

Symmetry theories, Wigner's function, compactification, and holography

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The global symmetry data of a D -dimensional absolute quantum field theory (QFT) can sometimes be packaged in terms of a $(D + 1)$ -dimensional bulk system obtained by extending along an interval, with a relative QFT _{D} at one end and suitable gapped or free boundary conditions at the other end. The partition function of the QFT _{D} can then be interpreted as a wave function depending on background fields. However, in some cases, it is not possible or simply cumbersome to fix an absolute form of the symmetry data. Additionally, it is also of interest to consider entangled and mixed states of relative QFTs as well as entangled and mixed states of gapped or free boundary conditions. We argue that Wigner's quasiprobabilistic function on phase space provides a physical interpretation of the symmetry data in all such situations. We illustrate these considerations in the case of string compactifications and holographic systems.

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I. INTRODUCTION

Symmetries provide important constraints on a broad range of quantum systems. One of the recent developments in the subject is the appearance of rich topological structures connected with the symmetries themselves [1]. This has by now led to vast generalizations of the standard paradigm of symmetries, which are still being uncovered.

A convenient way to capture many aspects of symmetries is by working with a higher-dimensional bulk system. For example, for a finite symmetry G of a D -dimensional QFT _{D} on a manifold M_D , there is a corresponding symmetry topological field theory SymTFT _{$D+1$} obtained by extending along an interval $I \times M_D$. At one end, we have a relative QFT _{D} , and, at the other, we have a gapped theory. This can be enlarged to cover situations such as continuous symmetries. One can visualize the two boundaries as specifying a physical state (i.e., a choice of relative QFT) $|Z\rangle$ and a gapped or free boundary condition $\langle B|$. The partition function of the absolute theory is then a wave function:

$$Z(B) = \langle B|Z\rangle. \quad (1.1)$$

There are now many bottom-up and top-down treatments of this basic framework.¹ See Fig. 1(a) for a depiction of the SymTFT formalism.

However, in some situations, this formulation can be somewhat awkward. For example, specifying a particular polarization and, thus, an absolute QFT _{D} is not always possible. Examples of this sort include many 6D superconformal field theories (SCFTs) where no polarization is available.² Another issue is that, especially in holographic systems, it is often convenient to entertain (at least semi-classically) an ensemble of theories³ and, thus, mixed states

¹See, e.g., [2–25] for a partial list of references to foundational early work, as well as more recent generalizations.

²Instead, one typically has a vector of partition functions (see, e.g., [26–28]). Formally speaking, one can always introduce by hand an auxiliary “center of mass” theory to produce a single partition function, but this can obscure some physical features. Adding this center of mass mode is straightforward, for example, for 6D (2,0) SCFTs constructed from M5-branes. For other 6D SCFTs, the same construction applies simply because the added center of mass edge mode (which would be added to the topological boundary condition) couples to the original system only via its 2-form symmetry.

³We remain agnostic as to whether such ensembles are an artifact of working in the IR or might simply reflect a necessary approximation in dealing with a highly complex system. For recent discussions of ensemble averaging in holography from bottom-up and top-down perspectives, see, e.g., [18,29–37].

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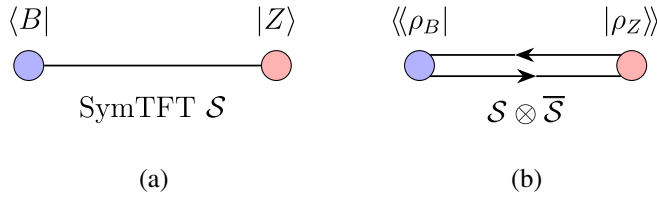


FIG. 1. (a) Schematic depiction of the SymTFT formalism and its wave function interpretation. We decompress an absolute QFT to a relative QFT $|Z\rangle$ and a topological or free boundary $\langle B|$ with bulk SymTFT or SymTh $\mathcal{S}$. The overlap is then $Z(B)$, the partition function with prescribed boundary conditions. (b) Similarly, mixed states of relative QFTs $\rho_Z = \sum_{ij} z_{ij} |Z_i\rangle \langle Z_j|$ and topological boundary conditions are decompressed into a $|\rho_Z\rangle\rangle$ and $\langle\langle \rho_B|$ which are ket and bra to the SymTFT and SymTh $\mathcal{S} \otimes \bar{\mathcal{S}}$.

of relative QFTs and their boundary conditions, e.g.,

$$\hat{\rho}_Z = \sum_{i,j} z_{ij} |Z_i\rangle \langle Z_j| \quad \text{and} \quad \hat{\rho}_B = \sum_{i,j} b_{ij} |B_i\rangle \langle B_j|. \quad (1.2)$$

In such situations, fixing a single polarization or absolute QFT can be cumbersome.

In this note, we address these issues by developing the SymTFT and SymTh formalism without reference to a fixed polarization.⁴ To do this, we formulate the relevant symmetry data in terms of Wigner’s quasiprobabilistic function [38]. This is a function that takes values on the phase space of a quantum mechanical theory. Integrating over a Lagrangian submanifold of the phase space then reduces to standard expressions with respect to a fixed polarization such as $|Z(B)|^2$, namely, the “square of the partition function.”⁵

The rest of this note is organized as follows. In Sec. II, we review Wigner’s function in quantum mechanics. In Sec. III, we extend this to the SymTFT and SymTh formalism and use it to formulate partition functions for entangled states and ensembles of relative QFTs and gapped or free boundary conditions. In particular, we use this setup to study entanglement of SymTFTs and SymThs without reference to a fixed polarization. In Sec. IV, we examine two examples motivated by higher-dimensional gravity—string compactification and holography.

⁴SymTh is an abbreviation for symmetry theory. By now there are many extra-dimensional constructions associated with the characterization of a system’s symmetries; see, e.g., [18–25], which generalize the original SymTFT discussion. In particular, these bulk theories need not be topological, and we will collectively refer to these constructions as SymThs.

⁵A perhaps helpful comment is that especially in intrinsically chiral systems it is often more straightforward to define the “partition function squared” rather than the partition function itself. Examples include many 2D and 6D conformal field theories.

II. WIGNER’S QUASIPROBABILISTIC DISTRIBUTION

We first review Wigner’s quasiprobabilistic distribution [38]. For our purposes, the main utility of this formalism is that it allows us to visualize a distribution function on all of phase space, namely, without fixing a particular polarization (i.e., a position or momentum basis for states). This will be useful in the context of the SymTFT formalism, since then we need not specify a particular absolute quantum field theory.

A. Continuum case

We begin by reviewing some general aspects of Wigner’s quasiprobabilistic function in the continuum [38]. The main ideas can already be conveyed in quantum mechanics. Introduce position and momentum operators $\hat{q}$ and $\hat{p}$ with commutation relation $[\hat{q}, \hat{p}] = i$. We label the eigenstates for these operators as $|q\rangle$ and $|p\rangle$. For an arbitrary state, with density matrix

$$\hat{\rho} = \sum_i \rho_i |i\rangle \langle i|, \quad (2.1)$$

we introduce the quasiprobability function:

$$W_\rho(q, p) = \frac{1}{\pi} \int dq' \langle q - q' | \hat{\rho} | q + q' \rangle e^{2ipq'}. \quad (2.2)$$

We can obtain the matrix elements for the density matrix by integrating against the position and momentum, respectively:

$$\int dp W_\rho(q, p) = \langle q | \hat{\rho} | q \rangle, \quad (2.3)$$

$$\int dq W_\rho(q, p) = \langle p | \hat{\rho} | p \rangle. \quad (2.4)$$

Given a classical function $g(q, p)$, we produce a corresponding Weyl-ordered operator $\hat{G}$. The expectation value of $\hat{G}$ is given by the integral over all of the phase space:

$$\text{Tr}(\hat{\rho} \hat{G}) = \int dq dp W_\rho(q, p) g(q, p). \quad (2.5)$$

We now give a slightly more general presentation of $W_\rho(q, p)$. To begin, we introduce the translation operators:

$$U_q = \exp(-iq\hat{p}) \quad \text{with} \quad U_q |q'\rangle = |q + q'\rangle, \quad (2.6)$$

$$V_p = \exp(+ip\hat{q}) \quad \text{with} \quad V_p |p'\rangle = |p + p'\rangle, \quad (2.7)$$

as well as the charge conjugation operator $\mathcal{C}$ that acts as

$$\mathcal{C} |q\rangle = |-q\rangle \quad \text{and} \quad \mathcal{C} |p\rangle = |-p\rangle. \quad (2.8)$$

Then, we can write $W_\rho(q, p)$ as an expectation value of operators:

$$W_\rho(q, p) = \frac{1}{\pi} \int dq' \langle q - q' | \hat{\rho} | q + q' \rangle e^{2ipq'} \quad (2.9)$$

$$= \frac{1}{\pi} \int dq' \langle -q' | V_p^\dagger U_q^\dagger \hat{\rho} U_q V_p | q' \rangle. \quad (2.10)$$

Now, since $\langle q' | \mathcal{C} = \langle -q' |$, we also have

$$W_\rho(q, p) = \frac{1}{\pi} \int dq' \langle q' | \mathcal{C} V_p^\dagger U_q^\dagger \hat{\rho} U_q V_p | q' \rangle \quad (2.11)$$

or

$$W_\rho(q, p) = \frac{1}{\pi} \text{Tr}(\hat{\rho} U_q V_p \mathcal{C} V_p^\dagger U_q^\dagger). \quad (2.12)$$

An interesting feature of this expression is that it allows us to treat the operators U and V on the same footing. In the context of partition functions and SymTFT expressions, this is helpful because one often has to specify a polarization. This is especially awkward in systems where an absolute polarization may not even be available.

Along these lines, it is helpful to repackage our expression slightly differently. Introduce the “doubled operator” which ranges over all of the phase space:

$$\mathbb{U}_{q,p} = \exp(-iq\hat{p} + ip\hat{q}). \quad (2.13)$$

Then, using the Baker-Campbell-Hausdorff formula, we have

$$U_q V_p = \mathbb{U}_{q,p} \exp\left(-\frac{i}{2}qp\right), \quad (2.14)$$

so it follows that

$$W_\rho(q, p) = \frac{1}{\pi} \text{Tr}(\hat{\rho} \mathbb{U}_{q,p} \mathcal{C} \mathbb{U}_{q,p}^\dagger), \quad (2.15)$$

because the c-number phase cancels out.

This last formulation is particularly suggestive. Let $\Phi^k = (q, p)$ denote a coordinate on the classical phase space with symplectic pairing⁶:

$$\mu_{kl} \Phi_{(a)}^k \Phi_{(b)}^l \equiv q_{(a)} p_{(b)} - p_{(a)} q_{(b)}. \quad (2.16)$$

Then, we have

$$\mathbb{U}_\Phi \equiv \mathbb{U}_{q,p} = \exp(-i\mu_{kl} \Phi^k \Phi^l), \quad (2.17)$$

⁶The following expression clearly generalizes to the case where we have a higher-dimensional phase space.

where $\hat{\Phi} \equiv (\hat{q}, \hat{p})$. Hence,

$$W_\rho(\Phi) = \frac{1}{\pi} \text{Tr}(\hat{\rho} \mathbb{U}_\Phi \mathcal{C} \mathbb{U}_\Phi^\dagger). \quad (2.18)$$

Specifying a Lagrangian submanifold L of the full phase space, integrating over L results in a standard probability distribution or expectation value on the conjugate variables.

B. Generalizations

In many cases, the observables of interest are actually discretized. This leads to some technical complications compared with the discussion given above. Nevertheless, one can retain much of the same formal structure, albeit at the expense of working with a somewhat more general class of Wigner functions. Here, we discuss relevant generalizations in the continuum case.

Indeed, the primary requirement we have for any quasiprobabilistic function on phase space is that we correctly produce the correct quantum mechanical expectation values upon integrating or summing over the appropriate set of values. Along these lines, one can introduce a generalized family of Wigner functions labeled by pairs $r, s \in \mathbb{R}$ given explicitly by⁷

$$W_\rho^{[r,s]}(q, p) = \frac{r+s}{2\pi} \int dq' \langle q - rq' | \hat{\rho} | q + sq' \rangle e^{(r+s)ipq'}. \quad (2.19)$$

In (2.2), we have $(r, s) = (1, 1)$.

This expression can also be recast in terms of an expectation value over suitable symmetry operators. The only difference is that we also have to introduce a rescaling operator Δ_λ with $\lambda \in \mathbb{R}$ that acts on the states as

$$\Delta_\lambda |q\rangle = |\lambda q\rangle. \quad (2.20)$$

Proceeding through a similar set of steps to those already outlined in the preceding subsection, we now have

$$W_\rho^{[r,s]}(q, p) = \frac{r+s}{2\pi} \text{Tr}(\mathcal{C} \Delta_r^\dagger V_p^\dagger U_q^\dagger \hat{\rho} U_q V_p \Delta_s). \quad (2.21)$$

Note that $\Delta_1 = \text{Id}$, the identity operator, and $\Delta_{-1} = \mathcal{C}$, the charge conjugation operator.

Following analogous steps that produced (2.18), we can favorably rewrite this expression further. Introduce the operator

$$\mathbb{U}_\Phi^{[r]} \equiv \mathbb{U}_\Phi \Delta_r, \quad (2.22)$$

⁷For further discussion on generalizations and extensions to Wigner's quasiprobability function, see Refs. [39–41]. The necessary nonuniqueness of Wigner's function, as quantified by r and s , was already noted by Wigner in his original work [38].

and we have

$$W_\rho^{[r,s]}(\Phi) = \frac{r+s}{2\pi} \text{Tr}(\hat{\rho} \mathbb{U}_\Phi^{[s]} \mathcal{C} \mathbb{U}_\Phi^{[r]\dagger}). \quad (2.23)$$

C. Discretized case

We now explain how to set up the formalism of Wigner's function in discrete setups. Instead of working directly with position and momentum variables, it is more natural to work with the exponentiated operators⁸:

$$U_e = \exp(i e \hat{p}) \quad \text{and} \quad V_m = \exp(-i m \hat{q}), \quad (2.24)$$

with

$$V_m U_e V_m^{-1} U_e^{-1} = \exp\left(\frac{2\pi i}{N} e m\right), \quad (2.25)$$

where the position and momenta eigenstates $|e\rangle$ and $|m\rangle$ naturally take values over the integers mod N . These vectors span the phase space $\mathbb{Z}_N \times \mathbb{Z}_N$. Compared with the continuum case, the only changes are that now we have finite sums, and the inner product between a position and a momentum eigenstate is

$$\langle e|m\rangle = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} e m\right). \quad (2.26)$$

Other than this, all of the manipulations from the continuum case carry over essentially unchanged. The only complication is that we must work with $W_\rho^{[r,s]}(e, m)$ with $r+s$ relatively prime to N ; otherwise, inversion of Fourier transforms breaks down. Furthermore, given such a good pair, there is a change in normalization⁹ compared to (2.23); we have

$$W_\rho^{[r,s]}(\Phi) = \frac{1}{N} \text{Tr}(\hat{\rho} \mathbb{U}_\Phi^{[s]} \mathcal{C} \mathbb{U}_\Phi^{[r]\dagger}) \quad (2.28)$$

in the discrete case.

D. Wave function interpretation for Wigner's functions

We now recast Wigner's function as a wave function. This reinterpretation of Wigner's function will later serve in

⁸Roughly speaking, the commutation relations for $\hat{q}$ and $\hat{p}$ are assumed to take the form $[\hat{q}, \hat{p}] = 2\pi i/N$. Of course, these are to be viewed as “mod N ” expressions, so it is really more appropriate to work in terms of their exponentiated analogs.

⁹Simply because for all units of $g \in \mathbb{Z}_N$, as characterized by $\text{gcd}(g, N) = 1$, we have

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{\frac{2\pi i}{N} g n(n'-n'')} = \delta_{n', n''}. \quad (2.27)$$

SymTFT and SymTh discussion in making the corresponding symmetry sandwich manifest, including its bulk topological theory, and also specify a natural basis to expand boundary conditions in. In preparation for that discussion, we concentrate on the discrete case here. The continuum case can be treated similarly.

Consider a discretized setup with N -dimensional Hilbert space $\mathcal{H}$ and the mixed state $\hat{\rho} = \sum_i \rho_i |i\rangle \langle i|$. Then we can consider the doubled Hilbert space $\mathcal{H} \otimes \bar{\mathcal{H}}$ where the overline indicates CPT or CRT relevant to $\mathcal{C}$. We define the pure states¹⁰

$$|\rho\rangle\rangle \equiv \mathcal{N}_\rho \sum_i \rho_i |i\rangle \otimes |\bar{i}\rangle,$$

$$\langle\langle \Phi^{[r,s]} | \equiv \mathcal{N}_{\Phi^{[r,s]}} \sum_{e'} \langle e' | \mathcal{C} \mathbb{U}_\Phi^{[r]\dagger} \otimes \langle e' | \mathbb{U}_\Phi^{[s]\dagger}, \quad (2.29)$$

where we have introduced two normalization factors $\mathcal{N}_\rho$ and $\mathcal{N}_{\Phi^{[r,s]}}$. We fix $\mathcal{N}_{\Phi^{[r,s]}}$ by requiring orthonormality on phase space:

$$\langle\langle \Phi_1^{[r,s]} | \Phi_2^{[r,s]} \rangle\rangle = \delta_{\Phi_1, \Phi_2}. \quad (2.30)$$

Next, note that the bra $\langle\langle \Phi^{[r,s]} |$ depends, after fixing the pair (r, s) , only on a point Φ of the original phase space. In contrast, the original $\langle e |$ depends on the existence of a Lagrangian subset of phase space (which distinguishes position and momentum states) and e then labels a point within that Lagrangian subset.

The bra-ket notation for $|\cdot\rangle\rangle$ and $|\cdot\rangle$ distinguishes between vectors of the doubled Hilbert space $\mathcal{H} \otimes \bar{\mathcal{H}}$ and the original Hilbert space $\mathcal{H}$, respectively. These two states contract to give the previously considered Wigner's function with respect to the mixed state ρ :

$$\langle\langle \Phi^{[r,s]} | \rho \rangle\rangle = W_\rho^{[r,s]}(\Phi). \quad (2.31)$$

Here, definitions are such that $\langle e | O | \bar{i} \rangle = \langle i | O^\dagger | e \rangle$ for operators O . For example, we have $|\bar{e}\rangle = |e\rangle$ for basis vectors and complex conjugation according to $\langle e | \bar{i} \rangle = \langle i | e \rangle = \overline{\langle e | i \rangle}$.

Given a classical phase space function $f(\Phi)$, we can now naturally associate it to the covector

$$\langle\langle f^{[r,s]} | \equiv \sum_\Phi f(\Phi) \langle\langle \Phi^{[r,s]} | \quad (2.32)$$

with the sum over all of phase space. When discussing SymTFTs and SymThs, this will allow us to map functions

¹⁰We comment that the pure states defined below are not the purifications of mixed states in the original Hilbert space. For example, the purification would have been $\sum_i \sqrt{\rho_i} |i\rangle \otimes |\bar{i}\rangle$. Rather, we have used the operator state map [42], resulting in the so-called Choi state [43].

on their phase space to mixed state boundary conditions. Here, we simply note that we have

$$\langle\langle f^{[r,s]}|\rho\rangle\rangle = \sum_{\Phi} f(\Phi) \langle\langle \Phi^{[r,s]}|\rho\rangle\rangle = \sum_{\Phi} f(\Phi) W_{\rho}^{[r,s]}(\Phi). \quad (2.33)$$

E. Illustrative example: Abelian Chern-Simons theory

We now discuss Abelian Chern-Simons theory as an illustrative example. The quantization of this theory can be somewhat subtle, since there is no single “split” of the phase space into position and momentum variables. See, in particular, [44–46] for a careful treatment. We also comment that these considerations naturally extend to theories with a $(2k+1)$ -form potential and a Chern-Simons-like action, as in Refs. [12,47–50]. In particular, the case of a three-form potential captures symmetry data in 6D superconformal field theories (see [12,28,50]).

Consider the level N action for a spin-Chern-Simons theory¹¹:

$$S_{3D} = \frac{N}{4\pi} \int_{M_3} AdA. \quad (2.34)$$

We consider the spacetime $M_3 = \mathbb{R}_{\tau} \times \Sigma$ with genus g Riemann surface Σ and time coordinate τ .

Next, we spell out the different quantities introduced in the earlier sections. After imposing the Gauss' law constraint, we canonically quantize the variables. This results in the conjugate variables $\hat{q} = \hat{a}_x$ and $\hat{p} = \hat{a}_y$ and the equal time commutator:

$$[\hat{a}_x(z_1), \hat{a}_y(z_2)] = \frac{2\pi i}{N} \delta^2(z_1 - z_2), \quad (2.35)$$

where locally $A|_{\Sigma} = a_x dx + a_y dy$ on some spatial patch of Σ with local coordinates $z = (x, y)$. The algebra implied by (2.35) for gauge-invariant Wilson loops is readily realized on a Hilbert space $\mathcal{H}(\Sigma)$ of complex dimension:

$$\dim_{\mathbb{C}} \mathcal{H}(\Sigma) = N^g. \quad (2.36)$$

The basis for this Hilbert space can be given explicitly using the symplectic basis of $H_1(\Sigma)$, as generated by A-cycles α_i and B-cycles β_j , which can be arranged to intersect such that

$$\alpha_i \cdot \beta_j = \epsilon_{ij}, \quad \alpha_i \cdot \alpha_j = 0, \quad \beta_i \cdot \beta_j = 0, \quad (2.37)$$

where $i, j = 1, \dots, g$ and $H_1(\Sigma) \cong \mathbb{Z}^{2g}$. Basis vectors of $\mathcal{H}(\Sigma)$ are then further characterized by quantized

holonomies about the A-cycles or B-cycles (or some mixture of the two).

We denote an orthonormal electric basis by $|i, e_i\rangle$ where $i = 1, \dots, g$ runs over the A-cycles and $n_i = 0, \dots, N-1$ labels the holonomy along the i th A-cycle α_i which is an N th root of unity. We denote an orthonormal magnetic basis by $|j, m_j\rangle$, where $j = 1, \dots, g$ runs over the B-cycles β_j and $n_k = 0, \dots, N-1$ labels the holonomy along the j th B-cycle, which is also an N th root of unity. Then we have the inner product

$$\langle i, e_i | j, m_j \rangle = \frac{1}{\sqrt{N}} \exp\left(\frac{2\pi i}{N} \delta_{ij} e_i m_j\right). \quad (2.38)$$

The conjugate variables take values in the phase space $\mathbb{Z}_N^g \times \mathbb{Z}_N^g$. Explicitly, we have the parametrization $e = (e_1, \dots, e_g)$ and $m = (m_1, \dots, m_g)$ with $e_i, m_j \in \mathbb{Z}_N$. Given the density matrix $\hat{\rho}$, we can now construct Wigner's functions. For simplicity, consider the example of the torus $\Sigma = T^2$ for which we have the phase space $\mathbb{Z}_N \times \mathbb{Z}_N$ with the Wigner functions

$$W_{\rho}^{[r,s]}(e, m) = \frac{1}{N} \sum_{e'=0}^{N-1} \langle e - re' | \hat{\rho} | e + se' \rangle e^{\frac{2\pi i}{N}(r+s)me'}. \quad (2.39)$$

Observe that given N we must restrict the pair (r, s) to satisfy $\gcd(N, r+s) = 1$ to obtain a good Wigner's function as otherwise the analogs of (2.3)–(2.5) receive additional contributions to their right-hand sides. For example, whenever N is odd, we can continue to use $r, s = 1$ (the standard presentation). However, when N is even, it is perhaps most minimal to use $r = 1$ and $s = 0$, but this is only a mild inconvenience.

III. WIGNER'S FUNCTION AND SYMTFTS OR SYMTHS

Our discussion so far has focused on some general features of Wigner's quasiprobability distribution in quantum mechanics. We now apply it to the SymTFT and SymTh formalism of QFTs.

The basic object of interest in this context is the partition function of a D -dimensional QFT $_D$. For some global symmetries G of an absolute quantum field theory, we can introduce background values e for the global symmetries in an electric polarization of the theory. Then, we can evaluate the partition function with respect to this choice of background values $Z(e)$. This can, in turn, be interpreted as the evaluation of a path integral for a gapped or free theory in the bulk with suitable electric boundary conditions dictating the overlap of states:

$$Z(e) = \langle e | Z \rangle. \quad (3.1)$$

We refer to $|Z\rangle$ as the physical boundary condition (namely, a relative quantum field theory) and $\langle e|$ as the gapped or

¹¹These considerations generalize to $2k+1$ -form potentials and their Chern-Simons-like actions. In this case, one demands a Wu structure (rather than Spin structure) on the manifold M_{4k+3} which is viewed as the boundary of a manifold M_{4k+4} .

free boundary condition. Instead of working with respect to this fixed polarization, we can instead work in a different one by gauging a nonanomalous symmetry. In the language of quantum mechanics, this gauging just involves summing over the choices of e to reach

$$Z(m) = \sum_e \langle m|e \rangle \langle e|Z \rangle; \quad (3.2)$$

i.e., we have a change of polarization.¹² Considering the pure state $\hat{\rho} = |Z\rangle\langle Z|$ in situations with an absolute polarization, we have

$$\sum_m W_Z(e, m) = |Z(e)|^2, \quad \sum_e W_Z(e, m) = |Z(m)|^2. \quad (3.3)$$

In what follows, we shall freely interchange between integration and summation notation when the context and local conventions are clear.

However, sometimes we cannot work with respect to an absolute quantum field theory. An example of this sort are most 6D SCFTs (see, e.g., [51,52] for reviews). In these cases, one typically has a vector of partition functions.¹³ An alternative is to introduce an auxiliary “center of mass” degree of freedom so as to have a single partition function. In any case, the presence of a nontrivial defect group [28] without a fixed notion of polarization makes the formulation in terms of an absolute quantum field theory somewhat more awkward. That being said, there is clearly a well-posed notion of the corresponding relative quantum field theories, as well as a notion of defects and symmetry operators, from both a bottom-up and top-down perspective.¹⁴

With this in mind, we now observe that, so long as we have a field configuration space, we can still speak of Wigner’s function and the associated basis of states $\langle\langle \Phi^{[r,s]} |$ of (2.29), and, therefore, speak, for example, of a corresponding square of a partition function.

More precisely, for some mixed state $\hat{\rho}_Z = \sum_{ij} z_{ij} |Z_i\rangle\langle Z_j|$ we can consider the associated doubled SymTFT with Hilbert space $\mathcal{H} \otimes \tilde{\mathcal{H}}$ following Sec. IID. The resulting SymTFT is the direct product of the original SymTFT with Hilbert space $\mathcal{H}$ and its conjugate. A mixed state boundary condition

¹²Sometimes this gauging procedure is obstructed in the sense that the absolute QFT may possess an anomaly. This is not much of a concern if we broaden our scope to simply consider a $(D+1)$ -dimensional bulk system with the relative QFT _{D} now viewed as an edge mode. In this broader setting, we simply have the mechanism of anomaly inflow. The anomaly then tells us that we cannot always compress the SymTFT _{$D+1$} interval.

¹³In some cases, however, one can still fix an absolute theory. See [53] for recent discussion on this point.

¹⁴See, e.g., [28,53–56].

$$\hat{\rho}_B = \sum_{ij} b_{ij} |e_i\rangle\langle e_j|, \quad (3.4)$$

formulated here with respect to an electric polarization, then corresponds to a classical function $p_B^{[r,s]}(e, m)$ on phase space defined by using the discrete analog of (2.5),

$$\text{Tr}(\hat{\rho}_B \hat{\rho}_Z) = \sum_{e,m} p_B^{[r,s]}(e, m) W_Z^{[r,s]}(e, m). \quad (3.5)$$

This allows us to define the SymTFT and SymTh boundary condition

$$\langle\langle \rho_B^{[r,s]} | \equiv \sum_{\mathbb{B}} p_B(\mathbb{B}) \langle\langle \mathbb{B}^{[r,s]} | \quad (3.6)$$

with phase space coordinate $\mathbb{B} = (e, m)$. The key point is now that this boundary condition does not depend on any choice of polarization. In principle, $p_B(\mathbb{B})$ may be any classical function on phase space,¹⁵ and we can use the above expressions to associate a boundary condition to it. Moreover, the above expression continues to hold with equal utility when $\mathbb{B}$ are drawn from a configuration space which does not split into position and momentum states, i.e., when no polarization exists.

Given the above boundary condition, we then have the overlap

$$\langle\langle \rho_B^{[r,s]} | \rho_Z \rangle\rangle = \sum_{\mathbb{B}} p_B(\mathbb{B}) W_Z^{[r,s]}(\mathbb{B}). \quad (3.7)$$

In particular, Wigner’s function

$$\langle\langle \mathbb{B}^{[r,s]} | \rho_Z \rangle\rangle = W_Z^{[r,s]}(\mathbb{B}) \quad (3.8)$$

is interpreted as specifying a natural set of partition functions to a mixed state $\hat{\rho}_Z$ of relative theories. Furthermore, $\langle\langle \rho_B^{[r,s]} | \rho_Z \rangle\rangle$ is then understood as a weighted sum of this basis of partition functions with weights specified by the classical distribution $p_B(\mathbb{B})$. Here, the index on p_B makes reference to only the “boundary.”

The above constructions rely on the doubled SymTFT with Hilbert space $\mathcal{H} \otimes \tilde{\mathcal{H}}$. Interestingly, given the sandwich corresponding to $\langle\langle \rho_B^{[r,s]} | \rho_Z \rangle\rangle$, we can “unfold it” and represent it as a sum over a disjoint family of sandwiches of the original SymTFT with Hilbert space $\mathcal{H}$. See also [57] for related discussion.

In order to demonstrate this, let us momentarily specialize to the pure state $\hat{\rho} = |Z\rangle\langle Z|$. Then, we have

¹⁵Observe that starting from a theory admitting a polarization and a density matrix $\hat{\rho}_B$ as in (3.4), we produce a function depending on $(\dim \mathcal{H}) - 1$ variables. Generic functions on phase space depend on $(\dim \mathcal{H})^2$ parameters.

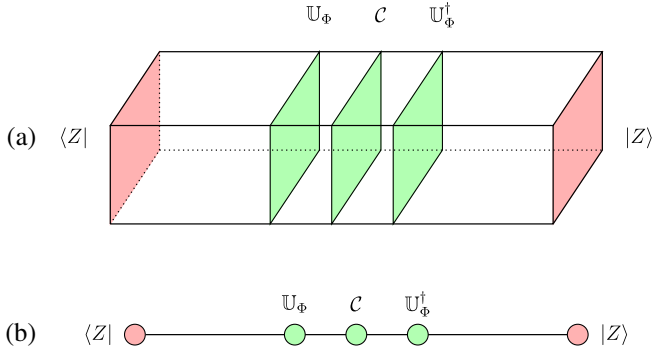


FIG. 2. Wigner's function for the pure state $\hat{\rho} = |Z\rangle\langle Z|$ avoids any reference to topological or free boundary conditions and can be rephrased purely in reference to two sets of physical boundary conditions (red). The operators $\mathbb{U}_\Phi, C, \mathbb{U}_\Phi^\dagger$ may be interpreted as (-1) -form symmetry operators which are codimension-one interfaces (green) in the SymTFT or SymTh slab. (a) and (b) show the same image, with the latter sketched in fewer dimensions.

$$W_Z^{[r,s]}(\mathbb{B}) = \langle\langle \mathbb{B}^{[r,s]} | \rho_Z \rangle\rangle = \langle Z | \mathbb{U}_\mathbb{B}^{[s]} C \mathbb{U}_\mathbb{B}^{[r]\dagger} | Z \rangle. \quad (3.9)$$

This quantity expresses Wigner's function as a matrix element of the original SymTFT. Furthermore, both boundaries are physical, avoiding reference to a polarization. In the SymTFT and SymTh formalism, we can view the $\mathbb{U}_\mathbb{B}$'s as (-1) -form symmetry operators which modify the choice of boundary conditions (see Fig. 2). More generally, with $\hat{\rho}_Z = \sum_{ij} z_{ij} |Z_i\rangle\langle Z_j|$ it follows that

$$\langle\langle \rho_B^{[r,s]} | \rho_Z \rangle\rangle = \sum_{\mathbb{B}, ij} z_{ij} p_B(\mathbb{B}) \langle Z_j | \mathbb{U}_\mathbb{B}^{[s]} C \mathbb{U}_\mathbb{B}^{[r]\dagger} | Z_i \rangle. \quad (3.10)$$

See Fig. 3 for a depiction.

Let us contrast the above to situations with an absolute polarization and pure state $\hat{\rho}_Z = |Z\rangle\langle Z|$ for which partition functions $Z(e) = \langle e | Z \rangle$ and $Z(m) = \langle m | Z \rangle$ are well defined. Here, we recover the expected partition function squared starting from (3.9). For example, summing over the magnetic degrees of freedom, we have

$$\sum_m W_Z^{[r,s]}(e, m) = |Z(e)|^2. \quad (3.11)$$

Similarly, we can still calculate expectation values of operators by integrating over the entire classical phase space using (2.5):

$$\langle Z | \hat{G} | Z \rangle = \sum_{e, m} W_Z^{[r,s]}(e, m) g(e, m). \quad (3.12)$$

For example, the electric polarization projection operator $\hat{E}_e = |e\rangle\langle e|$ yields

$$|Z(e)|^2 = \sum_{e', m'} W_Z^{[r,s]}(e', m') E_e(e', m'). \quad (3.13)$$

It is evident that $E_e(e', m') = \delta_{e, e'}$.

Let us also make a comment regarding the application of the formalism of Wigner's function in the SymTFT and SymTh framework in expressions such as the trace¹⁶:

$$W_Z^{[r,s]}(\mathbb{B}) \sim \text{Tr}_{\text{gap/free}}(\hat{\rho}_Z \mathbb{U}_\mathbb{B}^{[s]} C \mathbb{U}_\mathbb{B}^{[r]\dagger}) = \text{Tr}_\mathcal{H}(\hat{\rho}_Z \mathbb{U}_\mathbb{B}^{[s]} C \mathbb{U}_\mathbb{B}^{[r]\dagger}), \quad (3.14)$$

with $\mathbb{B}$ a choice of field configuration in the phase space. Here, the trace is actually restricted to those states associated with gapped or free boundary conditions, i.e., elements of $\mathcal{H}$, and of course does not cover, for example, enriched Neumann boundary conditions such as $|Z\rangle$. One can dispense with this restriction by instead inserting a suitable projection operator.

Finally,¹⁷ note that Wigner's function allows us to quantify how complex, from a quantum information theoretic perspective, the mixed state $|\rho_Z\rangle\rangle$ is. For this, return to the continuum Wigner function in (2.2) and recall that there, by a theorem of Hudson [58], one has that $W_\rho(p, q) \geq 0$ for a pure state $\rho = |\psi\rangle\langle\psi|$ only if $\psi(x)$ is Gaussian. The discrete analog of this result when Hilbert spaces are finite dimensional has $|\psi\rangle$ be a so-called stabilizer state [59]. Generalizing to mixed states, Wigner functions are non-negative whenever $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $|\psi_i\rangle$ stabilizer states. Another interesting result relating positivity to complexity is that Gaussian convolution of arbitrary quantum states eventually produces a strictly positive Wigner function; i.e., coarse-graining makes Wigner functions positive [60]. In contrast, magic states, in the sense of [61, 62], are pure states that are not stabilizer states and cannot be obtained by topological operations from stabilizer states. While exploring the characterization and consequences of negative Wigner functions for our setup of mixed states of partition vectors is beyond the scope of this work, we comment here that we expect these to organize orbits with respect to (-1) -form symmetry group actions of the underlying quantum system [as topological codimension-1 interfaces of a SymTFT may be interpreted as a (-1) -form symmetry operators with respect to its edge mode system] and leave further investigation for future work.

¹⁶That is, we trace over the space of states intrinsically defined by the SymTFT or SymTh treated as a QFT in its own right. Of course, one must enrich the space of state to include the physical states such as $|Z\rangle$.

¹⁷We thank the referee for an insightful question leading to the addition of this paragraph.

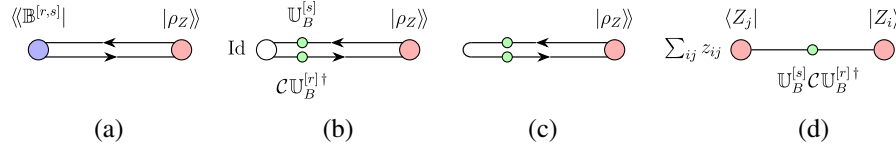


FIG. 3. We sketch steps relating “unfolding” the doubled SymTFT sandwich to a sum of SymTFT sandwiches with two physical boundary conditions. (a) Starting point, i.e., the initial doubled configuration with partition function given by Wigner’s function. Boundary conditions are determined by two pure states. (b) We separate out two operators from the boundary condition, which are indicated by green dots and are supported in one of the two systems tensored to give the overall doubled system. (c) The boundary condition trivializes to the identity operator Id (with respect to a single copy of the SymTFT) upon separating out these two operators. We can, therefore, reconnect the two SymTFT supports to give a single connected system. (d) Finally, we make $\rho_Z = \sum_{ij} z_{ij} |Z_i\rangle\langle Z_j|$ explicit and fuse the two green dots to $\mathbb{U}_B^{[s]} \mathcal{C} \mathbb{U}_B^{[r]\dagger}$.

A. Defects and symmetry operators

One of the important features of the SymTFT framework is that it provides a systematic way to organize the symmetry data of a system. In particular, one has the topological symmetry operators that link with defects of the theory. Recall that a defect stretches between the physical and gapped or free boundaries, whereas a symmetry operator is one that topologically links with this defect. Now, the precise spectrum of genuine defects and symmetry operators are those that can be specified without regard to any higher-dimensional operators. By contrast, nongenuine defects and symmetry operators are those that are better viewed as the boundaries of a higher-dimensional defect or operator.

How is this information taken into account by Wigner’s function? To begin, consider the case where the boundary and QFT state are both pure, i.e., $\hat{\rho}_B = |B\rangle\langle B|$ and $\hat{\rho}_Z = |Z\rangle\langle Z|$. We consider the effects of the operator $\mathbb{U}_B^{[r]}$ used to specify the background fields in the first place. This can be viewed as specifying a codimension-one defect in the bulk, i.e., a wall. From the perspective of the D -dimensional QFT, it fills all of the spacetime. Suppose we now consider a defect which stretches along the auxiliary bulk direction [see Fig. 4(a)]. As it passes through the $\mathbb{U}_B^{[r]}$ wall, it can be dressed by additional operators or defects. This dressing is the analog of making a defect genuine versus nongenuine. By the same token, a symmetry operator which crosses through the $\mathbb{U}_B^{[r]}$ wall may either pass through, remaining detached from the wall but possibly changing to a different operator, or instead attach back to the wall via a “flux tube” [see Figs. 4(b) and 4(c)]. The fully detached case corresponds to a genuine symmetry operator, and the case that attaches back to the wall is a nongenuine symmetry operator.

Now that we have explained how this works in the case of pure states, we can extend these considerations to more general mixed states of relative QFTs $\hat{\rho}_Z = \sum_i z_i |Z_i\rangle\langle Z_i|$ and mixed boundary conditions, as specified by the classical function $p_B(\mathbb{B})$. Returning to (3.10), recall that we have the overlap

$$\langle\langle \rho_B^{[r,s]} | \rho_Z \rangle\rangle = \sum_{\mathbb{B}, i} z_i p_B(\mathbb{B}) \langle Z_i | \mathbb{U}_B^{[s]} \mathcal{C} \mathbb{U}_B^{[r]\dagger} | Z_i \rangle. \quad (3.15)$$

Given a linking configuration of a defect $\mathcal{D}$ and symmetry operator $\mathcal{U}$, we can compute $\langle\langle \rho_B^{[r,s]} | \text{lnk}(\mathcal{D}, \mathcal{U}) | \rho_Z \rangle\rangle$ as follows. Inserting defects and symmetry operators follows by working with each summand individually; i.e., we evaluate expectation values with defects and symmetry

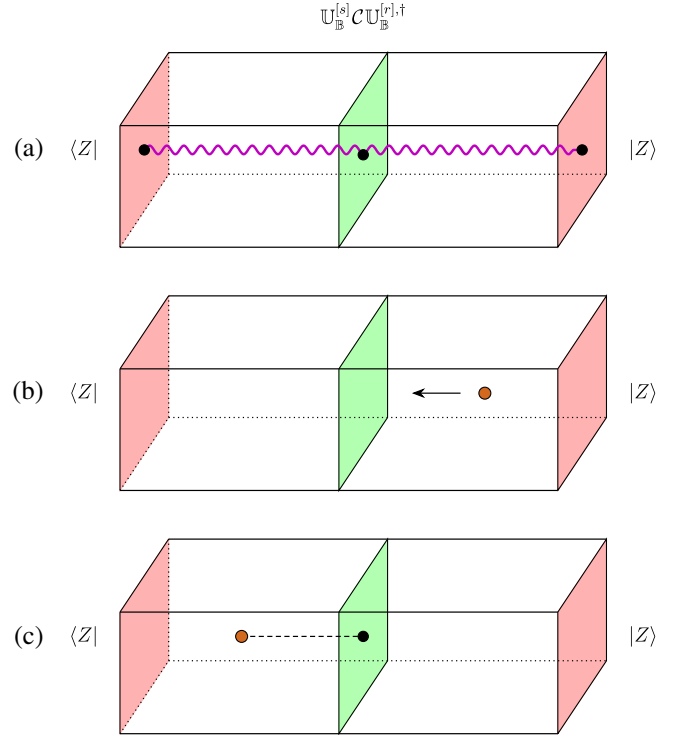


FIG. 4. We sketch defect and symmetry operators in the presence of a $\mathbb{U}_B^{[s]} \mathcal{C} \mathbb{U}_B^{[r]\dagger}$ insertion viewed as a codimension-one SymTFT operator or wall. (a) Defect operators piercing this wall may be dressed along their intersection with it (black dot). (b), (c) Deforming a symmetry operator across the wall, it may remain attached thereto (dashed line). When genuine operators are mapped onto genuine operators, the dashed line is absent or trivial.

operators (genuine or otherwise) inserted. We then perform a weighted sum of these answers with weights $p_B(\mathbb{B})z_i$.

B. Illustrative example: 6D SCFTs

To illustrate some of these points, consider a 6D SCFT with nontrivial defect group [28] with $\mathcal{N} = (2, 0)$ supersymmetry and labeled by some ADE Lie algebra.¹⁸ Similar considerations hold for $\mathcal{N} = (1, 0)$ SCFTs, but, in this case, one must also typically contend with possible 1-form and 0-form symmetries (see, e.g., [54,63,64]). For ease of exposition, we leave this extension implicit in what follows.

In the SymTFT formalism, treating the 2-form symmetries is rather awkward because there is often no polarization available for the defect group unless one makes a number of restrictive choices.¹⁹ Rather, one typically has to contend with a vector of partition functions. Of course, the dot product of two vectors returns a scalar. As such, we expect Wigner's function to make sense even in these situations.

In this case, the bulk SymTFT for the 2-form symmetries is governed by a Chern-Simons-like action for Abelian 3-form C^I potentials of the form (see, e.g., [12,18,50,65])

$$\mathcal{S}_{7D} = \frac{K_{IJ}}{4\pi} \int_{7D} C^I dC^J, \quad (3.16)$$

with integer level matrix $K_{IJ} \in \mathbb{Z}$ which is minus the Cartan matrix of the ADE Lie algebra associated to the 6D SCFT.²⁰ The defect group [28] is then the finite group

$$\mathbb{D} \cong \text{Coker} K_{IJ} \quad (3.17)$$

and equipped with a Dirac pairing induced by the inverse K^{IJ} . These SCFTs, on their tensor branch, famously contain 2-form gauge fields with anti-self-dual curvatures, and, consequently, there is no distinction between electric and magnetic defects; all defects are subsumed into $\mathbb{D}$. In structure is extremely similar to the Chern-Simons example discussed in Sec. II E and, therefore, computations will be essentially analogous, with the only core difference being that we will focus on noncompact flat spacetimes.

When the order $|\mathbb{D}|$ is not a square, then we cannot specify a polarization. For example, all SCFTs associated labeled by the Lie algebra $\mathfrak{su}_N$ with N not a perfect square are intrinsically relative in this sense. Nonetheless, the SymTFT with action $\mathcal{S}_{7D}$ makes sense, and the SCFT

realizes a physical boundary condition to it. See, for example, [66] for further discussion.

Furthermore, bulk topological operators can then be built from general periods of the C^I :

$$\mathcal{T}_{\nu, \Sigma} = \exp \left(i \int_{\Sigma} \nu_I \hat{C}^I \right), \quad (3.18)$$

where Σ is a 3-manifold and the hatted variable indicates the operator associated to field C by canonical quantization, similar to our discussion of Abelian Chern-Simons theory in Sec. II E. These satisfy the braiding relations²¹

$$\begin{aligned} \mathcal{T}_{\nu, \Sigma} \mathcal{T}_{\nu', \Sigma'} &= \exp \left(2\pi i \times \nu_I \left(\frac{1}{K} \right)^{IJ} \nu'_J \times \text{Ink}(\Sigma, \Sigma') \right) \\ &\times \mathcal{T}_{\nu', \Sigma'} \mathcal{T}_{\nu, \Sigma}. \end{aligned} \quad (3.19)$$

Here, $\text{Ink}(\Sigma_1, \Sigma_2)$ is the Gaussian linking between the 3-surfaces in the 7D SymTFT world volume. Observe that we can also split these operators up into defects and symmetry operators of the 6D SCFT, depending on whether they extend in the auxiliary bulk direction (as expected for the defects) or do not (as expected for the symmetry operators) of the SymTFT.

Nevertheless, it is clear that we can still specify a Wigner's function for the 2-form symmetries. To start, we comment that although the phase space of the 7D SymTFT depends on periods of the 3-form potential C , screening effects in the 6D SCFT tell us that in flat spacetimes it suffices to label the phase space operators as $\mathbb{U}_{\mu}^{[s]}$ with $\mu \in \mathbb{D}$ an element in the defect group [28]. We will set $s = 1$ going forward.

Indeed, given a $\mu \in \mathbb{D}$, we can produce a representative three-form potential $\mu_I = K_{IJ} C^J$ which is dual to the non-compact 3-cycle $\mathbb{R}^3$ in the flat 6D spacetime. Any two choices of such coordinate planes are gauge equivalent. The gauge invariant data of this background field are packaged in terms of its periods, as obtained by integrating over a basis of three-cycles. Owing to screening by dynamical states (see [28]), these periods are captured by elements $\mu \in \mathbb{D}$. By abuse of notation, we can then construct the operator

$$\mathbb{U}_{\mu} = \exp \left(-i K_{IJ} \int_{6D} C^I \hat{C}^J \right), \quad (3.20)$$

with 3-form chemical potential C ; namely, the phase is defined by integrating over the entire 6D spacetime. This expression should be compared to (2.17).

Next, consider a pure state $\hat{\rho}_Z = |Z\rangle\langle Z|$. Then we have Wigner's function

²¹See also the discussion in [12,28,50,66,67].

¹⁸See [51,52] for recent reviews of 6D SCFTs.

¹⁹For example, if the defect group splits as $\mathbb{D} = L \oplus \bar{L}$ with L a Lagrangian subgroup, then an absolute polarization can be produced, but there is no guarantee that this is always possible. For further discussion on this point, see Ref. [53].

²⁰In other words, the intersection form of two-cycles for the resolution of the ADE singularity $\mathbb{C}^2/\Gamma_{ADE}$ for $\Gamma_{ADE} \subset SU(2)$ a finite subgroup.

$$W_Z(\mu) = \langle\langle \mu | \rho_Z \rangle\rangle = \langle Z | U_\mu \mathcal{C} U_\mu^\dagger | Z \rangle, \quad (3.21)$$

which is the partition function of a doubled system of 6D SCFTs (which is absolute) with 2-form symmetry group

$$\Gamma^{(2\text{-form})} \cong \mathbb{D}^\vee, \quad (3.22)$$

where $\vee$ indicates the Pontryagin dual group. Since $\mathbb{D}$ is an Abelian group, we have the noncanonical isomorphism $\mathbb{D}^\vee \cong \mathbb{D}$.

IV. COMPACTIFICATION AND HOLOGRAPHY

These considerations become especially helpful in the context of systems where there is already a natural notion of a $(D+1)$ -dimensional bulk. In this section, we discuss two examples motivated by higher-dimensional gravity, namely, string compactification as well as holography. Of course, there are also examples that blend these considerations.

A. String compactification

In the context of string compactifications, one often engineers a D -dimensional QFT by starting with a higher-dimensional noncompact geometry with the degrees of freedom of the QFT localized at the tip of a singular conical geometry. In actual string backgrounds, one can often entertain multiple conical regions, all glued together by a bulk geometry. In this setting, when the bulk is large and D -dimensional gravity can be neglected, one can then speak of a more general notion of an entangled topological boundary condition for the whole multithroat system. See Fig. 5 for a depiction of a string compactification and its reduction to the SymTFT or SymTh.

To proceed, consider a string compactification with respect to an extra-dimensional space X and split it into the disjoint union of all conical regions X^{loc} as above and their complement $X^{\text{bulk}} = X \setminus X^{\text{loc}}$. In particular, we then have $X = X^{\text{loc}} \cup X^{\text{bulk}}$ and $X^{\text{loc}} = \sqcup_i X_i^{\text{loc}}$, where all $X_i^{\text{loc}}, X^{\text{bulk}}$ are connected subspaces of X with boundary. We denote the common boundary as $K = \partial X^{\text{loc}} = \partial X^{\text{bulk}}$, which has as many connected components as X^{loc} . Note that we take X^{bulk} to be completely smooth, and all singularities are contained in X^{loc} . The above setup should more precisely be associated with the “compressed” absolute system engineered by X . In order to decompress it and introduce the slab interval I of the SymTFT or SymTh, consider shrinking both X^{loc} and X^{bulk} using smooth deformation retractions. Abusing notation, we will continue to refer to these retracts as X^{loc} and X^{bulk} as they are homotopic. The space X now decomposes naturally as

$$X = X^{\text{loc}} \cup (I \times K) \cup X^{\text{bulk}}. \quad (4.1)$$

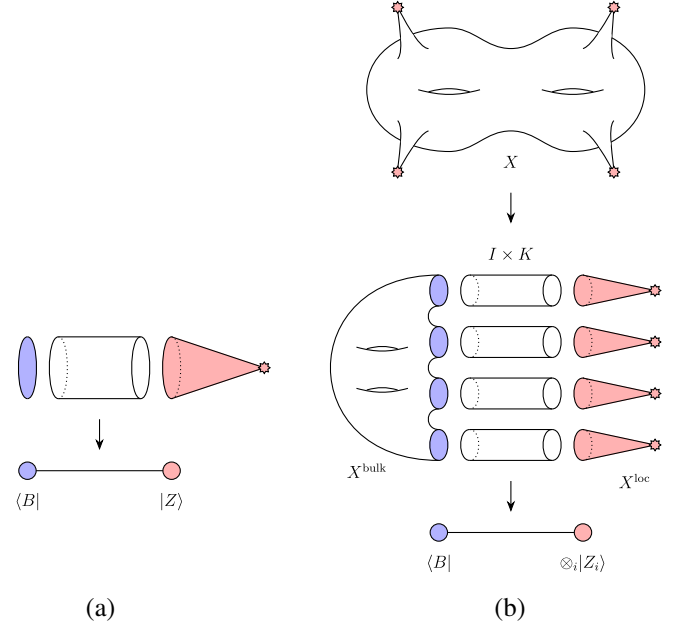


FIG. 5. Depiction of the SymTFT and SymTh sliver in a single (a) and multithroat (b) string background. In the multithroat configuration, the tensor product of the physical boundary conditions produces a pure state relative QFT and the bulk geometry results in a mixed state for topological boundary conditions.

The SymTFT or SymTh is then supported on I and is computed via compactification with respect to the compact cross section K . After compactifying, each local model X_i^{loc} engineers the SymTFT or SymTh state $|Z_i\rangle$, and to X^{loc} we therefore associate overall the pure state

$$|Z\rangle \equiv \bigotimes_i |Z_i\rangle. \quad (4.2)$$

Recall that K has multiple connected components, and, consequently, the SymTFT or SymTh constructed via compactification is a direct product of theories whose overall Hilbert space is simply the tensor product of the Hilbert spaces associated with the individual connected components of K . Completely analogously, X^{bulk} is associated with some boundary condition which we denote $|B\rangle$, and the overall partition function of the system is $\langle B | Z \rangle$. When X is compact, then $|B\rangle$ is usually a relative theory associated with supergravity bulk modes [68]. When X is noncompact, then $|B\rangle$ may be topological, but cases where these too support some relative theory associated with supergravity bulk modes can occur [18].

Consider now the case where the QFT has been decompressed onto the interval I which supports a SymTFT and expand the boundary condition $|B\rangle$ with respect to a basis of states

$$\begin{aligned}
|B\rangle &= \sum_{i_1, \dots, i_m} |B_{i_1, \dots, i_m}\rangle \langle B_{i_1, \dots, i_m} | B \rangle \\
&\equiv \sum_{i_1, \dots, i_m} b_{i_1, \dots, i_m}^* |B_{i_1, \dots, i_m}\rangle,
\end{aligned} \quad (4.3)$$

where m is the number of connected components of K and $|B_{i_1, \dots, i_m}\rangle = |B_{i_1}\rangle \otimes \dots \otimes |B_{i_m}\rangle$ and the coefficients $b_{i_1, \dots, i_m} = \langle B_{i_1, \dots, i_m} | B \rangle^\dagger$ are functions of the background fields. The overall partition function is, thus,

$$\langle B | Z \rangle = \sum_{i_1, \dots, i_m} b_{i_1, \dots, i_m} \langle B_{i_1} | Z_{i_1} \rangle \dots \langle B_{i_m} | Z_{i_m} \rangle, \quad (4.4)$$

which is an ensemble of direct products of absolute theories with partition functions $\langle B_{i_k} | Z_{i_k} \rangle = Z_{i_k}(B_{i_k})$ weighted by $b_{i_1, \dots, i_m}$.

Finally, observe that, once we trace over some of the $|B_i\rangle$ and $|Z_j\rangle$ states, we wind up with mixed states for the gapped or free and relative QFT boundaries. This tracing out is similar in spirit to that given in [32], and it would be interesting to make the connection between these operations more precise.

B. Holographic systems

Let us now turn to holographic examples featuring asymptotically AdS regions. In situations joined by worm-hole configurations [69,70], the physical boundary conditions of the SymTFT or SymTh also need to be extended to cover entangled and mixed states. Specifying a single polarization from the start is physically cumbersome compared with using Wigner's function.

Recall that, for a AdS/CFT pair with a single boundary, the SymTFT formalism can be viewed as defining a “topological sliver” of the AdS/CFT correspondence [20].²² In more detail, consider a large N holographic CFT _{D} on some D -dimensional manifold M_D . Then, the SymTFT or SymTh is supported on the manifold $I \times M_D$, with I an interval. On one boundary, we have the relative quantum field theory, as specified by a physical boundary condition $|Z\rangle$. On the other boundary, we have a topological or free boundary condition $\langle B|$ which specifies the polarization of the theory.

In the context of holographic systems, we can view the physical boundary condition as equivalently being captured by a gravitational theory on an asymptotically AdS _{$D+1$} spacetime. Then, the SymTFT or SymTh amounts to a small nongravitational sliver which is glued onto this asymptotic geometry [20]. See Fig. 6 for a depiction.

Consider the well-studied case of a stack of N D3-branes with semiclassical gravity dual AdS₅ $\times$ S^5 . Then $|Z\rangle$ is a relative $\mathfrak{su}_N$ gauge theory. The SymTFT was computed in [5], and the boundary condition $|B\rangle$ was discussed in [72].

²²See also [5,71,72].

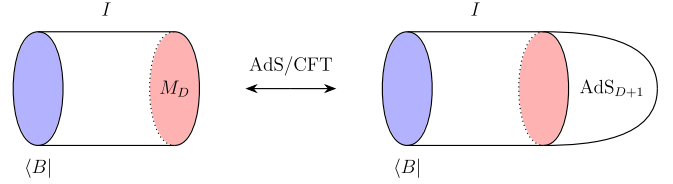


FIG. 6. Left: sketch of the SymTFT or SymTh slab for a CFT with spacetime M_D . Right: the relative CFT is recast as its dual semiclassical gravitational theory on AdS _{$D+1$} with boundary M_D . The SymTFT or SymTh is now naturally supported on an asymptotic sliver within the AdS _{$D+1$} gravitational spacetime.

The boundary condition $|B\rangle$ can be chosen to support a free $\mathfrak{u}_1$ gauge theory associated with the center of mass of the brane stack or taken to be topological or gapped.²³ In the former case $\langle B | Z \rangle$ is the partition function of an absolute $U(N)$ gauge theory, while in the latter the gauge group is $SU(N)/\mathbb{Z}_K$ with K dividing N . In the former, the free Abelian center of mass mode is coupled only topologically to the interacting degrees of freedom as specified by the SymTFT; they are edge modes to a common topological bulk.²⁴

We now generalize this to situations with more than one boundary. To this end, consider starting with a general $(D+1)$ -dimensional manifold M_{D+1} with no boundaries. We cut this into two pieces with the same boundary, namely, P_{D+1} and T_{D+1} such that $\partial P_{D+1} = \partial \overline{T_{D+1}} = M_D$. In general, M_D might consist of several components:

$$M_D = \bigsqcup_i M_D^{(i)}. \quad (4.5)$$

We specify a gravitational theory (with prescribed boundary conditions) on P_{D+1} and a SymTFT or SymTh (with prescribed boundary conditions) on T_{D+1} . From P_{D+1} and T_{D+1} we get pure states:

$$|Z\rangle \equiv \sum_{i_1, \dots, i_m} z_{i_1, \dots, i_m} |Z_{i_1}\rangle \otimes \dots \otimes |Z_{i_m}\rangle, \quad (4.6)$$

$$|B\rangle \equiv \sum_{i_1, \dots, i_m} b_{i_1, \dots, i_m} |B_{i_1}\rangle \otimes \dots \otimes |B_{i_m}\rangle, \quad (4.7)$$

and a corresponding overlap via $\langle B | Z \rangle = Z(B)$ (see Fig. 7). Abstractly, T_{D+1} can simply be viewed as a geometrization

²³Given the SymTFT action $N/2\pi \int B_2 dC_2$, the boundary condition $B_2 = *C_2$ admits the center of mass degree of freedom to be added as a boundary mode and realizes the overall gauge algebra $\mathfrak{u}_N$, while Neumann or Dirichlet boundary conditions freeze the center of mass mode and realize the overall gauge algebra $\mathfrak{su}_N$. As the bulk is topological, the center of mass of the brane stack is measured with respect to $|B\rangle$, and the distinguished cases correspond to an active or passive framing of coordinates. See [72,73] for further details.

²⁴One can also view the contribution from the center of mass degrees of freedom as adding a junction theory, in the sense of Ref. [18].

of the boundary condition; the SymTFT assigns a state to T_{D+1} that serves as a boundary condition. In top-down constructions, we can also think of the above setting as arising from, say, a compact geometry with multiple brane stacks dispersed. After taking near-horizon limits centered on each stack, multiple AdS throats develop, and after possibly tracing out some or all degrees of freedom, we find the bulk beyond the throats to specify gapped or free boundary conditions.

The state $|Z\rangle$ consists of a linear combination of tensor products of different relative theories. In the context of theories with a bulk gravity dual, we interpret this as a Euclidean wormhole connecting different large N theories. The state $|B\rangle$ consists of a linear combination of different topological and free boundary conditions. In the context of a string compactification, we interpret this as taking a local model with prescribed boundary conditions and gluing it to other local models via some bulk extra-dimensional geometry.

We can also fix our attention on just one of the factors in the tensor product. One way to do this is to perform a partial trace over the other sectors. In general, this results in a mixed state of relative theories and topological boundary conditions. In particular, this makes it somewhat awkward to specify a single absolute quantum field theory. It is, therefore, somewhat more natural to use Wigner's function so that we can deal with all possible choices of polarizations simultaneously.

With this in mind, we now consider Wigner's function for the physical state $|Z\rangle$ as specified above. Then, returning to (2.18), we have

$$W_Z^{[r,s]}(\mathbb{B}) = \langle Z | \mathbb{U}_{\mathbb{B}}^{[s]} \mathbb{C} \mathbb{U}_{\mathbb{B}}^{[r]\dagger} | Z \rangle, \quad (4.8)$$

where, here, the operator $\mathbb{U}_{\mathbb{B}}^{[s]} \mathbb{C} \mathbb{U}_{\mathbb{B}}^{[r]\dagger}$ is defined intrinsically in the SymTFT_{D+1} region of the geometry. As such, we can simultaneously handle all possible polarizations. Geometrically, we also get a linking that extends over all of M_{D+1} ; see Fig. 7.

1. AdS-Schwarzschild geometry

As a specific holographic example, we now turn to the eternal black hole in AdS [74,75]. In Euclidean signature, this background has a single boundary, but in Lorentzian signature, we have two boundaries. To leading order, one can interpret this bulk system as the gravity dual of two entangled Lorentzian signature CFTs, which we assume to be on manifolds of the form $\mathbb{R}_{\text{time}} \times S^{D-1} = M_D$. In this setting, the “partition function” just amounts to evaluating the path integral with prescribed boundary conditions in the past and future.

There are now two natural notions of time. First, there is t , the time of each boundary CFT. Second, we have r , the radial coordinate of the associated symmetry theory or

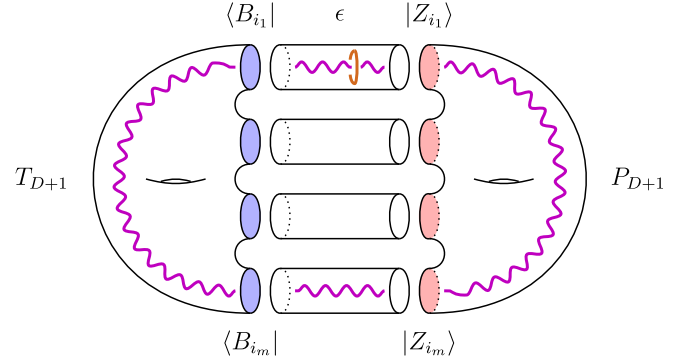


FIG. 7. Depiction of a more general entangled state in the SymTFT and SymTh formalism, embedded in a holographic system. In this setting, preparation of the general state involves a gravitational path integral over the physical region P_{D+1} and a topological or free path integral over the region T_{D+1} . We also depict a defect (purple) threading the geometry and its linking with a symmetry operator (brown) in the ϵ sliver.

gravity dual. To begin with, we shall mainly focus on the radial direction as the time coordinate but return to the other notion of time evolution later.

Observe that we now have a two-sided system, and, as such, we also get two SymTFTs or SymThs. Including the gravitational bulk, we observe that there is a natural path integral with left and right boundary conditions B_L and B_R :

$$Z_{\text{grav}}(B_L, B_R) = \int_{B_L}^{B_R} d\Phi e^{iS[\Phi]}, \quad (4.9)$$

where we leave implicit the boundary conditions in the past and future.

Let us now interpret this expression in the SymTFT framework. Along these lines, a general comment here is that the “state” we are discussing is that coming from the SymTFT and SymTh formalism; i.e., it is a state of a $(D+1)$ -dimensional system. This is to be contrasted with the states one would discuss of the corresponding entangled boundary CFTs. Indeed, while one views AdS-Schwarzschild as the purification of the thermofield double state of the CFT $_D$, that is a statement about a lower-dimensional system. In the present setting, building a linear combination of states is more akin to engineering an ensemble of different theories. Of course, changing the notion of time and, thus, the construction of states relates the two pictures via the path integral.

With this in mind, introduce the density matrices associated with SymTFT states:

$$\hat{\rho}_Z = \sum_i z_i |Z_i\rangle \langle Z_i| \quad \text{and} \quad \hat{\rho}_B = \sum_j b_j |B_j\rangle \langle B_j|. \quad (4.10)$$

Here, we have chosen to work in the respective eigenbases so that the density matrices are diagonal. By inspection, we can equivalently write the gravitational path integral as the

expectation value:

$$Z_{\text{grav}}(p_B) = \text{Tr}(\hat{\rho}_B \hat{\rho}_Z) = \sum_{i,j} Z_i(B_j) M_{ij} \overline{Z_i(B_j)}, \quad (4.11)$$

with $M_{ij} = z_i b_j$. We can also write this as the expectation value obtained from Wigner's function:

$$Z_{\text{grav}}(p_B) = \sum_{\mathbb{B}} W_Z^{[r,s]}(\mathbb{B}) p_B(\mathbb{B}), \quad (4.12)$$

where $p_B(\mathbb{B})$ is the classical function corresponding to the density matrix operator $\hat{\rho}_B$. Here, Wigner's function is given by

$$W_Z^{[r,s]}(\mathbb{B}) = \text{Tr}(\hat{\rho}_Z \mathbb{U}_{\mathbb{B}}^{[s]} \mathbb{C} \mathbb{U}_{\mathbb{B}}^{[r]\dagger}), \quad (4.13)$$

in the obvious notation.

2. Gluing to the thermofield double state

It is also of interest to now specify fixed boundary conditions at $t = 0$ as specified by the thermofield double state of a CFT at inverse temperature β . Recall from Ref. [75] that the AdS-Schwarzschild geometry can also be viewed as the CFT thermofield double state

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_{E_L=E_R=E} e^{-\beta E/2} |E_L, E_R\rangle; \quad (4.14)$$

namely, we work at a fixed time slice ($t = 0$) and prepare a corresponding state via the Euclidean path integral on the background $I_{\beta/2} \times S^{D-1}$ [75]. The SymTFT or SymTh for this setting consists of a single physical boundary condition and a single topological boundary condition. Additionally, we have to specify the boundary conditions for the path integral at the two ends of the interval, which we label as $\varphi_L^{(0)}$ and $\varphi_R^{(0)}$. We glue the left and right ends of this interval

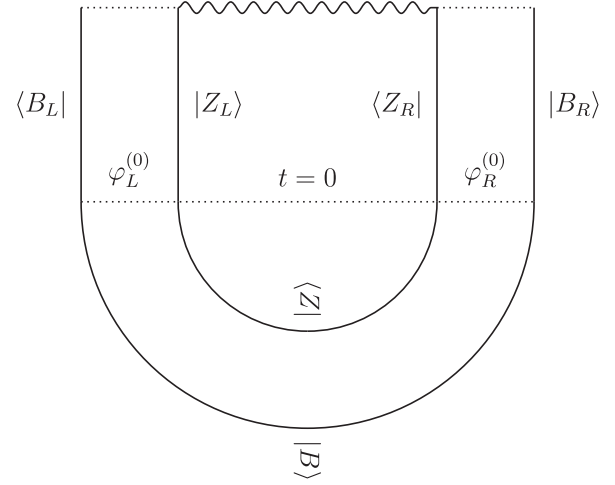


FIG. 8. Depiction of the preparation of the thermofield double state, its gravity dual, and the gluing to a SymTFT or SymTh sliver. Below the $t = 0$ dashed line, we have a single-sided gravity dual in Euclidean signature, which prepares the thermofield double. This is then glued to the two-sided SymTFT or SymTh geometry in Lorentzian signature (above $t = 0$). There are corresponding relative QFT and topological or free boundary conditions, as well as field configurations at $t = 0$ which are suitably glued together by the path integral defined in each region.

to the Lorentzian signature partition functions. Taking all of this into account, the wave function depends on a single topological boundary condition and two gluing conditions on the left and right. We write all of this as $Z_{\text{Euc}}(B; \varphi_L^{(0)}, \varphi_R^{(0)})$, which is prepared via the Euclidean path integral on $I \times (I_{\beta/2} \times S^{D-1})$.

Turning next to the Lorentzian signature path integrals, we now have a single fixed topological boundary condition $B = B_L = B_R$. In this patch, we have $Z_{\text{Lor}}(B_L = B, B_R = B, \varphi_L^{(0)}, \varphi_R^{(0)})$, in the obvious notation. Putting everything together, we get

$$Z_{\text{grav}}(B_L, B_R) = \begin{cases} \int d\varphi_L^{(0)} d\varphi_R^{(0)} Z_{\text{Lor}}(B, B, \varphi_L^{(0)}, \varphi_R^{(0)}) Z_{\text{Euc}}(B; \varphi_L^{(0)}, \varphi_R^{(0)}) & B_L = B_R = B, \\ 0 & B_L \neq B_R. \end{cases} \quad (4.15)$$

Here, the answer vanishes if $B_L \neq B_R$, because it is impossible to glue these topological boundary conditions to the Euclidean section. Observe that the topological boundary conditions on the two sides of the eternal black hole are now entangled, and this is a consequence of the geometrical connection between these topological boundaries. See Fig. 8 for a depiction. Thus, our construction provides an explicit example of the ER = EPR proposal [76] in the context of SymTFT and SymTh, wherein an ER

bridge connecting the topological boundaries results in the states on these boundaries being entangled.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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