

Lorentzian Gribov no-pole condition for Yang-Mills theory

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For over four decades, Gribov's no-pole condition has been almost exclusively explored in Euclidean space, where the elliptic nature of the Faddeev-Popov operator provides a clear spectral boundary. In physical Minkowski spacetime, however, this operator becomes a hyperbolic wave operator, and the Euclidean positivity criterion collapses. We show that the natural Lorentzian replacement is a real-time boundary-value problem: a gauge configuration remains inside the first Gribov region as long as the Faddeev-Popov wave equation admits no source-free solutions obeying the Feynman boundary condition. For backgrounds localized in time, this condition translates to the injectivity of the negative-frequency block of the classical ghost scattering map. For stationary backgrounds, it becomes a spatial bound-state or threshold-resonance problem under Fourier transformation. Using a Wronskian identity, we prove that pure frequency mixing in stable self-adjoint time-dependent channels is structurally protected and cannot by itself produce the obstruction. While static chromomagnetic backgrounds reproduce the familiar zero-frequency horizon crossing, static chromoelectric potentials reach the horizon at finite, nonvanishing frequency—a uniquely Lorentzian phenomenon arising because A_0 couples directly to the ghost time derivative. We also cast the condition in Fredholm form and show that the exact restriction is a functional determinant, which the naive local continuation of the Gribov-Zwanziger action fails to reproduce, leaving the construction of a genuine local real-time action as the central open problem.

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I. INTRODUCTION

The Faddeev-Popov prescription lies at the heart of gauge theory quantization, representing the gauge-fixing Jacobian through anticommuting ghost fields [1]. Yet, this elegant construct carries a classic vulnerability: beyond perturbation theory, local gauge-fixing conditions are rarely unique. Non-Abelian gauge orbits typically pierce the gauge-fixing slice multiple times—a global topological obstruction formalized by Singer [2] and analyzed dynamically by Gribov in Coulomb and Landau gauges [3]. In Euclidean Landau gauge, the Faddeev-Popov operator is

$$\mathcal{M}_E^{ab}[A] = -\partial_\mu D_\mu^{ab}[A], \quad (1)$$

on the slice $\partial_\mu A_\mu^a = 0$. Because this operator is elliptic and Hermitian on the standard functional domain, the first Gribov region can be cleanly mapped as the domain of spectral positivity:

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$$\Omega_E = \{A_\mu : \partial_\mu A_\mu = 0, \mathcal{M}_E[A] > 0\}. \quad (2)$$

The boundary of this region—the Gribov horizon—is reached when the operator develops its first zero mode, signaling that the gauge-fixing slice has become parallel to a gauge orbit. This zero mode manifests physically as a pole in the ghost propagator. Gribov's no-pole condition and Zwanziger's horizon condition are two mathematically equivalent formulations of this geometric boundary, verified perturbatively and proven to all orders in Euclidean Landau gauge [4,5]. Crucially, Zwanziger's formulation allows the restricted path integral to be localized into a renormalizable action using auxiliary fields [6,7].

This spectral definition, however, collapses when we transition to physical Minkowski space. In Lorenz gauge,

$$\partial^\mu A_\mu^a = 0, \quad (3)$$

the Faddeev-Popov operator becomes a hyperbolic wave operator,

$$\mathcal{M}_A^{ab} = -\partial^\mu D_\mu^{ab}[A], \quad (4)$$

rather than an elliptic operator. Because the spectrum of a wave operator is unbounded and typically continuous, spectral positivity loses its direct geometric meaning, and we can no longer define the Gribov region by demanding a positive-definite operator.

Gribov himself anticipated this challenge in Sec. V of his seminal work [3]: in Minkowski spacetime, the spectral positivity criterion must be replaced by a boundary-value prescription. Specifically, we must impose the Feynman boundary condition on the Faddeev-Popov operator—the same prescription that dictates the real-time ghost determinant in the path integral. The problematic gauge copies are then the source-free solutions of the wave equation that behave as negative-frequency modes in the far past and as positive-frequency modes in the far future. Gribov poetically characterized this phenomenon as the classical analog of ghost pair production in an external field.

We formalize this Lorentzian analogue by framing the no-pole condition as a boundary-value problem rather than a spectral inequality. Let $\text{Ker}_F \mathcal{M}_A$ be the vector space of nontrivial, source-free solutions to the copy equation

$$\mathcal{M}_A \omega = 0 \quad (5)$$

that satisfy the Feynman boundary prescription. The first Lorentzian Gribov region is then naturally defined as

$$\Omega_M^{(1)} = \{A_\mu : \partial^\mu A_\mu = 0, \text{Ker}_F \mathcal{M}_A = 0\}_0, \quad (6)$$

where the subscript $_0$ denotes the connected component containing the vacuum $A_\mu = 0$.

For gauge backgrounds that are localized in time (such as pulses), the boundary-value problem (5) becomes an asymptotic ghost scattering problem. Let $\mathcal{H}_\pm^{\text{in}}$ and $\mathcal{H}_\pm^{\text{out}}$ be the positive- and negative-frequency asymptotic modes in the far past and far future, and let $S_{\text{gh}}[A]$ be the classical ghost scattering matrix. The Feynman boundary condition demands that a ghost fluctuation contains only negative frequencies in the past and only positive frequencies in the future. The survival of such a mode is governed by the negative-to-negative block of the scattering matrix, $\alpha_{--} : \mathcal{H}_{--}^{\text{in}} \rightarrow \mathcal{H}_{--}^{\text{out}}$. The no-pole condition then takes the elegant algebraic form

$$\text{Ker } \alpha_{--}[A] = 0. \quad (7)$$

A temporally localized background thus breaches the Lorentzian horizon when α_{--} loses injectivity, meaning that an incoming negative-frequency ghost wave packet is completely converted or scattered away, leaving no negative-frequency component in the future. For stationary backgrounds, there is no asymptotic temporal scattering; after Fourier transforming in time, the no-pole condition (6) manifests as a spatial bound-state or threshold-resonance condition.

We summarize the physical and formal results of this paper below:

- (1) The boundary-value kernel condition in (6) provides a precise real-time Minkowski replacement for the traditional Euclidean spectral positivity of $\mathcal{M}_E$ in (2).

- (2) For temporally localized backgrounds, this condition is equivalent to the scattering injectivity condition (7). For stationary backgrounds, it becomes a frequency-domain bound-state or threshold-resonance problem in a spatial channel.
- (3) The condition admits a Fredholm Lippmann-Schwinger form:

$$\text{Det}_{\text{Fr}}(1 + G_F^0 V_A) \neq 0, \quad (8)$$

where G_F^0 is the free Feynman propagator and V_A is the derivative-coupled color potential. This determinant is the ghost-sector functional determinant in the background, and its zeros mark the Gribov horizon.

- (4) For temporally localized backgrounds, the singular values of α_{--} define a nonlocal flow horizon functional,

$$H_{\text{flow}}[A] = -\text{Tr}_{\mathcal{H}_{--}} \log(\alpha_{--}^\dagger[A] \alpha_{--}[A]), \quad (9)$$

which acts as a dynamical barrier, diverging logarithmically as a copy mode emerges.

- (5) Through a Wronskian conservation identity (ghost flux conservation), we prove that pure frequency mixing in stable self-adjoint temporal channels cannot produce the obstruction at finite frequency. Simple time-dependence does not ruin gauge-fixing.
- (6) Static *chromomagnetic* sheets reach the horizon through static, zero-frequency copy modes, reproducing the familiar Euclidean eigenvalue crossing. We demonstrate this explicitly with a smooth Pöschl-Teller magnetic sheet.
- (7) Static *chromoelectric* wells, by contrast, cross the horizon at a *finite, nonvanishing* frequency $\Omega_\star \neq 0$. Because the scalar potential A_0 couples directly to the time derivative in the wave operator, it acts as a frequency-dependent potential depth. The background is static, yet the copy mode it supports oscillates in time; this is a purely Lorentzian effect with no Euclidean analog.
- (8) The traditional Zwanziger horizon functional admits a natural real-time Feynman-inverse continuation, but it is *not* equivalent to the flow functional H_{flow} . We show in Appendix C that the two diverge on the horizon with different critical exponents and that for symmetric backgrounds the naive continuation misses the horizon altogether; the exact restriction is the functional determinant, and recovering it from a local action remains open.

Most applications of the Gribov-Zwanziger restriction originate in the Euclidean theory, relying on the analytic continuation of selected Green's functions to extract real-time signatures. In contrast, our approach defines the Gribov obstruction directly as a real-time, boundary-value problem in Minkowski spacetime, prior to any

continuation. To our knowledge, this is the first direct formulation of the Gribov horizon as a Minkowski Feynman-boundary problem. The closest existing real-time formulations are the canonical (Hamiltonian) analyses in Coulomb gauge [8,9], which remain spatial eigenvalue problems on a fixed-time slice rather than in-out wave scattering problems in time. A related but distinct Lorentz-covariant approach to the Gribov ambiguity is Slavnov's formulation using a gauge-invariant ghost Lagrangian [10].

We emphasize that this condition is gauge-fixed and does not, on its own, prove confinement. Its purpose is focused: it identifies the domain of classical Yang-Mills backgrounds for which the Lorenz-gauge Faddeev-Popov determinant becomes singular under the same Feynman boundary prescription that defines the real-time path integral. This provides a precise real-time counterpart of the Gribov restriction, fully aligned with the Gribov-Zwanziger physical picture in which elementary gluons exhibit a violation of reflection positivity and fail to admit a physical particle interpretation [11–13].

II. LORENZ-GAUGE FADDEEV-POPOV OPERATOR

We work in $d = 4$ Minkowski space with metric

$$\eta_{\mu\nu} = \text{diag}(+, -, -, -), \quad \square = \partial^\mu \partial_\mu. \quad (10)$$

The Yang-Mills gauge transformation is

$$\delta_\omega A_\mu^a = D_\mu^{ab}[A]\omega^b, \quad D_\mu^{ab}[A] = \delta^{ab}\partial_\mu - gf^{abc}A_\mu^c, \quad (11)$$

with the sign convention fixed by $[T^a, T^b] = if^{abc}T^c$. The Lorenz gauge functional is

$$F^a[A] = \partial^\mu A_\mu^a. \quad (12)$$

Its variation is

$$\delta_\omega F^a[A] = \partial^\mu D_\mu^{ab}[A]\omega^b. \quad (13)$$

We define

$$\mathcal{M}_A^{ab} = -\partial^\mu D_\mu^{ab}[A]. \quad (14)$$

On the gauge slice $\partial^\mu A_\mu^a = 0$, this becomes

$$\mathcal{M}_A^{ab} = -\delta^{ab}\square + gf^{abc}A_\mu^c\partial^\mu. \quad (15)$$

Thus, the ghost action is

$$S_{\text{gh}}[A, \bar{c}, c] = \int d^4x \bar{c}^a \mathcal{M}_A^{ab} c^b. \quad (16)$$

The Faddeev-Popov wave operator is not merely a tool for ghost loops; it is the mathematical sentinel guarding the

uniqueness of the gauge slice. An infinitesimal gauge transformation ω deforms a Lorenz-gauge background A_μ into a redundant copy $A_\mu + D_\mu[A]\omega$ that remains on the gauge slice to first order if and only if ω lies in the kernel of this operator:

$$\mathcal{M}_A \omega = 0. \quad (17)$$

Equation (17) is therefore both the classical field equation for the ghost in the background A_μ and the infinitesimal gauge-copy equation.

To study the solutions to this wave equation, we isolate the free propagation from the background interaction by splitting the operator as

$$\mathcal{M}_A = \mathcal{M}_0 + V_A, \quad (18)$$

where

$$\mathcal{M}_0^{ab} = -\delta^{ab}\square, \quad (V_A)^{ab} = gf^{abc}A_\mu^c\partial^\mu. \quad (19)$$

Here, $\mathcal{M}_0$ is the free wave operator, while V_A acts as a derivative-coupled color potential. Throughout this work, G_F^0 denotes the Green's-function inverse of the free differential operator $\mathcal{M}_0$ with Feynman boundary conditions:

$$G_F^{0,ab}(p) = \frac{\delta^{ab}}{p^2 + i0}. \quad (20)$$

We note that G_F^0 is the Green's function of the wave operator itself, which differs from the standard time-ordered ghost propagator by an overall factor of i . With this convention, the Lippmann-Schwinger operator $1 + G_F^0 V_A$ is defined unambiguously.

III. BOUNDARY-VALUE FORM OF THE LORENTZIAN NO-POLE CONDITION

Because the copy equation is a wave equation rather than an elliptic equation, the concept of a “zero mode” is mathematically incomplete until we specify boundary conditions. The natural boundary conditions are those dictated by the Feynman prescription of the ghost determinant in the real-time path integral. Let $\text{Ker}_F \mathcal{M}_A$ denote the vector space of source-free solutions to

$$\mathcal{M}_A \omega = 0, \quad (21)$$

subject to this Feynman prescription. For a background localized in time, this boundary condition requires the copy mode to be purely negative frequency in the far past and purely positive frequency in the far future. For a stationary background, the prescription translates to the $+i0$ frequency-space regularization, which manifests in the spatial channel as a bound-state or threshold-resonance condition.

Definition 1: Lorentzian no-pole condition. For any Lorenz-gauge background for which this boundary-value problem is defined, the Minkowski no-pole condition is

$$\text{Ker}_F \mathcal{M}_A = 0. \quad (22)$$

The first Lorentzian Gribov region is defined as the connected component containing the vacuum,

$$\Omega_M^{(1)} = \{A_\mu : \partial^\mu A_\mu = 0, \text{Ker}_F \mathcal{M}_A = 0\}_0, \quad (23)$$

where the subscript $_0$ denotes the connected component containing $A_\mu = 0$. The Lorentzian Gribov horizon is the failure boundary:

$$\partial\Omega_M = \{A_\mu : \partial^\mu A_\mu = 0, \text{Ker}_F \mathcal{M}_A \neq 0\}. \quad (24)$$

Crucially, this condition makes no reference to a lowest eigenvalue. Instead, it asks a purely dynamical question: does the Faddeev-Popov wave equation admit a source-free copy mode under the Feynman boundary prescription? In infinite volume, a threshold configuration can drive the horizon functionals (defined in Sec. VI) to divergence even in the absence of a strictly normalizable kernel vector. In finite volume, the kernel condition is literal.

The Feynman boundary condition is not an arbitrary choice. It is the prescription carried by the ghost determinant in the in-out generating functional—the very object whose zeros we are characterizing. Retarded or advanced prescriptions compute ghost response functions rather than the vacuum persistence amplitude and do not register the copy in the same way; a genuinely in-in, Schwinger-Keldysh treatment would replace the single Feynman propagator by a contour-ordered matrix of ghost propagators and would define its own, contour-dependent notion of horizon. We do not pursue those variants here: the region $\Omega_M^{(1)}$ defined above is the one relevant to the standard real-time, in-out path integral.

Assume now that $A_\mu(t, \mathbf{x})$ vanishes outside a finite time interval, or decays fast enough that free in and out decompositions exist. The free asymptotic ghost modes are

$$u_{\mathbf{p},\lambda}^{(+)}(x) = \frac{e^{-i|\mathbf{p}|t + i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2|\mathbf{p}|}} e_\lambda, \quad (25)$$

$$u_{\mathbf{p},\lambda}^{(-)}(x) = \frac{e^{+i|\mathbf{p}|t + i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2|\mathbf{p}|}} e_\lambda, \quad (26)$$

where e_λ is a color basis vector. Let $\mathcal{H}_+^{\text{in}}$ and $\mathcal{H}_-^{\text{in}}$ denote the positive- and negative-frequency Cauchy data at $t \rightarrow -\infty$, and define $\mathcal{H}_\pm^{\text{out}}$ similarly at $t \rightarrow +\infty$. To diagnose the propagation of these unphysical yet mathematically vital ghost fields, we equip the auxiliary asymptotic spaces with a positive-definite norm $\|\cdot\|$ defined by the sum of the

modulus squared of the free mode coefficients: if $f = \sum_{\mathbf{p},\lambda} c_{\mathbf{p},\lambda} u_{\mathbf{p},\lambda}^{(\pm)}$, then $\|f\|^2 = \sum_{\mathbf{p},\lambda} |c_{\mathbf{p},\lambda}|^2$. Geometrically, this is the Klein-Gordon inner product restricted to $\mathcal{H}_+$, and minus the Klein-Gordon product restricted to $\mathcal{H}_-$, both of which are positive-definite on their respective subspaces. We emphasize that this norm is purely a diagnostic tool defined on the auxiliary one-particle solution space; because ghosts are strictly unphysical, this norm does not correspond to any physical Hilbert space norm.

As the ghost wave packet propagates through the time-localized gauge background A_μ , the interaction maps incoming states to outgoing states. This evolution is described by the classical ghost scattering operator:

$$S_{\text{gh}}[A] : \mathcal{H}_+^{\text{in}} \oplus \mathcal{H}_-^{\text{in}} \longrightarrow \mathcal{H}_+^{\text{out}} \oplus \mathcal{H}_-^{\text{out}}. \quad (27)$$

Block decomposing this scattering operator yields

$$S_{\text{gh}}[A] = \begin{pmatrix} \alpha_{++} & \beta_{+-} \\ \beta_{-+} & \alpha_{--} \end{pmatrix}. \quad (28)$$

An incoming negative-frequency mode $v \in \mathcal{H}_-^{\text{in}}$ therefore scatters and evolves into

$$v \mapsto \beta_{+-} v + \alpha_{--} v, \quad \beta_{+-} v \in \mathcal{H}_+^{\text{out}}, \quad \alpha_{--} v \in \mathcal{H}_-^{\text{out}}. \quad (29)$$

The off-diagonal block β_{+-} measures the degree of frequency mixing. While in a physical field theory frequency mixing signals particle pair production, here it remains confined to the gauge-fixing sector: ghosts do not correspond to physical states. Instead, the mixing described by (29) indicates that the Faddeev-Popov Jacobian is being tested along a time-dependent gauge orbit.

Gribov's no-pole condition is a much stronger requirement than the simple presence of frequency mixing ($\beta_{+-} \neq 0$). A nonvanishing β_{+-} merely means that an incoming negative-frequency ghost disturbance is scattered partly into positive frequencies. The no-pole problem asks whether the wave equation admits a nontrivial source-free solution satisfying the Feynman endpoint conditions. This requires the future negative-frequency component to vanish completely:

$$\exists v \in \mathcal{H}_-^{\text{in}}, \quad v \neq 0, \quad \alpha_{--}[A]v = 0. \quad (30)$$

Such a solution exhibits the specialized asymptotics:

$$\omega(t \rightarrow -\infty) \in \mathcal{H}_-^{\text{in}}, \quad \omega(t \rightarrow +\infty) \in \mathcal{H}_+^{\text{out}}. \quad (31)$$

Thus, for temporally localized backgrounds, the no-pole condition (22) simplifies to the injectivity of the negative-to-negative scattering block:

$$\text{Ker } \alpha_{--}[A] = 0. \quad (32)$$

The Lorentzian horizon is not triggered by the onset of frequency mixing. Rather, it is marked by the loss of injectivity of the scattering block α_{--} , signaling the emergence of a physical Feynman-boundary copy mode.

IV. WRONSKIAN PROTECTION

A common concern when working in real time is that any time-dependence in a gauge background, by mixing positive and negative frequencies, might automatically trigger a Gribov copy. Fortunately, a powerful dynamical safeguard prevents this: in stable, self-adjoint *temporal* channels, the conservation of the Wronskian—physically the conservation of the auxiliary ghost flux—rules out the formation of copy modes through frequency mixing alone, at any finite frequency. The gauge slice remains structurally elastic, resisting copies under pure frequency-mixing oscillations. This is the Lorentzian analogue of the Euclidean statement that one remains strictly inside the first Gribov region until a zero mode appears.

Proposition 1 (Wronskian protection). Consider one charged ghost channel reduced to

$$\ddot{\phi}(t) + \Omega^2(t)\phi(t) = 0, \quad (33)$$

with real $\Omega^2(t)$ and asymptotic limits

$$\Omega(t \rightarrow -\infty) = \omega_-, \quad \Omega(t \rightarrow +\infty) = \omega_+, \quad \omega_{\pm} > 0. \quad (34)$$

Let a negative frequency in solution be expanded as

$$\phi(t \rightarrow -\infty) \sim \frac{e^{+i\omega_- t}}{\sqrt{2\omega_-}}, \quad (35)$$

$$\phi(t \rightarrow +\infty) \sim \alpha \frac{e^{+i\omega_+ t}}{\sqrt{2\omega_+}} + \beta \frac{e^{-i\omega_+ t}}{\sqrt{2\omega_+}}. \quad (36)$$

Then,

$$|\alpha|^2 - |\beta|^2 = 1. \quad (37)$$

In particular, $\alpha \neq 0$.

The proof follows from the conservation of the Wronskian

$$W[\phi, \phi^*] = \phi \dot{\phi}^* - \phi^* \dot{\phi}. \quad (38)$$

From (33), it follows that $\dot{W} = 0$. With the flux-normalized modes above, $W = -i$ in the far past and $W = -i(|\alpha|^2 - |\beta|^2)$ in the far future. This gives (37).

The same conclusion holds in the matrix case. If the reduced ghost equation is a stable self-adjoint oscillator

$$\ddot{\phi} + K(t)\phi = 0, \quad K(t) = K(t)^\dagger, \quad (39)$$

with positive asymptotic frequency matrices, conservation of the matrix Wronskian gives, in flux-normalized in and out bases,

$$\alpha^\dagger \alpha - \beta^\dagger \beta = \mathbf{1}. \quad (40)$$

If $\alpha v = 0$ for a nonzero vector v , then $-\| \beta v \|^2 = \| v \|^2$, which is impossible. Thus, $\text{Ker } \alpha = 0$ throughout this class. The derivation is given in Appendix A.

This is a statement about the auxiliary positive norm on asymptotic ghost mode coefficients introduced in Sec. III. It is not a physical ghost Hilbert space statement, and it does not use an indefinite ghost metric. It tells us when α_{--} cannot lose injectivity.

The result fixes the role of the time-dependent examples. A Cartan pulse with real nonzero asymptotic frequencies may have $\beta \neq 0$, but that does not put the background on the horizon. Stationary backgrounds require a different test because there is no temporal in-out block. The chromomagnetic examples below give zero-frequency copy modes, the direct Lorentzian image of the Euclidean lowest-eigenvalue crossing. The chromoelectric example reaches the same kernel condition at nonzero frequency because A_0 changes the spatial bound-state problem.

The assumptions of the Wronskian argument are restrictive, and it is instructive to see exactly how they can fail. For a general $SU(2)$, the Cartan background with only the color-3 components $A_\mu^3(x)$ nonzero, the charged-sector copy equation reduces to

$$\square \phi_s + i s g A_\mu^3 \partial^\mu \phi_s = 0, \quad A_\mu^3 \partial^\mu = A_0^3 \partial_t - \mathbf{A}^3 \cdot \nabla, \quad (41)$$

and, for a spatially homogeneous background $A_\mu^3 = A_\mu^3(t)$ with a spatial Fourier mode $\omega = e^{i\mathbf{p} \cdot \mathbf{x}} \phi_s(t) v_s$, to the single channel

$$\ddot{\phi}_s + i s g A_0^3(t) \dot{\phi}_s + Q_s(t) \phi_s = 0, \quad Q_s(t) = \mathbf{p}^2 + s g \mathbf{A}^3(t) \cdot \mathbf{p}, \quad s = \pm 1. \quad (42)$$

Eliminating the first-derivative term brings this to the form of an effective oscillator,

$$\ddot{\chi}_s + \left[Q_s + \frac{g^2}{4} (A_0^3)^2 - \frac{i}{2} s g \dot{A}_0^3 \right] \chi_s = 0. \quad (43)$$

Both reductions are carried out in Appendix A. Equation (43) shows what the Wronskian argument assumes. If A_0^3 is constant and Q_s is real, the field redefinition is a phase rotation and the reduced oscillator is self-adjoint. Then, α_{--} cannot develop a kernel. For the homogeneous channel (42), Lorenz gauge makes this automatic: when $\mathbf{A}^3$ is spatially constant, $\partial^\mu A_\mu^3 = \dot{A}_0^3 = 0$, so A_0^3 is constant.

The imaginary term in (43) can appear only after leaving this homogeneous setting. If $\dot{A}_0^3 \neq 0$ while Lorenz gauge is maintained, the background must also carry spatial dependence so that $\nabla \cdot \mathbf{A}^3$ can balance $\dot{A}_0^3$. At that point, the reduced temporal oscillator is no longer the whole problem. The invariant signal is the chromoelectric field, for example, $F_{0i}^3 = \partial_0 A_i^3 - \partial_i A_0^3$ in a Cartan sector.

A stationary chromoelectric scalar potential has no temporal scattering matrix α_{--} , but it still has a well-defined Feynman-boundary kernel problem after Fourier transforming in time. The example solved in Sec. VII reaches the horizon through that spatial bound-state problem, at nonzero frequency. Time-dependent chromoelectric backgrounds with both temporal and spatial structure remain to be analyzed.

V. FREDHOLM FORM OF THE KERNEL CONDITION

The same kernel condition can be written in Lippmann-Schwinger form. With $\mathcal{M}_A = \mathcal{M}_0 + V_A$, a Feynman-boundary solution satisfies

$$\mathcal{M}_0 \omega = -V_A \omega. \quad (44)$$

Acting with the free Feynman inverse gives

$$\omega = -G_F^0 V_A \omega, \quad (1 + G_F^0 V_A) \omega = 0. \quad (45)$$

Conversely, any solution of the second equation has the Feynman boundary prescription by construction and solves $\mathcal{M}_A \omega = 0$. Therefore,

$$\text{Ker}_F \mathcal{M}_A \simeq \text{Ker}(1 + G_F^0 V_A). \quad (46)$$

In a finite spatial volume with a ultraviolet cutoff, $1 + G_F^0 V_A$ reduces to a finite-dimensional matrix. The Minkowski no-pole condition then reads

$$\det(1 + G_F^0 V_A) \neq 0. \quad (47)$$

Here, the operator $1 + G_F^0 V_A$ acts on the color-valued spacetime function space $L^2(\mathbb{R}^{1,3}) \otimes \mathfrak{g}$, with G_F^0 acting as the Feynman propagator inverse of $\mathcal{M}_0$.

In the thermodynamic limit of infinite volume, this finite determinant must be replaced by a Fredholm determinant. Up to a background-independent normalization, $\text{Det}_{\text{Fr}}(1 + G_F^0 V_A)$ is the ghost-sector functional determinant in the background A_μ , that is, the ghost contribution

$\text{Det}_F \mathcal{M}_A / \text{Det} \mathcal{M}_0$ to the gauge-fixed weight. Its vanishing marks the Gribov horizon, where the gauge-fixing description breaks down, loosely analogous to the way zeros of a partition function locate a phase boundary.

To ensure functional-analytic rigor, we assume that the gauge background is short range in space and time, such that classical wave operators are well defined, in accordance with standard mathematical scattering theory [14,15]. We must, however, insert a word of caution regarding the definition of this determinant in $d = 4$. Since the derivative-coupled potential $V_A = g f^{abc} A_\mu^c \partial^\mu$ is a first-order differential operator, the product $G_F^0 V_A$ behaves asymptotically as $A \partial / \partial^2 \sim A/p$ at high momentum. This mild ultraviolet falloff means that $G_F^0 V_A$ is not trace class (it does not lie in the trace-class ideal $\mathcal{S}_1$), preventing us from defining the naive determinant $\text{Det}(1 + G_F^0 V_A)$ directly. To make it well defined, the determinant must be interpreted as a regularized determinant $\text{Det}_p(1 + G_F^0 V_A)$ with $G_F^0 V_A \in \mathcal{S}_p$ for a sufficiently large power p , or as a relative determinant defined with respect to the free evolution [16].

For the purposes of this paper, we focus on the robust and unambiguous kernel condition:

$$\text{Ker}(1 + G_F^0 V_A) = 0. \quad (48)$$

Wherever the Fredholm determinant can be properly regularized and defined, the horizon is marked by its vanishing:

$$\partial \Omega_M : \text{Det}_{\text{Fr}}(1 + G_F^0 V_A) = 0. \quad (49)$$

Within the connected component of the vacuum, this is equivalent to

$$A \in \Omega_M^{(1)} \Leftrightarrow -1 \notin \text{Spec}(G_F^0 V_A). \quad (50)$$

VI. RESTRICTED PATH INTEGRAL AND HORIZON FUNCTIONALS

A formal restriction of the Lorenz-gauge real-time path integral is

$$Z_{\Omega_M} = \int \mathcal{D}A \delta(\partial^\mu A_\mu) \text{Det}_F \mathcal{M}_A \chi_{\Omega_M^{(1)}}[A] e^{iS_{\text{YM}}[A]}. \quad (51)$$

Here, $\text{Det}_F \mathcal{M}_A$ means the determinant with Feynman boundary conditions, and

$$\chi_{\Omega_M^{(1)}}[A] = \begin{cases} 1, & \text{Ker}_F \mathcal{M}_A = 0 \text{ and } A \text{ belongs to the vacuum component,} \\ 0, & \text{otherwise.} \end{cases} \quad (52)$$

The characteristic function is nonlocal in time because it refers to the full boundary-value problem. In a temporally localized background, this means the full in-out evolution.

For temporally localized backgrounds, the singular values of α_{--} give the barrier functional

$$H_{\text{flow}}[A] = -\text{Tr}_{\mathcal{H}_-} \log(\alpha_{--}^\dagger[A] \alpha_{--}[A]). \quad (53)$$

In finite volume with a UV cutoff, this is an ordinary matrix trace. In infinite volume, it should be read as a relative determinant, and we assume

$$\alpha_{--}^\dagger \alpha_{--} - \mathbf{1} \in \mathcal{S}_1. \quad (54)$$

If this trace-class condition fails, one must replace H_{flow} by a regularized determinant. The boundary condition $\text{Ker} \alpha_{--} \neq 0$ still has a finite-volume meaning and gives the correct limiting criterion in the scattering problem. In finite volume, if $s_n[A]$ are the singular values of α_{--} , then

$$H_{\text{flow}}[A] = -\sum_n \log s_n^2[A]. \quad (55)$$

When a singular value vanishes, $H_{\text{flow}}[A] \rightarrow +\infty$. Thus, Eq. (53) detects the time-localized horizon.

The Fredholm form gives a spacetime determinant:

$$H_{\text{det}}[A] = -\log |\text{Det}_{\text{Fr}}(1 + G_F^0 V_A)|^2. \quad (56)$$

Both (53), when it is defined, and (56) diverge at a Feynman-boundary Faddeev-Popov zero mode, but they are built on different spaces: H_{flow} is a trace over the spatial asymptotic space $\mathcal{H}_-$, while H_{det} is a regularized determinant on the spacetime one-particle space of Sec. V. In the scattering setting, their equality is the standard relation between the in-out vacuum amplitude of an anticommuting field and its negative-frequency Bogoliubov block: for the Faddeev-Popov ghost, the in-out functional integral gives

$$\langle \text{out} | \text{in} \rangle \propto \text{Det}_F \mathcal{M}_A \propto \det \alpha_{--}, \quad (57)$$

up to a background-independent normalization [17]. Under the same assumptions,

$$H_{\text{flow}} \quad \text{and} \quad H_{\text{det}} \quad \text{define the same boundary } \partial\Omega_M. \quad (58)$$

A Lorentzian analogue of the Zwanziger horizon action would replace the Euclidean inverse by the Feynman inverse [6,7]. This is not part of the definition. The boundary condition defining $\Omega_M^{(1)}$ is the Feynman-boundary kernel condition; α_{--} and the Fredholm determinant are representations of it in the settings where they exist. A local real-time Gribov-Zwanziger action would have to reproduce this same boundary. As we show in Appendix C, the naive Feynman continuation of the Zwanziger functional does not; it diverges on the horizon with a different critical

exponent than H_{flow} , and for symmetric backgrounds, it can fail to diverge at all. A faithful local action must therefore reproduce the determinant structure of H_{flow} , not the tree-level resolvent bilinear.

VII. SOLVABLE EXAMPLES

We now illustrate these real-time boundary-value phenomena using three solvable configurations. The first is a stationary chromomagnetic background—a chromomagnetic shear layer modeled as a smooth Pöschl-Teller sheet—which reproduces the static Gribov picture: the horizon is breached by a zero-frequency, static copy mode. The second example consists of transient Cartan pulses, showing that real-time frequency mixing in a stable self-adjoint channel cannot drive the system to the horizon. The third example is a static chromoelectric well; here, the coupling of the scalar potential A_0 to the time derivative creates a frequency-dependent potential depth, forcing the Gribov horizon to be breached through a finite-frequency spatial bound state, with a static background but a time-oscillating copy mode.

A. Static Pöschl-Teller sheet

We model a chromomagnetic shear layer using a smooth Pöschl-Teller sheet. Take the static $SU(2)$ Cartan background:

$$A_0^3 = 0, \quad A_y^3(x) = A \text{sech}^2(\kappa x), \quad A_\mu^{1,2} = 0. \quad (59)$$

This configuration satisfies the Lorenz gauge because A_y^3 has no spatial y -dependence. In the charged color sector, we define:

$$i\epsilon v_s = s v_s, \quad s = \pm 1, \quad \epsilon^{12} = 1. \quad (60)$$

For a spatial copy mode of the form:

$$\omega^a = e^{-i\Omega t + i q y + i p_z z} \psi_s(x) v_s^a, \quad (61)$$

the copy equation reduces to the spatial differential equation:

$$\psi_s''(x) + [\Omega^2 - q^2 - p_z^2 - s g q A \text{sech}^2(\kappa x)] \psi_s(x) = 0. \quad (62)$$

This is a Schrödinger problem with a Pöschl-Teller well, whose normalizable levels are standard [18]. A bound state requires an attractive well, i.e., $-s g q A / \kappa^2 > 0$, which selects the charged channel with $s g q A < 0$; the opposite channel is repulsive and supports no copy. Writing the well strength as $\nu_s(\nu_s + 1)$, the level condition yields the dispersion relation

$$q^2 + p_z^2 - \Omega_n^2 = \kappa^2(\nu_s - n)^2, \quad n = 0, 1, \dots < \nu_s, \quad (63)$$

as detailed in Appendix B. Writing the static problem as $\mathcal{H}_s(A) \psi_s = \Omega_n^2 \psi_s$, the static Gribov horizon is the point

where the spatial Faddeev-Popov operator develops a zero eigenvalue. Thus the critical magnetic background is obtained by setting $\Omega_n = 0$. This gives the horizon condition:

$$q^2 + p_z^2 = \kappa^2(\nu_s - n)^2, \quad A_{\text{hor}}^{(n)} = -\frac{s\kappa^2}{gq}\nu_s(\nu_s + 1). \quad (64)$$

For the ground state ($n = 0$) with $p_z = 0$ and transverse momentum $q = \kappa\nu_s > 0$, the critical boundary amplitude is:

$$A_{\text{hor}} = -\frac{s\kappa}{g}(\nu_s + 1), \quad \omega^a = e^{i\kappa\nu_s y} \text{sech}^{\nu_s}(\kappa x) v_s^a. \quad (65)$$

This represents an explicit Faddeev-Popov zero mode. It does not describe a physical ghost particle; rather, it is a stationary, redundant copy direction along the Lorenz-gauge slice. In an infinite transverse volume, the transverse plane-wave factor represents a generalized zero mode; in a finite transverse box, it becomes a strictly normalizable zero mode.

B. Time-dependent Cartan pulses

To demonstrate the robustness of the Wronskian protection, we consider transient color-field flashes modeled as homogeneous spatial pulses:

$$A_i^3(t) = a_i \text{sech}^2(\kappa t), \quad A_0^3 = 0. \quad (66)$$

For a spatial Fourier mode, the copy equation simplifies to

$$\ddot{\phi}_s + [\mathbf{p}^2 + sg(\mathbf{a} \cdot \mathbf{p}) \text{sech}^2(\kappa t)]\phi_s = 0. \quad (67)$$

This is an exactly solvable, self-adjoint Pöschl-Teller wave equation in time. Because the temporal evolution is self-adjoint, the flux is perfectly conserved, and the Bogoliubov coefficients satisfy $|\alpha_s|^2 - |\beta_s|^2 = 1$ for all nonvanishing momenta. Consequently, the scattering coefficient α_s can never vanish. The apparent singularity in the formal infrared limit $|\mathbf{p}| \rightarrow 0$ is a threshold artifact; the coupling strength itself vanishes as $|\mathbf{p}| \rightarrow 0$, leaving a finite, nonzero transmission limit.

A similar dynamic protection is found for a color-field step (Sauter/Rosen-Morse step) [19,20],

$$A_y^3(t) = A \tanh(\kappa t), \quad (68)$$

which shifts the asymptotic frequencies of the ghost modes:

$$\begin{aligned} \omega_-^2 &= p_x^2 + p_z^2 + q^2 - sgqA, \\ \omega_+^2 &= p_x^2 + p_z^2 + q^2 + sgqA. \end{aligned} \quad (69)$$

As long as the asymptotic frequencies remain real and positive ($\omega_{\pm} > 0$), the flux-normalized coefficients satisfy

$|\alpha_s|^2 - |\beta_s|^2 = 1$. These solvable time-dependent examples prove that real-time frequency mixing does not equate to a Gribov copy; the gauge slice shakes but does not tear.

C. Chromoelectric well

While the magnetic configurations reproduce the zero-frequency horizon crossing of Euclidean theory, chromoelectric fields display a uniquely Lorentzian behavior: they can drive the gauge slice to the Gribov horizon at *finite*, *nonvanishing* frequency.

Consider a static, localized Cartan scalar potential,

$$A_0^3(z) = -A \text{sech}^2(\kappa z), \quad A_i^3 = 0, \quad (70)$$

which satisfies Lorenz gauge trivially ($\partial^\mu A_\mu^3 = \dot{A}_0^3 = 0$) and carries a localized chromoelectric field:

$$\begin{aligned} F_{0z}^3 &= -\partial_z A_0^3 = -2A\kappa \text{sech}^2(\kappa z) \tanh(\kappa z) \neq 0, \\ F_{ij}^3 &= 0. \end{aligned} \quad (71)$$

Separating the copy equation using the ansatz $\omega^a = e^{-i\Omega t + i\mathbf{p}_\perp \cdot \mathbf{x}_\perp} \chi_s(z) v_s^a$ in the charged sector ($ie v_s = s v_s$) yields a spatial Pöschl-Teller wave equation. Because A_0^3 multiplies the time derivative (∂_t) in the Faddeev-Popov operator, the scalar potential enters the reduced problem multiplied by the frequency Ω :

$$\chi_s'' + [\Omega^2 - \mathbf{p}_\perp^2 + sg\Omega A \text{sech}^2(\kappa z)]\chi_s = 0. \quad (72)$$

This spatial potential has a dynamically tuned depth $sg\Omega A$ proportional to the ghost frequency itself. At zero frequency ($\Omega = 0$), the well vanishes entirely, showing that static chromoelectric fields are completely transparent to static copies. For $\Omega \neq 0$, the normalizable bound states require an attractive channel ($sg\Omega A > 0$) and satisfy

$$\begin{aligned} \kappa^2 \nu_s(\nu_s + 1) &= sg\Omega A, \quad \mathbf{p}_\perp^2 - \Omega^2 = \kappa^2(\nu_s - n)^2, \\ n &= 0, 1, \dots < \nu_s, \end{aligned} \quad (73)$$

where $\nu_s(\nu_s + 1) = sg\Omega A/\kappa^2$.

For any nonzero transverse momentum, a normalizable solution exists in the attractive charged channel. For the ground state ($n = 0$), writing $\beta_s = \kappa\nu_s = \sqrt{\mathbf{p}_\perp^2 - \Omega^2}$ combines the two conditions into

$$\sqrt{\mathbf{p}_\perp^2 - \Omega^2}(\sqrt{\mathbf{p}_\perp^2 - \Omega^2} + \kappa) = sg\Omega A. \quad (74)$$

As Ω ranges from 0 to $|\mathbf{p}_\perp|$, the left-hand side falls from $|\mathbf{p}_\perp|(|\mathbf{p}_\perp| + \kappa) > 0$ to 0, while the right-hand side rises from 0 to $sgA|\mathbf{p}_\perp| > 0$. By the intermediate value theorem, a unique root $\Omega_\star \in (0, |\mathbf{p}_\perp|)$ must exist for every transverse momentum $|\mathbf{p}_\perp| > 0$, screening scale $\kappa > 0$, and attractive potential depth $sgA > 0$. The horizon crossing does not

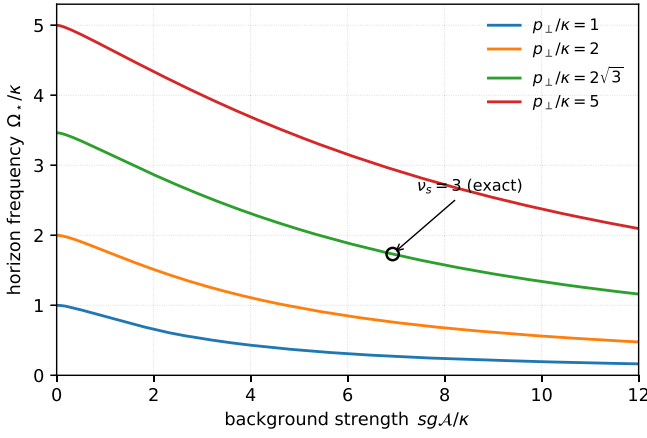


FIG. 1. Finite-frequency Lorentzian horizon of the static chromoelectric well. The horizon frequency Ω_* (ground state $n = 0$), obtained by numerically solving the horizon condition (74), in units of the inverse width κ , as a function of the background strength sgA/κ for several transverse momenta $|\mathbf{p}_\perp|/\kappa$. For every nonzero coupling, the horizon lies at a strictly positive Ω_* , falling from the threshold $\Omega_* \rightarrow |\mathbf{p}_\perp|$ at weak coupling to $\Omega_* \rightarrow 0$ at strong coupling. The open circle marks the exact reflectionless solution $\nu_s = 3$ of (76).

require a discrete tuning of the background amplitude A ; the dependence of Ω_* on the coupling and the transverse momentum is shown in Fig. 1.

For the reflectionless value $\nu_s = 3$, choosing the parameters

$$\kappa = \frac{|\mathbf{p}_\perp|}{2\sqrt{3}}, \quad sgA = 2|\mathbf{p}_\perp|, \quad (75)$$

we find the explicit, exact solution:

$$\Omega_* = \frac{1}{2}|\mathbf{p}_\perp| \neq 0, \quad \omega^a = e^{-\frac{1}{2}|\mathbf{p}_\perp|t + i\mathbf{p}_\perp \cdot \mathbf{x}_\perp} \text{sech}^3(\kappa z) v_s^a. \quad (76)$$

This is an explicit, exact normalizable Faddeev-Popov copy mode at a finite, nonzero frequency. The background lies precisely on the Lorentzian Gribov horizon ($\text{Ker}_F \mathcal{M}_A \neq 0$), where the Feynman ghost determinant vanishes ($\text{Det}_F \mathcal{M}_A = 0$) and the horizon functional diverges. Just as in the magnetic sheet, the plane wave factor describes a generalized mode in infinite volume, which becomes strictly normalizable in a finite transverse box.

This chromoelectric copy differs from the magnetic examples in two crucial ways:

- (1) It has no Euclidean counterpart. In Euclidean signature, A_0 and A_i enter the elliptic Faddeev-Popov operator on the same footing. The frequency-dependent well depth of (72) is therefore a purely Lorentzian phenomenon.
- (2) It does not violate Wronskian protection. The protection theorem of Sec. IV applies strictly to the temporal in-out scattering block. The chromo-

electric well is static and has no temporal scattering; the horizon crossing occurs as a spatial bound state, where the electric potential forces the copy mode to carry a nonzero resonant frequency $\Omega_* \neq 0$. The copy carries a spacelike invariant $\Omega_*^2 - \mathbf{p}_\perp^2 = -\kappa^2 \nu_s^2 < 0$, representing a localized, nonpropagating copy direction rather than an asymptotic on-shell ghost particle.

The absence of a Euclidean counterpart can be made precise by attempting the continuation directly. Wick rotating the frequency, $\Omega = i\omega_E$, turns $\Omega^2 \rightarrow -\omega_E^2$ in (72) but renders the well depth $sg\Omega A \rightarrow isg\omega_E A$ imaginary. The continued operator is no longer self-adjoint and supports no real bound state at $\omega_E \neq 0$, so the finite-frequency Lorentzian horizon is not the analytic continuation of any Euclidean horizon. The chromomagnetic crossing, by contrast, sits at $\Omega = 0 = \omega_E$ and continues trivially. The two static families therefore continue to the Euclidean problem in opposite ways—one faithfully and the other not at all.

VIII. GAUGE-FIXED INTERPRETATION

The Faddeev-Popov determinant acts as a geometric test, verifying whether the Lorenz-gauge slice remains strictly transverse to the gauge orbits. In the real-time path integral, this transversality is evaluated under the Feynman boundary conditions. If the wave operator develops a nontrivial kernel under these conditions,

$$\text{Ker}_F \mathcal{M}_A \neq 0, \quad (77)$$

then the gauge-fixing map develops a null direction with the same boundary prescription as the ghost propagator. This signals that the background has crossed the boundary of the Lorentzian Gribov region (23). We emphasize that this condition is gauge fixed and does not, on its own, serve as a gauge-invariant criterion for confinement.

For time-localized backgrounds, the no-pole condition takes on a scattering interpretation. An incoming negative-frequency ghost mode may scatter and mix with positive frequencies, but this frequency mixing is not by itself an obstruction. The horizon is reached only when the future negative-frequency component vanishes completely for a nonzero incoming mode. In a finite volume, this boundary is tracked by the smallest singular value of the negative-to-negative scattering block:

$$s_{\min}[A] = \inf_{\|v\|=1} \|\alpha_{--}[A]v\|. \quad (78)$$

The Gribov horizon is breached precisely when $s_{\min}[A] = 0$.

Proposition 1 guarantees that this loss of injectivity cannot occur at finite frequency within any stable, self-adjoint temporal channel with positive asymptotic

frequencies. Instead, the static chromoelectric well of Sec. VII shows where the finite-frequency horizon does occur: in a spatial channel, where the scalar potential A_0^3 couples to the time derivative and enters the bound-state equations multiplied by the frequency. The more general time-dependent case—a chromoelectric background with simultaneous temporal and spatial dependence—remains to be explored, as its reduced evolution is no longer restricted to a simple self-adjoint oscillator.

Finally, we address a tempting physical analogy. One might draw a parallel between this horizon crossing and Schwinger pair production in an external field [21]. This comparison must, however, be kept strictly qualitative. A “ghost pair” is not a physical state; it is a classical gauge copy mode satisfying the Feynman boundary conditions. Consequently, there is no physical particle-production rate associated with this transition. Because massless ghosts lack an intrinsic mass scale, the critical threshold for any specific configuration will depend heavily on its spatial profile, color orientation, volume regulation, and infrared regularization. One should not expect a universal critical field strength or a simple exponential rate formula.

A. First region versus the fundamental modular region

It is worth being explicit about which Euclidean object $\Omega_M^{(1)}$ is the analog of. In Euclidean space, the first Gribov region is not a fundamental domain; gauge orbits still intersect it more than once, and the standard remedy is the fundamental modular region, defined variationally as the set of absolute minima along each orbit of the norm $\|A^g\|^2 = \int d^4x A_\mu^g A_\mu^g$ [22,23]. That construction rests on the norm being positive definite. In Minkowski signature, the covariant norm $\int d^4x A_\mu A^\mu$ is indefinite and unbounded along gauge orbits, so the covariant minimization has no direct Lorentzian counterpart. The parallel with the rest of this paper is close: the same indefiniteness of the metric that deprives the Euclidean positivity condition $\mathcal{M}_E > 0$ of meaning also deprives the modular minimization principle of meaning. We stress that this concerns the *covariant* construction only. It does not exclude a modular region defined canonically on fixed-time slices, where the spatial norm $\int d^3x A_i A_i$ is positive definite, nor a Coulomb-gauge or contour-dependent alternative. Accordingly, what is constructed here is the Lorentzian analogue of the first Gribov region, not of a fundamental modular region, and the kernel condition $\text{Ker}_F \mathcal{M}_A = 0$ does not by itself guarantee the absence of finite or global copies.

B. Relation to other confinement criteria

Several other pictures of confinement—center vortices, Abelian monopoles obtained by Abelian projection, and the associated lattice criteria—are formulated in terms of specific classes of nonperturbative configurations rather than in terms of the gauge-fixing determinant [24,25]. It is

natural to ask whether such configurations sit on the Lorentzian horizon. For static configurations, the present framework answers this directly. If $A_0 = 0$ and the background is time independent, the Lorenz condition reduces to spatial transversality, $\partial_i A_i = 0$, and the Faddeev-Popov wave operator acting on a time-independent gauge parameter $\omega(t, \mathbf{x}) = \psi(\mathbf{x})$ collapses to the three-dimensional operator,

$$\mathcal{M}_A \psi(\mathbf{x}) = \partial_i D_i [A] \psi(\mathbf{x}). \quad (79)$$

Any static transverse background possessing a spatial Faddeev-Popov zero mode, $-\partial_i D_i \psi = 0$, therefore supplies a stationary $\Omega = 0$ Lorentzian copy mode, subject to the normalizability or threshold prescription discussed above. Configurations already known to lie on a Coulomb- or Landau-gauge horizon embed into the stationary Lorentzian horizon in this sense, and the chromomagnetic sheet of Sec. VII is an explicit realization: it is static and transverse and reaches the horizon precisely at $\Omega = 0$. The corresponding statement for generic vortex or monopole configurations, which are usually studied in other gauges and may carry time dependence or asymptotics outside the class assumed here, is a well-posed computation of $\text{Ker}_F \mathcal{M}_A$ that we have not carried out.

A separate caveat concerns the frequently used statement that configurations near the horizon dominate. In the Euclidean theory, this is a statement about a measure: the volume of the region concentrates near its boundary in a high dimension, and this is combined with the positive weight e^{-S} . In real time, the weight is the oscillatory phase $e^{iS_{\text{YM}}}$, there is no positive measure to concentrate, and it is cancellation rather than volume that decides which configurations control the integral. The near-horizon dominance argument therefore does not carry over as it stands, and we regard a real-time counterpart of it as an open question rather than as a consequence of the present construction.

C. Two dimensions

Yang-Mills theory in $1+1$ dimensions has no local propagating gauge polarizations, yet its gauge-orbit space, holonomies, electric-flux sectors, Faddeev-Popov determinant and Gribov regions remain nontrivial; explicit analyses exhibit Faddeev-Popov zero modes, a first Gribov region, and a fundamental modular region in that setting [26,27]. There is no tension with the present framework, because the Faddeev-Popov equation is an equation for the gauge parameter and not for physical polarizations; the copy problem concerns gauge redundancy and does not disappear merely because no transverse gluon modes exist. Confinement of external color sources in two dimensions, on the other hand, follows from Gauss’s law and the resulting linear Coulomb potential; it is kinematical and calls for no near-horizon mechanism. The coexistence of a

nontrivial Gribov problem with confinement in $1+1$ dimensions is thus perfectly consistent, but it does not by itself establish a causal link between them. We add one technical remark specific to our examples. The finite-frequency mechanism of Sec. VII relies on a transverse momentum, the localized copy mode existing because $\mathbf{p}_\perp^2 - \Omega^2 > 0$ supplies spatial decay at real Ω . In strict $1+1$ dimensions, there is no such momentum, so that particular real-frequency bound state is absent, and the minimal setting for it is $2+1$ dimensions. This limits the example, not the boundary-value framework.

IX. DISCUSSION

In this work, we have proposed a direct, real-time formulation of the Gribov horizon. The first Lorentzian Gribov region is defined as the connected component containing the vacuum,

$$\Omega_M^{(1)} = \{A: \partial^\mu A_\mu = 0, \text{Ker}_F(-\partial^\mu D_\mu[A]) = 0\}_0, \quad (80)$$

where Ker_F denotes the kernel of the Faddeev-Popov wave operator subject to the Feynman boundary prescription. For time-localized backgrounds, this boundary-value problem translates to the injectivity of the negative-to-negative Bogoliubov scattering block,

$$\text{Ker}\alpha_{--}[A] = 0, \quad (81)$$

while in the Fredholm representation, it corresponds to the emergence of a non-trivial kernel for the Lippmann-Schwinger operator:

$$\text{Ker}(1 + G_F^0 V_A) = 0. \quad (82)$$

Our findings establish that ordinary real-time frequency mixing must not be conflated with a Gribov copy. Within a stable, self-adjoint temporal channel, the Wronskian conservation identity (40) keeps the scattering block α_{--} strictly injective, even in the presence of strong frequency mixing ($\beta_{+-} \neq 0$). The Gribov horizon is reached through two distinct mechanisms in the solvable examples. Static chromomagnetic backgrounds reach the horizon at zero frequency ($\Omega = 0$), reproducing the familiar spectral crossing of the Euclidean theory. In contrast, static chromoelectric wells cross the horizon at a finite, nonvanishing frequency ($\Omega_\star \neq 0$), because the scalar potential A_0 couples directly to the time derivative in the wave operator. This finite-frequency copy mode has no direct Euclidean counterpart, as Wick rotation places A_0 and A_i on an identical spatial footing in the Euclidean operator.

We reiterate that this entire formulation is gauge fixed. Ghosts remain unphysical mathematical constructs; while Becchi-Rouet-Stora-Tyutin (BRST) cohomology defines physical states perturbatively, BRST symmetry in the presence of the Gribov horizon remains a delicate open

issue [28]. A Faddeev-Popov copy mode is an infinitesimal direction of gauge redundancy, and in Minkowski space, its selection is governed by the Feynman boundary prescription. Gribov's description of "ghost pair production" should be understood strictly within this geometric, gauge-fixing framework.

For backgrounds localized in time, the nonlocal barrier functional is given by the flow functional:

$$H_{\text{flow}}[A] = -\text{Tr}_{\mathcal{H}_-} \log(\alpha_{--}^\dagger \alpha_{--}). \quad (83)$$

By continuing the Euclidean Zwanziger horizon functional using the Feynman propagator, we can write a corresponding real-time expression:

$$H_Z^M[A] = -g^2 \int d^4x d^4y f^{adc} A_\mu^c(x) \times (\mathcal{M}_A^{-1})_F^{de}(x, y) f^{eab} A^{b\mu}(y). \quad (84)$$

Although both functionals are built from the same background resolvent $\mathcal{M}_A^{-1}$, they are not the same object, and the difference is instructive. In Appendix C, we show that near the horizon $H_{\text{flow}} = H_{\text{det}}$ diverges only logarithmically, while H_Z^M develops a simple pole, and that for the symmetric backgrounds of Sec. VII the naive Feynman continuation H_Z^M does not register the horizon at all, because the copy mode carries quantum numbers that the background cannot source. The exact restriction is thus a functional determinant, and a local Gribov-Zwanziger action must reproduce that $\text{Tr} \log$ structure rather than the tree-level bilinear H_Z^M . Ultimately, translating our real-time, boundary-value horizon condition into a local, renormalizable Lorentzian Gribov-Zwanziger action remains the most compelling frontier. Resolving this challenge is not merely a mathematical exercise; it is a natural step toward the real-time spectral properties of gluons and the Minkowski-space dynamics of confinement. A useful next step will be to explore a Lorenz-gauge background that carries both a scalar potential $A_0(t, \mathbf{x})$ and explicit time dependence, allowing us to compute α_{--} and track its smallest singular value $s_{\min}(t)$ dynamically and to test whether genuine time dependence sharpens the separation between H_{flow} and H_Z^M established in Appendix C.

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assistive aids: to edit and improve the language of the manuscript, to help explore and cross-check algebra and derivations, and to assist with code generation and script automation for mathematical visualization. All visual data representations are computed directly from the analytic equations derived in the text. No text, equation, or figure was incorporated without the author's review and verification. These tools do not meet the criteria for authorship and are not listed as authors; the author takes full responsibility for the entire manuscript.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: WRONSKIAN PROTECTION: DERIVATIONS

This Appendix collects the derivations underlying Sec. IV.

1. Matrix Wronskian identity

For the stable self-adjoint oscillator (39), the relevant conserved object is the sesquilinear form

$$W[\phi, \psi] = \phi^\dagger \dot{\psi} - \dot{\phi}^\dagger \psi, \quad (\text{A1})$$

defined on any two solutions. Using $\ddot{\phi} = -K\phi$, $\ddot{\psi} = -K\psi$ and $K = K^\dagger$,

$$\frac{d}{dt} W[\phi, \psi] = \phi^\dagger \ddot{\psi} - \ddot{\phi}^\dagger \psi = \phi^\dagger (K^\dagger - K)\psi = 0, \quad (\text{A2})$$

so W is conserved. Take a flux-normalized basis of negative-frequency in solutions ϕ_a with $\phi_a(t \rightarrow -\infty) \sim e_a e^{+i\omega_- t} / \sqrt{2\omega_-}$, written in the eigenbasis of the (positive) asymptotic frequency matrices, and out asymptotics

$$\phi_a(t \rightarrow +\infty) \sim \sum_b \left(\alpha_{ba} \frac{e^{+i\omega_+ t}}{\sqrt{2\omega_+}} + \beta_{ba} \frac{e^{-i\omega_+ t}}{\sqrt{2\omega_+}} \right) e_b. \quad (\text{A3})$$

Evaluating the conserved matrix $\mathcal{W}_{ab} = W[\phi_a, \phi_b]$ in the far past gives $\mathcal{W}^{-\infty} = i\mathbf{1}$, while in the far future, the oscillating cross terms drop by conservation, and the constant part gives $\mathcal{W}^{+\infty} = i(\alpha^\dagger \alpha - \beta^\dagger \beta)$. Equating the two yields the identity (40) quoted in the main text.

2. Reduction of the Cartan channel

Take an $SU(2)$ Cartan background with only the color-3 components $A_\mu^3(x)$ nonzero. In the charged sector, with $ie v_s = s v_s$, one has $(V_A \omega) = g e^{ab} A_\mu^3 \partial^\mu \phi_s v_s = -isg A_\mu^3 \partial^\mu \phi_s v_s$, so the copy equation $\mathcal{M}_A \omega = 0$ for $\omega =$

$\phi_s v_s$ reduces to (41). For a spatially homogeneous background $A_\mu^3 = A_\mu^3(t)$ and a spatial Fourier mode $\omega = e^{i\mathbf{p} \cdot \mathbf{x}} \phi_s(t) v_s$, this becomes exactly (42), with no further terms; the coefficient of $\dot{\phi}_s$ is fixed by the Faddeev-Popov operator $-\partial^\mu D_\mu$. For inhomogeneous backgrounds, the spatial gradients in (41) add further terms.

To reach (43), write (42) as

$$\ddot{\phi}_s + P_s(t) \dot{\phi}_s + Q_s(t) \phi_s = 0, \quad P_s(t) = isg A_0^3(t), \quad (\text{A4})$$

and set

$$\phi_s(t) = \exp \left[-\frac{1}{2} \int^t du P_s(u) \right] \chi_s(t), \quad (\text{A5})$$

which removes the first-derivative term and leaves the effective oscillator (43), with the standard combination $Q_s - \frac{1}{4} P_s^2 - \frac{1}{2} \dot{P}_s$.

APPENDIX B: DETAILS OF THE SOLVABLE REDUCTIONS

For the static Pöschl-Teller sheet of Sec. VII, the reduced equation (62) is brought to standard form by defining

$$\lambda_s = -\frac{sgqA}{\kappa^2}, \quad \rho^2 = \frac{q^2 + p_z^2 - \Omega^2}{\kappa^2}. \quad (\text{B1})$$

For an attractive well, $\lambda_s = \nu_s(\nu_s + 1) > 0$, the normalizable Pöschl-Teller levels are [18]

$$\rho_n = \nu_s - n, \quad n = 0, 1, \dots, \nu_s, \quad (\text{B2})$$

which is the dispersion relation (63) quoted in the main text. Setting $\Omega_n = 0$ there gives the horizon condition (64), and the ground state with $p_z = 0$ and $q = \kappa \nu_s$ gives the explicit zero mode (65).

The time-dependent pulses of Sec. VII are reduced by the same charged-sector projection, followed by a spatial Fourier transform, which turns the copy equation into a one-dimensional scattering problem in time; the Pöschl-Teller and Sauter/Rosen-Morse profiles quoted there are then the standard exactly solvable cases [18–20].

APPENDIX C: FLOW FUNCTIONAL VERSUS THE FEYNMAN-CONTINUED ZWANZIGER FUNCTIONAL

The main text introduced two functionals that both diverge on the Lorentzian horizon: the exact flow functional $H_{\text{flow}} = H_{\text{det}}$, and the Feynman-inverse continuation H_Z^M of Zwanziger's horizon function. It is tempting to read them as two faces of one object, much as Gribov's no-pole and Zwanziger's horizon conditions coincide in the Euclidean theory. They are not. Both are built from the

single background resolvent $\mathcal{M}_A^{-1}$, but they assemble it in genuinely different ways: $H_{\text{flow}} = H_{\text{det}} = -\log |\text{Det}_F \mathcal{M}_A / \text{Det} \mathcal{M}_0|^2$ is a closed ghost loop, the logarithm of a functional determinant, whereas $H_Z^M = -g^2 \int f A (\mathcal{M}_A^{-1})_F f A$ is a single open ghost line dressed by the background—a tree-level resolvent bilinear. This appendix makes the distinction quantitative and draws a concrete lesson from it.

1. Channel reduction of H_Z^M

It is convenient to first strip away the color algebra. For an $SU(2)$ Cartan background, the structure constants reduce to $f^{ad3} = \epsilon^{ad}$, and the charged sector is diagonalized by the eigenvectors $i\epsilon v_s = s v_s$, so that $\epsilon v_s = -i s v_s$. The color contraction in H_Z^M then collapses to a single sum over the two charged channels, since $\epsilon^{ad} (\mathcal{M}_A^{-1})^{de} \epsilon^{ea} = (-is)^2 \sum_s G_A^{(s)} = -\sum_s G_A^{(s)}$. This leaves

$$H_Z^M[A] = g^2 \sum_{s=\pm} \int d^4 x d^4 y A_\mu^3(x) G_A^{(s)}(x, y) A^{3\mu}(y),$$

$$G_A^{(s)} = (-\square - is g A_\mu^3 \partial^\mu)_F^{-1}, \quad (\text{C1})$$

where $G_A^{(s)}$ is the Feynman resolvent of the charged-channel wave operator of Eq. (41)—the very operator whose kernel defines the horizon.

2. Same boundary, different critical exponents

Approach the horizon along a one-parameter family of backgrounds, and let ψ_0 be the reduced copy mode that is about to appear, with $\mathcal{M}_A^{(s)} \psi_0 = \lambda_0 \psi_0$ and a simple crossing $\lambda_0(A) \simeq \lambda'_0(A_{\text{hor}} - A)$, where $\lambda'_0 = \langle \psi_0 | \partial_A \mathcal{M}_A^{(s)} | \psi_0 \rangle \neq 0$. The two functionals feel this crossing very differently. The determinant carries the vanishing eigenvalue as a simple factor, $\text{Det}_F \mathcal{M}_A \propto \lambda_0$, while the resolvent carries it as a simple pole, $\mathcal{M}_A^{(s)-1} = |\psi_0\rangle\langle\psi_0|/\lambda_0 + (\text{regular})$. As a result,

$$H_{\text{det}} \sim -2 \log |A_{\text{hor}} - A|, \quad H_Z^M \sim \frac{g^2 |\langle A^3 | \psi_0 \rangle|^2}{\lambda'_0 (A_{\text{hor}} - A)}. \quad (\text{C2})$$

Both diverge on the horizon $\partial\Omega_M$, but the flow functional does so only logarithmically, while the Zwanziger functional

develops a simple pole. The two also part company well away from the boundary: already at second order in the background, $H_{\text{det}}^{(2)} = \text{ReTr}(G_F^0 V_A G_F^0 V_A)$ is a momentum loop, whereas $H_Z^{M(2)} = g^2 \sum_s \langle A^3 | G_F^0 | A^3 \rangle$ is a tree. The only structural bridge between them is the identity $\partial_A H_{\text{det}} = -2 \text{ReTr}(\mathcal{M}_A^{-1} \partial_A V_A)$: the gradient of the log-determinant is a trace of the same resolvent of which H_Z^M is a single matrix element.

3. Overlap selection rule

That last observation has a sharp consequence. Because H_{det} is governed by a *trace* of $\mathcal{M}_A^{-1}$, its divergence is automatic: the trace always contains the critical term λ'_0/λ_0 , which blows up whenever an eigenvalue genuinely crosses zero. The pole of H_Z^M , in contrast, is weighted by how strongly the background itself overlaps the emerging copy mode, through the factor $|\langle A^3 | \psi_0 \rangle|^2$; when that overlap vanishes, H_Z^M stays finite even as the determinant diverges. This is exactly what happens for the symmetric backgrounds of Sec. VII, where momentum conservation forces the overlap to zero. For the Pöschl-Teller sheet the copy mode, $\psi_0 \propto e^{i\kappa\nu_s y} \text{sech}^{\nu_s}(\kappa x)$ carries transverse momentum $p_y = \kappa\nu_s \neq 0$, while the background $A_y^3(x)$ is independent of y , so the overlap vanishes; equivalently, the coupling is proportional to p_y , the background decouples at the injected momentum $p_y = 0$, and (C1) reduces to its free, A -independent value. The chromoelectric well behaves the same way: its copy mode carries $(\Omega_\star, \mathbf{p}_\perp) \neq 0$, while $A_0^3(z)$ is independent of t and $\mathbf{x}_\perp$. In both cases, the flow functional diverges on the horizon, as it must, while the naive Zwanziger functional passes through unaware.

4. Lesson for a local action

The naive Feynman continuation of the Zwanziger horizon function is therefore not a faithful representation of the Lorentzian restriction. Where it does diverge, it carries the wrong critical exponent, and for the most symmetric—and most calculable—backgrounds, it can miss the horizon altogether. A local real-time Gribov-Zwanziger action must reproduce the determinant, the Tr log structure of H_{flow} , and not merely the tree-level bilinear H_Z^M . This is the precise form of the open problem raised in Sec. IX.

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