

Second-order spin hydrodynamics from Zubarev's nonequilibrium statistical operator formalism

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Using the Zubarev's nonequilibrium statistical operator formalism, we derive the second-order expression for the dissipative tensors in relativistic spin hydrodynamics, *viz.* rotational stress tensor ($\tau_{\mu\nu}$), boost heat vector (q_μ), shear stress tensor ($\pi_{\mu\nu}$), and bulk viscous pressure (Π). The first two ($\tau_{\mu\nu}$ and q_μ) emerge due to the inclusion of the antisymmetric part in the energy-momentum tensor, which, in turn, governs the conservation of spin angular momentum ($\Sigma^{\alpha\mu\nu}$). As a result, new thermodynamic forces, generated due to the antisymmetric part of $T_{\mu\nu}$, contain the spin chemical potential. In this work, we have also taken the spin density ($S^{\mu\nu}$) as an independent thermodynamic variable, in addition to the energy density and particle density, thereby resulting in two novel transport coefficients given by the correlation between spin density tensor and rotational stress tensor and vice versa. Additionally, the newly found terms in $\pi_{\mu\nu}$ and Π are the artifacts of the new thermodynamic forces that arise due to the antisymmetric part of $T^{\mu\nu}$. Finally, we have derived the evolution equations for the aforesaid tensors: $\tau_{\mu\nu}$, q_μ , $\pi_{\mu\nu}$, and Π .

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I. INTRODUCTION

Ever since the experiments [1–3] measuring the spin polarization of Λ hyperons, there has been a growing interest in studying spin-dependent observables (see [4,5] for a recent review on spin polarization). Also, the study of spin transport phenomena in condensed matter systems, known as spintronics [6], has become a prominent research area. In addition to this, despite the success of relativistic hydrodynamics in various areas, it has been unable to provide a comprehensive explanation for the experimental data on the differential spin polarization of Λ hyperons. This problem is known as the “spin sign problem” [2,3] (see [7,8] for review). An effective way to address these findings is by incorporating the spin degree of freedom. Significant efforts have been made to explain the spin polarization in heavy ion collisions [9–25] and to develop relativistic hydrodynamics that includes spin, known as spin hydrodynamics (see [26] for a recent review on spin hydrodynamics).

The intriguing feature of spin hydrodynamics is captured in recent studies [27,28], where two new transport

coefficients arise, reflecting the physical effect of spin. These new transport coefficients namely boost heat conductivity and rotational viscosity, originate at first-order, and tell the slow dynamics of spinful fluid systems [27]. These transport coefficients emerge due to the inclusion of new dissipative tensors via the antisymmetric part of the energy-momentum tensor.

Transport coefficients are crucial, as they govern the evolution of dissipative tensors present in the medium. Each dissipative tensor evolves over time, capturing particular physical aspects of the medium. At the second order, new transport coefficients arise, and the effect of spin can be seen in the traditional dissipative tensors, such as bulk viscous pressure and shear stress tensor. Moreover, the first-order theory is acausal, and in order to restore causality, a second-order theory is needed. Also, a numerical implementation of causal spin hydrodynamic theory may give information about the spin sign problem in heavy-ion collisions. These results strongly motivate the development of a theoretical framework that describes spin transport in relativistic plasma.

Various approaches have been proposed to incorporate spin into hydrodynamics, including the quantum kinetic theory approach [29–37], approaches based on the second law of thermodynamics [27,28,38–40], the effective Lagrangian approach [41–44], the equilibrium partition function approach [45–47], entropy current analysis [48,49], and the quantum statistical density operator [50–54].

This paper employs Zubarev's nonequilibrium statistical operator (NESO) [55,56] to develop a framework for second-order spin hydrodynamics. Zubarev's NESO

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method describes nonequilibrium processes and incorporates irreversibility in the description. The introduction of irreversibility is achieved by incorporating an infinitesimal source term into the Liouville equation. The NESO is an extension of the Gibbs canonical ensemble, i.e. the Gibbs relation remains locally valid. By utilizing this formalism, we can distinctly separate the equilibrium and nonequilibrium components. We have derived the second-order expression for rotational stress tensor ($\tau_{\mu\nu}$), boost heat vector (q_μ), shear stress tensor ($\pi_{\mu\nu}$), and bulk viscous pressure (Π) by incorporating the antisymmetric part in the energy-momentum tensor, in addition to the symmetric part. Moreover, using this method, the transport coefficients of the system can be naturally obtained in the form of a Kubo-type formula.

This paper is structured as follows: Section II deals with the equation of motions for the energy-momentum tensor, charge, and angular momentum. We also discuss the entropy production using the extended thermodynamic relation involving the spin chemical potential and spin density, in addition to the conventional thermodynamic variables. In Sec. III, we introduce the nonequilibrium statistical operator and calculate the nonequilibrium part of the nonequilibrium statistical operator in terms of hydrodynamic gradient expansion. In Sec. IV, we revisit the first-order spin hydrodynamics and derived the transport coefficients that emerge in first-order spin hydrodynamics. In Sec. V, we derive the second-order expressions and the evolution equations for the rotational stress tensor ($\tau_{\mu\nu}$), boost heat vector (q_μ), shear stress tensor ($\pi_{\mu\nu}$), and bulk viscous pressure (Π). Finally, we conclude with a summary and discussion in Sec. VI.

II. REVIEW OF RELATIVISTIC SPIN HYDRODYNAMICS

Throughout this paper, we employ the metric tensor $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The projection operator is defined as $\Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$, with the normalization condition for the fluid velocity given by $u_\mu u^\mu = 1$. Given our focus on investigating the spin effect, we adopt the Landau frame where $h_\mu = 0$. In addition, the projectors and other shorthand notations used in this paper are

$$\Delta_{\mu\nu}^{\lambda\delta} = \frac{1}{2}(\Delta_\mu^\lambda \Delta_\nu^\delta + \Delta_\nu^\lambda \Delta_\mu^\delta) - \frac{1}{3} \Delta^{\lambda\delta} \Delta_{\mu\nu},$$

$$\Delta_{\mu\nu}^{\lambda\delta} = \frac{1}{2}(\Delta_\mu^\lambda \Delta_\nu^\delta - \Delta_\nu^\lambda \Delta_\mu^\delta), \quad (1)$$

$$A^{(\mu\nu)} = \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu}), \quad A^{[\mu\nu]} = \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu}), \quad (2)$$

$$A^{(\mu\nu)} = \Delta_{\lambda\delta}^{\mu\nu} A^{\lambda\delta}, \quad A^{\langle\mu\nu\rangle} = \Delta_{\lambda\delta}^{\mu\nu} A^{\lambda\delta}. \quad (3)$$

A. Equation of motions

To start with the derivation of hydrodynamic equations of motion, we assume that the energy-momentum tensor $T^{\mu\nu}$, charge current N^μ , and total angular momentum $J^{\mu\alpha\beta}$ are conserved for the system of interest [27,39]. The conservation laws are given by

$$\partial_\mu T^{\mu\nu} = 0, \quad (4)$$

$$\partial_\mu N^\mu = 0, \quad (5)$$

$$\partial_\mu J^{\mu\alpha\beta} = 0. \quad (6)$$

The orbital and the spin part contribute to the total angular momentum. Microscopically, it is known that

$$J^{\mu\alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + \Sigma^{\mu\alpha\beta}, \quad (7)$$

where $\Sigma^{\mu\alpha\beta}$ is the intrinsic spin degree of freedom. $J^{\mu\alpha\beta}$ and $\Sigma^{\mu\alpha\beta}$ are antisymmetric in the last two indices. The conservation law for $J^{\mu\alpha\beta}$ restrict $\Sigma^{\mu\alpha\beta}$ to follow

$$\partial_\mu \Sigma^{\mu\alpha\beta} = -(T^{\alpha\beta} - T^{\beta\alpha}). \quad (8)$$

The next step involves writing a systematic gradient expansion for $T^{\mu\nu}$, N^μ , and $\Sigma^{\mu\alpha\beta}$. A general tensor decomposition is given by

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - (p + \Pi) \Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} + q^\mu u^\nu - q^\nu u^\mu + \tau^{\mu\nu}, \quad (9)$$

$$N^\mu = n u^\mu + j^\mu, \quad (10)$$

$$\Sigma^{\mu\alpha\beta} = u^\mu S^{\alpha\beta}, \quad (11)$$

where ε is the energy density, n is the particle density, $S^{\alpha\beta}$ is the spin density, and p is the equilibrium pressure. We have adopted the phenomenological form of the spin tensor, given by $\Sigma^{\mu\alpha\beta} = u^\mu S^{\alpha\beta}$. This choice is motivated by its intuitive physical interpretation: it represents the flow of spin density, where the spin density $S^{\alpha\beta}$ is defined in the local rest frame of the fluid. Moreover, this form is consistent with the structure of the global equilibrium solution of thermal vorticity [8,57]. Furthermore, it is important to note that u^μ , ε , p , n , $\Delta_{\mu\nu}$ are zeroth order term $\mathcal{O}(\partial^0)$ in gradient expansion, whereas, the dissipative terms Π , h^μ , q^μ , j^μ , $\pi^{\mu\nu}$, $\tau^{\mu\nu}$ in general, contain the gradient correction up to all orders, and they satisfy the conditions $u_\mu h^\mu = 0$, $u_\mu q^\mu = 0$, $u_\mu j^\mu = 0$, $u_\mu \pi^{\mu\nu} = 0$, $u_\mu \tau^{\mu\nu} = 0$, $\pi^{\mu\nu} = \pi^{\nu\mu}$, $\pi_\mu^\mu = 0$, and $\tau^{\mu\nu} = -\tau^{\nu\mu}$.

To calculate the equations of motion, first, we substitute the gradient expansion (9) in conservation equation (4) and take the projection of (4) along u_ν and Δ_ν^α . We obtain

$$D\varepsilon + (p + \Pi + \varepsilon)\theta + \partial_\mu h^\mu + u_\mu D h^\mu + u_\nu \partial_\mu \pi^{\mu\nu} + \partial_\mu q^\mu - u_\mu D q^\mu + u_\nu \partial_\mu \tau^{\mu\nu} = 0, \quad (12)$$

$$h D u^\alpha + \Pi D u^\alpha - \nabla^\alpha (p + \Pi) + h^\alpha \theta + h^\mu \partial_\mu u^\alpha + \Delta_\nu^\alpha D h^\nu + \Delta_\nu^\alpha \partial_\mu \pi^{\mu\nu} - q^\alpha \theta + q^\mu \partial_\mu u^\alpha - \Delta_\nu^\alpha D q^\nu + \Delta_\nu^\alpha \partial_\mu \tau^{\mu\nu} = 0. \quad (13)$$

Here, we defined $\theta \equiv \partial_\mu u^\mu$ and $D \equiv u^\mu \partial_\mu$. Further, we use (10) and (11) in (5) and (8), respectively. Thus we obtain

$$Dn + n\theta + \partial_\mu j^\mu = 0, \quad (14)$$

$$D S^{\alpha\beta} + S^{\alpha\beta} \theta + 2(q^\alpha u^\beta - q^\beta u^\alpha + \tau^{\alpha\beta}) = 0. \quad (15)$$

B. Entropy production

In order to incorporate the spin into the description, we treat $\Sigma^{\mu\alpha\beta}$ to be an independent hydrodynamic field, and the thermodynamic relations near local thermodynamics equilibrium get modified as [27]

$$T ds = d\varepsilon - \mu dn - \frac{1}{2} \omega_{\mu\nu} dS^{\mu\nu} \quad (\text{first law}), \quad (16)$$

$$dp = s dT + n d\mu + \frac{1}{2} S^{\mu\nu} d\omega_{\mu\nu} \quad (\text{Gibbs-Duhem}), \quad (17)$$

$$\varepsilon + p = Ts + \mu n + \frac{1}{2} \omega_{\mu\nu} S^{\mu\nu} \quad (\text{Euler's relation}). \quad (18)$$

Here, s is the entropy density, T is the temperature, and μ is the chemical potential conjugate to the particle density n . Similarly, $\omega_{\mu\nu}$ is the spin chemical potential conjugate to spin density $S^{\mu\nu}$.

The above thermodynamic relations are essential to write the entropy current. However, it has been shown that the form of entropy current can be derived from the first principle, using the quantum statistical density operator [58,59]. The form of entropy current, treating spin tensor $\Sigma^{\mu\alpha\beta}$ as an independent hydrodynamic field, is given by

$$S^\mu = p\beta^\mu + \beta_\nu T^{\mu\nu} - \alpha N^\mu - \frac{1}{2} \Omega_{\alpha\beta} \Sigma^{\mu\alpha\beta}. \quad (19)$$

The parameters β^μ , α , and $\Omega_{\alpha\beta}$ are defined by

$$\beta^\mu = \beta u^\mu, \quad \alpha = \beta\mu, \quad \Omega_{\alpha\beta} = \beta\omega_{\alpha\beta}, \quad (20)$$

where, β is the inverse temperature, and u^μ is the fluid four velocity.

According to the second law of thermodynamics, the divergence of entropy current, i.e., the entropy production rate, should be positive for a dissipative system; therefore,

$$\partial_\mu S^\mu \geq 0. \quad (21)$$

Using the gradient expansions for $T^{\mu\nu}$, N^μ , and $\Sigma^{\mu\alpha\beta}$ from (9), (10), and (11), respectively, the divergence of entropy current is given by

$$\begin{aligned} \partial_\mu S^\mu = & -\Pi\beta\theta + \beta h^\mu (\beta^{-1} \partial_\mu \beta + D u_\mu) - j^\mu \partial_\mu \alpha + \beta q^\mu M_\mu \\ & + \beta \pi^{\mu\nu} \sigma_{\mu\nu} + \beta \tau^{\mu\nu} \xi_{\mu\nu}, \end{aligned} \quad (22)$$

Here, $M_\mu \equiv (\beta^{-1} \partial_\mu \beta - D u_\mu + 2\beta^{-1} \Omega_{\mu\nu} u^\nu)$, $\sigma_{\mu\nu} \equiv \Delta_{\mu\nu}^{\lambda\delta} \partial_\lambda u_\delta$, and $\xi_{\mu\nu} \equiv \Delta_{\mu\nu}^{\lambda\delta} (\partial_\lambda u_\delta + \beta^{-1} \Omega_{\lambda\delta})$. The $\Delta_{\mu\nu}^{\lambda\delta} \partial_\lambda u_\delta$ term in $\xi_{\mu\nu}$ is known as the vorticity tensor ($\bar{\omega}_{\mu\nu}$).

We would like to emphasize that the hydrodynamic gradient orderings assigned to $\Omega_{\alpha\beta}$ and the spin density $S^{\alpha\beta}$ vary across the literature. For instance, in [39], both $\Omega_{\alpha\beta}$ and $S^{\alpha\beta}$ are treated as leading-order quantities, i.e., $\mathcal{O}(\partial^0)$. In contrast, other works such as [40,60] adopt the convention that both are of order $\mathcal{O}(\partial)$. In the present work, we assign $\Omega_{\alpha\beta} \sim \mathcal{O}(\partial^1)$ and $S^{\alpha\beta} \sim \mathcal{O}(\partial^0)$ within the framework of the hydrodynamic gradient expansion. The choice of $S^{\alpha\beta}$ as a zeroth-order quantity is motivated by analogy with the energy density ε and particle density n , both of which are considered to be of $\mathcal{O}(\partial^0)$. On the other hand, it is well established that in global equilibrium, the spin chemical potential $\omega_{\alpha\beta}$ can be expressed in terms of the thermal vorticity, which is of $\mathcal{O}(\partial)$. Consequently, the natural ordering for both $\omega_{\alpha\beta}$ or $\Omega_{\alpha\beta}$ is $\mathcal{O}(\partial)$. This physically motivated ordering scheme is widely employed in the literature [27,28,49,54]. A natural question that arises is how to relate $S^{\alpha\beta}$ and $\Omega^{\alpha\beta}$ when they are assigned different orders. An illustrative example is provided in [61]. However, if one aims to describe a system with strong vorticity, the ordering scheme $\Omega_{\alpha\beta} \sim \mathcal{O}(\partial^1)$ and $S^{\alpha\beta} \sim \mathcal{O}(\partial^0)$ becomes inadequate. In such cases, it is necessary to treat the vorticity as a leading-order term [26].

Having clarified that point, we now return to the calculations. Employing the ordering scheme $\Omega_{\alpha\beta} \sim \mathcal{O}(\partial^1)$ and $S^{\alpha\beta} \sim \mathcal{O}(\partial^0)$ and enforcing the condition (21) on (22), we obtain the relation between the dissipative tensors and hydrodynamic forces up to first order:

$$\begin{aligned} \Pi &= -\zeta\theta, \\ \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu}, \\ q^\mu &= -\lambda M^\mu, \\ \tau^{\mu\nu} &= 2\gamma\xi^{\mu\nu}. \end{aligned} \quad (23)$$

The transport coefficients ζ and η are known as bulk viscosity and shear viscosity, respectively. The other two transport coefficients, λ and γ , are new in spin hydrodynamics. They are known as boost heat conductivity and rotational viscosity, respectively [27].

III. NONEQUILIBRIUM STATISTICAL OPERATOR

This method starts by defining a nonequilibrium statistical operator $\hat{\rho}(t)$ for a system that is in a hydrodynamic regime [55,56]. For a system containing spin degrees of freedom, the operator $\hat{\rho}(t)$ is defined as [54,62]

$$\hat{\rho}(t) = Q^{-1} \exp \left[- \int d^3x \hat{Z}(\vec{x}, t) \right], \quad \text{with} \quad (24)$$

$$Q = \text{Tr} \exp \left[- \int d^3x \hat{Z}(\vec{x}, t) \right],$$

where

$$\hat{Z}(\vec{x}, t) = \epsilon \int_{-\infty}^t dt_1 e^{\epsilon(t_1-t)} [\beta^\nu(\vec{x}, t_1) \hat{T}_{0\nu}(\vec{x}, t_1) - \alpha(\vec{x}, t_1) \hat{N}^0(\vec{x}, t_1) - \frac{1}{2} \Omega_{\alpha\beta}(\vec{x}, t_1) \hat{\Sigma}^{0\alpha\beta}(\vec{x}, t_1)]. \quad (25)$$

Here, ϵ is the small infinitesimal parameter and $\hat{T}^{\mu\nu}$, $\hat{N}^\mu$, and $\hat{\Sigma}^{\mu\alpha\beta}$ are operator-valued functions. Their statistical averages give the corresponding hydrodynamic fields (9)–(11). Zubarev's prescription for taking the statistical average is given by

$$\langle \hat{O}(\vec{x}, t) \rangle = \lim_{\epsilon \rightarrow 0^+} \lim_{V \rightarrow \infty} \text{Tr}[\hat{\rho}(t) \hat{O}(\vec{x}, t)]. \quad (26)$$

The main advantage of the NESO is that it can be decomposed into equilibrium and nonequilibrium parts. This can be achieved by integrating $\hat{Z}(\vec{x}, t)$ by parts, which gives

$$\hat{\rho}(t) = Q^{-1} \exp(-\hat{A} + \hat{B}), \quad (27)$$

where

$$\hat{A}(t) = \int d^3x \left[\beta^\nu(\vec{x}, t) \hat{T}_{0\nu}(\vec{x}, t) - \alpha(\vec{x}, t) \hat{N}^0(\vec{x}, t) - \frac{1}{2} \Omega_{\alpha\beta}(\vec{x}, t) \hat{\Sigma}^{0\alpha\beta}(\vec{x}, t) \right], \quad (28)$$

$$\hat{B}(t) = \int d^3x \int_{-\infty}^t dt_1 e^{\epsilon(t_1-t)} \hat{C}(\vec{x}, t_1), \quad (29)$$

$$\hat{C}(\vec{x}, t) = \hat{T}_{\mu\nu}(\vec{x}, t) \partial^\mu \beta^\nu(\vec{x}, t) - \hat{N}^\mu(\vec{x}, t) \partial_\mu \alpha(\vec{x}, t) - \frac{1}{2} \hat{\Sigma}^{\mu\alpha\beta}(\vec{x}, t) \partial_\mu \Omega_{\alpha\beta}(\vec{x}, t) - \frac{1}{2} \Omega_{\alpha\beta}(\vec{x}, t) (\hat{T}^{\beta\alpha}(\vec{x}, t) - \hat{T}^{\alpha\beta}(\vec{x}, t)). \quad (30)$$

Here we used the conservation laws to write the nonequilibrium part in a covariant form. If we consider the nonequilibrium part $\hat{B}$ as a perturbation, the density matrix $\hat{\rho}(t)$ can be expanded around the local equilibrium. This expansion allows us to systematically account for deviations from local equilibrium through higher-order terms. Thus, the statistical average of any operator, when expanded up to the second order around local equilibrium, can be expressed as follows [62]:

$$\langle \hat{O}(x) \rangle = \langle \hat{O}(x) \rangle_l + \int d^4x_1 (\hat{O}(x), \hat{C}(x_1)) + \int d^4x_1 d^4x_2 (\hat{O}(x), \hat{C}(x_1), \hat{C}(x_2)), \quad (31)$$

where $\int d^4x_1 = \int d^3x_1 \int_{-\infty}^t dt_1 e^{\epsilon(t_1-t)}$ and the statistical averaging in local equilibrium is denoted by the subscript l . The two-point and three-point correlation functions are defined by

$$(\hat{O}(x), \hat{X}(x_1)) = \int_0^1 d\tau \langle \hat{O}(x) [X_\tau(x_1) - \langle X_\tau(x_1) \rangle_l] \rangle_l, \quad (32)$$

$$(\hat{O}(x), \hat{X}(x_1), \hat{Y}(x_2)) = \frac{1}{2} \int_0^1 d\tau \int_0^1 d\lambda \langle \tilde{T} \{ \hat{O}(x) [\hat{X}_\lambda(x_1) \hat{Y}_\tau(x_2) - \langle \hat{X}_\lambda(x_1) \rangle_l \langle \hat{Y}_\tau(x_2) \rangle_l - \hat{X}_\lambda(x_1) \langle \hat{Y}_\tau(x_2) \rangle_l - \langle \tilde{T} \hat{X}_\lambda(x_1) \hat{Y}_\tau(x_2) \rangle_l + 2 \langle \hat{X}_\lambda(x_1) \rangle_l \langle \hat{Y}_\tau(x_2) \rangle_l] \} \rangle_l, \quad (33)$$

respectively. Here, $X_\tau = e^{-\tau A} X e^{\tau A}$ and $\tilde{T}\{X_\lambda Y_\tau\}$ is the antichronological time-ordering operator with respect to variables τ and λ .

Next, we calculate the statistical average of various dissipative terms ($\hat{\tau}$, $\hat{q}$, $\hat{\pi}$, Π) up to second order in gradient expansion using the NESO. We proceed by writing a tensor decomposition of operators $\hat{T}^{\mu\nu}$, $\hat{N}^\mu$, and $\hat{\Sigma}^{\mu\alpha\beta}$. The straightforward decomposition is given by

$$\hat{T}^{\mu\nu} = \hat{\epsilon} u^\mu u^\nu - \hat{p} \Delta^{\mu\nu} + \hat{h}^\mu u^\nu + \hat{h}^\nu u^\mu + \hat{\pi}^{\mu\nu} + \hat{q}^\mu u^\nu - \hat{q}^\nu u^\mu + \hat{\tau}^{\mu\nu}, \quad (34)$$

$$\hat{N}^\mu = \hat{n}u^\mu + \hat{j}^\mu, \quad (35)$$

$$\hat{\Sigma}^{\mu\alpha\beta} = u^\mu \hat{S}^{\alpha\beta}, \quad (36)$$

where $\hat{p}$ is the actual isotropic pressure. All the operators, except the pressure ($\hat{p}$), can be directly matched with the corresponding hydrodynamic quantities by taking the statistical average using the full nonequilibrium statistical operator. That means $\varepsilon \equiv \langle \hat{\varepsilon} \rangle$, $h^\mu \equiv \langle \hat{h}^\mu \rangle$, $\pi^{\mu\nu} \equiv \langle \hat{\pi}^{\mu\nu} \rangle$, $q^\mu \equiv \langle \hat{q}^\mu \rangle$, $\tau^{\mu\nu} \equiv \langle \hat{\tau}^{\mu\nu} \rangle$, $n \equiv \langle \hat{n} \rangle$, $j^\mu \equiv \langle \hat{j}^\mu \rangle$, $S^{\mu\nu} \equiv \langle \hat{S}^{\mu\nu} \rangle$. Whereas, the statistical average of $\hat{p}$, $\langle \hat{p} \rangle \equiv p + \Pi$. The operators can be written in terms of the projections of $\hat{T}^{\mu\nu}$ and $\hat{N}^\mu$,

$$\begin{aligned} \hat{\varepsilon} &= u^\mu u_\nu \hat{T}^{\mu\nu}, & \hat{p} &= -\frac{1}{3} \Delta_{\mu\nu} \hat{T}^{\mu\nu}, & \hat{h}^\alpha + \hat{q}^\alpha &= \Delta_\mu^\alpha u_\nu \hat{T}^{\mu\nu}, & \hat{h}^\alpha - \hat{q}^\alpha &= u_\mu \Delta_\nu^\alpha \hat{T}^{\mu\nu}, \\ \hat{\pi}^{\mu\nu} &= \Delta_{\alpha\beta}^{\mu\nu} \hat{T}^{\alpha\beta}, & \hat{\tau}^{\mu\nu} &= \Delta_{\alpha\beta}^{\mu\nu} \hat{T}^{\alpha\beta}, & \hat{n} &= u_\mu \hat{N}^\mu, & \hat{j}^\mu &= \Delta_\nu^\mu \hat{N}^\nu, & \hat{S}^{\alpha\beta} &= u_\mu \hat{\Sigma}^{\mu\alpha\beta}. \end{aligned} \quad (37)$$

The crucial quantity to evaluate for the statistical average is $\hat{C}$. To calculate $\hat{C}(x, t)$, we use the gradient decomposition of $\hat{T}^{\mu\nu}$, $\hat{N}^\mu$, and $\hat{\Sigma}^{\mu\alpha\beta}$ from (34)–(36), respectively. Neglecting the third-order gradient terms, we can rewrite (30) as

$$\begin{aligned} \hat{C} &= \hat{\varepsilon} D\beta - \hat{n} D\alpha - \frac{1}{2} \hat{S}^{\alpha\beta} D\Omega_{\alpha\beta} - \hat{p} \beta \theta + \hat{h}^\mu (\partial_\mu \beta + \beta D u_\mu) - \hat{j}^\mu \partial_\mu \alpha + \hat{q}^\mu (\partial_\mu \beta - \beta D u_\mu + 2\Omega_{\mu\nu} u^\nu) \\ &\quad + \beta \hat{\pi}^{\mu\nu} \partial_\mu u_\nu + \hat{\tau}^{\mu\nu} (\beta \partial_\mu u_\nu + \Omega_{\mu\nu}), \end{aligned} \quad (38)$$

The terms $D\beta$, $D\alpha$, and $D\Omega_{\alpha\beta}$ can be eliminated using the equations of motion. To achieve this, we choose the independent variables to be ε , n , and $S^{\alpha\beta}$. Consequently, any thermodynamic quantity, such as β , can be written as a function of these variables, i.e., $\beta \equiv \beta(\varepsilon, n, S^{\alpha\beta})$, and any change in that thermodynamic quantity can be given as

$$D\beta = \left. \frac{\partial \beta}{\partial \varepsilon} \right|_{n, S^{\alpha\beta}} D\varepsilon + \left. \frac{\partial \beta}{\partial n} \right|_{\varepsilon, S^{\alpha\beta}} Dn + \left. \frac{\partial \beta}{\partial S^{\alpha\beta}} \right|_{\varepsilon, n} DS^{\alpha\beta}. \quad (39)$$

The values of $D\varepsilon$, Dn , and $DS^{\alpha\beta}$ up to second order are substituted from the equations of motion (12), (14), and (15), in which we fix that the terms Π , h^μ , q^μ , j^μ , $\pi^{\mu\nu}$, and $\tau^{\mu\nu}$ only contain $\mathcal{O}(\partial^1)$ corrections. Thus, the full expression of $D\beta$, keeping the terms up to second order in gradient expansion, is given by

$$\begin{aligned} D\beta &= -\theta \left[(\varepsilon + p) \frac{\partial \beta}{\partial \varepsilon} + n \frac{\partial \beta}{\partial n} + S^{\alpha\beta} \frac{\partial \beta}{\partial S^{\alpha\beta}} \right] - \frac{\partial \beta}{\partial n} \partial_\mu j^\mu - \frac{\partial \beta}{\partial S^{\alpha\beta}} (2\tau^{\alpha\beta} + 4q^\alpha u^\beta) - \frac{\partial \beta}{\partial \varepsilon} (\Pi\theta + \partial_\mu h^\mu + u_\mu D h^\mu + u_\nu \partial_\mu \pi^{\mu\nu} \\ &\quad + \partial_\mu q^\mu - u_\mu D q^\mu + u_\nu \partial_\mu \tau^{\mu\nu}), \end{aligned} \quad (40)$$

It is important to note that by excluding the third-order and higher-order terms, we posit that the gradient ordering of the term $\frac{\partial \beta}{\partial S^{\alpha\beta}}$ is $\mathcal{O}(\partial^1)$. This assertion can be understood by deriving the Maxwell relations from (16). The first step involves rewriting (16) in terms of β , α , and $\Omega_{\mu\nu}$, utilizing the connection provided by (20). Thus, we can write

$$ds = \beta d\varepsilon - \alpha dn - \frac{1}{2} \Omega_{\mu\nu} dS^{\mu\nu}. \quad (41)$$

Using (41), we can find the relations

$$\left. \frac{\partial \beta}{\partial n} \right|_{\varepsilon, S^{\alpha\beta}} = - \left. \frac{\partial \alpha}{\partial \varepsilon} \right|_{n, S^{\alpha\beta}}, \quad \left. \frac{\partial \beta}{\partial S^{\alpha\beta}} \right|_{\varepsilon, n} = - \frac{1}{2} \left. \frac{\partial \Omega_{\alpha\beta}}{\partial \varepsilon} \right|_{n, S^{\alpha\beta}}, \quad \left. \frac{\partial \alpha}{\partial S^{\alpha\beta}} \right|_{\varepsilon, n} = \frac{1}{2} \left. \frac{\partial \Omega_{\alpha\beta}}{\partial n} \right|_{\varepsilon, S^{\alpha\beta}}. \quad (42)$$

Given that $\Omega_{\alpha\beta} \sim \mathcal{O}(\partial^1)$, it follows that the partial derivative $\frac{\partial \Omega_{\alpha\beta}}{\partial \varepsilon}$ is also of the order $\mathcal{O}(\partial^1)$.

Similarly, we can also calculate $D\alpha$ and $D\Omega_{\alpha\beta}$ given by

$$D\alpha = -\theta \left[(\varepsilon + p) \frac{\partial \alpha}{\partial \varepsilon} + n \frac{\partial \alpha}{\partial n} + S^{\alpha\beta} \frac{\partial \alpha}{\partial S^{\alpha\beta}} \right] - \frac{\partial \alpha}{\partial n} \partial_\mu j^\mu - \frac{\partial \alpha}{\partial S^{\alpha\beta}} (2\tau^{\alpha\beta} + 4q^\alpha u^\beta) - \frac{\partial \alpha}{\partial \varepsilon} (\Pi\theta + \partial_\mu h^\mu + u_\mu D h^\mu + u_\nu \partial_\mu \pi^{\mu\nu} + \partial_\mu q^\mu - u_\mu D q^\mu + u_\nu \partial_\mu \tau^{\mu\nu}), \quad (43)$$

and

$$D\Omega_{\alpha\beta} = -\theta \left[(\varepsilon + p) \frac{\partial \Omega_{\alpha\beta}}{\partial \varepsilon} + n \frac{\partial \Omega_{\alpha\beta}}{\partial n} + S^{\mu\nu} \frac{\partial \Omega_{\alpha\beta}}{\partial S^{\mu\nu}} \right] - \frac{\partial \Omega_{\alpha\beta}}{\partial n} \partial_\mu j^\mu - \frac{\partial \Omega_{\alpha\beta}}{\partial S^{\mu\nu}} (2\tau^{\mu\nu} + 4q^\mu u^\nu) - \frac{\partial \Omega_{\alpha\beta}}{\partial \varepsilon} (\Pi\theta + \partial_\mu h^\mu + u_\mu D h^\mu + u_\nu \partial_\mu \pi^{\mu\nu} + \partial_\mu q^\mu - u_\mu D q^\mu + u_\nu \partial_\mu \tau^{\mu\nu}). \quad (44)$$

Using the values of $D\beta$, $D\alpha$, and $D\Omega_{\alpha\beta}$ together with the relations (42), and extracting the value of Du_α from (13), we can express the nonequilibrium part ($\hat{C}$) of the nonequilibrium statistical operator in the Landau frame. Thus, we obtain

$$\begin{aligned} \hat{C} = & -\hat{P}^* \beta \theta - \hat{j}^\mu \partial_\mu \alpha + \hat{q}^\mu (\partial_\mu \beta - \beta h^{-1} \nabla_\mu p + 2\Omega_{\mu\nu} u^\nu) + \beta \hat{\pi}^{\mu\nu} \partial_\mu u_\nu + \hat{\tau}^{\mu\nu} (\beta \partial_\mu u_\nu + \Omega_{\mu\nu}) \\ & + \hat{q}^\alpha \beta h^{-1} [\Pi D u_\alpha - \nabla_\alpha \Pi + \Delta_{\alpha\nu} \partial_\mu \pi^{\mu\nu} - q_\alpha \theta + q^\mu \partial_\mu u_\alpha - \Delta_{\alpha\nu} D q^\nu + \Delta_{\alpha\nu} \partial_\mu \tau^{\mu\nu}] \\ & - \hat{\beta}^* (\Pi\theta + u_\nu \partial_\mu \pi^{\mu\nu} + \partial_\mu q^\mu - u_\mu D q^\mu + u_\nu \partial_\mu \tau^{\mu\nu}) + \hat{\alpha}^* \partial_\mu j^\mu + \hat{\Omega}_{\mu\nu}^* (\tau^{\mu\nu} + 2q^\mu u^\nu). \end{aligned} \quad (45)$$

Here, we set $h_\mu = 0$ for the Landau frame, and we have defined

$$\hat{P}^* = (\hat{p} - \hat{\varepsilon}\Gamma - \hat{n}\delta) - \hat{S}^{\alpha\beta} \mathcal{K}_{\alpha\beta} = \hat{p}^* - \hat{S}^{\alpha\beta} \mathcal{K}_{\alpha\beta}, \quad (46)$$

and the new parameters as

$$\begin{aligned} \hat{\beta}^* &= \hat{\varepsilon} \frac{\partial \beta}{\partial \varepsilon} + \hat{n} \frac{\partial \beta}{\partial n} + \hat{S}^{\alpha\beta} \frac{\partial \beta}{\partial S^{\alpha\beta}}, \\ \hat{\alpha}^* &= \hat{\varepsilon} \frac{\partial \alpha}{\partial \varepsilon} + \hat{n} \frac{\partial \alpha}{\partial n} + \hat{S}^{\alpha\beta} \frac{\partial \alpha}{\partial S^{\alpha\beta}}, \\ \hat{\Omega}_{\mu\nu}^* &= \hat{\varepsilon} \frac{\partial \Omega_{\mu\nu}}{\partial \varepsilon} + \hat{n} \frac{\partial \Omega_{\mu\nu}}{\partial n} + \hat{S}^{\alpha\beta} \frac{\partial \Omega_{\mu\nu}}{\partial S^{\alpha\beta}}. \end{aligned} \quad (47)$$

We have also defined

$$\Gamma = \frac{\partial p}{\partial \varepsilon} \Big|_{n, S^{\mu\nu}}, \quad \delta = \frac{\partial p}{\partial n} \Big|_{\varepsilon, S^{\mu\nu}} \quad \text{and} \quad \mathcal{K}_{\alpha\beta} = \frac{\partial p}{\partial S^{\alpha\beta}} \Big|_{\varepsilon, n}. \quad (48)$$

Next, we separate $\hat{C}$ into two parts, $\hat{C} = \hat{C}_F + \hat{C}_S$, containing first-order and second-order gradient terms, respectively. $\hat{C}_F$ and $\hat{C}_S$ are

$$\hat{C}_F = -\hat{p}^* \beta \theta - \hat{j}^\mu \nabla_\mu \alpha + \beta \hat{q}^\mu M_\mu + \beta \hat{\pi}^{\mu\nu} \sigma_{\mu\nu} + \beta \hat{\tau}^{\mu\nu} \xi_{\mu\nu}, \quad (49)$$

$$\hat{C}_S = \beta \sum_{i=1,2} [(\hat{\mathfrak{D}}_i \partial_{\varepsilon n}^i \beta) \mathcal{X} + (\hat{\mathfrak{D}}_i \partial_{\varepsilon n}^i \alpha) \mathcal{Y} + (\hat{\mathfrak{D}}_i \partial_{\varepsilon n}^i \Omega_{\mu\nu}) \mathcal{Z}^{\mu\nu}] + \beta \hat{q}^\alpha \mathcal{Q}_\alpha + \hat{S}^{\mu\nu} \mathcal{K}_{\mu\nu} \beta \theta. \quad (50)$$

Here, $\hat{\mathfrak{D}}_i \equiv (\hat{\varepsilon}, \hat{n})$ and $\partial_{\varepsilon n}^i \equiv (\frac{\partial}{\partial \varepsilon}, \frac{\partial}{\partial n})$ and we defined

$$\mathcal{Q}_\alpha = h^{-1} [\Pi D u_\alpha - \nabla_\alpha \Pi + \Delta_{\alpha\nu} \partial_\mu \pi^{\mu\nu} - q_\alpha \theta + q^\mu \partial_\mu u_\alpha - \Delta_{\alpha\nu} D q^\nu + \Delta_{\alpha\nu} \partial_\mu \tau^{\mu\nu}], \quad (51)$$

$$\mathcal{X} = -\beta^{-1} (\Pi\theta + u_\nu \partial_\mu \pi^{\mu\nu} + \partial_\mu q^\mu - u_\mu D q^\mu + u_\nu \partial_\mu \tau^{\mu\nu}), \quad (52)$$

$$\mathcal{Y} = \beta^{-1} \partial_\mu j^\mu, \quad (53)$$

$$\mathcal{Z}^{\mu\nu} = \beta^{-1} (\tau^{\mu\nu} + 2q^\mu u^\nu). \quad (54)$$

Employing (49) and (50), we can rewrite the statistical average of any operator in terms of gradient expansion as

$$\langle \hat{O}(x) \rangle = \langle \hat{O}(x) \rangle_l + \langle \hat{O}(x) \rangle_1 + \langle \hat{O}(x) \rangle_2, \quad (55)$$

where $\langle \hat{O}(x) \rangle_1$ and $\langle \hat{O}(x) \rangle_2$ are the first-order and second-order gradient terms, respectively. The first-order contribution is calculated assuming that the thermodynamic forces remain constant within the correlation length. This yields the local contribution from the two-point correlation function. The resulting expression is given by

$$\langle \hat{O}(x) \rangle_1 = \int d^4x_1 (\hat{O}(x), \hat{C}_F(x_1))|_{\text{local}}. \quad (56)$$

Whereas, the second-order contribution to the statistical average can originate from three possible ways, given by

$$\langle \hat{O}(x) \rangle_2 = \langle \hat{O}(x) \rangle_2^{(2),NL} + \langle \hat{O}(x) \rangle_2^{(2),ET} + \langle \hat{O}(x) \rangle_2^{(3)}, \quad (57)$$

where

$$\langle \hat{O}(x) \rangle_2^{(2),NL} = \int d^4x_1 (\hat{O}(x), \hat{C}_F(x_1)) - \langle \hat{O}(x) \rangle_1, \quad (58)$$

$$\langle \hat{O}(x) \rangle_2^{(2),ET} = \int d^4x_1 (\hat{O}(x), \hat{C}_S(x_1))|_{\text{local}}, \quad (59)$$

$$\langle \hat{O}(x) \rangle_2^{(3)} = \int d^4x_1 d^4x_2 (\hat{O}(x), \hat{C}_F(x_1), \hat{C}_F(x_2))|_{\text{local}}, \quad (60)$$

here, $\langle \hat{O}(x) \rangle_2^{(2),NL}$ is the nonlocal (*NL*) correction from the two-point correlation function, $\langle \hat{O}(x) \rangle_2^{(2),ET}$ is the local correction from the two-point correlation function using external thermodynamic forces (*ET*), and $\langle \hat{O}(x) \rangle_2^{(3)}$ is the local correction from the three-point correlation function. The meaning of local and nonlocal are explained in the next section.

IV. FIRST-ORDER SPIN HYDRODYNAMICS

The transport coefficients emerging from first-order spin hydrodynamics have been calculated using the density operator formalism [54]. In this section, we provide a concise overview of the calculations pertinent to first-order spin hydrodynamics. We begin by calculating the average value for the rotational stress tensor $\tau_{\mu\nu}$, the antisymmetric counterpart of the $\pi_{\mu\nu}$. Using (56), the first-order gradient correction to $\langle \hat{\tau}_{\mu\nu}(x) \rangle$ is given by

$$\langle \hat{\tau}_{\mu\nu}(x) \rangle_1 = \int d^4x_1 (\hat{\tau}_{\mu\nu}(x), \hat{C}_F(x_1))|_{\text{local}}. \quad (61)$$

Now, we take the help of Curie's principle, that the correlation between the operators of different ranks vanishes in an isotropic media. Hence, the only nonzero correlation for the $\langle \hat{\tau}_{\mu\nu}(x) \rangle_1$ is

$$\begin{aligned} \langle \hat{\tau}_{\mu\nu}(x) \rangle_1 &= \int d^4x_1 (\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1)) \beta(x_1) \xi^{\rho\sigma}(x_1)|_{\text{local}} \\ &+ \int d^4x_1 (\hat{\tau}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1)) \beta(x_1) \sigma^{\rho\sigma}(x_1)|_{\text{local}}. \end{aligned} \quad (62)$$

Also, in an isotropic medium, we can write [see Appendix C Eq. (C4)]

$$(\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1)) = \frac{1}{3} \Delta_{\mu\nu\rho\sigma} (\hat{\tau}^{\lambda\eta}(x), \hat{\tau}_{\lambda\eta}(x_1)), \quad (63)$$

$$(\hat{\tau}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1)) = 0. \quad (64)$$

Further, we use the Taylor expansion to expand $\beta(x_1) \xi^{\rho\sigma}(x_1)$ around $x_1 = x$. We have

$$\begin{aligned} \beta(x_1) \xi^{\rho\sigma}(x_1) &= \beta(x) \xi^{\rho\sigma}(x) + (x_1 - x)^\alpha \frac{\partial}{\partial x_1^\alpha} [\beta(x_1) \xi^{\rho\sigma}(x_1)]|_x \\ &+ \dots \end{aligned} \quad (65)$$

Here, we like to emphasize the gradient ordering in the second term. In this context, $(x_1 - x)^\alpha$ represents the correlation length, which is comparable to the microscopic length, i.e., the mean free path of the particles (λ). Consequently, the term $(x_1 - x)^\alpha \frac{\partial}{\partial x_1^\alpha} \sim \frac{\lambda}{L}$, where L is the macroscopic scale. The ratio $\frac{\lambda}{L}$ known as Knudsen number. Therefore, the second term in (65) is of the second order in hydrodynamic gradient expansion. This term is neglected in first-order hydrodynamics calculation. The first term in (65), evaluated at $x_1 = x$ is the local term. All the higher-order terms are called nonlocal terms. Substituting (63)–(65) in (62), and keeping only the first-order gradient terms, we obtain

$$\tau_{\mu\nu}(x) \equiv \langle \hat{\tau}_{\mu\nu}(x) \rangle_1 = \frac{1}{3} \beta(x) \xi_{\mu\nu}(x) \int d^4x_1 (\hat{\tau}^{\lambda\eta}(x), \hat{\tau}_{\lambda\eta}(x_1)). \quad (66)$$

By employing Curie's principle, a similar approach is applied to the remaining dissipative quantities. Thus, we can write

$$q_\mu(x) \equiv \langle \hat{q}_\mu(x) \rangle_1 = \frac{1}{3} \beta(x) M_\mu(x) \int d^4 x_1 (\hat{q}^\lambda(x), \hat{q}_\lambda(x_1)), \quad (67)$$

$$\pi_{\mu\nu}(x) \equiv \langle \hat{\pi}_{\mu\nu}(x) \rangle_1 = \frac{1}{5} \beta(x) \sigma_{\mu\nu}(x) \int d^4 x_1 (\hat{\pi}^{\lambda\eta}(x), \hat{\pi}_{\lambda\eta}(x_1)). \quad (68)$$

However, the bulk viscous pressure (Π) is defined by

$$\Pi = \langle \hat{p} \rangle - p(\varepsilon, n, S^{\alpha\beta}), \quad (69)$$

and up to the first order in gradient expansion, we obtain [see Appendix D Eq. (D4)]

$$\Pi = \langle \hat{p}^*(x) \rangle_1, \quad (70)$$

which is given by

$$\Pi = \langle \hat{p}^*(x) \rangle_1 = -\beta(x) \theta(x) \int d^4 x_1 (\hat{p}^*(x), \hat{p}^*(x_1)). \quad (71)$$

Here, we followed a procedure similar to the one outlined while calculating (66). Comparing the equations (66)–(68), and (71) with (23), we can deduce the first-order transport coefficients, given by

$$\gamma = \frac{\beta}{6} \int d^4 x_1 (\hat{\tau}^{\lambda\eta}(x), \hat{\tau}_{\lambda\eta}(x_1)), \quad (72)$$

$$\lambda = -\frac{\beta}{3} \int d^4 x_1 (\hat{q}^\lambda(x), \hat{q}_\lambda(x_1)), \quad (73)$$

$$\eta = \frac{\beta}{10} \int d^4 x_1 (\hat{\pi}^{\lambda\eta}(x), \hat{\pi}_{\lambda\eta}(x_1)), \quad (74)$$

$$\zeta = \beta \int d^4 x_1 (\hat{p}^*(x), \hat{p}^*(x_1)). \quad (75)$$

V. SECOND-ORDER SPIN HYDRODYNAMICS

Relativistic hydrodynamics, which considers only the symmetric canonical energy-momentum tensor and excludes the spin degree of freedom, has been extensively studied up to the second-order using the density operator formalism [62]. We extend this formalism to include the antisymmetric part of the energy-momentum tensor and derive the second-order spin hydrodynamics. We start by calculating the second-order corrections to the rotational stress tensor.

A. Rotational stress tensor ($\tau_{\mu\nu}$)

The second-order correction to $\langle \hat{\tau}_{\mu\nu}(x) \rangle$ can be decomposed into three parts [see (56)], given by

$$\langle \hat{\tau}_{\mu\nu}(x) \rangle_2 = \langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2),NL} + \langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2),ET} + \langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(3)}. \quad (76)$$

The general definition of these three terms is defined in (58)–(60). Since we are interested in the correlation in isotropic media, these terms can be further simplified using Curie's principle. Thus,

$$\begin{aligned} \langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2),NL} &= \int d^4 x_1 (\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\alpha\beta}(x_1)) \beta(x_1) \xi^{\alpha\beta}(x_1) \\ &\quad - \langle \hat{\tau}_{\mu\nu}(x) \rangle_1, \end{aligned} \quad (77)$$

$$\begin{aligned} \langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2),ET} &= \int d^4 x_1 (\hat{\tau}_{\mu\nu}(x), \hat{S}_{\alpha\beta}(x_1)) \mathcal{K}^{\alpha\beta}(x_1) \\ &\quad \times \beta(x_1) \theta(x_1)|_{\text{local}}, \end{aligned} \quad (78)$$

$$\langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(3)} = \int d^4 x_1 d^4 x_2 (\hat{\tau}_{\mu\nu}(x), \hat{C}_F(x_1), \hat{C}_F(x_2))|_{\text{local}}, \quad (79)$$

where $\hat{C}_F$ is given by Eq. (49).

1. Nonlocal correction from two-point correlation function: $\langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2),NL}$

To calculate the nonlocal correction from the two-point correlation function, we first generalize Eq. (63), which can be given by

$$(\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1)) = \frac{1}{3} \Delta_{\mu\nu\rho\sigma}(x, x_1) (\hat{\tau}^{\lambda\eta}(x), \hat{\tau}_{\lambda\eta}(x_1)). \quad (80)$$

This is the nonlocal generalization of two-point correlators, with the nonlocal projector defined as (see A 2)

$$\Delta_{\mu\nu\rho\sigma}(x, x_1) = \Delta_{\mu\nu\alpha\beta}(x) \Delta_{\rho\sigma}^{\alpha\beta}(x_1). \quad (81)$$

In order to extract nonlocal terms from (77), we have to express $\hat{\tau}^{\lambda\sigma}$ and $\hat{\tau}_{\lambda\sigma}$ in terms of the energy-momentum tensor, and then we expand all the hydrodynamic quantities around $x_1 = x$. From (37), we can write

$$\hat{\tau}^{\lambda\eta}(x) = \Delta_{\gamma\delta}^{\lambda\eta}(x) \hat{T}^{\gamma\delta}(x), \quad (82)$$

$$\hat{\tau}_{\lambda\eta}(x_1) = \Delta_{\lambda\eta\alpha\beta}(x_1) \hat{T}^{\alpha\beta}(x_1). \quad (83)$$

Substituting (82) and (83) together with (80) and (81) in (77), we obtain

$$\begin{aligned} \langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2),NL} &= \frac{1}{3} \Delta_{\mu\nu\rho\sigma}(x) \int d^4 x_1 (\hat{T}^{\gamma\delta}(x), \hat{T}^{\alpha\beta}(x_1)) \\ &\quad \times \Delta_{\gamma\delta\alpha\beta}(x, x_1) \beta(x_1) \xi^{\rho\sigma}(x_1) - \langle \hat{\tau}_{\mu\nu}(x) \rangle_1. \end{aligned} \quad (84)$$

Furthermore, the Taylor's expansion of $\Delta_{\gamma\delta\alpha\beta}(x, x_1)\beta(x_1)\xi^{\rho\sigma}(x_1)$ around $x_1 = x$ is given by

$$\begin{aligned} \Delta_{\gamma\delta\alpha\beta}(x, x_1)\beta(x_1)\xi^{\rho\sigma}(x_1) &= \Delta_{\gamma\delta\alpha\beta}(x)\beta(x)\xi^{\rho\sigma}(x) + (x_1 - x)^\lambda \left[\Delta_{\gamma\delta\alpha\beta}(x) \frac{\partial}{\partial x_1^\lambda} \{\beta(x_1)\xi^{\rho\sigma}(x_1)\} \Big|_{x_1=x} \right. \\ &\quad \left. + \beta(x)\xi^{\rho\sigma}(x) \frac{\partial}{\partial x_1^\lambda} \{\Delta_{\gamma\delta\alpha\beta}(x, x_1)\} \Big|_{x_1=x} \right], \end{aligned} \quad (85)$$

where the first-order derivative of $\Delta_{\gamma\delta\alpha\beta}(x, x_1)$ is given by [see Eq. (A17) in A 2]

$$\frac{\partial}{\partial x_1^\lambda} \Delta_{\mu\nu\rho\sigma}(x, x_1) \Big|_{x_1=x} = -[\Delta_{\mu\nu\rho\tau}(x)u_\sigma(x) + \Delta_{\mu\nu\tau\sigma}(x)u_\rho(x)] \frac{\partial u^\tau(x_1)}{\partial x_1^\lambda} \Big|_{x_1=x}. \quad (86)$$

Substituting (85) in (84), we find that the leading-order term in (85) cancels $\langle \hat{\tau}_{\mu\nu}(x) \rangle_1$ in (84), with the help of (66). Thus, we obtain

$$\begin{aligned} \langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2),NL} &= \left[\frac{1}{3} \Delta_{\mu\nu\rho\sigma}(x) \Delta_{\gamma\delta\alpha\beta}(x) \frac{\partial}{\partial x_1^\lambda} \{\beta(x_1)\xi^{\rho\sigma}(x_1)\} \Big|_{x_1=x} \int d^4x_1 (\hat{T}^{\gamma\delta}(x), \hat{T}^{\alpha\beta}(x_1))(x_1 - x)^\lambda \right] \\ &\quad - \left[\frac{1}{3} \beta(x)\xi_{\mu\nu}(x) \frac{\partial u^\tau(x_1)}{\partial x_1^\lambda} \Big|_{x_1=x} \{\Delta_{\gamma\delta\alpha\tau}(x)u_\beta(x) + \Delta_{\gamma\delta\tau\beta}(x)u_\alpha(x)\} \int d^4x_1 (\hat{T}^{\gamma\delta}(x), \hat{T}^{\alpha\beta}(x_1))(x_1 - x)^\lambda \right]. \end{aligned} \quad (87)$$

Next, we approximate $u_\beta(x) \simeq u_\beta(x_1)$ using the leading-order approximation, because the terms in (87) are already multiplied with the second-order terms. Thus, using (37) in the Landau frame, we find that the second part of Eq. (87) gives the two types of correlation,

$$\int d^4x_1 (\hat{\tau}_{\alpha\tau}(x), \hat{q}^\alpha(x_1)), \quad \text{and} \quad \int d^4x_1 (\hat{\tau}_{\alpha\tau}(x), \hat{e}(x_1)u^\alpha(x_1)), \quad (88)$$

which can be safely set to zero, by revoking Curie's principle and using the property $u^\alpha \hat{\tau}_{\alpha\tau} = 0$. Also, in the first-order gradient approximation, we can write

$$\Delta_{\gamma\delta\alpha\beta}(x) \int d^4x_1 (\hat{T}^{\gamma\delta}(x), \hat{T}^{\alpha\beta}(x_1)) = \int d^4x_1 (\hat{\tau}^{\lambda\sigma}(x), \hat{\tau}_{\lambda\sigma}(x_1)). \quad (89)$$

Substituting (89) in (87) and using (88), we get

$$\langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2),NL} = 2[\beta(x)]^{-1} \Delta_{\mu\nu\rho\sigma}(x) L^\lambda(x) \frac{\partial}{\partial x_1^\lambda} \{\beta(x_1)\xi^{\rho\sigma}(x_1)\} \Big|_{x_1=x}. \quad (90)$$

Here, we defined

$$L^\lambda(x) = \frac{\beta(x)}{6} \int d^4x_1 (\hat{\tau}^{\alpha\beta}(x), \hat{\tau}_{\alpha\beta}(x_1))(x_1 - x)^\lambda, \quad (91)$$

which can be further simplified in the following form [see Eq. (B22) in Appendix B]

$$L^\lambda = \gamma_\omega u^\lambda, \quad (92)$$

where

$$\gamma_\omega = i \lim_{\omega \rightarrow 0} \frac{d}{d\omega} \gamma_{\hat{\tau}\hat{\tau}}(\omega). \quad (93)$$

Here $\gamma_{\hat{\tau}\hat{\tau}}(\omega)$ is the frequency-dependent transport coefficient, given by (see Appendix B)

$$\gamma_{\hat{\tau}\hat{\tau}}(\omega) = \frac{\beta}{6} \int d^4x_1 e^{i\omega(t-t_1)} (\hat{\tau}^{\alpha\beta}(x), \hat{\tau}_{\alpha\beta}(x_1)). \quad (94)$$

Combining (90)–(92), and using the first-order approximation $D\beta \simeq \Gamma\beta\theta$, we obtain

$$\langle \hat{\tau}_{\mu\nu} \rangle_2^{(2),NL} = 2\gamma_\omega [\Delta_{\mu\nu\rho\sigma} D\xi^{\rho\sigma} + \theta \Gamma \xi_{\mu\nu}], \quad (95)$$

where we removed the argument x from all quantities.

2. Local correction from two-point correlation function using extended thermodynamic forces: $\langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2).ET}$

Since $\hat{C}_S$ contains an antisymmetric tensor operator $\hat{S}_{\alpha\beta}$, the term (78) is nonzero for a rotational stress tensor. To solve this term, we can expand $\mathcal{K}^{\alpha\beta}(x_1)\beta(x_1)\theta(x_1)$ around $x_1 = x$, in a manner similar to how we solved for $\langle \hat{\tau}_{\mu\nu}(x) \rangle_1$. From (78), we have

$$\begin{aligned} \langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(2).ET} &= \int d^4x_1 (\hat{\tau}_{\mu\nu}(x), \hat{S}_{\alpha\beta}(x_1)) \mathcal{K}^{\alpha\beta}(x_1) \\ &\quad \times \beta(x_1) \theta(x_1) |_{\text{local}}, \\ &= \gamma_{\tau S}(x) \mathcal{K}_{\mu\nu}(x) \theta(x), \end{aligned} \quad (96)$$

where we defined

$$\gamma_{\tau S}(x) = \frac{\beta(x)}{3} \int d^4x_1 (\hat{\tau}_{\lambda\sigma}(x), \hat{S}^{\lambda\sigma}(x_1)). \quad (97)$$

Also, we have used (see Appendix C),

$$(\hat{\tau}_{\mu\nu}(x), \hat{S}_{\alpha\beta}(x_1)) = \frac{1}{3} \Delta_{\mu\nu\alpha\beta}(\hat{\tau}_{\lambda\sigma}(x), \hat{S}^{\lambda\sigma}(x_1)). \quad (98)$$

3. Local correction from three-point correlation function: $\langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(3)}$

Using (60), this correction term is given by

$$\langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(3)} = \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{C}_F(x_1), \hat{C}_F(x_2)) |_{\text{local}}, \quad (99)$$

where

$$\hat{C}_F(x) = -\hat{p}^* \beta \theta - \hat{j}_\mu \nabla^\mu \alpha + \beta \hat{q}_\mu M^\mu + \beta \hat{\pi}_{\mu\nu} \sigma^{\mu\nu} + \beta \hat{\tau}_{\mu\nu} \xi^{\mu\nu}, \quad (100)$$

and $M^\mu = \beta^{-1} \partial^\mu \beta - h^{-1} \nabla^\mu p + 2\beta^{-1} \Omega^{\mu\nu} u_\nu$.

In order to calculate the local contribution from (99), we expand the thermodynamic forces around $x_1 = x$ and $x_2 = x$ and collect only the leading-order terms, which is equivalent to replacing the argument of thermodynamic forces with x . Thus, substituting the value of $\hat{C}_F$ in (99), we get ten types of nonzero correlations, given by

$$\mathcal{A}_{1\mu\nu}(x) = \beta^2(x) \xi^{\rho\sigma}(x) \xi^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\tau}_{\alpha\beta}(x_2)), \quad (101)$$

$$\mathcal{A}_{2\mu\nu}(x) = \beta^2(x) \sigma^{\rho\sigma}(x) \sigma^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2)), \quad (102)$$

$$\mathcal{A}_{3\mu\nu}(x) = \beta^2(x) \sigma^{\rho\sigma}(x) \xi^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\tau}_{\alpha\beta}(x_2)), \quad (103)$$

$$\mathcal{A}_{4\mu\nu}(x) = \beta^2(x) \xi^{\rho\sigma}(x) \sigma^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2)), \quad (104)$$

$$\mathcal{A}_{5\mu\nu}(x) = -\beta^2(x) \xi^{\rho\sigma}(x) \theta(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{p}^*(x_2)), \quad (105)$$

$$\mathcal{A}_{6\mu\nu}(x) = -\beta^2(x) \xi^{\alpha\beta}(x) \theta(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{p}^*(x_1), \hat{\tau}_{\alpha\beta}(x_2)), \quad (106)$$

$$\mathcal{A}_{7\mu\nu}(x) = \beta^2(x) M^\alpha(x) M^\beta(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{q}_\alpha(x_1), \hat{q}_\beta(x_2)), \quad (107)$$

$$\mathcal{A}_{8\mu\nu}(x) = \nabla^\alpha \alpha(x) \nabla^\beta \alpha(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{j}_\alpha(x_1), \hat{j}_\beta(x_2)), \quad (108)$$

$$\mathcal{A}_{9\mu\nu}(x) = -\beta(x) \nabla^\alpha \alpha(x) M^\beta(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{j}_\alpha(x_1), \hat{q}_\beta(x_2)), \quad (109)$$

$$\mathcal{A}_{10\mu\nu}(x) = -\beta(x) M^\alpha(x) \nabla^\beta \alpha(x) \int d^4x_1 d^4x_2 (\hat{\tau}_{\mu\nu}(x), \hat{q}_\alpha(x_1), \hat{j}_\beta(x_2)). \quad (110)$$

Now, we write the correlation functions in the form of projectors and traces. We start by writing the most general expression for the three-point correlation function, involving three second-rank tensors and satisfying the orthogonality condition $u_\mu \hat{\pi}^{\mu\nu}$ and $u_\mu \hat{\tau}^{\mu\nu}$, given by

$$\begin{aligned} (\hat{X}_{\mu\nu}(x), \hat{Y}_{\rho\sigma}(x_1), \hat{Z}_{\alpha\beta}(x_2)) = & a_1 \Delta_{\mu\nu} \Delta_{\rho\alpha} \Delta_{\sigma\beta} + a_2 \Delta_{\mu\nu} \Delta_{\rho\beta} \Delta_{\sigma\alpha} + b_1 \Delta_{\rho\sigma} \Delta_{\mu\alpha} \Delta_{\nu\beta} + b_2 \Delta_{\rho\sigma} \Delta_{\mu\beta} \Delta_{\nu\alpha} \\ & + c_1 \Delta_{\alpha\beta} \Delta_{\mu\rho} \Delta_{\nu\sigma} + c_2 \Delta_{\alpha\beta} \Delta_{\mu\sigma} \Delta_{\nu\rho} + d \Delta_{\mu\nu} \Delta_{\rho\sigma} \Delta_{\alpha\beta} \\ & + e_1 \Delta_{\mu\rho} \Delta_{\nu\alpha} \Delta_{\sigma\beta} + e_2 \Delta_{\mu\rho} \Delta_{\nu\beta} \Delta_{\sigma\alpha} + e_3 \Delta_{\mu\sigma} \Delta_{\nu\alpha} \Delta_{\rho\beta} + e_4 \Delta_{\mu\sigma} \Delta_{\nu\beta} \Delta_{\rho\alpha} \\ & + e_5 \Delta_{\mu\alpha} \Delta_{\nu\rho} \Delta_{\sigma\beta} + e_6 \Delta_{\mu\alpha} \Delta_{\nu\sigma} \Delta_{\rho\beta} + e_7 \Delta_{\mu\beta} \Delta_{\nu\rho} \Delta_{\sigma\alpha} + e_8 \Delta_{\mu\beta} \Delta_{\nu\sigma} \Delta_{\rho\alpha}, \end{aligned} \quad (111)$$

where $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ can either be $\hat{\pi}$ or $\hat{\tau}$. The coefficients $a_1, a_2, b_1, b_2, c_1, c_2, d, e_1 - e_8$ are obtained by imposing the symmetric and antisymmetric properties of $\hat{\pi}_{\mu\nu}$ and $\hat{\tau}_{\mu\nu}$, respectively (see Appendix C). Thus, we find that the three-point correlation functions among different two-rank tensors can be written as

$$(\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\tau}_{\alpha\beta}(x_2)) = \frac{1}{3} [\Delta_{\rho\alpha} \Delta_{\mu\nu\sigma\beta} - \Delta_{\rho\beta} \Delta_{\mu\nu\sigma\alpha} - \Delta_{\sigma\alpha} \Delta_{\mu\nu\rho\beta} + \Delta_{\sigma\beta} \Delta_{\mu\nu\rho\alpha}] (\hat{\tau}_\lambda^\delta(x), \hat{\tau}_\delta^\eta(x_1), \hat{\tau}_\eta^\lambda(x_2)), \quad (112)$$

$$(\hat{\tau}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2)) = \frac{-1}{15} [\Delta_{\rho\alpha} \Delta_{\mu\nu\sigma\beta} + \Delta_{\rho\beta} \Delta_{\mu\nu\sigma\alpha} + \Delta_{\sigma\alpha} \Delta_{\mu\nu\rho\beta} + \Delta_{\sigma\beta} \Delta_{\mu\nu\rho\alpha}] (\hat{\tau}_\lambda^\delta(x), \hat{\pi}_\delta^\eta(x_1), \hat{\pi}_\eta^\lambda(x_2)), \quad (113)$$

$$\begin{aligned} (\hat{\tau}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\tau}_{\alpha\beta}(x_2)) = & \frac{1}{5} \left[-\Delta_{\rho\alpha} \Delta_{\mu\nu\sigma\beta} + \Delta_{\rho\beta} \Delta_{\mu\nu\sigma\alpha} - \Delta_{\sigma\alpha} \Delta_{\mu\nu\rho\beta} + \Delta_{\sigma\beta} \Delta_{\mu\nu\rho\alpha} + \frac{4}{3} \Delta_{\rho\sigma} \Delta_{\mu\nu\alpha\beta} \right] \\ & \times (\hat{\tau}_\lambda^\delta(x), \hat{\pi}_\delta^\eta(x_1), \hat{\tau}_\eta^\lambda(x_2)), \end{aligned} \quad (114)$$

$$\begin{aligned} (\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2)) = & \frac{-1}{5} \left[-\Delta_{\rho\alpha} \Delta_{\mu\nu\sigma\beta} - \Delta_{\rho\beta} \Delta_{\mu\nu\sigma\alpha} + \Delta_{\sigma\alpha} \Delta_{\mu\nu\rho\beta} + \Delta_{\sigma\beta} \Delta_{\mu\nu\rho\alpha} - \frac{4}{3} \Delta_{\alpha\beta} \Delta_{\mu\nu\rho\sigma} \right] \\ & \times (\hat{\tau}_\lambda^\delta(x), \hat{\tau}_\delta^\eta(x_1), \hat{\pi}_\eta^\lambda(x_2)). \end{aligned} \quad (115)$$

The other correlations including two second-rank tensors are given by (see Appendix C)

$$(\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{p}^*(x_2)) = \frac{1}{3} \Delta_{\mu\nu\rho\sigma} (\hat{\tau}_{\delta\eta}(x), \hat{\tau}^{\delta\eta}(x_1), \hat{p}^*(x_2)), \quad (116)$$

$$(\hat{\tau}_{\mu\nu}(x), \hat{p}^*(x_1), \hat{\tau}_{\alpha\beta}(x_2)) = \frac{1}{3} \Delta_{\mu\nu\alpha\beta} (\hat{\tau}_{\delta\eta}(x), \hat{p}^*(x_1), \hat{\tau}^{\delta\eta}(x_2)), \quad (117)$$

$$(\hat{\tau}_{\mu\nu}(x), \hat{q}_\alpha(x_1), \hat{q}_\beta(x_2)) = \frac{1}{3} \Delta_{\mu\nu\alpha\beta} (\hat{\tau}_{\delta\eta}(x), \hat{q}^\delta(x_1), \hat{q}^\eta(x_2)), \quad (118)$$

$$(\hat{\tau}_{\mu\nu}(x), \hat{j}_\alpha(x_1), \hat{j}_\beta(x_2)) = \frac{1}{3} \Delta_{\mu\nu\alpha\beta} (\hat{\tau}_{\delta\eta}(x), \hat{j}^\delta(x_1), \hat{j}^\eta(x_2)) \quad (119)$$

$$(\hat{\tau}_{\mu\nu}(x), \hat{j}_\alpha(x_1), \hat{q}_\beta(x_2)) = \frac{1}{3} \Delta_{\mu\nu\alpha\beta} (\hat{\tau}_{\delta\eta}(x), \hat{j}^\delta(x_1), \hat{q}^\eta(x_2)), \quad (120)$$

$$(\hat{\tau}_{\mu\nu}(x), \hat{q}_\alpha(x_1), \hat{j}_\beta(x_2)) = \frac{1}{3} \Delta_{\mu\nu\alpha\beta} (\hat{\tau}_{\delta\eta}(x), \hat{q}^\delta(x_1), \hat{j}^\eta(x_2)). \quad (121)$$

The symmetry property of the three-point correlation function indicates that $\mathcal{A}_{3\mu\nu}(x) = \mathcal{A}_{4\mu\nu}(x)$, $\mathcal{A}_{5\mu\nu}(x) = \mathcal{A}_{6\mu\nu}(x)$, and $\mathcal{A}_{9\mu\nu}(x) = \mathcal{A}_{10\mu\nu}(x)$. Substituting the correlation functions from (112)–(121) in (101)–(110), we obtain the second-order gradient correction from the three-point correlation function to the rotational stress tensor, given by

$$\langle \hat{\tau}_{\mu\nu}(x) \rangle_2^{(3)} = \gamma_{\tau\tau\tau} \xi_{\alpha\leftarrow\mu} \xi_{\nu\rightarrow}^\alpha + \gamma_{\tau\pi\pi} \sigma_{\alpha\leftarrow\mu} \sigma_{\nu\rightarrow}^\alpha + 2\gamma_{\tau\pi\tau} \sigma_{\alpha\leftarrow\mu} \xi_{\nu\rightarrow}^\alpha + 2\gamma_{\tau\tau\pi} \xi_{\mu\leftarrow} \theta + \gamma_{\tau q q} M_{\leftarrow\mu} M_{\nu\rightarrow} + \gamma_{\tau j j} \beta^{-1} \nabla_{\leftarrow\mu} \alpha \nabla_{\nu\rightarrow} \alpha + 2\gamma_{\tau q j} M_{\leftarrow\mu} \nabla_{\nu\rightarrow} \alpha, \quad (122)$$

where we used the shorthand notations (3) and the new transport coefficients are defined by

$$\begin{aligned}\gamma_{\tau\tau\tau} &= \frac{-4}{3}\beta^2 \int d^4x_1 d^4x_2 (\hat{\tau}_\lambda^\delta(x), \hat{\tau}_\delta^\eta(x_1), \hat{\tau}_\eta^\lambda(x_2)), \\ \gamma_{\tau\pi\pi} &= \frac{-4}{15}\beta^2 \int d^4x_1 d^4x_2 (\hat{\tau}_\lambda^\delta(x), \hat{\pi}_\delta^\eta(x_1), \hat{\pi}_\eta^\lambda(x_2)), \\ \gamma_{\tau\pi\tau} &= \frac{4}{5}\beta^2 \int d^4x_1 d^4x_2 (\hat{\tau}_\lambda^\delta(x), \hat{\pi}_\delta^\eta(x_1), \hat{\tau}_\eta^\lambda(x_2)), \\ \gamma_{\tau\tau p} &= \frac{-1}{3}\beta^2 \int d^4x_1 d^4x_2 (\hat{\tau}_{\delta\eta}(x), \hat{\tau}^{\delta\eta}(x_1), \hat{p}^*(x_2)), \\ \gamma_{\tau q q} &= \frac{1}{3}\beta^2 \int d^4x_1 d^4x_2 (\hat{\tau}_{\delta\eta}(x), \hat{q}^\delta(x_1), \hat{q}^\eta(x_2)), \\ \gamma_{\tau j j} &= \frac{1}{3}\beta \int d^4x_1 d^4x_2 (\hat{\tau}_{\delta\eta}(x), \hat{j}^\delta(x_1), \hat{j}^\eta(x_2)), \\ \gamma_{\tau q j} &= \frac{-1}{3}\beta \int d^4x_1 d^4x_2 (\hat{\tau}_{\delta\eta}(x), \hat{q}^\delta(x_1), \hat{j}^\eta(x_2)).\end{aligned}$$

4. Final expression

Using (66) and (72) for the first-order correction and (95), (96), and (122) for the second-order corrections, the complete expression for the rotational stress tensor i.e. antisymmetric counterpart of the shear stress tensor, up to second order, is given by

$$\begin{aligned}\langle \hat{\tau}_{\mu\nu}(x) \rangle &= 2\gamma\xi_{\mu\nu} + 2\gamma_\omega[\Delta_{\mu\nu\rho\sigma}D\xi^{\rho\sigma} + \theta\Gamma\xi_{\mu\nu}] + \gamma_{\tau S}\mathcal{K}_{\mu\nu}\theta \\ &+ \gamma_{\tau\tau\tau}\xi_{\alpha\leftarrow\mu}\xi_{\nu\rightarrow}^\alpha + \gamma_{\tau\pi\pi}\sigma_{\alpha\leftarrow\mu}\sigma_{\nu\rightarrow}^\alpha + 2\gamma_{\tau\pi\tau}\sigma_{\alpha\leftarrow\mu}\xi_{\nu\rightarrow}^\alpha \\ &+ 2\gamma_{\tau\tau p}\xi_{\mu\nu}\theta + \gamma_{\tau q q}M_{\leftarrow\mu}M_{\nu\rightarrow} \\ &+ \gamma_{\tau j j}\beta^{-1}\nabla_{\leftarrow\mu}\alpha\nabla_{\nu\rightarrow}\alpha + 2\gamma_{\tau q j}M_{\leftarrow\mu}\nabla_{\nu\rightarrow}\alpha.\end{aligned}\quad (123)$$

Following [62,63], the evolution equation for the rotational stress tensor can be obtained by replacing $\xi^{\rho\sigma} \sim \frac{1}{2\gamma}\tau^{\rho\sigma}$ in the term $D\xi^{\rho\sigma}$, by employing (23), the first-order Navier-Stokes relation. Thus, we obtain

$$2\gamma_\omega \Delta_{\mu\nu\rho\sigma}D\xi^{\rho\sigma} \simeq -\tau_\tau\dot{\tau}_{\mu\nu} + \tilde{\gamma}_\tau\theta\tau_{\mu\nu}, \quad (124)$$

where

$$\tau_\tau = -\gamma_\omega\gamma^{-1}, \quad (125)$$

$$\tilde{\gamma}_\tau = \tau_\tau\gamma^{-1}\beta\left(\Gamma\frac{\partial\gamma}{\partial\beta} - \delta\frac{\partial\gamma}{\partial\alpha}\right). \quad (126)$$

Therefore, the evolution equation for the rotational stress tensor is given by

$$\begin{aligned}\tau_{\mu\nu} + \tau_\tau\dot{\tau}_{\mu\nu} &= 2\gamma\xi_{\mu\nu} + \tilde{\gamma}_\tau\tau_{\mu\nu}\theta + 2\gamma_\omega\Gamma\xi_{\mu\nu}\theta + \gamma_{\tau S}\mathcal{K}_{\mu\nu}\theta \\ &+ 2\gamma_{\tau\tau p}\xi_{\mu\nu}\theta + \gamma_{\tau\tau\tau}\xi_{\alpha\leftarrow\mu}\xi_{\nu\rightarrow}^\alpha + \gamma_{\tau\pi\pi}\sigma_{\alpha\leftarrow\mu}\sigma_{\nu\rightarrow}^\alpha \\ &+ 2\gamma_{\tau\pi\tau}\sigma_{\alpha\leftarrow\mu}\xi_{\nu\rightarrow}^\alpha + \gamma_{\tau q q}M_{\leftarrow\mu}M_{\nu\rightarrow} \\ &+ \gamma_{\tau j j}\beta^{-1}\nabla_{\leftarrow\mu}\alpha\nabla_{\nu\rightarrow}\alpha + 2\gamma_{\tau q j}M_{\leftarrow\mu}\nabla_{\nu\rightarrow}\alpha.\end{aligned}\quad (127)$$

B. Boost heat vector (q_μ)

1. Nonlocal correction from two-point correlation function: $\langle \hat{q}_\mu(x) \rangle_2^{(2),NL}$

Invoking the Curie's principle for the isotropic medium and using (58) and (49), we have

$$\begin{aligned}\langle \hat{q}_\mu(x) \rangle_2^{(2),NL} &= \int d^4x_1 (\hat{q}_\mu(x), \hat{q}_\nu(x_1))\beta(x_1)M^\nu(x_1) \\ &- \langle \hat{q}_\mu(x) \rangle_1.\end{aligned}\quad (128)$$

In an isotropic medium, the correlator can be written as

$$(\hat{q}_\mu(x), \hat{q}_\nu(x_1)) = \frac{1}{3}\Delta_{\mu\nu}(\hat{q}^\sigma(x), \hat{q}_\sigma(x_1)). \quad (129)$$

Similar to the calculation of a nonlocal correction to $\langle \hat{\tau}_{\mu\nu}(x) \rangle_2$, we have to first express $\hat{q}^\mu$ in terms of the energy-momentum tensor, and then we expand all the hydrodynamic quantities around $x_1 = x$. From (37), in the Landau frame, we can write

$$\hat{q}_\sigma(x_1) = \Delta_{\sigma\alpha}(x_1)u_\beta(x_1)\hat{T}^{\alpha\beta}(x_1). \quad (130)$$

Using the Taylor expansion, we expand all the hydrodynamic quantities $\Delta_{\sigma\alpha}(x_1)$ and $u_\beta(x_1)$ (not operators) around $x_1 = x$, and keeping terms only up to first-order in gradient expansion, we have

$$\hat{q}_\sigma(x_1) = [\hat{q}_\sigma(x_1)]_x + (x_1 - x)^\tau \left[\frac{\partial \hat{q}_\sigma(x_1)}{\partial x_1^\tau} \right]_x, \quad (131)$$

where

$$[\hat{q}(x_1)]_x = \Delta_{\sigma\alpha}(x)u_\beta(x)\hat{T}^{\alpha\beta}(x), \quad (132)$$

$$\begin{aligned}\left[\frac{\partial \hat{q}_\sigma(x_1)}{\partial x_1^\tau} \right]_x &= -\hat{q}^\alpha(x_1)u_\sigma(x)\frac{\partial u_\alpha(x_1)}{\partial x_1^\tau} \Big|_x - \hat{\varepsilon}(x_1)\frac{\partial u_\sigma(x_1)}{\partial x_1^\tau} \Big|_x \\ &- \hat{p}(x_1)\frac{\partial u_\sigma(x_1)}{\partial x_1^\tau} \Big|_x + [\hat{\pi}_\sigma^\beta(x_1) \\ &+ \hat{\tau}_\sigma^\beta(x_1)]\frac{\partial u_\beta(x_1)}{\partial x_1^\tau} \Big|_x,\end{aligned}\quad (133)$$

defining $M'^\lambda = \beta M^\lambda$, using the Taylor expansion for $M'^\lambda(x_1)$, given by

$$M'^\lambda(x_1) = M'^\lambda(x) + (x_1 - x)^\tau \frac{\partial M'^\lambda(x_1)}{\partial x_1^\tau} \Big|_x + \dots \quad (134)$$

Substituting (129), (131), and (134) in (128) and, collecting all the terms up to second-order in gradient expansion, we obtain

$$\begin{aligned} \langle \hat{q}_\mu(x) \rangle_2^{(2),NL} &= \frac{1}{3} \Delta_{\mu\lambda}(x) \int d^4x_1 [(\hat{q}^\sigma(x), [\hat{q}_\sigma(x_1)]_x) M'^\lambda(x) \\ &\quad + (\hat{q}^\sigma(x), [\hat{q}_\sigma(x_1)]_x) (x_1 - x)^\tau \frac{\partial M'^\lambda(x_1)}{\partial x_1^\tau} \Big|_x \\ &\quad + \left(\hat{q}^\sigma(x), \left[\frac{\partial q_\sigma(x_1)}{\partial x_1^\tau} \right]_x \right) (x_1 - x)^\tau M'^\lambda(x) \\ &\quad - \langle \hat{q}_\mu(x) \rangle_1. \end{aligned} \quad (135)$$

With the help of (67), the first term in the bracket cancels $\langle \hat{q}_\mu(x) \rangle_1$. Thus, we obtain

$$\begin{aligned} \langle \hat{q}_\mu(x) \rangle_2^{(2),NL} &= \frac{1}{3} \Delta_{\mu\lambda}(x) \partial_\tau M'^\lambda(x) \int d^4x_1 (\hat{q}^\sigma(x), [\hat{q}_\sigma(x_1)]_x) (x_1 - x)^\tau \\ &\quad + \frac{1}{3} M'_\mu(x) \int d^4x_1 \left(\hat{q}^\sigma(x), \left[\frac{\partial q_\sigma(x_1)}{\partial x_1^\tau} \right]_x \right) (x_1 - x)^\tau. \end{aligned} \quad (136)$$

Observing (133), we can conclude that the second term in (136) vanishes, on account of Curie's principle and the orthogonal property $\hat{q}^\sigma u_\sigma = 0$. Thus, (136) can be simplified to

$$\langle \hat{q}_\mu(x) \rangle_2^{(2),NL} = [\beta(x)]^{-1} \Delta_{\mu\lambda}(x) R^\tau(x) \partial_\tau M'^\lambda(x), \quad (137)$$

where we have approximated $[\hat{q}(x_1)]_x \simeq \hat{q}(x_1)$, because (137) already contains a second-order term. We also defined

$$R^\tau(x) = \frac{\beta(x)}{3} \int d^4x_1 (\hat{q}^\sigma(x), \hat{q}_\sigma(x_1)) (x_1 - x)^\tau, \quad (138)$$

which can be simplified in the following form [see Eq. (B22) in Appendix B]

$$R^\tau = \lambda_\omega u^\tau, \quad (139)$$

where

$$\lambda_\omega = i \lim_{\omega \rightarrow 0} \frac{d}{d\omega} \lambda_{\hat{q}\hat{q}}(\omega). \quad (140)$$

Here $\lambda_{\hat{q}\hat{q}}(\omega)$ is defined by (see Appendix B)

$$\lambda_{\hat{q}\hat{q}}(\omega) = \frac{\beta}{3} \int d^4x_1 e^{i\omega(t-t_1)} (\hat{q}^\sigma(x), \hat{q}_\sigma(x_1)). \quad (141)$$

Finally, solving (137) using (139), and using the first-order approximation $D\beta \simeq \Gamma\beta\theta$, we obtain

$$\langle \hat{q}_\mu(x) \rangle_2^{(2),NL} = \lambda_\omega [M_\mu \Gamma \theta + \Delta_{\mu\lambda} D M^\lambda]. \quad (142)$$

2. Local correction from two-point correlation function using extended thermodynamic forces: $\langle \hat{q}_\mu(x) \rangle_2^{(2),ET}$

From (59), we can write

$$\langle \hat{q}_\mu(x) \rangle_2^{(2),ET} = \int d^4x_1 (\hat{q}_\mu(x), \hat{C}_S(x_1))|_{\text{local}}, \quad (143)$$

where $\hat{C}_S$ is given by (50). $\hat{C}_S$ has only one vector term, then by the help of Curie's principle, and expanding all hydrodynamic quantities around $x_1 = x$, we can extract the local terms in (143), given by

$$\langle \hat{q}_\mu(x) \rangle_2^{(2),ET} = -\lambda(x) \mathcal{Q}_\mu(x), \quad (144)$$

where

$$\lambda(x) = \frac{-\beta}{3} \int d^4x_1 (\hat{q}^a(x), \hat{q}_a(x_1)). \quad (145)$$

3. Local correction from three-point correlation function: $\langle \hat{q}_\mu(x) \rangle_2^{(3)}$

From (60), we can write

$$\langle \hat{q}_\mu(x) \rangle_2^{(3)} = \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{C}_F(x_1), \hat{C}_F(x_2))|_{\text{local}}, \quad (146)$$

where $\hat{C}_F$ is given by (49). Substituting (49) in (146) and employing Curie's principle, the nonvanishing correlations are given by

$$\mathcal{B}_{1\mu}(x) = -\beta^2(x) M^\nu(x) \theta(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{q}_\nu(x_1), \hat{p}^*(x_2)), \quad (147)$$

$$\mathcal{B}_{2\mu}(x) = -\beta^2(x) M^\nu(x) \theta(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{p}^*(x_1), \hat{q}_\nu(x_2)), \quad (148)$$

$$\mathcal{B}_{3\mu}(x) = \beta^2(x) M^\nu(x) \xi^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{q}_\nu(x_1), \hat{\tau}_{\alpha\beta}(x_2)), \quad (149)$$

$$\mathcal{B}_{4\mu}(x) = \beta^2(x) M^\nu(x) \xi^{\rho\sigma}(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{q}_\nu(x_2)), \quad (150)$$

$$\mathcal{B}_{5\mu}(x) = \beta^2(x) M^\nu(x) \sigma^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{q}_\nu(x_1), \hat{\pi}_{\alpha\beta}(x_2)), \quad (151)$$

$$\mathcal{B}_{6\mu}(x) = \beta^2(x) M^\nu(x) \sigma^{\rho\sigma}(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{q}_\nu(x_2)), \quad (152)$$

$$\mathcal{B}_{7\mu}(x) = -\beta(x) \nabla^\nu \alpha(x) \xi^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{j}_\nu(x_1), \hat{\tau}_{\alpha\beta}(x_2)), \quad (153)$$

$$\mathcal{B}_{8\mu}(x) = -\beta(x) \nabla^\nu \alpha(x) \xi^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{\tau}_{\alpha\beta}(x_1), \hat{j}_\nu(x_2)), \quad (154)$$

$$\mathcal{B}_{9\mu}(x) = -\beta(x) \nabla^\nu \alpha(x) \sigma^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{j}_\nu(x_1), \hat{\pi}_{\alpha\beta}(x_2)), \quad (155)$$

$$\mathcal{B}_{10\mu}(x) = -\beta(x) \nabla^\nu \alpha(x) \sigma^{\alpha\beta}(x) \int d^4x_1 d^4x_2 (\hat{q}_\mu(x), \hat{\pi}_{\alpha\beta}(x_1), \hat{j}_\nu(x_2)). \quad (156)$$

Again the symmetric property of the three-point function leads to $\mathcal{B}_{1\mu}(x) = \mathcal{B}_{2\mu}(x)$, $\mathcal{B}_{3\mu}(x) = \mathcal{B}_{4\mu}(x)$, $\mathcal{B}_{5\mu}(x) = \mathcal{B}_{6\mu}(x)$, $\mathcal{B}_{7\mu}(x) = \mathcal{B}_{8\mu}(x)$, and $\mathcal{B}_{9\mu}(x) = \mathcal{B}_{10\mu}(x)$. In the isotopic medium, the three-point correlation functions can be written as

$$(\hat{q}_\mu(x), \hat{q}_\nu(x_1), \hat{p}^*(x_2)) = \frac{1}{3} \Delta_{\mu\nu}(\hat{q}_\lambda(x), \hat{q}^\lambda(x_1), \hat{p}^*(x_2)), \quad (157)$$

$$(\hat{q}_\mu(x), \hat{q}_\nu(x_1), \hat{\tau}_{\alpha\beta}(x_2)) = \frac{1}{3} \Delta_{\mu\nu\alpha\beta}(\hat{q}^\lambda(x), \hat{q}^\delta(x_1), \hat{\tau}_{\lambda\delta}(x_2)), \quad (158)$$

$$(\hat{q}_\mu(x), \hat{q}_\nu(x_1), \hat{\pi}_{\alpha\beta}(x_2)) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta}(\hat{q}^\lambda(x), \hat{q}^\delta(x_1), \hat{\pi}_{\lambda\delta}(x_2)), \quad (159)$$

$$(\hat{q}_\mu(x), \hat{j}_\nu(x_1), \hat{\tau}_{\alpha\beta}(x_2)) = \frac{1}{3} \Delta_{\mu\nu\alpha\beta}(\hat{q}^\lambda(x), \hat{j}^\delta(x_1), \hat{\tau}_{\lambda\delta}(x_2)), \quad (160)$$

$$(\hat{q}_\mu(x), \hat{j}_\nu(x_1), \hat{\pi}_{\alpha\beta}(x_2)) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta}(\hat{q}^\lambda(x), \hat{j}^\delta(x_1), \hat{\pi}_{\lambda\delta}(x_2)). \quad (161)$$

Substituting (157)–(161) in (147)–(156), we get the final expression for the second-order correction from the three-point correlation function in $\langle \hat{q}_\mu \rangle$, given by

$$\langle \hat{q}_\mu \rangle_2^{(3)} = \lambda_{qqp} M_\mu \theta + \lambda_{qq\tau} \xi_{\mu\nu} M^\nu + \lambda_{qq\pi} \sigma_{\mu\nu} M^\nu + \lambda_{qj\tau} \xi_{\mu\nu} \nabla^\nu \alpha + \lambda_{qj\pi} \sigma_{\mu\nu} \nabla^\nu \alpha, \quad (162)$$

where the new transport coefficients are defined by

$$\lambda_{qqp} = \frac{-2}{3} \beta^2 \int d^4x_1 d^4x_2 (\hat{q}_\lambda(x), \hat{q}^\lambda(x_1), \hat{p}^*(x_2)), \quad (163)$$

$$\lambda_{qq\tau} = \frac{2}{3} \beta^2 \int d^4x_1 d^4x_2 (\hat{q}^\lambda(x), \hat{q}^\delta(x_1), \hat{\tau}_{\lambda\delta}(x_2)), \quad (164)$$

$$\lambda_{qq\pi} = \frac{2}{5} \beta^2 \int d^4x_1 d^4x_2 (\hat{q}^\lambda(x), \hat{q}^\delta(x_1), \hat{\pi}_{\lambda\delta}(x_2)), \quad (165)$$

$$\lambda_{qj\tau} = -\frac{2}{3}\beta \int d^4x_1 d^4x_2 (\hat{q}^\lambda(x), \hat{j}^\delta(x_1), \hat{\tau}_{\lambda\delta}(x_2)), \quad (166)$$

$$\lambda_{qj\pi} = -\frac{2}{5}\beta \int d^4x_1 d^4x_2 (\hat{q}^\lambda(x), \hat{j}^\delta(x_1), \hat{\pi}_{\lambda\delta}(x_2)). \quad (167)$$

4. Final expression

Combining (67), (73), (142), (144), and (162), the complete second-order expression for boost heat vector, i.e., the antisymmetric counterpart of the heat flow vector, is given by

$$\langle \hat{q}_\mu \rangle = -\lambda M_\mu + \lambda_\omega [M_\mu \Gamma \theta + \Delta_{\mu\lambda} D M^\lambda] - \lambda \mathcal{Q}_\mu + \lambda_{qqp} M_\mu \theta + \lambda_{qq\tau} \xi_{\mu\nu} M^\nu + \lambda_{qq\pi} \sigma_{\mu\nu} M^\nu + \lambda_{qj\tau} \xi_{\mu\nu} \nabla^\nu \alpha + \lambda_{qj\pi} \sigma_{\mu\nu} \nabla^\nu \alpha. \quad (168)$$

Approximating, $M^\lambda \sim \lambda^{-1} q^\lambda$, by using the first-order Navier-Stokes relation (23), in the term $D M^\lambda$. We get the evolution equation for the boost heat vector,

$$q_\mu + \tau_q \dot{q}_\mu = -\lambda M_\mu + \tilde{\lambda}_\tau \theta q_\mu + \lambda_\omega \Gamma M_\mu \theta - \lambda \mathcal{Q}_\mu + \lambda_{qqp} M_\mu \theta + \lambda_{qq\tau} \xi_{\mu\nu} M^\nu + \lambda_{qq\pi} \sigma_{\mu\nu} M^\nu + \lambda_{qj\tau} \xi_{\mu\nu} \nabla^\nu \alpha + \lambda_{qj\pi} \sigma_{\mu\nu} \nabla^\nu \alpha, \quad (169)$$

where

$$\tau_q = \lambda_\omega \lambda^{-1} \quad (170)$$

$$\tilde{\lambda}_\tau = \tau_q \lambda^{-1} \beta \left(\Gamma \frac{\partial \lambda}{\partial \beta} - \delta \frac{\partial \lambda}{\partial \alpha} \right). \quad (171)$$

C. Shear stress tensor ($\pi_{\mu\nu}$)

1. Nonlocal correction from two-point correlation function: $\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(2),NL}$

Invoking Curie's principle and using (58) with (49), we can write

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(2),NL} = \int d^4x_1 (\hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\alpha\beta}(x_1)) \beta(x_1) \sigma^{\alpha\beta}(x_1) - \langle \hat{\pi}_{\mu\nu}(x) \rangle_1. \quad (172)$$

In this case, the nonlocal generalization of the two-point correlation function is given by

$$(\hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\alpha\beta}(x_1)) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta}(x, x_1) (\hat{\pi}^{\lambda\eta}(x), \hat{\pi}_{\lambda\eta}(x_1)). \quad (173)$$

Next, we write $\hat{\pi}^{\lambda\eta}$ and $\hat{\pi}_{\lambda\eta}$ in terms of energy-momentum tensor. Using (37), we have

$$\hat{\pi}^{\lambda\eta}(x) = \Delta_{\gamma\delta}^{\lambda\eta}(x) \hat{T}^{\gamma\delta}(x), \quad (174)$$

$$\hat{\pi}_{\lambda\eta}(x_1) = \Delta_{\lambda\eta\alpha\beta}(x_1) \hat{T}^{\alpha\beta}(x_1). \quad (175)$$

Substituting (173) with (174) and (175) in (172), we obtain

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(2),NL} = \frac{1}{5} \Delta_{\mu\nu\rho\sigma}(x) \int d^4x_1 (\hat{T}^{\gamma\delta}(x), \hat{T}^{\alpha\beta}(x_1)) \Delta_{\gamma\delta\alpha\beta}(x, x_1) \beta(x_1) \sigma^{\rho\sigma}(x_1) - \langle \hat{\pi}_{\mu\nu}(x) \rangle_1. \quad (176)$$

The procedure for the calculation of nonlocal correction from two-point correlation function to $\langle \hat{\pi}_{\mu\nu}(x) \rangle_2$ is similar to that of $\langle \hat{\tau}_{\mu\nu}(x) \rangle_2$, with the antisymmetric fourth-rank projector replaced by the symmetric fourth-rank projector. Hence, as discussed in Sec. VA 1, expanding all the hydrodynamic quantities around $x_1 = x$ in (176), then collecting all the second-order terms, we obtain

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(2),NL} = 2 \Delta_{\mu\nu\rho\sigma}(x) [\beta(s)]^{-1} K^\lambda(x) \frac{\partial}{\partial x_1^\lambda} [\beta(x_1) \sigma^{\rho\sigma}(x_1)]|_{x_1=x}. \quad (177)$$

Here we defined

$$K^\lambda(x) = \frac{\beta(x)}{10} \int d^4x_1 (\hat{\pi}_{\alpha\beta}(x), \hat{\pi}^{\alpha\beta}(x_1)) (x_1 - x)^\lambda, \quad (178)$$

which can be written as [see Eq. (B22) in Appendix B]

$$K^\lambda = \eta_\omega u^\lambda, \quad (179)$$

where

$$\eta_\omega = i \lim_{\omega \rightarrow 0} \frac{d}{d\omega} \eta_{\hat{\pi}\hat{\pi}}(\omega). \quad (180)$$

The frequency-dependent transport coefficient $\eta_{\hat{\pi}\hat{\pi}}(\omega)$ is given by (see Appendix B)

$$\eta_{\hat{\pi}\hat{\pi}}(\omega) = \frac{\beta}{10} \int d^4x_1 e^{i\omega(t-t_1)} (\hat{\pi}^{\alpha\beta}(x), \hat{\pi}_{\alpha\beta}(x_1)). \quad (181)$$

Solving (177) with (179) and using $D\beta \simeq \Gamma\beta\theta$, we obtain

$$\langle \hat{\pi}_{\mu\nu} \rangle_2^{(2),NL} = 2\eta_\omega [\Delta_{\mu\nu\rho\sigma} D\sigma^{\rho\sigma} + \theta\Gamma\sigma_{\mu\nu}]. \quad (182)$$

2. Local correction from two-point correlation function using extended thermodynamic forces: $\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(2),ET}$

With the help of (59), we can write

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(2),ET} = \int d^4x_1 (\hat{\pi}_{\mu\nu}(x), \hat{C}_S(x_1))|_{\text{local}}, \quad (183)$$

where $\hat{C}_S$ is given by (50). Since (50) does not contain any term involving a symmetric second-rank operator, all the correlations vanish by employing Curie's principle. Therefore,

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(2),ET} = 0. \quad (184)$$

3. Local correction from three-point correlation function: $\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(3)}$

From (60), we can write

$$\langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(3)} = \int d^4x_1 d^4x_2 (\hat{\pi}_{\mu\nu}(x), \hat{C}_F(x_1), \hat{C}_F(x_2))|_{\text{local}}, \quad (185)$$

where $\hat{C}_F$ is given by (49). Here, we follow the procedure given in Sec. VA 3. Substituting (49) in (185), the non-vanishing correlations, involving three second-rank tensor operators are given by (see Appendix C)

$$(\hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2)) = \frac{1}{35} [3(\Delta_{\rho\alpha}\Delta_{\mu\nu\sigma\beta} + \Delta_{\rho\beta}\Delta_{\mu\nu\sigma\alpha} + \Delta_{\sigma\alpha}\Delta_{\mu\nu\rho\beta} + \Delta_{\sigma\beta}\Delta_{\mu\nu\rho\alpha}) - 4(\Delta_{\rho\sigma}\Delta_{\mu\nu\alpha\beta} + \Delta_{\alpha\beta}\Delta_{\mu\nu\rho\sigma})](\hat{\pi}_\lambda^\delta(x), \hat{\pi}_\delta^\eta(x_1), \hat{\pi}_\eta^\lambda(x_2)), \quad (186)$$

$$(\hat{\pi}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\tau}_{\alpha\beta}(x_2)) = \frac{-1}{5} [\Delta_{\rho\alpha}\Delta_{\mu\nu\sigma\beta} - \Delta_{\rho\beta}\Delta_{\mu\nu\sigma\alpha} - \Delta_{\sigma\alpha}\Delta_{\mu\nu\rho\beta} + \Delta_{\sigma\beta}\Delta_{\mu\nu\rho\alpha}](\hat{\pi}_\lambda^\delta(x), \hat{\tau}_\delta^\eta(x_1), \hat{\tau}_\eta^\lambda(x_2)), \quad (187)$$

$$(\hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{\tau}_{\alpha\beta}(x_2)) = \frac{-1}{15} [-\Delta_{\rho\alpha}\Delta_{\mu\nu\sigma\beta} + \Delta_{\rho\beta}\Delta_{\mu\nu\sigma\alpha} - \Delta_{\sigma\alpha}\Delta_{\mu\nu\rho\beta} + \Delta_{\sigma\beta}\Delta_{\mu\nu\rho\alpha}](\hat{\pi}_\lambda^\delta(x), \hat{\pi}_\delta^\eta(x_1), \hat{\tau}_\eta^\lambda(x_2)), \quad (188)$$

$$(\hat{\pi}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\pi}_{\alpha\beta}(x_2)) = \frac{1}{15} [-\Delta_{\rho\alpha}\Delta_{\mu\nu\sigma\beta} - \Delta_{\rho\beta}\Delta_{\mu\nu\sigma\alpha} + \Delta_{\sigma\alpha}\Delta_{\mu\nu\rho\beta} + \Delta_{\sigma\beta}\Delta_{\mu\nu\rho\alpha}](\hat{\pi}_\lambda^\delta(x), \hat{\tau}_\delta^\eta(x_1), \hat{\pi}_\eta^\lambda(x_2)), \quad (189)$$

whereas, other nonvanishing correlations are given by

$$(\hat{\pi}_{\mu\nu}(x), \hat{\pi}_{\rho\sigma}(x_1), \hat{p}^*(x_2)) = \frac{1}{5} \Delta_{\mu\nu\rho\sigma} (\hat{\pi}_{\delta\eta}(x), \hat{\pi}^{\delta\eta}(x_1), \hat{p}^*(x_2)), \quad (190)$$

$$(\hat{\pi}_{\mu\nu}(x), \hat{p}^*(x_1), \hat{\pi}_{\alpha\beta}(x_2)) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta} (\hat{\pi}_{\delta\eta}(x), \hat{p}^*(x_1), \hat{\pi}^{\delta\eta}(x_2)), \quad (191)$$

$$(\hat{\pi}_{\mu\nu}(x), \hat{q}_\alpha(x_1), \hat{q}_\beta(x_2)) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta} (\hat{\pi}_{\delta\eta}(x), \hat{q}^\delta(x_1), \hat{q}^\eta(x_2)), \quad (192)$$

$$(\hat{\pi}_{\mu\nu}(x), \hat{j}_\alpha(x_1), \hat{j}_\beta(x_2)) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta} (\hat{\pi}_{\delta\eta}(x), \hat{j}^\delta(x_1), \hat{j}^\eta(x_2)), \quad (193)$$

$$(\hat{\pi}_{\mu\nu}(x), \hat{j}_\alpha(x_1), \hat{q}_\beta(x_2)) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta} (\hat{\pi}_{\delta\eta}(x), \hat{j}^\delta(x_1), \hat{q}^\eta(x_2)), \quad (194)$$

$$(\hat{\pi}_{\mu\nu}(x), \hat{q}_\alpha(x_1), \hat{j}_\beta(x_2)) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta} (\hat{\pi}_{\delta\eta}(x), \hat{q}^\delta(x_1), \hat{j}^\eta(x_2)). \quad (195)$$

Finally, with the help of correlation functions given in (186)–(195), approximating all the thermodynamics forces at x , we obtain

$$\begin{aligned} \langle \hat{\pi}_{\mu\nu}(x) \rangle_2^{(3)} = & \eta_{\pi\pi\pi} \sigma_{\alpha\langle\mu} \sigma_{\nu\rangle}^\alpha + \eta_{\pi\tau\tau} \xi_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha + 2\eta_{\pi\pi\tau} \sigma_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha + 2\eta_{\pi\pi p} \sigma_{\mu\nu} \theta \\ & + \eta_{\pi q q} M_{\langle\mu} M_{\nu\rangle} + \eta_{\pi j j} \beta^{-1} \nabla_{\langle\mu} \alpha \nabla_{\nu\rangle} \alpha + 2\eta_{\pi q j} M_{\langle\mu} \nabla_{\nu\rangle} \alpha, \end{aligned} \quad (196)$$

where the various transport coefficients are defined by

$$\begin{aligned} \eta_{\pi\pi\pi} &= \frac{12}{35} \beta^2 \int d^4x_1 d^4x_2 (\hat{\pi}_\lambda^\delta(x), \hat{\pi}_\delta^\eta(x_1), \hat{\pi}_\eta^\lambda(x_2)), \\ \eta_{\pi\tau\tau} &= \frac{4}{5} \beta^2 \int d^4x_1 d^4x_2 (\hat{\pi}_\lambda^\delta(x), \hat{\tau}_\delta^\eta(x_1), \hat{\tau}_\eta^\lambda(x_2)), \\ \eta_{\pi\pi\tau} &= \frac{-4}{15} \beta^2 \int d^4x_1 d^4x_2 (\hat{\pi}_\lambda^\delta(x), \hat{\pi}_\delta^\eta(x_1), \hat{\tau}_\eta^\lambda(x_2)), \\ \eta_{\pi\pi p} &= \frac{-1}{5} \beta^2 \int d^4x_1 d^4x_2 (\hat{\pi}_{\delta\eta}(x), \hat{\pi}^{\delta\eta}(x_1), \hat{p}^*(x_2)), \\ \eta_{\pi q q} &= \frac{1}{5} \beta^2 \int d^4x_1 d^4x_2 (\hat{\pi}_{\delta\eta}(x), \hat{q}^\delta(x_1), \hat{q}^\eta(x_2)), \\ \eta_{\pi j j} &= \frac{1}{5} \beta \int d^4x_1 d^4x_2 (\hat{\pi}_{\delta\eta}(x), \hat{j}^\delta(x_1), \hat{j}^\eta(x_2)), \\ \eta_{\pi q j} &= \frac{-1}{5} \beta \int d^4x_1 d^4x_2 (\hat{\pi}_{\delta\eta}(x), \hat{q}^\delta(x_1), \hat{j}^\eta(x_2)). \end{aligned}$$

4. Final expression

Combining (68) and (74) with (182), (184), and (196), we get the full expression of the shear stress tensor up to second order in gradient expansion, given by

$$\begin{aligned} \langle \hat{\pi}_{\mu\nu}(x) \rangle = & 2\eta\sigma_{\mu\nu} + 2\eta_\omega [\Delta_{\mu\nu\rho\sigma} D\sigma^{\rho\sigma} + \theta\Gamma\sigma_{\mu\nu}] \\ & + \eta_{\pi\pi\pi} \sigma_{\alpha\langle\mu} \sigma_{\nu\rangle}^\alpha + \eta_{\pi\tau\tau} \xi_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha + 2\eta_{\pi\pi\tau} \sigma_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha \\ & + 2\eta_{\pi\pi p} \sigma_{\mu\nu} \theta + \eta_{\pi q q} M_{\langle\mu} M_{\nu\rangle} + \eta_{\pi j j} \beta^{-1} \nabla_{\langle\mu} \alpha \nabla_{\nu\rangle} \alpha \\ & + 2\eta_{\pi q j} M_{\langle\mu} \nabla_{\nu\rangle} \alpha. \end{aligned} \quad (197)$$

Using (23), and approximating $\sigma^{\rho\sigma} \sim \frac{\pi^{\rho\sigma}}{2\eta}$ in the term $D\sigma^{\rho\sigma}$, we get the evolution equation for the shear stress tensor, given by

$$\begin{aligned} \pi_{\mu\nu} + \tau_\pi \dot{\pi}_{\mu\nu} = & 2\eta\sigma_{\mu\nu} + \tilde{\eta}_\tau \theta \pi_{\mu\nu} + 2\eta_\omega \Gamma \sigma_{\mu\nu} \theta + 2\eta_{\pi\pi p} \sigma_{\mu\nu} \theta \\ & + \eta_{\pi\pi\pi} \sigma_{\alpha\langle\mu} \sigma_{\nu\rangle}^\alpha + \eta_{\pi\tau\tau} \xi_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha + 2\eta_{\pi\pi\tau} \sigma_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha \\ & + \eta_{\pi q q} M_{\langle\mu} M_{\nu\rangle} + \eta_{\pi j j} \beta^{-1} \nabla_{\langle\mu} \alpha \nabla_{\nu\rangle} \alpha \\ & + 2\eta_{\pi q j} M_{\langle\mu} \nabla_{\nu\rangle} \alpha, \end{aligned} \quad (198)$$

where

$$\tau_\pi = -\eta_\omega \eta^{-1}, \quad (199)$$

$$\tilde{\eta}_\tau = \tau_\pi \eta^{-1} \beta \left(\Gamma \frac{\partial \eta}{\partial \beta} - \delta \frac{\partial \eta}{\partial \alpha} \right). \quad (200)$$

D. Bulk viscous pressure

From Appendix D Eq. (D4), we have

$$\begin{aligned} \Pi = & \langle \hat{p}^* \rangle_1 + \langle \hat{p}^* \rangle_2 + \frac{1}{2} (\langle \hat{\varepsilon} \rangle_1)^2 \frac{\partial^2 p}{\partial \varepsilon^2} + \frac{1}{2} (\langle \hat{n} \rangle_1)^2 \frac{\partial^2 p}{\partial n^2} \\ & + \langle \hat{\varepsilon} \rangle_1 \langle \hat{n} \rangle_1 \frac{\partial^2 p}{\partial n \partial \varepsilon} - \mathcal{K}_{\mu\nu} \langle \hat{S}^{\mu\nu} \rangle_1. \end{aligned} \quad (201)$$

The first-order corrections to energy density, particle density, and spin density are given by

$$\langle \hat{\varepsilon} \rangle_1 = -\zeta_{\varepsilon p} \theta, \quad (202)$$

$$\langle \hat{n} \rangle_1 = -\zeta_{np} \theta, \quad (203)$$

$$\langle \hat{S}^{\mu\nu} \rangle_1 = \zeta_{s\tau} \xi^{\mu\nu}. \quad (204)$$

Here we used (58) and (49) and applied the Curie's principle to simplify Eqs. (202)–(204). The transport coefficients $\zeta_{\varepsilon p}$, ζ_{np} , and $\zeta_{s\tau}$ are defined by

$$\zeta_{\varepsilon p} = \beta \int d^4x_1 (\hat{\varepsilon}(x), \hat{p}^*(x_1)), \quad (205)$$

$$\zeta_{np} = \beta \int d^4x_1 (\hat{n}(x), \hat{p}^*(x_1)), \quad (206)$$

$$\zeta_{s\tau} = \frac{1}{3} \beta \int d^4x_1 (\hat{S}^{\lambda\sigma}(x), \hat{\tau}_{\lambda\sigma}(x_1)). \quad (207)$$

Using (202), (203), and (204), we can rewrite (201) as

$$\begin{aligned} \Pi = & -\zeta\theta + \left[\frac{1}{2} \zeta_{\varepsilon p}^2 \frac{\partial^2 p}{\partial \varepsilon^2} + \frac{1}{2} \zeta_{np}^2 \frac{\partial^2 p}{\partial n^2} + \zeta_{\varepsilon p} \zeta_{np} \frac{\partial^2 p}{\partial n \partial \varepsilon} \right] \theta^2 \\ & - \zeta_{s\tau} \mathcal{K}_{\mu\nu} \xi^{\mu\nu} + \langle \hat{p}^* \rangle_2, \end{aligned} \quad (208)$$

where we used $\langle \hat{p}^* \rangle_1 = -\zeta\theta$, by employing (75) and (71). To complete the calculation of bulk viscous pressure, we need to evaluate $\langle \hat{p}^* \rangle_2$.

1. Nonlocal correction from the two-point correlation function: $\langle \hat{p}^*(x) \rangle_2^{(2),NL}$

Invoking Curie's principle and using (58) and (49), we have

$$\langle \hat{p}^*(x) \rangle_2^{(2),NL} = - \int d^4x_1 (\hat{p}^*(x), \hat{p}^*(x_1)) \beta(x_1) \theta(x_1) - \langle \hat{p}^*(x) \rangle_1. \quad (209)$$

As we have calculated the other nonlocal correction terms, we write $\hat{p}^*$ in terms of the energy-momentum tensor and the particle current vector, then proceed by expanding all the hydrodynamic quantities around $x_1 = x$. From (46) and (37), we have

$$\hat{p}^*(x_1) = -\frac{1}{3} \Delta_{\mu\nu}(x_1) \hat{T}^{\mu\nu}(x_1) - \Gamma(x_1) u_\mu(x_1) u_\nu(x_1) \hat{T}^{\mu\nu}(x_1) - \delta(x_1) u_\mu(x_1) \hat{N}^\mu(x_1). \quad (210)$$

Expanding all the hydrodynamic quantities (not operators) around $x_1 = x$, and neglecting the second-order and higher-order terms, we have

$$\hat{p}^*(x_1) = [\hat{p}^*(x_1)]_x + (x_1 - x)^\lambda \left[\frac{\partial \hat{p}^*(x_1)}{\partial x_1^\lambda} \right]_x, \quad (211)$$

where

$$[\hat{p}^*(x_1)]_x = -\frac{1}{3} \Delta_{\mu\nu}(x) \hat{T}^{\mu\nu}(x_1) - \Gamma(x) u_\mu(x) u_\nu(x) \hat{T}^{\mu\nu}(x_1) - \delta(x) u_\mu(x) \hat{N}^\mu(x_1), \quad (212)$$

$$\left[\frac{\partial \hat{p}^*(x_1)}{\partial x_1^\lambda} \right]_x = -\hat{j}^\mu(x_1) \delta(x) \frac{\partial u_\mu(x_1)}{\partial x_1^\lambda} \Big|_x - \hat{\varepsilon}(x_1) \frac{\partial \Gamma(x_1)}{\partial x_1^\lambda} \Big|_x - \hat{n}(x_1) \frac{\partial \delta(x_1)}{\partial x_1^\lambda} \Big|_x. \quad (213)$$

Since we are working in Landau frame, we have set $\hat{h}^\mu = 0$ while calculating (211). We can see that there is no contribution from the antisymmetric counterpart of heat flow vector, i.e. boost heat vector $\hat{q}^\mu$. Substituting (211) in (209) and expanding $\beta\theta$ around $x_1 = x$, we get

$$\begin{aligned} \langle \hat{p}^*(x) \rangle_2^{(2),NL} &= - \int d^4x_1 (\hat{p}^*(x), [\hat{p}^*(x_1)]_x + (x_1 - x)^\lambda \left[\frac{\partial \hat{p}^*(x_1)}{\partial x_1^\lambda} \right]_x) \left[\beta(x) \theta(x) + (x_1 - x)^\lambda \frac{\partial(\beta\theta)}{\partial x_1^\lambda} \Big|_x \right] - \langle \hat{p}^*(x) \rangle_1 \\ &= - \frac{\partial(\beta\theta)}{\partial x_1^\lambda} \Big|_x \int d^4x_1 (\hat{p}^*(x), [\hat{p}^*(x_1)]_x) (x_1 - x)^\lambda - \beta(x) \theta(x) \int d^4x_1 \left(\hat{p}^*(x), \left[\frac{\partial \hat{p}^*(x_1)}{\partial x_1^\lambda} \right]_x \right) (x_1 - x)^\lambda, \end{aligned} \quad (214)$$

where the leading-order term cancels $\langle \hat{p}^*(x) \rangle_1$ using (71). Further, we approximate the first term in (214) with $[\hat{p}^*(x_1)]_x = \hat{p}^*(x_1)$, because it already has a second-order term. Also, we use (213) to solve the second term in (214). Thus, we obtain

$$\langle \hat{p}^*(x) \rangle_2^{(2),NL} = -\partial_\lambda(\beta\theta) \beta^{-1} K_p^\lambda + \theta(\partial_\lambda \Gamma) K_\varepsilon^\lambda + \theta(\partial_\lambda \delta) K_n^\lambda, \quad (215)$$

where, we have omitted the argument x from all quantities. Also, we defined

$$K_p^\lambda = \beta(x) \int d^4x_1 (\hat{p}^*(x), \hat{p}^*(x_1)) (x_1 - x)^\lambda = \zeta_{pp}^\omega u^\lambda, \quad (216)$$

$$K_\varepsilon^\lambda = \beta(x) \int d^4x_1 (\hat{p}^*(x), \hat{\varepsilon}(x_1)) (x_1 - x)^\lambda = \zeta_{p\varepsilon}^\omega u^\lambda, \quad (217)$$

$$K_n^\lambda = \beta(x) \int d^4x_1 (\hat{p}^*(x), \hat{n}(x_1)) (x_1 - x)^\lambda = \zeta_{pn}^\omega u^\lambda, \quad (218)$$

where ζ_{pp}^ω , $\zeta_{p\varepsilon}^\omega$, and, ζ_{pn}^ω are related to the frequency-dependent transport coefficients [see Eq. (B22) in Appendix B]. Substituting, (216), (217), and (218) in (215), we find

$$\langle \hat{p}^* \rangle_2^{(2),NL} = -D(\beta\theta)\beta^{-1}\zeta_{pp}^\omega + \theta(D\Gamma)\zeta_{pe}^\omega + \theta(D\delta)\zeta_{pn}^\omega. \quad (219)$$

We have three independent thermodynamic variables, hence, we can calculate the comoving derivatives, $D\Gamma$ and $D\delta$, in a similar way as we calculated $D\beta$ in (39). Further, we use the value of $D\epsilon$, Dn , and, $DS^{\alpha\beta}$ from the equations of motion, up to the first order in gradient. Thus, we obtain

$$\langle \hat{p}^* \rangle_2^{(2),NL} = -\zeta_{pp}^\omega D\theta - \theta^2[\Gamma\zeta_{pp}^\omega + \tilde{\Gamma}\zeta_{pe}^\omega + \tilde{\delta}\zeta_{pn}^\omega], \quad (220)$$

where $\tilde{\Gamma}$ and $\tilde{\delta}$ are defined as

$$\begin{aligned} \tilde{\Gamma} &= h \frac{\partial \Gamma}{\partial \epsilon} + n \frac{\partial \Gamma}{\partial n} + S^{\alpha\beta} \frac{\partial \Gamma}{\partial S^{\alpha\beta}} \quad \text{and} \\ \tilde{\delta} &= h \frac{\partial \delta}{\partial \epsilon} + n \frac{\partial \delta}{\partial n} + S^{\alpha\beta} \frac{\partial \delta}{\partial S^{\alpha\beta}}. \end{aligned} \quad (221)$$

2. Local correction from two-point correlation function using extended thermodynamic forces: $\langle \hat{p}^*(x) \rangle_2^{(2),ET}$

Using the definition (59) with (50) and employing Curie's principle, we can write

$$\langle \hat{p}^*(x) \rangle_2^{(2),ET} = \int d^4x_1 \left(\hat{p}^*(x), \beta(x_1) \sum_{i=1,2} [(\hat{\mathfrak{S}}_i \partial_{en}^i \beta) \mathcal{X} + (\hat{\mathfrak{S}}_i \partial_{en}^i \alpha) \mathcal{Y} + (\hat{\mathfrak{S}}_i \partial_{en}^i \Omega_{\mu\nu}) \mathcal{Z}^{\mu\nu}] \right) |_{\text{local}}. \quad (222)$$

Since we have to extract the local terms from (222), which can be achieved by expanding all the hydrodynamic quantities around $x_1 = x$, we get

$$\langle \hat{p}^*(x) \rangle_2^{(2),ET} = \sum_{i=1,2} \zeta_{p\mathfrak{D}_i}(x) [(\partial_{en}^i \beta)_x \mathcal{X}(x) + (\partial_{en}^i \alpha)_x \mathcal{Y}(x) + (\partial_{en}^i \Omega_{\mu\nu})_x \mathcal{Z}^{\mu\nu}(x)], \quad (223)$$

where we defined

$$\zeta_{p\mathfrak{D}_i}(x) = \beta(x) \int d^4x_1 (\hat{p}^*(x), \hat{\mathfrak{D}}_i(x_1)). \quad (224)$$

3. Local correction from three-point correlation function: $\langle \hat{p}^*(x) \rangle_2^{(3)}$

From (60), we can write

$$\langle \hat{p}^*(x) \rangle_2^{(3)} = \int d^4x_1 d^4x_2 (\hat{p}^*(x), \hat{C}_F(x_1), \hat{C}_F(x_2)) |_{\text{local}}, \quad (225)$$

where $\hat{C}_F$ is given by the Eq. (49). The nonvanishing correlators are

$$(\hat{p}^*(x), \hat{p}^*(x_1), \hat{p}^*(x_2)) = (\hat{p}^*(x), \hat{p}^*(x_1), \hat{p}^*(x_2)), \quad (226)$$

$$(\hat{p}^*(x), \hat{j}_\mu(x_1), \hat{j}_\nu(x_2)) = \frac{1}{3} \Delta_{\mu\nu} (\hat{p}^*(x), \hat{j}^\alpha(x_1), \hat{j}_\alpha(x_2)), \quad (227)$$

$$(\hat{p}^*(x), \hat{q}_\mu(x_1), \hat{q}_\nu(x_2)) = \frac{1}{3} \Delta_{\mu\nu} (\hat{p}^*(x), \hat{q}^\alpha(x_1), \hat{q}_\alpha(x_2)), \quad (228)$$

$$(\hat{p}^*(x), \hat{\pi}_{\mu\nu}(x_1), \hat{\pi}_{\alpha\beta}(x_2)) = \frac{1}{5} \Delta_{\mu\nu\alpha\beta} (\hat{p}^*(x), \hat{\pi}^{\lambda\sigma}(x_1), \hat{\pi}_{\lambda\sigma}(x_2)), \quad (229)$$

$$(\hat{p}^*(x), \hat{\tau}_{\mu\nu}(x_1), \hat{\tau}_{\alpha\beta}(x_2)) = \frac{1}{3} \Delta_{\mu\nu\alpha\beta} (\hat{p}^*(x), \hat{\tau}^{\lambda\sigma}(x_1), \hat{\tau}_{\lambda\sigma}(x_2)). \quad (230)$$

All other correlators vanish in an isotropic medium. Substituting (49) in (225) and using (226)–(230), we obtain

$$\langle \hat{p}^* \rangle_2^{(3)} = \zeta_{ppp} \theta^2 + \zeta_{pqq} M_\mu M^\mu + \zeta_{p\pi\pi} \sigma_{\mu\nu} \sigma^{\mu\nu} + \zeta_{p\tau\tau} \xi_{\mu\nu} \xi^{\mu\nu} + \beta^{-1} \zeta_{pj\alpha} \nabla_\mu \alpha \nabla^\mu \alpha, \quad (231)$$

where the transport coefficients are defined by

$$\zeta_{ppp} = \beta^2 \int d^4x_1 d^4x_2 (\hat{p}^*(x), \hat{p}^*(x_1), \hat{p}^*(x_2)), \quad (232)$$

$$\zeta_{pqq} = \frac{\beta^2}{3} \int d^4x_1 d^4x_2 (\hat{p}^*(x), \hat{q}^\alpha(x_1), \hat{q}_\alpha(x_2)), \quad (233)$$

$$\zeta_{p\pi\pi} = \frac{\beta^2}{5} \int d^4x_1 d^4x_2 (\hat{p}^*(x), \hat{\pi}^{\lambda\sigma}(x_1), \hat{\pi}_{\lambda\sigma}(x_2)), \quad (234)$$

$$\zeta_{p\tau\tau} = \frac{\beta^2}{3} \int d^4x_1 d^4x_2 (\hat{p}^*(x), \hat{\tau}^{\lambda\sigma}(x_1), \hat{\tau}_{\lambda\sigma}(x_2)), \quad (235)$$

$$\zeta_{pjj} = \frac{\beta}{3} \int d^4x_1 d^4x_2 (\hat{p}^*(x), \hat{j}^\alpha(x_1), \hat{j}_\alpha(x_2)). \quad (236)$$

4. Final expression

Combining (220), (223), and (231), we get

$$\begin{aligned} \langle \hat{p}^* \rangle_2 = & -\zeta_{pp}^\omega D\theta - \theta^2 [\Gamma \zeta_{pp}^\omega + \tilde{\Gamma} \zeta_{p\varepsilon}^\omega + \tilde{\delta} \zeta_{pn}^\omega] + \sum_{i=1,2} \zeta_{p\mathfrak{D}_i} [(\partial_{\varepsilon n}^i \beta) \mathcal{X} + (\partial_{\varepsilon n}^i \alpha) \mathcal{Y} + (\partial_{\varepsilon n}^i \Omega_{\mu\nu}) \mathcal{Z}^{\mu\nu}] \\ & + \zeta_{ppp} \theta^2 + \zeta_{pqq} M_\mu M^\mu + \zeta_{p\pi\pi} \sigma_{\mu\nu} \sigma^{\mu\nu} + \zeta_{p\tau\tau} \xi_{\mu\nu} \xi^{\mu\nu} + \beta^{-1} \zeta_{pjj} \nabla_\mu \alpha \nabla^\mu \alpha. \end{aligned} \quad (237)$$

Substituting (237) in (208), we obtain

$$\begin{aligned} \Pi = & -\zeta\theta + \left[\frac{1}{2} \zeta_{\varepsilon p} 2 \frac{\partial^2 p}{\partial \varepsilon^2} + \frac{1}{2} \zeta_{np} 2 \frac{\partial^2 p}{\partial n^2} + \zeta_{\varepsilon p} \zeta_{np} \frac{\partial^2 p}{\partial n \partial \varepsilon} \right] \theta^2 - \zeta_{S\tau} \mathcal{K}_{\mu\nu} \xi^{\mu\nu} \\ & - \zeta_{pp}^\omega D\theta - \theta^2 [\Gamma \zeta_{pp}^\omega + \tilde{\Gamma} \zeta_{p\varepsilon}^\omega + \tilde{\delta} \zeta_{pn}^\omega] + \sum_{i=1,2} \zeta_{p\mathfrak{D}_i} [(\partial_{\varepsilon n}^i \beta) \mathcal{X} + (\partial_{\varepsilon n}^i \alpha) \mathcal{Y} + (\partial_{\varepsilon n}^i \Omega_{\mu\nu}) \mathcal{Z}^{\mu\nu}] \\ & + \zeta_{ppp} \theta^2 + \zeta_{pqq} M_\mu M^\mu + \zeta_{p\pi\pi} \sigma_{\mu\nu} \sigma^{\mu\nu} + \zeta_{p\tau\tau} \xi_{\mu\nu} \xi^{\mu\nu} + \beta^{-1} \zeta_{pjj} \nabla_\mu \alpha \nabla^\mu \alpha. \end{aligned} \quad (238)$$

To get the evolution equation for the bulk viscous pressure, we approximate $\theta \sim -\zeta^{-1} \Pi$, in the term $\zeta_{pp}^\omega D\theta$. Thus, we obtain

$$\begin{aligned} \Pi + \tau_\Pi \dot{\Pi} = & -\zeta\theta + \tilde{\zeta}_\tau \Pi \theta + \left[\frac{1}{2} \zeta_{\varepsilon p} 2 \frac{\partial^2 p}{\partial \varepsilon^2} + \frac{1}{2} \zeta_{np} 2 \frac{\partial^2 p}{\partial n^2} + \zeta_{\varepsilon p} \zeta_{np} \frac{\partial^2 p}{\partial n \partial \varepsilon} \right] \theta^2 - \zeta_{S\tau} \mathcal{K}_{\mu\nu} \xi^{\mu\nu} \\ & - \theta^2 [\Gamma \zeta_{pp}^\omega + \tilde{\Gamma} \zeta_{p\varepsilon}^\omega + \tilde{\delta} \zeta_{pn}^\omega] + \sum_{i=1,2} \zeta_{p\mathfrak{D}_i} [(\partial_{\varepsilon n}^i \beta) \mathcal{X} + (\partial_{\varepsilon n}^i \alpha) \mathcal{Y} + (\partial_{\varepsilon n}^i \Omega_{\mu\nu}) \mathcal{Z}^{\mu\nu}] \\ & + \zeta_{ppp} \theta^2 + \zeta_{pqq} M_\mu M^\mu + \zeta_{p\pi\pi} \sigma_{\mu\nu} \sigma^{\mu\nu} + \zeta_{p\tau\tau} \xi_{\mu\nu} \xi^{\mu\nu} + \beta^{-1} \zeta_{pjj} \nabla_\mu \alpha \nabla^\mu \alpha, \end{aligned} \quad (239)$$

where

$$\tau_\Pi = -\zeta_{pp}^\omega \zeta^{-1} \quad (240)$$

$$\tilde{\zeta}_\tau = \tau_\Pi \zeta^{-1} \beta \left(\Gamma \frac{\partial \zeta}{\partial \beta} - \delta \frac{\partial \zeta}{\partial \alpha} \right) \quad (241)$$

VI. DISCUSSION

The evolution equations for the rotational stress tensor ($\tau_{\mu\nu}$), shear stress tensor ($\pi_{\mu\nu}$), boost heat vector (q_μ), and bulk viscous pressure (Π) are given in (127), (198), (169), and (239), respectively, which get reduced for the chargeless case,

$$\begin{aligned} \tau_\tau \dot{\tau}_{\mu\nu} + \tau_{\mu\nu} = & 2\gamma \xi_{\mu\nu} + \tilde{\gamma}_\tau \tau_{\mu\nu} \theta + 2\gamma_\omega \Gamma \xi_{\mu\nu} \theta + \gamma_{\tau S} \mathcal{K}_{\mu\nu} \theta + 2\gamma_{\tau p} \xi_{\mu\nu} \theta + \gamma_{\tau\tau\tau} \xi_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha + \gamma_{\tau\pi\pi} \sigma_{\alpha\langle\mu} \sigma_{\nu\rangle}^\alpha \\ & + 2\gamma_{\tau\pi\tau} \sigma_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha + \gamma_{\tau q q} M_{\langle\mu} M_{\nu\rangle}, \end{aligned} \quad (242)$$

$$\tau_\pi \dot{\pi}_{\mu\nu} + \pi_{\mu\nu} = 2\eta\sigma_{\mu\nu} + \tilde{\eta}_\tau \theta \pi_{\mu\nu} + 2\eta_\omega \Gamma \sigma_{\mu\nu} \theta + 2\eta_{\pi p} \sigma_{\mu\nu} \theta + \eta_{\pi\pi} \sigma_{\alpha\langle\mu} \sigma_{\nu\rangle}^\alpha + \eta_{\pi\tau\tau} \xi_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha + 2\eta_{\pi\pi\tau} \sigma_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha + \eta_{\pi qq} M_{\langle\mu} M_{\nu\rangle}, \quad (243)$$

$$\tau_q \dot{q}_\mu + q_\mu = -\lambda M_\mu + \tilde{\lambda}_\tau \theta q_\mu + \lambda_\omega \Gamma M_\mu \theta - \lambda \mathcal{Q}_\mu + \lambda_{qqp} M_\mu \theta + \lambda_{qq\tau} \xi_{\mu\nu} M^\nu + \lambda_{qq\pi} \sigma_{\mu\nu} M^\nu, \quad (244)$$

$$\begin{aligned} \tau_\Pi \dot{\Pi} + \Pi = & -\zeta \theta + \tilde{\zeta}_\tau \Pi \theta - \theta^2 [\Gamma \zeta_{pp}^\omega + \tilde{\Gamma} \zeta_{pe}^\omega] + \frac{1}{2} \zeta_{\varepsilon p} 2 \frac{\partial^2 p}{\partial \varepsilon^2} \theta^2 - \zeta_{s\tau} \mathcal{K}_{\mu\nu} \xi^{\mu\nu} + \zeta_{p\mathfrak{D}_1} [(\partial_{\varepsilon n}^1 \beta) \mathcal{X} + (\partial_{\varepsilon n}^1 \Omega_{\mu\nu}) \mathcal{Z}^{\mu\nu}] \\ & + \zeta_{ppp} \theta^2 + \zeta_{pqq} M_\mu M^\mu + \zeta_{p\pi\pi} \sigma_{\mu\nu} \sigma^{\mu\nu} + \zeta_{p\tau\tau} \xi_{\mu\nu} \xi^{\mu\nu}. \end{aligned} \quad (245)$$

The second and third terms on the right-hand side in the first line of each evolution equation arise due to the nonlocal correction. That is why they contain a finite relaxation time. All the terms written in the last line of each evolution equation originate from the second-order corrections due to the local contribution from the three-point correlation function. They represent all possible combinations of different thermodynamic forces ($\sigma_{\mu\nu}$, $\xi_{\mu\nu}$, θ and M_μ) present in the system. Transport coefficients coupled to these terms are given by the three-point correlation function. Comparing to the previous study [62], the extra terms, $\eta_{\pi\tau\tau} \xi_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha$, $2\eta_{\pi\pi\tau} \sigma_{\alpha\langle\mu} \xi_{\nu\rangle}^\alpha$ and $\eta_{\pi qq} M_{\langle\mu} M_{\nu\rangle}$, appearing in the evolution of the shear stress tensor (243) are the artifacts of the new thermodynamic forces ($\xi_{\mu\nu}$ and M_μ) that arise due to the addition of antisymmetric terms in the energy-momentum tensor.

Additionally, we find that the structure of the evolution equations for the shear stress tensor and the rotational stress tensor are similar, except for an extra term, $\gamma_{\tau s} \mathcal{K}_{\mu\nu} \theta$, which appears in the evolution equation of the rotational stress tensor. This term arises from corrections due to the extended thermodynamic forces via the two-point correlation function. Since we consider the spin density as an independent thermodynamic variable, which is an antisymmetric tensor of rank two, it generates the possibility of a two-point correlation in the presence of the rotational stress tensor—something that is absent in the case of the shear stress tensor.

Moreover, we identify a term, $\zeta_{s\tau} \mathcal{K}_{\mu\nu} \xi^{\mu\nu}$, involving the transport coefficient, given by the correlation of spin density with the rotational stress tensor, in the evolution of bulk viscous pressure. The origin of this term can be traced to the fact that the system now includes an additional independent thermodynamic variable, namely the spin density. Consequently, the equilibrium pressure depends on the spin density. Therefore, any deviation from equilibrium, which is nothing but the bulk viscous pressure, must also depend on the spin density.

Apart from that, within the framework of our current ordering scheme, we observe that the spin density operator ($\hat{S}^{\mu\nu}$) correlates only with the rotational stress tensor ($\hat{\tau}^{\mu\nu}$) through the two-point correlation function. This correlation gives rise to two novel transport coefficients, $\gamma_{\tau s}$ and $\zeta_{s\tau}$. These coefficients, which emerge at the second order, encapsulate the interplay between the antisymmetric

component of the energy-momentum tensor and the spin density tensor. Their existence can be attributed to the inclusion of spin density as an independent thermodynamic variable in the system. The introduction of these transport coefficients is significant, as they provide critical insights into the role and influence of spin density on the medium. Their study is essential for understanding the spin-induced transport properties of the system.

Further, we compare our results for the shear-stress tensor and the bulk viscous pressure with the results obtained for nonconformal fluids in [64]. The most general second-order expressions for the shear-stress tensor and bulk viscous pressure of a nonconformal fluid at zero particle chemical potential ($\mu = 0$) and zero spin chemical potential ($\omega_{\alpha\beta} = 0$) in flat space-time are given by

$$\begin{aligned} \pi_{\mu\nu} = & 2\eta\sigma_{\mu\nu} - 2\eta\tau_\pi \left[\Delta_{\mu\nu\alpha\beta} D\sigma^{\alpha\beta} + \frac{\theta}{3} \sigma_{\mu\nu} \right] - 2\eta\tau_\pi^* \frac{\theta}{3} \sigma_{\mu\nu} \\ & + \lambda_1 \sigma_{\alpha\langle\mu} \sigma_{\nu\rangle}^\alpha + \lambda_2 \sigma_{\alpha\langle\mu} \bar{\omega}_{\nu\rangle}^\alpha + \lambda_3 \bar{\omega}_{\alpha\langle\mu} \bar{\omega}_{\nu\rangle}^\alpha \\ & + \lambda_4 \nabla_{\langle\mu} \ln s \nabla_{\nu\rangle} \ln s, \end{aligned} \quad (246)$$

$$\begin{aligned} \Pi = & -\zeta \theta + \zeta_{\tau\Pi} D\theta + \xi_2 \theta^2 + \xi_1 \sigma_{\mu\nu} \sigma^{\mu\nu} + \xi_3 \bar{\omega}_{\mu\nu} \bar{\omega}^{\mu\nu} \\ & + \xi_4 \nabla_\mu \ln s \nabla^\mu \ln s, \end{aligned} \quad (247)$$

respectively. Here, $\bar{\omega}_{\mu\nu} = \Delta_{\mu\nu}^{\alpha\beta} \partial_\alpha u_\beta$ is the vorticity tensor.

Next, we rewrite our results for the shear stress tensor (197) and the bulk viscous pressure (238) for the case of a chargeless and spinless fluid with particle chemical potential $\mu = 0$ and spin chemical potential $\omega_{\mu\nu} = 0$. Thus we obtain

$$\begin{aligned} \pi_{\mu\nu} = & 2\eta\sigma_{\mu\nu} + 2\eta_\omega [\Delta_{\mu\nu\rho\sigma} D\sigma^{\rho\sigma} + \theta \Gamma \sigma_{\mu\nu}] + 2\eta_{\pi\pi p} \sigma_{\mu\nu} \theta \\ & + \eta_{\pi\pi\pi} \sigma_{\alpha\langle\mu} \sigma_{\nu\rangle}^\alpha + 2\eta_{\pi\pi\tau} \sigma_{\alpha\langle\mu} \bar{\omega}_{\nu\rangle}^\alpha + \eta_{\pi\tau\tau} \bar{\omega}_{\alpha\langle\mu} \bar{\omega}_{\nu\rangle}^\alpha \\ & + \eta_{\pi qq} M_{\langle\mu} M_{\nu\rangle} \end{aligned} \quad (248)$$

$$\begin{aligned} \Pi = & -\zeta \theta - \zeta_{pp}^\omega D\theta + [\zeta_\theta + \zeta_{ppp}] \theta^2 + \zeta_{p\pi\pi} \sigma_{\mu\nu} \sigma^{\mu\nu} \\ & + \zeta_{p\tau\tau} \bar{\omega}_{\mu\nu} \bar{\omega}^{\mu\nu} + \zeta_{pqq} M_\mu M^\mu + \zeta_{pe} \frac{\partial \beta}{\partial \varepsilon} \mathcal{X}, \end{aligned} \quad (249)$$

where we defined $\zeta_\theta = \frac{1}{2} \zeta_{\varepsilon p}^2 \frac{\partial^2 p}{\partial \varepsilon^2} - \Gamma \zeta_{pp}^\omega - \tilde{\Gamma} \zeta_{pe}^\omega$. Also, we want to emphasize on the point that, for a chargeless and

spinless case, M_μ can be written as $M_\mu = \frac{2(\varepsilon+p)}{\beta} \frac{\partial \beta}{\partial \varepsilon} \nabla_\mu \ln s$, by employing the ideal hydrodynamic equation $(\varepsilon + p)Du_\mu = \nabla_\mu p$ together with the thermodynamics relations, $Tds = d\varepsilon$, $dp = sdT$, and $\varepsilon + p = Ts$.

Now, It can be seen that Eq. (248) produces all the terms contained in (246), and the corresponding transport coefficients in each equation can be compared. However, for the case of bulk viscous pressure (249), there is an additional term $\zeta_{pe} \frac{\partial \beta}{\partial \varepsilon} \mathcal{X}$ which originates from the two-point correlation function using the extended thermodynamic forces.

VII. SUMMARY AND OUTLOOK

We have derived the second-order expressions for the dissipative currents $\tau_{\mu\nu}$, q_μ , $\pi_{\mu\nu}$, and Π using Zubarev's nonequilibrium statistical operator formalism, incorporating the full energy-momentum tensor up to the first order, including both symmetric and antisymmetric terms. Previously, the second-order expressions for $\pi_{\mu\nu}$ and Π were obtained using only the symmetric energy-momentum tensor [62]. Our calculations extend these second-order expressions for $\pi_{\mu\nu}$ and Π , introducing additional terms that account for the effects of spin degrees of freedom present in the system. Additionally, we have treated the spin density as an independent thermodynamic variable in our analysis. Our findings indicate that the spin density primarily influences $\tau_{\mu\nu}$ and Π . This influence manifests only in the second order, which is attributed to the gradient scheme of the spin chemical potential, which is considered first-order in the gradient expansion in this work. Furthermore, new transport coefficients arise in the second order, and each transport coefficient is related to a particular correlation function. We have presented a formula for each transport coefficient in terms of a correlation function. Finally, the second-order expressions of each dissipative tensor are used to find the evolution equation of the respective tensors.

As an outlook, there are a few research directions along the lines of this work. One can calculate the transport coefficients using quantum field theory with the help of Kubo formulas. Recently, work has been performed to calculate the Kubo formulas, taking the spin chemical

potential as a leading order term [65]. Also, one can rederive the second-order spin hydrodynamics using the full antisymmetric spin tensor that naturally includes the pseudogauge freedom. Recently, we came across [66], where a second-order spin hydrodynamics theory was investigated using a generalized form of the spin tensor. Furthermore, it would be interesting to see the cross-effect in second order by enforcing a more general positivity condition on entropy production [67]. Studying the second-order spin hydrodynamics and the evolution of dissipative tensors, combined with the numerical simulation, can give a better understanding of the evolution of the relativistic medium containing spin degrees of freedom.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: PROJECTION TENSORS

1. Fourth rank projection tensors

The two fourth-rank projection tensor used in this work are the symmetric traceless projector ($\Delta_{\mu\nu\alpha\beta}$) and the antisymmetric projector ($\Delta_{\mu\nu\alpha\beta}$). They both are orthogonal to fluid velocity, and defined by

$$\Delta_{\mu\nu\alpha\beta} = \frac{1}{2}(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\mu\beta}\Delta_{\nu\alpha}) - \frac{1}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta}, \quad (A1)$$

$$\Delta_{\mu\nu\alpha\beta} = \frac{1}{2}(\Delta_{\mu\alpha}\Delta_{\nu\beta} - \Delta_{\mu\beta}\Delta_{\nu\alpha}). \quad (A2)$$

Their properties are

$$\begin{aligned} u^\mu \Delta_{\mu\nu\alpha\beta} &= 0, & \Delta_{\mu\nu\alpha\beta} &= \Delta_{\nu\mu\alpha\beta} = \Delta_{\alpha\beta\mu\nu}, & \Delta_\rho^\mu \Delta_{\mu\nu\alpha\beta} &= \Delta_{\rho\nu\alpha\beta}, \\ \Delta_{\mu\nu\alpha\beta} \Delta_{\rho\sigma}^{\alpha\beta} &= \Delta_{\mu\nu\rho\sigma}, & \Delta_{\mu\alpha\beta}^\mu &= 0, & \Delta_{\mu\nu\beta}^\nu &= \frac{5}{3}\Delta_{\mu\beta}, & \Delta_{\mu\nu}^{\mu\nu} &= 5, \\ u^\mu \Delta_{\mu\nu\alpha\beta} &= 0, & \Delta_{\mu\nu\alpha\beta} &= -\Delta_{\nu\mu\alpha\beta} = \Delta_{\alpha\beta\mu\nu}, & \Delta_\rho^\mu \Delta_{\mu\nu\alpha\beta} &= \Delta_{\rho\nu\alpha\beta}, \\ \Delta_{\mu\nu\alpha\beta} \Delta_{\rho\sigma}^{\alpha\beta} &= \Delta_{\mu\nu\rho\sigma}, & \Delta_{\mu\alpha\beta}^\mu &= 0, & \Delta_{\mu\nu\beta}^\nu &= -\Delta_{\mu\beta}, & \Delta_{\mu\nu}^{\mu\nu} &= 3. \end{aligned} \quad (A3)$$

2. Nonlocal projectors

The nonlocal projectors are defined by

$$\Delta_{\mu\nu}(x, y) = \Delta_{\mu\lambda}(x) \Delta_{\nu}^{\lambda}(y), \quad (\text{A4})$$

$$\Delta_{\mu\nu\alpha\beta}(x, y) = \Delta_{\mu\nu\rho\sigma}(x) \Delta_{\alpha\beta}^{\rho\sigma}(y), \quad (\text{A5})$$

$$\Delta_{\mu\nu\alpha\beta}(x, y) = \Delta_{\mu\nu\rho\sigma}(x) \Delta_{\alpha\beta}^{\rho\sigma}(y), \quad (\text{A6})$$

with some straightforward orthogonal properties, given by

$$u^{\mu}(x) \Delta_{\mu\nu}(x, y) = \Delta_{\mu\nu}(x, y) u^{\nu}(y) = 0, \quad (\text{A7})$$

$$u^{\mu}(x) \Delta_{\mu\nu\alpha\beta}(x, y) = u^{\nu}(x) \Delta_{\mu\nu\alpha\beta}(x, y) = \Delta_{\mu\nu\alpha\beta}(x, y) u^{\alpha}(y) = \Delta_{\mu\nu\alpha\beta}(x, y) u^{\beta}(y) = 0, \quad (\text{A8})$$

$$u^{\mu}(x) \Delta_{\mu\nu\alpha\beta}(x, y) = u^{\nu}(x) \Delta_{\mu\nu\alpha\beta}(x, y) = \Delta_{\mu\nu\alpha\beta}(x, y) u^{\alpha}(y) = \Delta_{\mu\nu\alpha\beta}(x, y) u^{\beta}(y) = 0. \quad (\text{A9})$$

In order to write a hydrodynamic gradient expansion for the projectors, we expand the projectors around a fixed point. We start by the Taylor expansion for the fluid velocity and the second-rank projection tensor, given by

$$u_{\mu}(y) = u_{\mu}(x) + (y - x)^{\alpha} \left. \frac{\partial u_{\mu}(y)}{\partial y^{\alpha}} \right|_{y=x} + \dots, \quad (\text{A10})$$

$$\Delta_{\mu\nu}(y) = \Delta_{\mu\nu}(x) - (y - x)^{\alpha} \left[u_{\mu}(x) \left. \frac{\partial u_{\nu}(y)}{\partial y^{\alpha}} \right|_{y=x} + u_{\nu}(x) \left. \frac{\partial u_{\mu}(y)}{\partial y^{\alpha}} \right|_{y=x} \right] + \dots. \quad (\text{A11})$$

As discussed in Sec. IV, the second term in (A10) and (A11) are of first order in hydrodynamic gradient expansion. We can conclude that up to first order $u_{\mu}(y) u^{\mu}(x) \approx 1$. With the help of (A10) and using the definition of $\Delta_{\nu}^{\lambda}(y)$, we can rewrite (A4) as

$$\Delta_{\mu\nu}(x, y) = \Delta_{\mu\nu}(x) - u_{\nu}(x) (y - x)^{\alpha} \left. \frac{\partial u_{\mu}(y)}{\partial y^{\alpha}} \right|_{y=x} + \dots. \quad (\text{A12})$$

Now, we can derive some useful properties of nonlocal projectors which are approximated up to the first order in gradient correction. Using (A12), we obtain

$$\Delta_{\mu\nu}(x, y) \Delta_{\lambda}^{\nu}(x) \approx \Delta_{\mu\lambda}(x), \quad \Delta_{\alpha\beta}(x) \Delta^{\alpha\beta}(y) \approx 3. \quad (\text{A13})$$

Here, these properties are only valid in the first-order gradient approximation. In the same approximation, the nonlocal fourth rank symmetric traceless projector (A5), can be written as

$$\Delta_{\mu\nu\alpha\beta}(x, y) \approx \frac{1}{2} [\Delta_{\mu\alpha}(x, y) \Delta_{\nu\beta}(x, y) + \Delta_{\mu\beta}(x, y) \Delta_{\nu\alpha}(x, y)] - \frac{1}{3} \Delta_{\mu\nu}(x) \Delta_{\alpha\beta}(y), \quad (\text{A14})$$

whereas, the nonlocal fourth-rank antisymmetric projector (A6) can be rewritten as

$$\Delta_{\mu\nu\alpha\beta}(x, y) = \frac{1}{2} [\Delta_{\mu\alpha}(x, y) \Delta_{\nu\beta}(x, y) - \Delta_{\mu\beta}(x, y) \Delta_{\nu\alpha}(x, y)]. \quad (\text{A15})$$

Here, this equation is valid up to all orders in gradient expansion.

Using (A12) in (A14) and (A15), we obtain

$$\Delta_{\mu\nu\alpha\beta}(x, y) = \Delta_{\mu\nu\alpha\beta}(x) - (y - x)^{\rho} [\Delta_{\mu\nu\alpha\lambda}(x) u_{\beta}(x) + \Delta_{\mu\nu\lambda\beta}(x) u_{\alpha}(x)] \left. \frac{\partial u^{\lambda}(y)}{\partial y^{\rho}} \right|_{y=x} + \dots, \quad (\text{A16})$$

$$\Delta_{\mu\nu\alpha\beta}(x, y) = \Delta_{\mu\nu\alpha\beta}(x) - (y - x)^{\rho} [\Delta_{\mu\nu\alpha\lambda}(x) u_{\beta}(x) + \Delta_{\mu\nu\lambda\beta}(x) u_{\alpha}(x)] \left. \frac{\partial u^{\lambda}(y)}{\partial y^{\rho}} \right|_{y=x} + \dots. \quad (\text{A17})$$

APPENDIX B: CORRELATION FUNCTION

The derivation given in this section is similar as the one presented in [62,68,69]. Starting from the (32), we have

$$\langle \hat{X}(\vec{x}, t), \hat{Y}(\vec{x}_1, t_1) \rangle = \int_0^1 d\tau \langle \hat{X}(\vec{x}, t) [e^{-\hat{A}\tau} \hat{Y}(\vec{x}_1, t_1) e^{\hat{A}\tau} - \langle e^{-\hat{A}\tau} \hat{Y}(\vec{x}_1, t_1) e^{\hat{A}\tau} \rangle_I] \rangle_I. \quad (\text{B1})$$

Inferring to (28), we can conclude that $\hat{A} = \beta \hat{H}$, where $\hat{H}$ is the Hamiltonian of a rotating system in a grand canonical ensemble. Therefore, we obtain the time evolution of any operator in the Heisenberg picture in imaginary time as

$$\hat{Y}(\vec{x}, t + i\tau') = e^{-\hat{H}\tau'} \hat{Y}(\vec{x}, t) e^{\hat{H}\tau'}, \quad (\text{B2})$$

which leads to the Kubo-Martin-Schwinger relations, given by

$$\langle \hat{Y}(\vec{x}, t + i\tau') \rangle_I = \langle \hat{Y}(\vec{x}, t) \rangle_I, \quad (\text{B3})$$

$$\langle \hat{X}(\vec{x}, t) \hat{Y}(\vec{x}_1, t_1 + i\beta) \rangle_I = \langle \hat{Y}(\vec{x}_1, t_1) \hat{X}(\vec{x}, t) \rangle_I. \quad (\text{B4})$$

Using (B3) and (B4), in (B1) and assuming that the correlations vanish at $t \rightarrow \infty$, we can establish the relation

$$\langle \hat{X}(\vec{x}, t), \hat{Y}(\vec{x}_1, t_1) \rangle = -\frac{1}{\beta} \int_{-\infty}^{t_1} dt' \mathcal{G}_{\hat{X}\hat{Y}}^R(\vec{x} - \vec{x}_1, t - t'), \quad (\text{B5})$$

where

$$\mathcal{G}_{\hat{X}\hat{Y}}^R(\vec{x} - \vec{x}_1, t - t') = -i\theta(t - t') \langle [\hat{X}(\vec{x}, t), \hat{Y}(\vec{x}_1, t')] \rangle_I. \quad (\text{B6})$$

Now we consider a general form of frequency-dependent transport coefficient, involving a two-point correlation function, given by

$$\mathcal{I}_{\hat{X}\hat{Y}}(\omega) = \beta \int d^3x_1 \int_{-\infty}^t dt_1 e^{\epsilon(t_1 - t)} \times e^{i\omega(t - t_1)} \langle \hat{X}(\vec{x}, t), \hat{Y}(\vec{x}_1, t_1) \rangle. \quad (\text{B7})$$

The retarded Green's function obtained in (B6) is translationally invariant both in space as well as in time. We can exploit this property to rewrite the integral (B7) by setting $(\vec{x}, t) = (0, 0)$. Substituting (B5) in (B7), we obtain

$$\mathcal{I}_{\hat{X}\hat{Y}}(\omega) = - \int_{-\infty}^0 dt_1 e^{(\epsilon - i\omega)t_1} \int_{-\infty}^{t_1} dt' \int d^3x_1 \mathcal{G}_{\hat{X}\hat{Y}}^R(-\vec{x}_1, -t'). \quad (\text{B8})$$

Considering the Fourier transformation

$$\mathcal{G}_{\hat{X}\hat{Y}}^R(\vec{x}_1, t') = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{-i(\omega't' - \vec{k} \cdot \vec{x})} \mathcal{G}_{\hat{X}\hat{Y}}^R(\vec{k}, \omega'), \quad (\text{B9})$$

and using the definition of the Dirac delta function, we can write

$$\int d^3x \mathcal{G}_{\hat{X}\hat{Y}}^R(-\vec{x}, -t') = \lim_{\vec{k} \rightarrow 0} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} e^{i\omega't'} \mathcal{G}_{\hat{X}\hat{Y}}^R(\vec{k}, \omega'). \quad (\text{B10})$$

Substituting (B10) in (B8) and solving the integral over t' by shifting $\omega' \rightarrow \omega' - i\delta$, where $\delta \rightarrow 0^+$, we obtain

$$\mathcal{I}_{\hat{X}\hat{Y}}(\omega) = \lim_{\delta \rightarrow 0^+} \frac{i}{\omega + i\epsilon} \oint \frac{d\omega'}{2\pi i} \times \left[\frac{1}{\omega' - (\omega + i\epsilon + i\delta)} - \frac{1}{\omega' - i\delta} \right] \mathcal{G}_{\hat{X}\hat{Y}}^R(\omega'). \quad (\text{B11})$$

Here, $\mathcal{G}_{\hat{X}\hat{Y}}^R(\omega') \equiv \lim_{\vec{k} \rightarrow 0} \mathcal{G}_{\hat{X}\hat{Y}}^R(\vec{k}, \omega')$. In order to solve (B11), we first complexify ω' , and use Cauchy's integral formula together with the residue theorem. Inferring (B11), we obtain two poles at $\omega' = \omega + i\epsilon + i\delta$ and $\omega' = i\delta$. Also, we close the contour in the upper half plane and assume that, at infinity, $\mathcal{G}_{\hat{X}\hat{Y}}^R(\omega')$ vanishes sufficiently faster than the denominator ($\frac{1}{\omega'}$). Thus, we have

$$\mathcal{I}_{\hat{X}\hat{Y}}(\omega) = \lim_{\delta \rightarrow 0^+} \frac{i}{\omega + i\epsilon} [\mathcal{G}_{\hat{X}\hat{Y}}^R(\omega + i\epsilon + i\delta) - \mathcal{G}_{\hat{X}\hat{Y}}^R(i\delta)]. \quad (\text{B12})$$

Now we can take the limits $\delta \rightarrow 0^+$ and $\epsilon \rightarrow 0^+$, with ϵ being the irreversibility parameter, and which must be set to zero at the end of the calculations. Thus, we obtain

$$\mathcal{I}_{\hat{X}\hat{Y}}(\omega) = \frac{i}{\omega} [\mathcal{G}_{\hat{X}\hat{Y}}^R(\omega) - \mathcal{G}_{\hat{X}\hat{Y}}^R(0)]. \quad (\text{B13})$$

For $\omega \rightarrow 0$, (B13) can be simplified to

$$\mathcal{I}_{\hat{X}\hat{Y}}(0) = i \lim_{\omega \rightarrow 0} \frac{d}{d\omega} \mathcal{G}_{\hat{X}\hat{Y}}^R(\omega) = i \lim_{\omega \rightarrow 0} \lim_{\vec{k} \rightarrow 0} \frac{\partial}{\partial \omega} \mathcal{G}_{\hat{X}\hat{Y}}^R(\vec{k}, \omega). \quad (\text{B14})$$

Now, since the operators $\hat{X}$ and $\hat{Y}$ are Hermitian, we have

$$\langle [\hat{X}(\vec{x}, t), \hat{Y}(\vec{x}_1, t_1)] \rangle_I^* = -\langle [\hat{X}(\vec{x}, t), \hat{Y}(\vec{x}_1, t_1)] \rangle_I, \quad (\text{B15})$$

which indicates that $\langle [\hat{X}(\vec{x}, t), \hat{Y}(\vec{x}_1, t_1)] \rangle_I$ is a purely imaginary quantity, and by (B6), we can conclude that the retarded Green's function $\mathcal{G}_{\hat{X}\hat{Y}}^R$ and $\langle \hat{X}(\vec{x}, t), \hat{Y}(\vec{x}_1, t_1) \rangle$ are purely real quantities. Therefore, from (B7), we get the

conclusion that $\mathcal{I}_{\hat{X}\hat{Y}}(0)$ should be a purely real quantity. Thus we can rewrite (B14) as

$$\begin{aligned}\mathcal{I}_{\hat{X}\hat{Y}}(0) &= \beta \int d^4x_1 (\hat{X}(x), \hat{Y}(x_1)) = -\lim_{\omega \rightarrow 0} \frac{d}{d\omega} \text{Im} \mathcal{G}_{\hat{X}\hat{Y}}^R(\omega) \\ &= -\lim_{\omega \rightarrow 0} \lim_{\vec{k} \rightarrow 0} \frac{\partial}{\partial \omega} \text{Im} \mathcal{G}_{\hat{X}\hat{Y}}^R(\vec{k}, \omega).\end{aligned}\quad (\text{B16})$$

The other type of dissipative current obtained in this work is of the form

$$\mathcal{J}_{\hat{X}\hat{Y}}^\mu(\omega) = \beta \int d^4x_1 e^{i\omega(t-t_1)} (\hat{X}(x), \hat{Y}(x_1)) (x_1 - x)^\mu. \quad (\text{B17})$$

The temporal component is given by

$$\begin{aligned}\mathcal{J}_{\hat{X}\hat{Y}}^0(\omega) &= i\beta \frac{d}{d\omega} \int d^4x_1 e^{i\omega(t-t_1)} (\hat{X}(x), \hat{Y}(x_1)) \\ &= i \frac{d}{d\omega} \mathcal{I}_{\hat{X}\hat{Y}}(\omega).\end{aligned}\quad (\text{B18})$$

The spatial component of $\mathcal{J}_{\hat{X}\hat{Y}}^\mu(\omega)$ vanishes in the local rest frame, since the correlator $(\hat{X}(x), \hat{Y}(x_1))$ evaluated in the local rest frame depends on $|\vec{x} - \vec{x}_1|$, which makes $\mathcal{J}_{\hat{X}\hat{Y}}^i(\omega)$ an odd function of $\vec{x} - \vec{x}_1$. Furthermore, considering a Taylor expansion of $\mathcal{G}_{\hat{X}\hat{Y}}^R(\omega)$ around $\omega = 0$, we have

$$\begin{aligned}\mathcal{G}_{\hat{X}\hat{Y}}^R(\omega) &= \mathcal{G}_{\hat{X}\hat{Y}}^R(0) + \omega \frac{d}{d\omega} \mathcal{G}_{\hat{X}\hat{Y}}^R(\omega)|_{\omega=0} \\ &\quad + \frac{\omega^2}{2!} \frac{d^2}{d\omega^2} \mathcal{G}_{\hat{X}\hat{Y}}^R(\omega)|_{\omega=0} + \dots,\end{aligned}\quad (\text{B19})$$

which leads to

$$\begin{aligned}\mathcal{I}_{\hat{X}\hat{Y}}(\omega) &= \frac{i}{\omega} [\mathcal{G}_{\hat{X}\hat{Y}}^R(\omega) - \mathcal{G}_{\hat{X}\hat{Y}}^R(0)] = i \frac{d}{d\omega} \mathcal{G}_{\hat{X}\hat{Y}}^R(\omega)|_{\omega=0} \\ &\quad + \frac{i\omega}{2!} \frac{d^2}{d\omega^2} \mathcal{G}_{\hat{X}\hat{Y}}^R(\omega)|_{\omega=0} + \dots.\end{aligned}\quad (\text{B20})$$

Substituting (B20) in (B18) and taking the limit $\omega \rightarrow 0$, we have

$$\begin{aligned}\mathcal{J}_{\hat{X}\hat{Y}}^0(0) &= -\frac{1}{2} \lim_{\omega \rightarrow 0} \frac{d^2}{d\omega^2} \mathcal{G}_{\hat{X}\hat{Y}}^R(\omega) = -\frac{1}{2} \lim_{\omega \rightarrow 0} \frac{d^2}{d\omega^2} \text{Re} \mathcal{G}_{\hat{X}\hat{Y}}^R(\omega) \\ &= -\frac{1}{2} \lim_{\omega \rightarrow 0} \lim_{\vec{k} \rightarrow 0} \frac{\partial^2}{\partial \omega^2} \text{Re} \mathcal{G}_{\hat{X}\hat{Y}}^R(\vec{k}, \omega).\end{aligned}\quad (\text{B21})$$

Now, we assume $\mathcal{J}_{\hat{X}\hat{Y}}^0(0) = J$, then we can write

$$\beta \int d^4x_1 (\hat{X}(x), \hat{Y}(x_1)) (x_1 - x)^\mu = \mathcal{J}_{\hat{X}\hat{Y}}^\mu(0) = Ju^\mu. \quad (\text{B22})$$

APPENDIX C: CORRELATION FUNCTION IN TERMS OF PROJECTORS AND TRACES

The most general decomposition for a two-point correlation function involving two second-rank tensors is given by

$$(\hat{X}_{\mu\nu}(x), \hat{Y}_{\alpha\beta}(x_1)) = a_1 \Delta_{\mu\nu} \Delta_{\alpha\beta} + a_2 \Delta_{\mu\alpha} \Delta_{\nu\beta} + a_3 \Delta_{\mu\beta} \Delta_{\nu\alpha}. \quad (\text{C1})$$

A similar decomposition can also be given for a three-point correlation function involving one second-rank tensor and two vectors,

$$\begin{aligned}(\hat{X}_{\mu\nu}(x), \hat{Y}_\alpha(x_1), \hat{Z}_\beta(x_2)) &= b_1 \Delta_{\mu\nu} \Delta_{\alpha\beta} + b_2 \Delta_{\mu\alpha} \Delta_{\nu\beta} \\ &\quad + b_3 \Delta_{\mu\beta} \Delta_{\nu\alpha}.\end{aligned}\quad (\text{C2})$$

Here, we present the derivation of (63). All the other relations, similar to (C1) and (C2), can be derived in a similar way. Substituting $\hat{X} = \hat{\tau}$ and $\hat{Y} = \hat{\tau}$ in (C1), we demand that (C1) should be antisymmetric in the exchange of $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$. We obtain

$$a_1 = 0, \quad a_2 = -a_3 = a \quad (\text{say}). \quad (\text{C3})$$

Thus, we have

$$(\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\alpha\beta}(x_1)) = a [\Delta_{\mu\alpha} \Delta_{\nu\beta} - \Delta_{\mu\beta} \Delta_{\nu\alpha}] = 2a \Delta_{\mu\nu\alpha\beta}, \quad (\text{C4})$$

where a can be obtained by taking the trace of (C4),

$$a = \frac{1}{6} (\hat{\tau}_{\lambda\eta}(x), \hat{\tau}^{\lambda\eta}(x_1)). \quad (\text{C5})$$

Furthermore, we present the derivation for (112). All the other three-point correlation functions involving three second-rank tensors, given in (112)–(115) and (186)–(189), can be calculated in a similar way. Using (111), we can write

$$\begin{aligned}(\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\tau}_{\alpha\beta}(x_2)) &= a_1 \Delta_{\mu\nu} \Delta_{\rho\alpha} \Delta_{\sigma\beta} + a_2 \Delta_{\mu\nu} \Delta_{\rho\beta} \Delta_{\sigma\alpha} + b_1 \Delta_{\rho\sigma} \Delta_{\mu\alpha} \Delta_{\nu\beta} + b_2 \Delta_{\rho\sigma} \Delta_{\mu\beta} \Delta_{\nu\alpha} \\ &\quad + c_1 \Delta_{\alpha\beta} \Delta_{\mu\rho} \Delta_{\nu\sigma} + c_2 \Delta_{\alpha\beta} \Delta_{\mu\sigma} \Delta_{\nu\rho} + d \Delta_{\mu\nu} \Delta_{\rho\sigma} \Delta_{\alpha\beta} \\ &\quad + e_1 \Delta_{\mu\rho} \Delta_{\nu\alpha} \Delta_{\sigma\beta} + e_2 \Delta_{\mu\rho} \Delta_{\nu\beta} \Delta_{\sigma\alpha} + e_3 \Delta_{\mu\sigma} \Delta_{\nu\alpha} \Delta_{\rho\beta} + e_4 \Delta_{\mu\sigma} \Delta_{\nu\beta} \Delta_{\rho\alpha} \\ &\quad + e_5 \Delta_{\mu\alpha} \Delta_{\nu\rho} \Delta_{\sigma\beta} + e_6 \Delta_{\mu\alpha} \Delta_{\nu\sigma} \Delta_{\rho\beta} + e_7 \Delta_{\mu\beta} \Delta_{\nu\rho} \Delta_{\sigma\alpha} + e_8 \Delta_{\mu\beta} \Delta_{\nu\sigma} \Delta_{\rho\alpha}.\end{aligned}\quad (\text{C6})$$

This expression should be antisymmetric in the exchange of $\mu \leftrightarrow \nu$, $\rho \leftrightarrow \sigma$, and $\alpha \leftrightarrow \beta$. This concludes that the coefficients of projectors with indices $\mu\nu$, $\rho\sigma$, and $\alpha\beta$ must vanish. Thus, we have

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = d = 0. \quad (C7)$$

Furthermore, exchanging $\alpha \leftrightarrow \beta$ in (C6) and then comparing the coefficients of similar projector terms in $(\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\tau}_{\beta\alpha}(x_2))$ with $-(\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\tau}_{\alpha\beta}(x_2))$, we deduce that

$$e_1 = -e_2, \quad e_3 = -e_4, \quad e_6 = -e_8, \quad e_5 = -e_7. \quad (C8)$$

Similarly, exchanging $\rho \leftrightarrow \sigma$, we have

$$e_1 = -e_3, \quad e_5 = -e_6, \quad (C9)$$

and exchanging $\mu \leftrightarrow \nu$, we have

$$e_1 = -e_5 = e \text{ (say)}. \quad (C10)$$

Thus, we obtain

$$\begin{aligned} (\hat{\tau}_{\mu\nu}(x), \hat{\tau}_{\rho\sigma}(x_1), \hat{\tau}_{\alpha\beta}(x_2)) &= e(\Delta_{\mu\rho}\Delta_{\nu\alpha}\Delta_{\sigma\beta} - \Delta_{\mu\rho}\Delta_{\nu\beta}\Delta_{\sigma\alpha} - \Delta_{\mu\sigma}\Delta_{\nu\alpha}\Delta_{\rho\beta} + \Delta_{\mu\sigma}\Delta_{\nu\beta}\Delta_{\rho\alpha} \\ &\quad - \Delta_{\mu\alpha}\Delta_{\nu\rho}\Delta_{\sigma\beta} + \Delta_{\mu\alpha}\Delta_{\nu\sigma}\Delta_{\rho\beta} + \Delta_{\mu\beta}\Delta_{\nu\rho}\Delta_{\sigma\alpha} - \Delta_{\mu\beta}\Delta_{\nu\sigma}\Delta_{\rho\alpha}) \\ &= 2e[\Delta_{\rho\sigma}\Delta_{\mu\nu\sigma\beta} - \Delta_{\rho\beta}\Delta_{\mu\nu\sigma\alpha} - \Delta_{\sigma\alpha}\Delta_{\mu\nu\rho\beta} + \Delta_{\sigma\beta}\Delta_{\mu\nu\rho\alpha}]. \end{aligned} \quad (C11)$$

In order to calculate the value of e , we take the trace and use the properties (A3). Thus, we obtain

$$e = \frac{1}{6}(\hat{\tau}_\lambda^\delta(x), \hat{\tau}_\delta^\eta(x_1), \hat{\tau}_\eta^\lambda(x_2)). \quad (C12)$$

APPENDIX D: BULK VISCOUS PRESSURE

The nonequilibrium correction to the pressure is given by

$$\begin{aligned} \Pi &= \langle \hat{p} \rangle - p(\varepsilon, n, S^{\mu\nu}), \\ &= \langle \hat{p} \rangle_l + \langle \hat{p} \rangle_1 + \langle \hat{p} \rangle_2 - p(\varepsilon, n, S^{\mu\nu}). \end{aligned} \quad (D1)$$

Here, $p(\varepsilon, n, S^{\mu\nu})$ is the equilibrium pressure given by the equation of the state. The average of the pressure operator over the local equilibrium can be expanded around the equilibrium pressure; mathematically, it can be given by

$$\begin{aligned} \langle \hat{p} \rangle_l &= p(\varepsilon, n, S^{\mu\nu}) - \Gamma\Delta\varepsilon - \delta\Delta n - \mathcal{K}_{\mu\nu}\Delta S^{\mu\nu} + \frac{1}{2}(\Delta\varepsilon)^2 \frac{\partial^2 p}{\partial \varepsilon^2} + \frac{1}{2}(\Delta n)^2 \frac{\partial^2 p}{\partial n^2} + \frac{1}{2}\Delta S^{\mu\nu}\Delta S^{\alpha\beta} \frac{\partial}{\partial S^{\alpha\beta}} \left(\frac{\partial p}{\partial S^{\mu\nu}} \right) \\ &\quad + \frac{1}{2} \times 2(\Delta\varepsilon)(\Delta n) \frac{\partial^2 p}{\partial n \partial \varepsilon} + \frac{1}{2} \times 2(\Delta\varepsilon)(\Delta S^{\mu\nu}) \frac{\partial}{\partial \varepsilon} \left(\frac{\partial p}{\partial S^{\mu\nu}} \right) + \frac{1}{2} \times 2(\Delta n)(\Delta S^{\mu\nu}) \frac{\partial}{\partial n} \left(\frac{\partial p}{\partial S^{\mu\nu}} \right) + \dots \end{aligned} \quad (D2)$$

The corrections $\Delta\varepsilon$, Δn , and $\Delta S^{\mu\nu}$ are taken up to the second order. Thus,

$$\Delta\varepsilon = \langle \hat{\varepsilon} \rangle_1 + \langle \hat{\varepsilon} \rangle_2, \quad \Delta n = \langle \hat{n} \rangle_1 + \langle \hat{n} \rangle_2 \quad \text{and} \quad \Delta S^{\mu\nu} = \langle \hat{S}^{\mu\nu} \rangle_1 + \langle \hat{S}^{\mu\nu} \rangle_2. \quad (D3)$$

Using these in (D2) and neglecting all the third and higher order terms, we have

$$\langle \hat{p} \rangle_l = p(\varepsilon, n, S^{\mu\nu}) - \Gamma\langle \hat{\varepsilon} \rangle_1 - \Gamma\langle \hat{\varepsilon} \rangle_2 - \delta\langle \hat{n} \rangle_1 - \delta\langle \hat{n} \rangle_2 + \frac{1}{2}(\langle \hat{\varepsilon} \rangle_1)^2 \frac{\partial^2 p}{\partial \varepsilon^2} + \frac{1}{2}(\langle \hat{n} \rangle_1)^2 \frac{\partial^2 p}{\partial n^2} + \langle \hat{\varepsilon} \rangle_1 \langle \hat{n} \rangle_1 \frac{\partial^2 p}{\partial n \partial \varepsilon} - \mathcal{K}_{\mu\nu} \langle \hat{S}^{\mu\nu} \rangle_1.$$

Finally, the expression for the bulk viscous pressure, up to second order in hydrodynamic gradient expansion, is given by

$$\Pi = \langle \hat{p}^* \rangle_1 + \langle \hat{p}^* \rangle_2 + \frac{1}{2}(\langle \hat{\varepsilon} \rangle_1)^2 \frac{\partial^2 p}{\partial \varepsilon^2} + \frac{1}{2}(\langle \hat{n} \rangle_1)^2 \frac{\partial^2 p}{\partial n^2} + \langle \hat{\varepsilon} \rangle_1 \langle \hat{n} \rangle_1 \frac{\partial^2 p}{\partial n \partial \varepsilon} - \mathcal{K}_{\mu\nu} \langle \hat{S}^{\mu\nu} \rangle_1. \quad (D4)$$

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