

Celestial kinematical interpretation for an extended BMS_4 algebra

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Motivated by the work of Longhi and Materassi, who constructed a realization of the (centerless) BMS_4 algebra for the massive Klein-Gordon field in $3 + 1$ dimensions, we build a realization of the (centerless) massless BMS_4 algebra including superrotations. This realization depends only on the momenta in the light cone expressed in celestial coordinates without any reference to the Klein-Gordon field. The quadratic Casimir of the Lorentz algebra is written in terms of a second order differential operator and the volume form plays an essential role in this construction. The BMS_4 algebra in terms of vector fields shows its kinematical nature, like the Poincaré algebra. We also construct a dynamical realization of BMS_4 from the symplectic structure on the solutions of the massless four-dimensional Klein-Gordon field in terms of quadratic expressions of the Fourier modes and plane waves invariant under translations. Using the Mellin transform, we rewrite the Klein-Gordon field in terms of the boost invariant basis, and write down the corresponding BMS_4 realization. We also provide the relation with spherical harmonics, linking our results with the solutions of Longhi-Materassi, which are in fact a subset of ours.

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I. INTRODUCTION

The BMS_4 algebra was introduced in [1,2] as the asymptotic symmetry algebra of a four-dimensional flat spacetime at null infinity. An extended BMS_4 algebra, which includes the Poincaré supertranslations and superrotations, was proposed by [3]. A comprehensive presentation of recent applications of BMS symmetries can be found in [4]. Longhi and Materassi [5] found a canonical realization of the BMS_4 algebra in terms of the natural symplectic structure on the Fourier modes of a free massive Klein-Gordon (KG) field in four-dimensional Minkowski spacetime, extending the translation algebra to include supertranslations. In their construction a fundamental role is played by one of the two Casimir operators of the Lorentz

algebra, through the general solution of the eigenfunctions of the Casimir operator for a critical eigenvalue.

In the first part of this paper we extend this approach, by constructing a realization of supertranslations and superrotations in the momentum space for the massless case. It is particularly convenient to make use of celestial coordinates, which play a crucial role in celestial holography (see the reviews [6–9] and references therein). The extended BMS algebra realization, built in terms of differential operators of degrees zero and one, shows in this way its kinematic character, like the Poincaré algebra. This construction should be valid for any free relativistic massless theory. As an example we derive the BMS generators in the case of the KG field, both in the plane-wave and boost basis.

We also study the relation of our solutions with those obtained by Longhi and Materassi in terms of spherical harmonics, and it turns out that our construction yields more general solutions.

The rest of the paper is organized as follows. In Sec. II we write the expression of the Lorentz generators in terms of light-cone celestial coordinates. Section III contains our main results, the construction of the generators of extended BMS_4 and the essential role of the quadratic Lorentz Casimir differential operator. Section IV provides a realization of BMS_4 in terms of a massless Klein-Gordon field

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using the standard plane-wave representation, while the boost basis representation, related to the former by the Mellin transform, is presented in Sec. V. Section VI discusses the relation of our work with what would be the massless case of the Longhi-Materassi construction. We present our conclusions and outlook in Sec. VII. The Appendixes present some technical results not included in the main text.

II. LORENTZ GENERATORS IN CELESTIAL COORDINATES

In this section we will derive a kinematical representation of the extended BMS algebra for a lightlike momentum ($p^2 = 0$), by using the celestial coordinates. In this basis, the four-momentum is given by

$$p^\mu = \omega q^\mu, \quad q^\mu = [1 + z\bar{z}; z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}], \quad (1)$$

where ω is the light-cone energy and $z, \bar{z}$ are the complex dynamical variables that determine the momentum directions [10]:

$$\omega = \frac{1}{2}(p^0 + p^3), \quad z = \frac{p^1 + ip^2}{p^0 + p^3}. \quad (2)$$

On the other hand, in the Penrose description of Minkowski space-time, at the null boundary there is a Riemann sphere parametrized by complex coordinates. The starting point to construct celestial amplitudes is the identification of the kinematical variables $z, \bar{z}$ as the coordinates of this sphere. Note that the four-dimensional Lorentz group, which is isomorphic to $PSL(2, \mathbb{C})$, acts on the celestial sphere via conformal transformations (see, e.g., [11]).

In momentum space, the Lorentz generators

$$M_{0i} = -ip^0 \partial / \partial p^i, \quad M_{ij} = i(p_i \partial / \partial p^j - p_j \partial / \partial p^i), \quad (3)$$

where $p^0 = \omega(1 + z\bar{z})$, can be reexpressed in celestial coordinates as

$$\begin{aligned} M_{01} &= -\frac{i}{2}[\omega(z + \bar{z})\partial_\omega - (-1 + z^2)\partial_z - (-1 + \bar{z}^2)\partial_{\bar{z}}], \\ M_{02} &= -\frac{i}{2}[\omega(z - \bar{z})\partial_\omega - (1 + z^2)\partial_z + (1 + \bar{z}^2)\partial_{\bar{z}}], \\ M_{03} &= i[z\partial_z + \bar{z}\partial_{\bar{z}} - \omega\partial_\omega], \end{aligned} \quad (4)$$

and

$$\begin{aligned} M_{12} &= -(z\partial_z - \bar{z}\partial_{\bar{z}}), \\ M_{23} &= -\frac{1}{2}[\omega(-z + \bar{z})\partial_\omega + (-1 + z^2)\partial_z - (-1 + \bar{z}^2)\partial_{\bar{z}}], \\ M_{31} &= \frac{i}{2}[-\omega(z + \bar{z})\partial_\omega + (1 + z^2)\partial_z + (1 + \bar{z}^2)\partial_{\bar{z}}]. \end{aligned} \quad (5)$$

It is convenient to introduce the L_n and $\bar{L}_n$ $n = 0, \pm 1$ operators defined as

$$\begin{aligned} L_0 &= \frac{1}{2}(M_{12} + iM_{03}), \\ L_{-1} &= \frac{1}{2}(-M_{23} + iM_{31} - iM_{01} - M_{02}), \\ L_1 &= \frac{1}{2}(M_{23} + iM_{31} + iM_{01} - M_{02}), \\ \bar{L}_0 &= \frac{1}{2}(-M_{12} + iM_{03}), \\ \bar{L}_{-1} &= \frac{1}{2}(M_{23} + iM_{31} - iM_{01} + M_{02}), \\ \bar{L}_1 &= \frac{1}{2}(-M_{23} + iM_{31} + iM_{01} + M_{02}). \end{aligned} \quad (6)$$

With these definitions we get the explicit expressions

$$\begin{aligned} L_0 &= -z\partial_z + \frac{1}{2}\omega\partial_\omega, \\ L_{-1} &= -\partial_z, \quad L_1 = -z(z\partial_z - \omega\partial_\omega), \end{aligned} \quad (7)$$

and the corresponding ones for $\bar{L}_n$, $n = 0, \pm 1$ which satisfy the following Lorentz algebra:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m - n)\bar{L}_{m+n}, \\ [L_m, \bar{L}_n] &= 0, \quad m, n = 0, \pm 1. \end{aligned} \quad (8)$$

To consider the Poincaré algebra we introduce also the momenta, as in [12],

$$\begin{aligned} P_{-1/2, -1/2} &= P_0 + P_3, \quad P_{-1/2, +1/2} = P_1 - iP_2, \\ P_{1/2, -1/2} &= P_1 + iP_2, \quad P_{1/2, 1/2} = P_0 - P_3. \end{aligned} \quad (9)$$

In the representation given in Eq. (1), we get

$$\begin{aligned} P_{-1/2, -1/2} &= 2\omega, \quad P_{-1/2, +1/2} = 2\omega\bar{z}, \\ P_{1/2, -1/2} &= 2\omega z, \quad P_{1/2, 1/2} = 2\omega z\bar{z}. \end{aligned} \quad (10)$$

The Poincaré algebra is satisfied,

$$\begin{aligned} [P_{i,j}, P_{k,l}] &= 0, \\ [L_n, P_{k,l}] &= \left(\frac{n}{2} - k\right)P_{n+k,l}, \\ [\bar{L}_n, P_{k,l}] &= \left(\frac{n}{2} - l\right)P_{k,n+l}, \quad i, j, k, l = \pm \frac{1}{2}, \quad n = 0, \pm 1. \end{aligned} \quad (11)$$

III. LORENTZ CASIMIR AND EXTENDED BMS ALGEBRA

A. BMS algebra

To build an extension of the Poincaré algebra containing supertranslations, we start, as shown in [5], from the Casimir operator

$$\begin{aligned} C_2 &= \sum_i (M_{0i}^2) - M_{12}^2 - M_{23}^2 - M_{31}^2 \\ &= -\omega(3\partial_\omega + \omega\partial_\omega^2), \end{aligned} \quad (12)$$

where we have used the explicit expression of the Lorentz generators in celestial coordinates given in Eqs. (4) and (5). The second Casimir operator turns out to be zero, $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M^{\mu\nu}M^{\rho\sigma} = 0$. Since we know that a general four-vector is an eigenfunction of the C_2 operator with eigenvalue -3 , we consider all such eigenfunctions with the same eigenvalue:

$$\begin{aligned} C_2 f(\omega, z, \bar{z}) &= -\omega(3\partial_\omega + \omega\partial_\omega^2)f(\omega, z, \bar{z}) \\ &= -3f(\omega, z, \bar{z}). \end{aligned} \quad (13)$$

Since the Casimir operator does not contain derivatives with respect to $z, \bar{z}$, the general solution takes the form $f(\omega, z, \bar{z}) = \omega g(z, \bar{z}) + \omega^{-3}h(z, \bar{z})$, for some differentiable but otherwise arbitrary functions $g(z, \bar{z})$ and $h(z, \bar{z})$. Demanding regularity sets $h(z, \bar{z}) = 0$ and hence the regular solutions are given by

$$f(\omega, z, \bar{z}) = \omega g(z, \bar{z}) \quad (14)$$

with $g(z, \bar{z})$ differentiable but otherwise arbitrary.

Since the solutions given in Eq. (14) are degenerate and since the commutator of C_2 with the helicity M_{12} is zero, we now require that the solution be also eigenfunction of the helicity as in [5], or

$$M_{12}g(z, \bar{z}) = (z\partial_z - \bar{z}\partial_{\bar{z}})g(z, \bar{z}) = \lambda g(z, \bar{z}). \quad (15)$$

Notice that the solution of the homogeneous equation

$$M_{12}g(z, \bar{z}) = 0 \quad (16)$$

is an arbitrary function of the modulus $|z|$, since $M_{12}z = z, M_{12}\bar{z} = -\bar{z}$. This means, since M_{12} is a first-order differential operator, that we can multiply any solution corresponding to a given eigenvalue λ by a function of $|z|$ without changing the eigenvalue. Taking this into account, the whole set of independent eigenfunctions of the Casimir with eigenvalue -3 and with definite eigenvalue of M_{12} is spanned by functions of the form (14) with $g(z, \bar{z}) = z^p \bar{z}^q$, with $p, q \in \mathbb{Z}$, and which have $\lambda = p - q$.

Let us define the supertranslations as

$$P_{-1/2+p, -1/2+q}(\omega, z, \bar{z}) = 2\omega z^p \bar{z}^q, \quad p, q \in \mathbb{Z}, \quad (17)$$

or equivalently

$$P_{k,l}(\omega, z, \bar{z}) = 2\omega z^{k+1/2} \bar{z}^{l+1/2}, \quad k, l \in \mathbb{Z} + \frac{1}{2}, \quad (18)$$

with

$$\begin{aligned} M_{12}P_{-1/2+p, -1/2+q}(\omega, z, \bar{z}) \\ = (p - q)P_{-1/2+p, -1/2+q}(\omega, z, \bar{z}). \end{aligned} \quad (19)$$

The P 's in Eq. (10) correspond to the choices $p, q = (0, 0), (1, 0), (0, 1), (1, 1)$. One can check that the BMS algebra given in Eqs. (8) and (11) is satisfied for $k, l \in \mathbb{Z} + \frac{1}{2}$.

B. Extended BMS algebra

Since the Lorentz generators are first-order differential operators we consider the general holomorphic first-order differential operator

$$\xi = \xi^z(\omega, z)\partial_z + \xi^\omega(\omega, z)\partial_\omega. \quad (20)$$

The condition $[C_2, \xi] = 0$ implies

$$\xi^z(\omega, z) = \xi^z(z), \quad \xi^\omega(\omega, z) = \omega C(z). \quad (21)$$

A second condition comes by requiring that the Lorentz-invariant volume form

$$\Omega = \omega d\omega \wedge dz \wedge d\bar{z} \quad (22)$$

be invariant also under superrotations, $\mathcal{L}_\xi \Omega = 0$, or equivalently, due to

$$\mathcal{L}_\xi \Omega = (\text{div } \xi)\Omega, \quad (23)$$

that ξ be divergenceless. Using Cartan's identity for the Lie derivative $\mathcal{L}_\xi$ with respect to the vector field ξ we get

$$\left(\partial_\omega + \frac{1}{\omega}\right)\xi^\omega + \partial_z \xi^z + \partial_{\bar{z}} \xi^{\bar{z}} = 0. \quad (24)$$

Taking into account only the holomorphic part and plugging (21) into (24) one gets the relation

$$2C(z) = -\partial_z \xi^z(z). \quad (25)$$

If we consider the functions ξ^z and C to be expressed in Laurent series

$$\xi^z(z) = \sum_n a_n z^n, \quad C(z) = \sum_n c_n z^n, \quad (26)$$

we obtain the general relation

$$c_n = -\frac{n+1}{2} a_n. \quad (27)$$

Using this, the generalization of the Lorentz generators $L_n, n = 0, \pm 1$ for any $n \in \mathbb{Z}$ is given by

$$L_n = -z^{n+1} \partial_z + \frac{n+1}{2} z^n \omega \partial_\omega, \quad n \in \mathbb{Z}, \quad (28)$$

and similarly, for the antiholomorphic part,

$$\bar{L}_n = -\bar{z}^{n+1} \partial_{\bar{z}} + \frac{n+1}{2} \bar{z}^n \omega \partial_\omega, \quad n \in \mathbb{Z}. \quad (29)$$

Note that the first term in the superrotations is the standard representation of the Witt algebra (centerless Virasoro algebra) in two-dimensional conformal field theory. In a different context, this form of the superrotations was also considered in [13], defining, together with the supertranslations, what the authors of that paper call the *plane* representation of BMS_4 .

The supertranslation and superrotations operators satisfy the algebra given by Eq. (11) for $m, n \in \mathbb{Z}$ and $i, j, k, l \in \mathbb{Z} + \frac{1}{2}$.

We have then shown that the Poincaré algebra has an infinite-dimensional extension that we call extended BMS algebra. This algebra has two interesting infinite-dimensional subalgebras: the first one obtained extending the Poincaré algebra to BMS and the second one extending the Lorentz algebra to the Witt algebra. The extended BMS algebra is the semidirect sum of supertranslations and Witt.

Similar results were obtained by different methods by considering magnetic Carrollian conformal scalar field theories in $2+1$ dimensions [14–16].

IV. KLEIN-GORDON REPRESENTATION OF EXTENDED BMS GENERATORS

Let us consider the expansion of a real Klein-Gordon field in the $z, \bar{z}, \omega$ coordinates:

$$\phi(x) = \frac{1}{(2\pi)^3} \int \omega d\omega d^2 z [a(\omega, z, \bar{z}) e^{i\omega q^\mu x_\mu} + \text{H.c.}]. \quad (30)$$

We can express the field also in Poincaré transformed variables

$$x'^\mu = \Lambda^{-1\mu}_\nu (x^\nu - b^\nu) \sim x^\mu - b^\mu - \lambda^\mu_\nu x^\nu, \quad (31)$$

$$p'_\mu = (\Lambda^{-1})^\nu_\mu p_\nu = p_\nu \Lambda^\nu_\mu \sim p_\mu + p_\nu \lambda^\nu_\mu,$$

with $p'_\mu x'^\mu = p_\mu (x^\mu - b^\mu)$. Since ϕ is a scalar field

$$\begin{aligned} \phi(x) &= \phi'(x) \\ &= \frac{1}{(2\pi)^3} \int \omega' d\omega' d^2 z' [a'(\omega', z', \bar{z}') e^{i\omega' q'^\mu x'_\mu} + \text{H.c.}], \end{aligned} \quad (32)$$

with the transformed variables $\omega', z', \bar{z}'$ given by Eqs. (B3)–(B5). Since the integration measure is invariant under these transformations

$$\omega' d\omega' d^2 z' = \omega d\omega d^2 z, \quad (33)$$

we get

$$\begin{aligned} \phi(x) &= \frac{1}{(2\pi)^3} \int \omega d\omega d^2 z [a(\omega, z, \bar{z}) e^{i\omega q^\mu x_\mu} + \text{H.c.}] \\ &= \frac{1}{(2\pi)^3} \int \omega d\omega d^2 z [a'(\omega', z', \bar{z}') e^{i\omega' q'^\mu x'_\mu} + \text{H.c.}] \\ &\sim \frac{1}{(2\pi)^3} \int \omega d\omega d^2 z [a'(\omega', z', \bar{z}') e^{i\omega q^\mu x_\mu} \\ &\quad \times (1 - i\omega q^\mu b_\mu) + \text{H.c.}], \end{aligned} \quad (34)$$

where the $\sim$ in the last equation means that we have considered an infinitesimal transformation. Making use of the result in Appendix A, we obtain

$$\delta a(\omega, z, \bar{z}) = i\omega q^\mu b_\mu a(\omega, z, \bar{z}) - \frac{\partial a}{\partial \omega} \delta \omega - \frac{\partial a}{\partial z} \delta z - \frac{\partial a}{\partial \bar{z}} \delta \bar{z}, \quad (35)$$

which generalizes the result in (2.21) of [17] to include supertranslations and superrotations.

Using Eqs. (B3)–(B5), the infinitesimal variation of δa can be rewritten in terms of the generators $P_{k,l}, L_n, n = -1, 0, 1$ as

$$\begin{aligned} \delta a(\omega, z, \bar{z}) &= \left(\frac{i}{2} b_{k,l} P_{k,l} + \beta L_{-1} + 2\alpha L_0 + \gamma L_1 \right. \\ &\quad \left. + \bar{\beta} \bar{L}_{-1} + 2\bar{\alpha} \bar{L}_0 + \bar{\gamma} \bar{L}_1 \right) a(\omega, z, \bar{z}), \end{aligned} \quad (36)$$

when $k, l = \pm 1/2$ and where the translation parameters have been redefined as

$$\begin{aligned} b_{-1/2, -1/2} &= b^3 - b^0, & b_{1/2, 1/2} &= -b^0 - b^3, \\ b_{-1/2, 1/2} &= b^1 + ib^2, & b_{1/2, -1/2} &= b^1 - ib^2. \end{aligned} \quad (37)$$

The generalization of $\delta a(\omega, z, \bar{z})$ to supertranslations and superrotations is straightforward:

$$\delta a(\omega, z, \bar{z}) = \left(\frac{i}{2} b_{k,l} P_{k,l} + \epsilon_n L_n + \bar{\epsilon}_n \bar{L}_n \right) a(\omega, z, \bar{z}), \quad (38)$$

with $k, l \in \mathbb{Z} + \frac{1}{2}, n \in \mathbb{Z}$, and with a suitable choice of $\epsilon_n, n = -1, 0, 1$ to reproduce Eq. (36).

In conclusion, assuming that the $\delta a(\omega, z, \bar{z})$ is obtained, for example in the case of the supertranslations, by making use of

$$\delta a(\omega, z, \bar{z}) = b_{k,l} \{ \mathcal{P}_{k,l}, a(\omega, z, \bar{z}) \}, \quad (39)$$

the generators of the extended BMS algebra in the KG case are

$$\begin{aligned} \mathcal{P}_{k,l} &= \int \omega d\omega d^2z P_{k,l} a^*(\omega, z, \bar{z}) a(\omega, z, \bar{z}), \\ \mathcal{L}_n &= \int \omega d\omega d^2z a^*(\omega, z, \bar{z}) L_n a(\omega, z, \bar{z}), \\ \bar{\mathcal{L}}_n &= \int \omega d\omega d^2z a^*(\omega, z, \bar{z}) \bar{L}_n a(\omega, z, \bar{z}), \end{aligned} \quad (40)$$

with $k, l \in \mathbb{Z} + \frac{1}{2}, n \in \mathbb{Z}$ and $P_{k,l}, L_n, \bar{L}_n$ given by Eqs. (18), (28), and (29), if one assumes for the canonical Poisson bracket

$$\begin{aligned} \{ a(\omega, z, \bar{z}), a^*(\omega', z', \bar{z}') \} \\ = -i(2\pi)^3 \frac{1}{\omega} \delta(\omega - \omega') \delta^2(z - z'). \end{aligned} \quad (41)$$

Note that the transformation of $a(\omega, z, \bar{z})$ induces a transformation of the KG field via (30) which is nonlocal except for the ordinary translations and Lorentz transformations.

An expression for the supertranslations in terms of the Fourier modes of a Klein Gordon field was derived also in [18].

V. KLEIN-GORDON FIELD IN THE BOOST BASIS

The scalar field can be expanded in the boost basis as

$$\begin{aligned} \phi(x) &= \frac{1}{(2\pi)^3} \int d^2z \int_{c-i\infty}^{c+i\infty} \frac{d\Delta}{i2\pi} [\phi_{\Delta}^+(x, z, \bar{z}) a_{2-\Delta}(z, \bar{z}) \\ &+ \phi_{\Delta}^-(x, z, \bar{z}) a_{2-\Delta}^{\dagger}(z, \bar{z})], \end{aligned} \quad (42)$$

where we have used the inverse Mellin transform formula

$$\int_{c-i\infty}^{c+i\infty} \frac{\omega^{-\Delta} \Gamma(\Delta)}{(\epsilon \mp i q \cdot x)^{\Delta}} \frac{d\Delta}{i2\pi} = e^{\pm i \omega q \cdot x}, \quad (43)$$

with $\epsilon > 0$ a regulator to avoid the singularity at $q \cdot x = 0$ and we have defined, as in [19],

$$a_{\Delta}(z, \bar{z}) = \int_0^{\infty} \omega^{\Delta-1} a(\omega, z, \bar{z}) d\omega, \quad (44)$$

$$a_{\Delta}^{\dagger}(z, \bar{z}) \equiv \int_0^{\infty} \omega^{\Delta-1} a^{\dagger}(\omega z, \bar{z}) d\omega. \quad (45)$$

They obey

$$\begin{aligned} \{ a_{\Delta}(z, \bar{z}), (a_{\Delta'}(z', \bar{z}'))^{\dagger} \} \\ = -i(2\pi)^4 \delta^2(z - z') \delta(i(\Delta + \Delta'^* - 2)), \end{aligned} \quad (46)$$

with the generalized delta function defined as in [20].

The wave functions $\phi_{\Delta}^{\pm}$ appearing in (42) are the Mellin transforms of the plane waves

$$\phi_{\Delta}^{\pm}(x; z, \bar{z}) = \int_0^{\infty} d\omega \omega^{\Delta-1} e^{i(\pm q^{\mu} x_{\mu} + i\epsilon)\omega}. \quad (47)$$

The integral in the right-hand side of (47) gives $\phi_{\Delta}^{\pm} = (\mp i)^{\Delta} \Gamma(\Delta) \varphi_{\Delta}^{\pm}$, where

$$\varphi_{\Delta}^{\pm}(x, z, \bar{z}) = \frac{1}{(-q \cdot x_{\pm})^{\Delta}}, \quad (48)$$

where $x_{\pm}^{\mu} = (x^0 \mp i\epsilon, x^1, x^2, x^3)$. Using the boost differential operators in space-time $M_{0i} = -x^0 \partial_i - x_i \partial^0$ and that

$$q^i M_{0i}(q^{\mu} x_{\mu}) = -x^0 (q^i)^2 - x_i q^i q^0 = -q^0 q^{\mu} x_{\mu}, \quad (49)$$

where $(q^i)^2 = (q^0)^2$ has been used, one can prove that

$$q^i M_{0i} \phi_{\Delta}^{\pm}(x; z, \bar{z}) = q^0 \Delta \phi_{\Delta}^{\pm}(x; z, \bar{z}), \quad (50)$$

and hence that the space-time functions $\phi_{\Delta}^{\pm}(x; z, \bar{z})$ are eigenfunctions of a boost transformation in the q^i direction, with eigenvalue $q^0 \Delta$.

We can now compute, by using Eqs. (35), (44), and (45), the infinitesimal variation $\delta a_{\Delta}(z, \bar{z})$

$$\begin{aligned} \delta a_{\Delta}(z, \bar{z}) &= \int_0^{\infty} d\omega \omega^{\Delta-1} \delta a(\omega, z, \bar{z}) \\ &= \int_0^{\infty} d\omega \omega^{\Delta-1} \\ &\quad \times \left(\frac{i}{2} P_{k,l} b_{k,l} + L_n \epsilon_n + \bar{L}_n \bar{\epsilon}_n \right) a(\omega, z, \bar{z}) \\ &= i z^k \bar{z}^l b_{k,l} a_{\Delta+1} + \int_0^{\infty} d\omega (L_n \epsilon_n + \bar{L}_n \bar{\epsilon}_n) a(\omega, z, \bar{z}). \end{aligned} \quad (51)$$

Let us now evaluate

$$\begin{aligned}
& \int_0^\infty d\omega L_n a(\omega, z, \bar{z}) \\
&= \int_0^\infty d\omega \omega^{\Delta-1} \left(-z^{n+1} \partial_z + \frac{n+1}{2} z^n \omega \partial_\omega \right) a(\omega, z, \bar{z}) \\
&= -z^{n+1} \partial_z a_\Delta + \frac{n+1}{2} z^n \int_0^\infty d\omega \omega^\Delta \partial_\omega a(\omega, z, \bar{z}) \\
&= -z^{n+1} \partial_z a_\Delta + \frac{n+1}{2} z^n \int_0^\infty d\omega \omega^\Delta e^{-\epsilon \omega} \partial_\omega a(\omega, z, \bar{z}) \\
&= -z^{n+1} \partial_z a_\Delta - \Delta \frac{n+1}{2} z^n \int_0^\infty d\omega \omega^{\Delta-1} e^{-\epsilon \omega} a(\omega, z, \bar{z}) \\
&= -z^{n+1} \partial_z a_\Delta - \Delta \frac{n+1}{2} z^n a_\Delta(z, \bar{z}), \tag{52}
\end{aligned}$$

where we have added a convergence term. A similar procedure can be followed for the $\bar{L}_n$ term. Summing up, the infinitesimal variation $\delta a_\Delta(z, \bar{z})$ under supertranslations and superrotations is

$$\begin{aligned}
\delta a_\Delta(z, \bar{z}) &= i z^k \bar{z}^l b_{k,l} a_{\Delta+1} \\
&\quad - \epsilon_n \left[z^{n+1} \partial_z a_\Delta + \Delta \frac{n+1}{2} z^n a_\Delta(z, \bar{z}) \right] \\
&\quad - \bar{\epsilon}_n \left[\bar{z}^{n+1} \partial_{\bar{z}} a_\Delta + \Delta \frac{n+1}{2} \bar{z}^n a_\Delta(z, \bar{z}) \right] \tag{53}
\end{aligned}$$

which, by identifying,

$$T = z^k \bar{z}^l b_{k,l}, \quad \mathcal{Y}^z = \epsilon_n z^{n+1}, \quad \mathcal{Y}^{\bar{z}} = \bar{\epsilon}_n \bar{z}^{n+1} \tag{54}$$

and taking into account that for a scalar field $h = \bar{h} = \Delta$, agrees with Eq. (4.50) of [8] and extends it to include supertranslations and superrotations.

VI. RELATION WITH THE CONSTRUCTION IN LONGHI-MATERASSI

In [5], Longhi and Materassi give an explicit expression of the canonical realization of supertranslations for the massive case in terms of hypergeometric functions for the radial part and spherical harmonics for the angular dependence, and the relation of their expressions with the ones that we have obtained using the celestial coordinates should be investigated. Since we only have studied here the massless case, we will assume that the massless Longhi-Materassi construction has the same form as the massive one, replacing the complicated radial dependence by the simple form present in our solutions.

In order to make the connection more transparent, it is convenient to use a different parametrization to the one in (1), namely

$$p^\mu = \varpi \pi^\mu, \quad \pi^\mu = \left[1; \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}} \right], \tag{55}$$

where

$$\varpi = \omega(1+z\bar{z}). \tag{56}$$

In this case $(\pi^1)^2 + (\pi^2)^2 + (\pi^3)^2 = 1$ so that (π^1, π^2, π^3) are euclidean coordinates for points on the unit sphere. It will prove convenient to change coordinates. To this end, let us define

$$\zeta = \frac{2z}{1+z\bar{z}}. \tag{57}$$

In these new variables,

$$\pi^\mu = \left[1; \frac{1}{2}(\zeta + \bar{\zeta}), -\frac{i}{2}(\zeta - \bar{\zeta}), \sqrt{1 - \zeta\bar{\zeta}} \right] \tag{58}$$

and we have

$$\begin{aligned}
\partial_\omega &= (1+z\bar{z}) \partial_\varpi, \\
\partial_z &= \omega \bar{z} \partial_\varpi + \frac{2}{(1+z\bar{z})^2} \partial_\zeta - \frac{2\bar{z}^2}{(1+z\bar{z})^2} \partial_{\bar{\zeta}}, \\
\partial_{\bar{z}} &= \omega z \partial_\varpi + \frac{2}{(1+z\bar{z})^2} \partial_{\bar{\zeta}} - \frac{2z^2}{(1+z\bar{z})^2} \partial_\zeta. \tag{59}
\end{aligned}$$

Changing coordinates from $(\omega, z, \bar{z})$ to $(\varpi, \zeta, \bar{\zeta})$, neither the second Casimir

$$\begin{aligned}
C_2 &= \sum_i M_{0i}^2 - M_{12}^2 - M_{23}^2 - M_{31}^2 \\
&= -\omega(3\partial_\omega + \omega\partial_\omega^2) \\
&= -\varpi(3\partial_\varpi + \varpi\partial_\varpi^2), \tag{60}
\end{aligned}$$

nor the rotation generator

$$M_{12} = z\partial_z - \bar{z}\partial_{\bar{z}} = \zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}} \tag{61}$$

change form. Therefore in the new variables the common eigenfunctions of C_2 (regular and with eigenvalue -3) and M_{12} take the form

$$\mathcal{P}_{p,q} := \varpi \zeta^p \bar{\zeta}^q f(|\zeta|) = 2^{p+q} \frac{\omega z^p \bar{z}^q}{(1+|z|^2)^{p+q-1}} f\left(\frac{2|z|}{1+|z|^2}\right) \tag{62}$$

with $p, q \in \mathbb{Z}$. The dependence on the arbitrary differentiable function $f(|\zeta|)$ is due to the fact that it solves the homogeneous equation (16) and taking Eq. (61) into account. These functions $\mathcal{P}_{p,q}$ can be related to the $P_{k,l}$ in Sec. II by $P_{k,l} = \mathcal{P}_{k+\frac{1}{2}, l+\frac{1}{2}}$.

We are now in a position to make contact with spherical harmonics. Spherical harmonics are a basis for the space of square-integrable functions on the sphere and this condition constrains the function $f(|\zeta|)$ above. Indeed, viewing the sphere as the unit sphere in three-dimensional Euclidean space, spherical harmonics are the restriction to the sphere of complex-valued homogeneous harmonic polynomials in the Euclidean coordinates. More explicitly and up to normalization, the spherical harmonic $Y_{\ell m}$ is the restriction to the sphere of a complex-valued homogeneous polynomial $\Pi_{\ell m}$ in (π^1, π^2, π^3) of degree ℓ , which is harmonic

$$\Delta \Pi_{\ell m} = 0, \quad (63)$$

where Δ is the Euclidean Laplacian

$$\Delta = \partial_{\pi^1}^2 + \partial_{\pi^2}^2 + \partial_{\pi^3}^2, \quad (64)$$

and such that $\Pi_{\ell m}$ is an eigenvalue of M_{12} with eigenvalue $m \in \{-\ell, -\ell + 1, \dots, \ell - 1, \ell\}$:

$$M_{12} \Pi_{\ell m} = m \Pi_{\ell m}. \quad (65)$$

Homogeneous harmonic polynomials in (π^1, π^2, π^3) of degree ℓ are given by

$$a_{i_1 \dots i_\ell} \pi^{i_1} \dots \pi^{i_\ell}, \quad (66)$$

where $a_{i_1 \dots i_\ell}$ is totally symmetric and traceless [21]:

$$\delta^{ij} a_{ij i_1 \dots i_{\ell-2}} = 0. \quad (67)$$

To impose the condition (65), it is convenient to reexpress $\Pi_{\ell m}$ in terms of $\zeta = \pi^1 + i\pi^2$, $\bar{\zeta} = \pi^1 - i\pi^2$ and π^3 , since these coordinates diagonalize M_{12} :

$$M_{12} \zeta = \zeta, \quad M_{12} \bar{\zeta} = -\bar{\zeta} \quad \text{and} \quad M_{12} \pi^3 = 0. \quad (68)$$

The Euclidean Laplacian in those coordinates is given by

$$\Delta = 4\partial_{\zeta} \partial_{\bar{\zeta}} + \partial_{\pi^3}^2. \quad (69)$$

The first few harmonic homogeneous polynomials are easy to write down:

$$\begin{aligned} \Pi_{00} &= 1 & \Pi_{22} &= \zeta^2 \\ \Pi_{11} &= \zeta & \Pi_{21} &= \zeta \pi^3 \\ \Pi_{10} &= \pi^3 & \Pi_{20} &= \zeta \bar{\zeta} - 2(\pi^3)^2, \end{aligned} \quad (70)$$

with the convention that $\Pi_{\ell, -m} = \bar{\Pi}_{\ell m}$. Their restriction to the sphere gives the first few spherical harmonics, up to normalization.

In terms of the original coordinates $(z, \bar{z})$, the common *normalizable* eigenfunctions of C_2 (regular and with eigenvalue -3) and M_{12} take the form

$$\mathcal{P}_{\ell m} = \varpi \Pi_{\ell m}(\zeta, \bar{\zeta}) = \omega(1 + |z|^2) \Pi_{\ell m} \left(\frac{2z}{1 + |z|^2}, \frac{2\bar{z}}{1 + |z|^2} \right), \quad (71)$$

where we have already restricted the polynomials $\Pi_{\ell m}$ to the sphere. The first few are then

$$\begin{aligned} \mathcal{P}_{00} &= \omega(1 + |z|^2), & \mathcal{P}_{22} &= \frac{4\omega z^2}{1 + |z|^2}, \\ \mathcal{P}_{11} &= \frac{1}{2}\omega z, & \mathcal{P}_{21} &= \frac{2\omega z(1 - |z|^2)}{1 + |z|^2}, \\ \mathcal{P}_{10} &= \omega(1 - |z|^2), & \mathcal{P}_{20} &= -\frac{2\omega(|z|^4 - 4|z|^2 + 1)}{1 + |z|^2}, \end{aligned} \quad (72)$$

which, up to normalization, are the massless analogs of the solutions found by Longhi and Materassi in the massive case. Expanding the expressions in (72) in powers of $z\bar{z}$ one can obtain infinite series expressions in terms of positive powers of z and $\bar{z}$, which are a subset of the solutions $P_{k,l}$ given in (18).

It is also worth mentioning that, unlike the solutions in terms of spherical harmonics, our supertranslations are not, except for the ordinary space-time translations, eigenfunctions of the angular momentum differential operator,

$$\begin{aligned} J^2 &= M_{12}^2 + M_{23}^2 + M_{31}^2 \\ &= -\omega^2 z \bar{z} \partial_{\omega}^2 + \omega z(1 + z \bar{z}) \partial_{\omega} \partial_z + \omega \bar{z}(1 + z \bar{z}) \partial_{\omega} \partial_{\bar{z}} \\ &\quad - (1 + z \bar{z})^2 \partial_z \partial_{\bar{z}} + \omega(1 - z \bar{z}) \partial_{\omega}. \end{aligned} \quad (73)$$

VII. CONCLUSIONS AND OUTLOOK

In this work we have seen that the extended BMS algebra can be realized in terms of differential operators of degrees zero and one in celestial coordinates on the light cone, showing its kinematical nature, like the Poincaré group. As an example, we have built the realization of the extended BMS algebra for a massless Klein-Gordon field. The same procedure should work for other massless free fields. The relation to the original construction of Longhi and Materassi, in terms of spherical harmonics, has also been investigated. Notice that the solutions (62) are more general than the ones corresponding to spherical harmonics, since the later are obtained with non-negative p, q . This is related to the fact that the spherical harmonics provide a basis for L^2 functions on the sphere, which is a requirement that has not been imposed in our construction.

One can try to extend the construction described in this paper to the massive case, where the celestial coordinates are parametrized as

$$p^\mu = \frac{m}{2y} [1 + y^2 + z \bar{z}; z + \bar{z}, -i(z - \bar{z}), 1 - y^2 - z \bar{z}] \quad (74)$$

with inverse relation

$$y = \frac{m}{p^0 + p^3}, \quad z = \frac{p^1 + ip^2}{p^0 + p^3}, \quad (75)$$

and Casimir operator given by

$$C_2 = -y^2 \frac{\partial}{\partial y^2} + y \frac{\partial}{\partial y} - 4y^2 \partial_z \partial_{\bar{z}}. \quad (76)$$

The construction of the supertranslations in these coordinates should now recover the original results of Longhi and Materassi, given in terms of hypergeometric functions and spherical harmonics. However, if one tries to construct superrotations as first-order differential operators that commute with the Casimir and are divergentless as vector fields, one has that the only solutions are the ordinary Lorentz generators. This is due to the fact that any vector field commuting with the Laplace-Beltrami operator associated to a metric is necessarily Killing and the space of Killing vectors is bounded above by $n(n+1)/2$ for an n -dimensional manifold. The Lorentz Casimir acting on functions is the Laplace-Beltrami operator on functions on the mass hyperboloid (which is a hyperbolic space with $n = 3$ in $3+1$), and hence the only vector fields which commute with it are the isometries of the mass hyperboloid (see, e.g., [22], Proposition 4.2), which in this case is a six-dimensional Lie algebra spanned by the Lorentz generators. Hence, no superrotations are allowed in the massive case. Problems with the existence of superrotations in the massive case in $2+1$ were also pointed out in [23], using a different approach.

It would be interesting to understand the role of the Lorentz Casimir, which in our derivation is crucial, in the analysis of celestial amplitudes. Due to the isomorphism between extended BMS and conformal Carroll symmetries, a better understanding of the relation between our approach and the magnetic Carrollian field theories in $2+1$ dimensions [14–16] deserves further study.

Another application of our approach would be to consider the $w_{1+\infty}$ algebra [24], where the ω independent part is already known [25].

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: COMMENT ON ZERO MODES OF THE ON-SHELL FOURIER TRANSFORM

Let us first study the equation

$$0 = \int \widetilde{d\vec{p}} [e^{ip^\mu x_\mu} f(\vec{p}) + \text{H.c.}]; \quad (\text{A1})$$

by differentiating with respect to x^0 we get

$$0 = \int \widetilde{d\vec{p}} (ip^0) [e^{ip^\mu x_\mu} f(\vec{p}) - \text{H.c.}]. \quad (\text{A2})$$

Let us multiply the first equation by $(ip^0)e^{-ip^\mu x_\mu}$ and the second by $e^{-ip^\mu x_\mu}$, consider the difference and integrate over d^3x . We obtain

$$\begin{aligned} 0 &= \int d^3x \int \widetilde{d\vec{p}} \{ ip^0 [e^{i(p^\mu - p'^\mu)x_\mu} f(\vec{p}) + e^{-i(p^\mu + p'^\mu)x_\mu} f^\dagger(\vec{p})] + ip^0 [e^{i(p^\mu - p'^\mu)x_\mu} f(\vec{p}) - e^{-i(p^\mu + p'^\mu)x_\mu} f^\dagger(\vec{p})] \} \\ &= (2\pi)^3 \int \widetilde{d\vec{p}} \{ ip^0 [\delta^3(p - p') e^{-i(p^0 - p'^0)x^0} f(\vec{p}) + \delta^3(p + p') e^{i(p^0 + p'^0)x^0} f^\dagger(\vec{p})] \\ &\quad + ip^0 [\delta^3(p - p') e^{-i(p^0 - p'^0)x^0} f(\vec{p}) - \delta^3(p + p') e^{i(p^0 + p'^0)x^0} f^\dagger(\vec{p})] \} \\ &= (2\pi)^3 if(\vec{p}') \end{aligned} \quad (\text{A3})$$

where we have used

$$\int \widetilde{d\vec{p}} = \int \frac{d^3p}{2p^0} \quad (\text{A4})$$

and $p^0 = \sqrt{\bar{p}^2}$. In conclusion from Eq. (A1) we obtain

$$f(\vec{p}) = 0. \quad (\text{A5})$$

This computation can be also repeated in the celestial basis, making use of

$$d^3 p = 2\omega^2(1 + \bar{z}z)d\omega d^2 z \quad (\text{A6})$$

and

$$\delta^3(p' - p) = \frac{1}{2\omega^2(1 + \bar{z}z)} \delta(\omega' - \omega) \delta^2(z' - z). \quad (\text{A7})$$

Then, from

$$0 = \int \omega d\omega d^2 z [e^{i\omega q^\mu x_\mu} f(\omega, z, \bar{z}) + \text{H.C.}], \quad (\text{A8})$$

one can deduce

$$f(\omega, z, \bar{z}) = 0. \quad (\text{A9})$$

APPENDIX B: LORENTZ AND $PSL(2, \mathbb{C})$ TRANSFORMATIONS

The generic matrix $M \in SL(2, \mathbb{C})$ is written as

$$M = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}, \quad (\text{B1})$$

with the condition $\det M = ad - bc = 1$. For an infinitesimal transformation we parametrize M as

$$M = \begin{pmatrix} 1 + \alpha & -\gamma \\ -\beta & 1 - \alpha \end{pmatrix}. \quad (\text{B2})$$

$SL(2, \mathbb{C})$ acts on the space of momenta (identified with the space of Hermitian 2×2 matrices) via Lorentz transformations $X \mapsto MXM^\dagger$. Since $\pm I$ act trivially, (the identity component of) the Lorentz group is isomorphic to $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm I\}$.

Using that isomorphism, we can compute the infinitesimal Lorentz transformations of $\omega, z, \bar{z}$:

$$\delta\omega = \frac{1}{2}(\delta p^0 + \delta p^3) = \omega[-(\alpha + \bar{\alpha}) + \gamma z + \bar{\gamma} \bar{z}] \quad (\text{B3})$$

and

$$\delta z = \delta \left(\frac{p^1 + ip^2}{p^0 + p^3} \right) \sim 2\alpha z - \gamma z^2 + \beta \quad (\text{B4})$$

that coincide with the results of [7,20] when expanding the finite transformations for infinitesimal parameters.

In an analogous way we can obtain

$$\delta\bar{z} = 2\bar{\alpha} \bar{z} - \bar{\gamma} \bar{z}^2 + \bar{\beta}. \quad (\text{B5})$$

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