpha^2 \phi^{-\alpha} \phi^\alpha \\ \left. + \frac{\xi}{R^2} \{ (\tilde{J}^{(-\alpha)2} \chi^{-\alpha}) (\tilde{J}^{(\alpha)2} \chi^\alpha) + 2ik_\alpha (\tilde{J}^{(-\alpha)2} \chi^{-\alpha}) \phi^\alpha + k_\alpha^2 \phi^{-\alpha} \phi^\alpha \} \right\}. \quad (41)$$

The KK expansions of the Cartan components ϕ^i and χ^i are given by

$$\phi^i(x, \theta, \varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{1}{\sqrt{l(l+1)}} \phi_{lm}^i(x) Y_{lm}(\theta, \varphi), \quad (42)$$

$$\phi^\alpha(x, \theta, \varphi) = \sum_{j=1}^{\infty} \sum_{m=-j}^j \frac{1}{\sqrt{j(j+1)}} \phi_{jm}^\alpha(x) \mathcal{Y}_{0jm}(z, \varphi), \quad (44)$$

$$\chi^i(x, \theta, \varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{1}{\sqrt{l(l+1)}} \chi_{lm}^i(x) Y_{lm}(\theta, \varphi). \quad (43)$$

$$\chi^\alpha(x, \theta, \varphi) = \sum_{j=1}^{\infty} \sum_{m=-j}^j \frac{1}{\sqrt{j(j+1)}} \chi_{jm}^\alpha(x) \mathcal{Y}_{0jm}(z, \varphi), \quad (45)$$

Also, the KK expansions of the root components ϕ^α and χ^α for $k_\alpha = 0$ are given by

and the KK expansions of the root components ϕ^α and χ^α for $|k_\alpha| > 0$ are given by

$$\phi^\alpha(x, \theta, \varphi) = \sum_{j=|k_\alpha|}^{\infty} \sum_{m=-j}^j \frac{1}{\sqrt{j(j+1) - k_\alpha^2}} \phi_{jm}^\alpha(x) \mathcal{Y}_{k_\alpha jm}(z, \varphi), \quad (46)$$

$$\chi^\alpha(x, \theta, \varphi) = \sum_{j=|k_\alpha|}^{\infty} \sum_{m=-j}^j \frac{1}{\sqrt{j(j+1) - k_\alpha^2}} \chi_{jm}^\alpha(x) \mathcal{Y}_{k_\alpha jm}(z, \varphi). \quad (47)$$

We note that the modes of $l = 0$ for ϕ^i, χ^i and those of $j = 0$ for ϕ^α, χ^α with $k_\alpha = 0$ do not appear in the Lagrangian. The quantum numbers l, j, m , and k_α are exactly the same as in the case of A_μ . By substituting Eqs. (42)–(47) into Eq. (41) and integrating over θ and φ , we obtain the kinetic and mass terms of the extra-dimensional gauge fields

$$\begin{aligned} S_{\text{gauge}}^{\text{quadratic}} = & \int d^4x \left(\sum_{l=1}^{\infty} \sum_{m=-l}^l \left[-\frac{1}{2} \phi_{l,-m}^i \left\{ \square + \frac{l(l+1)}{R^2} \right\} \phi_{lm}^i - \frac{1}{2} \chi_{l,-m}^i \left\{ \square + \xi \frac{l(l+1)}{R^2} \right\} \chi_{lm}^i \right] \right. \\ & + \sum_{\alpha \text{ with } k_\alpha=0} \sum_{j=1}^{\infty} \sum_{m=-j}^j \left[-\frac{1}{2} \phi_{j,-m}^{-\alpha} \left\{ \square + \frac{j(j+1)}{R^2} \right\} \phi_{jm}^\alpha \right. \\ & \quad \left. - \frac{1}{2} \chi_{j,-m}^{-\alpha} \left\{ \square + \xi \frac{j(j+1)}{R^2} \right\} \chi_{jm}^\alpha \right] \\ & + \sum_{\alpha \text{ with } |k_\alpha|>0} \sum_{j=|k_\alpha|}^{\infty} \sum_{m=-j}^j \left[-\frac{1}{2} \phi_{j,-m}^{-\alpha} \left\{ \square + \frac{(j(j+1) - k_\alpha^2)^2 - k_\alpha^2 + \xi k_\alpha^2}{R^2(j(j+1) - k_\alpha^2)} \right\} \phi_{jm}^\alpha \right. \\ & \quad \left. - \frac{1}{2} \chi_{j,-m}^{-\alpha} \left\{ \square + \xi \frac{j(j+1) - k_\alpha^2}{R^2} \right\} \chi_{jm}^\alpha \right. \\ & \quad \left. + ik_\alpha \frac{1}{j(j+1) - k_\alpha^2} \phi_{j,-m}^{-\alpha} \square \chi_{jm}^\alpha + ik_\alpha \xi \frac{1}{R^2} \phi_{j,-m}^{-\alpha} \chi_{jm}^\alpha \right] \Bigg). \quad (48) \end{aligned}$$

To realize ϕ^i and χ^i as well as A_μ^i , we translate them as

$$\phi_{lm}^i = \frac{\phi_1^{(ilm)} - i\phi_2^{(ilm)}}{\sqrt{2}} = (\phi_{l,-m}^i)^*, \quad \phi_{l0}^i = \phi^{(il0)}, \quad (49)$$

$$\chi_{lm}^i = \frac{\chi_1^{(ilm)} - i\chi_2^{(ilm)}}{\sqrt{2}} = (\chi_{l,-m}^i)^*, \quad \chi_{l0}^i = \chi^{(il0)}, \quad (50)$$

where $m > 0$. Also, to realize ϕ^α and χ^α , we translate them as

$$\begin{aligned} \phi_{jm}^\alpha &= \frac{\phi_1^{(\alpha jm)} - i\phi_2^{(\alpha jm)}}{\sqrt{2}} = (\phi_{j,-m}^{-\alpha})^*, \\ \chi_{jm}^\alpha &= \frac{\chi_1^{(\alpha jm)} - i\chi_2^{(\alpha jm)}}{\sqrt{2}} = (\chi_{j,-m}^{-\alpha})^*, \end{aligned} \quad (51)$$

with positive roots α and $m = -j, -j+1, \dots, 0, \dots, j$. Then, the Lagrangian are realized in the following way:

$$\begin{aligned}
S_{\text{gauge}}^{\text{quadratic}} = & \int d^4x \left(\sum_{l=1}^{\infty} \sum_{m=1}^l \sum_{r=1,2} \left[-\frac{1}{2} \phi_r^{(ilm)} \left\{ \square + \frac{l(l+1)}{R^2} \right\} \phi_r^{(ilm)} - \frac{1}{2} \chi_r^{(ilm)} \left\{ \square + \xi \frac{l(l+1)}{R^2} \right\} \chi_r^{(ilm)} \right] \right. \\
& + \sum_{l=1}^{\infty} \left[-\frac{1}{2} \phi^{(i0)} \left\{ \square + \frac{l(l+1)}{R^2} \right\} \phi^{(i0)} - \frac{1}{2} \chi^{(i0)} \left\{ \square + \xi \frac{l(l+1)}{R^2} \right\} \chi^{(i0)} \right] \\
& + \sum_{\substack{\alpha > 0 \\ \text{with } k_\alpha = 0}} \sum_{j=1}^{\infty} \sum_{m=-j}^j \sum_{r=1,2} \left[-\frac{1}{2} \phi_r^{(\alpha jm)} \left\{ \square + \frac{j(j+1)}{R^2} \right\} \phi_r^{(\alpha jm)} \right. \\
& \left. - \frac{1}{2} \chi_r^{(\alpha jm)} \left\{ \square + \xi \frac{j(j+1)}{R^2} \right\} \chi_r^{(\alpha jm)} \right] \\
& + \sum_{\substack{\alpha > 0 \\ \text{with } |k_\alpha| > 0}} \sum_{j=|k_\alpha|}^{\infty} \sum_{m=-j}^j \sum_{r=1,2} \left[-\frac{1}{2} \phi_r^{(\alpha jm)} \left\{ \square + \frac{(j(j+1) - k_\alpha^2) - k_\alpha^2 + \xi k_\alpha^2}{R^2(j(j+1) - k_\alpha^2)} \right\} \phi_r^{(\alpha jm)} \right. \\
& \left. - \frac{1}{2} \chi_r^{(\alpha jm)} \left\{ \square + \xi \frac{j(j+1) - k_\alpha^2}{R^2} \right\} \chi_r^{(\alpha jm)} \right] \\
& \left. + \sum_{\substack{\alpha > 0 \\ \text{with } |k_\alpha| > 0}} \sum_{j=|k_\alpha|}^{\infty} \sum_{m=-j}^j \sum_{r,s=1,2} \frac{k_\alpha \epsilon^{rs}}{j(j+1) - k_\alpha^2} \phi_r^{(\alpha jm)} \left\{ \square + \xi \frac{j(j+1) - k_\alpha^2}{R^2} \right\} \chi_s^{(\alpha jm)} \right), \quad (52)
\end{aligned}$$

where ϵ^{rs} is Levi-Civita symbol. Then, we can understand almost all components of ϕ are physical scalars and some parts of ϕ and the all of χ are Nambu-Goldstone bosons. In the case $k_\alpha = 0$, that is, the gauge fields commute with the background fields, the mass spectrum of $\phi_r^{(\alpha jm)}$ coincides with that of $\phi_r^{(ilm)}$. Note that since massless χ clearly plays the role of the Nambu-Goldstone boson, we treat the cross kinetic terms between $\phi_r^{(\alpha jm)}$ and $\chi_r^{(\alpha jm)}$ perturbatively.

Since ϕ may possess massless scalar modes, we should consider not only the quadratic terms but also the cubic and quartic terms of the gauge fields as well as the couplings to fermion fields. Such interactions can generate radiative corrections and effective mass terms called Coleman-Weinberg potential [53]. Massless scalar modes of ϕ may therefore have important phenomenological implications for physics beyond the SM [54–67].

IV. APPLICATION

In this section, we show the gauge symmetry breaking of SU(3) and SU(5) as illustrative examples.

A. Gauge symmetry breaking of SU(3)

First, we consider the case in which the gauge group is SU(3). Let α^1, α^2 denote the simple roots of SU(3) and μ^1, μ^2 denote their fundamental weights: $\mu^i \cdot \alpha^j = \frac{1}{2} \delta^{ij}$. Then, we define $\alpha^3 = \alpha^1 + \alpha^2$ as depicted in Fig. 1. When the background field takes the form $\mu = \mu^1 - \mu^2$, we can easily calculate $k_{\alpha^1} \neq 0, k_{\alpha^2} \neq 0, k_{\alpha^3} = 0$. In this case, all modes of $A_{\mu,r}^{(\alpha^1 jm)}$ and $A_{\mu,r}^{(\alpha^2 jm)}$ are massive and the symmetries are

broken, while $A_{\mu,r}^{(\alpha^3 00)}$ are massless modes and the symmetry remains in the α^3 -direction. This implies that the higher-dimensional gauge symmetry SU(3) is spontaneously broken to SU(2) $\times$ U(1) including Cartan direction. In contrast, when $\mu = \mu^1 + \mu^2$, all k_α are nonzero, the symmetry is broken to only Cartan directions U(1) $\times$ U(1).

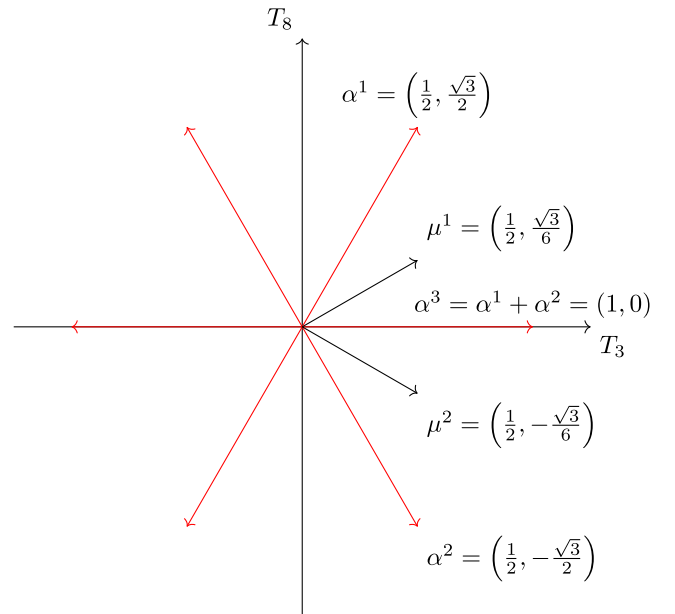


FIG. 1. Diagram of the simple roots α^1, α^2 of SU(3) and the fundamental weights μ^1, μ^2 . Here, T_3 and T_8 are the Cartan generators of SU(3).

B. Gauge symmetry breaking of SU(5)

Next, let the gauge group be SU(5), and denote the simple roots $\alpha^1, \alpha^2, \alpha^3, \alpha^4$. We denote the fundamental weights $\mu^1, \mu^2, \mu^3, \mu^4$ such that $\mu^i \cdot \alpha^j = \frac{1}{2} \delta^{ij}$. When $\mu = \mu^1$, the vector μ is orthogonal to α^2, α^3 and α^4 . Therefore, the gauge group is broken to $SU(4) \times U(1)$. On the other hand, when $\mu = \mu^2$, μ is orthogonal to α^1, α^3 , and α^4 . Therefore, the gauge group is broken to $SU(3) \times SU(2) \times U(1)$ in four dimensions. To discuss gauge symmetry breaking of larger groups, it is convenient to expand μ with fundamental weights μ^i .

V. SUMMARY

In this paper, we introduce a two-dimensional sphere S^2 as an extra dimensional space in addition to the four-dimensional Minkowski spacetime M^4 and analyze a gauge theory formulated on this six-dimensional spacetime.

One of the central features of this model is the nontrivial background field configuration $\langle A_\varphi \rangle = \mu \cdot H \cos \theta$. This setup allows us to explore mechanisms of spontaneous gauge symmetry breaking through the curvature and topology of S^2 , without conventional Higgs-like scalar fields. As any function on S^2 can be expressed in terms of eigenfunctions of the angular momentum operators, the gauge fields commuting with the background field can be expanded in the eigenfunctions of L_i , while those not commuting can be expanded in the eigenfunctions of an alternative SO(3) operator, $J_i^{(\alpha)}$. From the corresponding eigenvalues, we obtain the Kaluza-Klein mass spectrum. The nontrivial background on S^2 therefore induces spontaneous gauge symmetry breaking. Those gauge generators which are orthogonal to the background direction remain symmetry in four dimensions. On the contrary, those not orthogonal are broken generators. Additionally, performing the KK expansion of the extra-dimensional gauge components, we identify a physical scalar ϕ and a Nambu-Goldstone boson χ .

Before conclusion, let us add a few remarks. The background fields raise the gauge fields energy and might therefore appear to be undesirable. However, thanks to this background field, we could obtain massless fermions and stabilize the entire model [7,41,44,52]. Also, we may have obtained massless modes of ϕ . Then, we should consider an argument similar to Coleman-Weinberg potential [53], in

which one performs loop calculations of the scalar quartic self-interaction and the scalar-fermion coupling to consider an effective mass term. Based on these two considerations, we need to introduce fermions. We leave this study for future work.

Our framework for spontaneous gauge symmetry breaking on S^2 can be applied to GUT or GHU to construct explicit models that resolve the issues of the SM.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: EIGENFUNCTION OF ANGULAR OPERATOR $J^{(\alpha)}$

In this appendix, we solve Eqs. (22) and (23) under the condition that the surface terms vanish, which yields the relation Eq. (21) between j and k_α . Hereafter, we abbreviate k_α as k for simplicity.

Since the eigenfunctions of $J_3^{(\alpha)} = -i\partial_\varphi$ take the form $e^{im\varphi}$ with $m = -j, -j+1, \dots, j$, the eigenfunctions of $J^{(\alpha)2}$ can be expressed as

$$\mathcal{Y}_{kjm}(\theta, \varphi) = \Theta_{jkm}(\theta) e^{im\varphi}. \quad (\text{A1})$$

By substituting this into Eq. (17) and defining $z = \cos \theta$, we obtain

$$\left((1-z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} - \frac{2mkz + m^2 + k^2}{1-z^2} + j(j+1) \right) \Theta_{jkm} = 0. \quad (\text{A2})$$

Furthermore, by changing the variables as [68,69]

$$\Theta_{jkm} = (1-z)^{\frac{|m+k|}{2}} (1+z)^{\frac{|m-k|}{2}} w_{jkm}, \quad (\text{A3})$$

the differential equation for w_{jkm} becomes

$$\left[(1-z^2) \frac{d^2}{dz^2} - \{(|m+k| + |m-k| + 2)z + |m+k| - |m-k|\} \frac{d}{dz} + \left\{ j - \frac{|m+k|}{2} - \frac{|m-k|}{2} \right\} \left\{ j - \frac{|m+k|}{2} - \frac{|m-k|}{2} + |m+k| + |m-k| + 1 \right\} \right] w_{jkm} = 0. \quad (\text{A4})$$

By carefully examining this equation, we observe that it corresponds to the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ as

$$\left[(1-z^2) \frac{d^2}{dz^2} - \{(\alpha + \beta + 2)z + \alpha - \beta\} \frac{d}{dz} + n(n + \alpha + \beta + 1) \right] P_n^{(\alpha,\beta)}(z) = 0, \quad (\text{A5})$$

where

$$\alpha = |m + k|, \quad \beta = |m - k|, \quad n = j - \frac{|m + k|}{2} - \frac{|m - k|}{2}. \quad (\text{A6})$$

Note that n denotes the degree of the Jacobi polynomials and must be a non-negative integer to ensure the Jacobi polynomials remain finite. In the following, by imposing the condition that the surface term of A_μ vanishes, we demonstrate that the Jacobi polynomials must remain finite, i.e., that n must be a non-negative integer. A similar argument holds for the other fields.

The Lagrangian of A_μ in Eq. (29) is followed by the surface term

$$S_{\text{surface}, A_\mu} = \int d^4x d\varphi \left\{ \left[\frac{1}{2} \sin \theta A_{\mu,i} (\partial_\theta A^{\mu i}) \right]_{\theta=0}^{\theta=\pi} + \left[\frac{1}{2} \sin \theta A_\mu^{-\alpha} (\partial_\theta A^{\mu\alpha}) \right]_{\theta=0}^{\theta=\pi} \right\}. \quad (\text{A7})$$

Expanding A_μ by Eqs. (30) and (32), we have

$$\begin{aligned} -2S_{\text{surface}, A_\mu} &= \int d^4x d\varphi \left(- \sum_{l,l',m,m'} A_{\mu i, lm} A_{l' m'}^{\mu i} N_{lm} N_{l' m'} \left[Y_{lm}(z) (1-z^2) \frac{d}{dz} Y_{l' m'}(z) \right]_{z=-1}^{z=1} \right. \\ &\quad - \sum_{j,j',m,m'} A_{\mu, jm}^{(\alpha)} A_{j' m'}^{(\alpha)\mu} C_{kjm} C_{k j' m'} \left[(1-z)^{\frac{|m+k|}{2}} (1+z)^{\frac{|m-k|}{2}} P_{j - \frac{|m+k|}{2} - \frac{|m-k|}{2}}^{(|m+k|, |m-k|)}(z) \frac{e^{im\varphi}}{\sqrt{2\pi}} \right. \\ &\quad \times (1-z^2) \frac{d}{dz} \left\{ (1-z)^{\frac{|m'+k|}{2}} (1+z)^{\frac{|m'-k|}{2}} P_{j' - \frac{|m'+k|}{2} - \frac{|m'-k|}{2}}^{(|m'+k|, |m'-k|)}(z) \right\} \frac{e^{im'\varphi}}{\sqrt{2\pi}} \Big|_{z=-1}^1 \Big) \\ &\equiv \int d^4x \left(- \sum_{l,l',m,m'} A_{\mu i, lm} A_{l' -m}^{\mu i} N_{lm} N_{l' -m} \sigma_{ll'm} - \sum_{j,j',m} A_{\mu, jm}^{(\alpha)} A_{j' -m}^{(\alpha)\mu} C_{kjm} C_{k j' -m} \sigma_{jj'm}^{(\alpha)} \right). \end{aligned} \quad (\text{A8})$$

Since $A_{\mu lm}^i$ and $A_{\mu jm}^{(\alpha)}$ are independent for each l and m , $S_{\text{surface}, A_\mu}$ can vanish if $\sigma_{jj'm}^{(\alpha)} = \sigma_{ll'm} = 0$. We use the following recurrence relation for the Jacobi polynomials:

$$(1-z^2) \frac{dP_n^{(a,b)}}{dz} - (n+a+b+1) \left(z - \frac{b-a}{2n+a+b+2} \right) P_n^{(a,b)}(z) = - \frac{2(n+1)(n+a+b+1)}{2n+a+b+2} P_{n+1}^{(a,b)}(z). \quad (\text{A9})$$

By setting $a = |m - k|$, $b = |m + k|$, $n = j' - k^*$, and $k^* = \frac{|m+k|}{2} + \frac{|m-k|}{2}$, we can express $\sigma_{jj'm}^{(\alpha)}$ by powers of $(1-z)$ and $(1+z)$, and the Jacobi function as

$$\begin{aligned} \sigma_{jj'm}^{(\alpha)} &= \left[- \frac{|m-k|}{2} (1-z)^{k^*} (1+z)^{k^*+1} P_{j-k^*}^{(|m+k|, |m-k|)}(z) P_{j'-k^*}^{(|m-k|, |m+k|)}(z) \right. \\ &\quad + \frac{|m+k|}{2} (1-z)^{k^*+1} (1+z)^{k^*} P_{j-k^*}^{(|m+k|, |m-k|)}(z) P_{j'-k^*}^{(|m-k|, |m+k|)}(z) + (1-z)^{k^*} (1+z)^{k^*} (n+a+b+1) \\ &\quad \times \left\{ \left(z - \frac{b-a}{2n+a+b+2} \right) P_{j-k^*}^{(|m+k|, |m-k|)}(z) P_{j'-k^*}^{(|m-k|, |m+k|)}(z) - \frac{2(n+1)}{2n+a+b+2} P_{j-k^*}^{(|m+k|, |m-k|)}(z) P_{j'-k^*+1}^{(|m-k|, |m+k|)}(z) \right\} \Big]_{z=-1}^{z=1}. \end{aligned} \quad (\text{A10})$$

Therefore, when $k^* \neq 0$ (i.e., $k \neq 0$ or $m \neq 0$), the Jacobi function must remain finite at $z = \pm 1$ to ensure $\sigma_{jj'm}^{(\alpha)} = 0$. In contrast, when $k^* = 0$ (i.e., $k = m = 0$), we have

$$\sigma_{jj'0}^{(\alpha)} = (j' - j)\{1 + (-1)^{j+j'}\}, \quad (\text{A11})$$

and it follows that $\sigma_{j'jm}^{(\alpha)} = -\sigma_{jj'm}^{(\alpha)}$. Therefore, the surface terms automatically vanish upon summing over j, j' .

$$\sum_{j,j'} \underbrace{\sum_{\substack{\alpha: \text{ roots} \\ k_\alpha=0}} A_{\mu,j0}^{(\alpha)} A_{j'0}^{(\alpha)\mu} C_{kj0} C_{kj'0}}_{\text{symmetric under } j \leftrightarrow j'} \sigma_{jj'0}^{(\alpha)} = 0. \quad (\text{A12})$$

Similarly, by setting $k = 0$, the same argument holds for $\sigma_{ll'm}$. From the above argument, n must be a non-negative integer for the surface terms to vanish. Consequently, we obtain the relation Eq. (21) between j and k . Moreover, since j is a non-negative integer by SU(2) algebra and the single-valuedness of the Lagrangian, k must also be a non-negative integer.

The normalization constant is chosen to be

$$C_{jkm} = \sqrt{\frac{2j+1}{2^{|m+k|+|m-k|+1}}} \frac{\Gamma\left(j - \frac{|m+k|}{2} - \frac{|m-k|}{2} + 1\right) \Gamma\left(j + \frac{|m+k|}{2} + \frac{|m-k|}{2} + 1\right)}{\Gamma\left(j + \frac{|m+k|}{2} - \frac{|m-k|}{2} + 1\right) \Gamma\left(j - \frac{|m+k|}{2} + \frac{|m-k|}{2} + 1\right)}, \quad (\text{A13})$$

so that the orthogonality relation as follows,

$$\int d\theta d\varphi \sin\theta \mathcal{Y}_{kjm}^* \mathcal{Y}_{kj'm'} = \delta_{jj'} \delta_{mm'}. \quad (\text{A14})$$

Here, the eigenfunction $\mathcal{Y}_{kjm}(\theta, \varphi)$ is given by

$$\mathcal{Y}_{kjm}(\theta, \varphi) = C_{jkm} (1-z)^{\frac{|m+k|}{2}} (1+z)^{\frac{|m-k|}{2}} P_{j-\frac{|m+k|}{2}, \frac{|m-k|}{2}}^{(|m+k|, |m-k|)}(\cos\theta) \frac{e^{im\varphi}}{\sqrt{2\pi}}. \quad (\text{A15})$$

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