

Probing the vacuum as a chiral medium

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We study the *circular birefringence* experienced by linearly polarized photons colliding with a circularly polarized background creating a vacuum of definite chirality (handedness). For this scenario the standard Heisenberg-Euler approach fails and must be supplemented by derivative corrections which we match to known Hilbert series. Choosing a plane wave background, we find equivalence in the low-energy limit between three approaches: (i) adding derivative corrections to the Heisenberg-Euler Lagrangian, (ii) improving the locally constant field approximation to the one-loop polarization tensor, and (iii) performing a low-energy expansion of the direct $2 \rightarrow 2$ QED photon-photon scattering amplitude. Going beyond plane-wave backgrounds, we analyze an example of a circularly polarized standing wave sensitive to derivative corrections. We find a parameter regime where these corrections could be probed in experiments.

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I. INTRODUCTION

Vacuum birefringence is a long-standing prediction of quantum electrodynamics (QED), which still remains to be tested in intense fields. Its history and current status has recently been reviewed in [1]. We thus only reemphasize that despite tantalizing hints from astrophysics [2], photon-scattering experiments at colliders [3–8], and long-running dedicated cavity experiments [9,10], vacuum birefringence has so far remained elusive. However, an experimental confirmation is actively pursued using the high intensity of powerful optical laser pulses [1,11,12] to coherently enhance the very small cross section [13] for vacuum birefringence. (For reviews on photon-photon scattering and strong-field QED see, e.g., [14–18].)

The leading contribution to vacuum birefringence is from a single electron-positron loop, dressed by the electromagnetic background. For photon-laser and laser-laser collisions, the center-of-mass energy is typically much less than the electron rest mass. In this case, an effective approach can be used to calculate experimental observables, based on an effective Lagrangian which schematically can be written as

$$\mathcal{L}_{\text{eff}} = \sum_{n,i} \mathcal{L}_{n,i}, \quad \mathcal{L}_{n,i} \sim c_{n,i} \partial^{(i)} \mathcal{O}^{(n)}. \quad (1)$$

Here $c_{n,i}$ are low-energy constants with the integer n denoting the number of electromagnetic field tensors, F , appearing in the operator $\mathcal{O}^{(n)}$ and the index i counting the number of derivatives¹ acting on $\mathcal{O}^{(n)}$. (Alternative approaches to including derivative corrections also exist [19].) The many scenarios considered in the literature until now (as reviewed in [1]), study vacuum *linear* birefringence, for which the leading order (LO) contribution stems from $\mathcal{L}_{4,0}$, i.e., from the terms without extra derivatives. However, for vacuum *circular* birefringence [20,21], which can arise in a circularly polarized (CP) laser, we will show the leading contribution is from the next-to-leading order (NLO) (in derivatives), i.e., from $\mathcal{L}_{4,2}$. Circular birefringence therefore provides an experimental test of higher-dimensional operators in the effective field theory.

We begin in Sec. II by reviewing derivative corrections of the form $\mathcal{L}_{4,n}$ and show how they are related to helicity amplitudes expressed through Mandelstam variables. In Sec. III, we introduce our setup and an intuitive formalism based on the language of two-level systems (Jones vectors). In Sec. IV, we focus on the explicit example of a plane-wave background, and demonstrate the equivalence of three approaches to calculate the helicity flip probabilities: (i) adding derivative corrections to the locally constant field approximation (LCFA), (ii) adding derivative corrections to the HE Lagrangian, and (iii) evaluating the low-energy limit of QED 4-photon amplitudes. In Sec. V we use the effective approach to calculate circular birefringence in

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¹For a given number of derivatives and field strengths, there are typically several inequivalent terms, $\mathcal{L}_{n,i,j}$, each with its own coupling, $c_{n,i,j}$, $j = 1, 2, \dots, k$. To avoid index inflation, we suppress the subscript j .

a scenario replacing the highly symmetric plane wave by a standing circular wave. Section VI presents our conclusions.

II. DERIVATIVE CORRECTIONS TO LOW-INTENSITY HEISENBERG-EULER LAGRANGIAN

At energies below the pair threshold, QED (in $d = 4$, which we consider throughout) is well described by augmenting the Maxwell Lagrangian with the Heisenberg-Euler (HE) correction [22], $\mathcal{L}_{\text{HE}} \equiv \sum_n \mathcal{L}_{n,0}$. To LO in field strength ($n = 4$) this correction takes on the well-known explicit form

$$\begin{aligned} \mathcal{L}_{4,0} &\equiv \frac{\alpha^2}{360m^4} [4(\text{tr}F^2)^2 + 7(\text{tr}F\tilde{F})^2] \\ &\equiv \frac{\alpha^2}{180m^4} [-5(\text{tr}F^2)^2 + 14\text{tr}F^4]. \end{aligned} \quad (2)$$

Note that there are *two* independent terms at this order. In accordance with (1), the subscripts 4,0 signal that each term in the LO HE Lagrangian contains $n = 4$ factors of the Maxwell field strength, F , and *no* derivatives. This latter restriction ($i = 0$) will be relaxed shortly. For the second equality, we have used the identity [23]

$$(\text{tr}F\tilde{F})^2 = 4\text{tr}F^4 - 2(\text{tr}F^2)^2, \quad (3)$$

which allows one to write $\mathcal{L}_{4,0}$ entirely in terms of the field strength, F , without employing its dual, $\tilde{F}$, with components $\tilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. The Lagrangian (2), first considered by Heisenberg's students Euler and Kockel [24], is the leading term in a weak-field expansion that is known to all orders in F and $\tilde{F}$ [22], hence in resummed form. Due to Lorentz and gauge invariance, the complete HE Lagrangian only depends on the scalar and pseudoscalar invariants

$$\mathcal{S} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \equiv \frac{1}{4}\text{tr}F^2, \quad (4)$$

$$\mathcal{P} = -\frac{1}{4}F^{\mu\nu}\tilde{F}_{\mu\nu} \equiv \frac{1}{4}\text{tr}F\tilde{F}. \quad (5)$$

(In what follows, we will often omit the trace symbol to save space.) As QED conserves parity, only even powers of $\mathcal{P}$ (or $\tilde{F}$) are present. Allowing for parity violation would induce odd powers of $\mathcal{P}$ and hence enlarge the operator basis. At LO, for instance, a third term, proportional to $\mathcal{S}\mathcal{P}$, would appear.

Physically, the LO Lagrangian (2) describes the self-interactions of photons, hence photon-photon scattering, induced by the QED one-loop box diagram. At low center-of-mass energy squared, $s \ll m^2$, the fermion loop is replaced by an effective vertex of order α^2/m^4 . At NLO

in field strength corresponding to $\mathcal{L}_{6,0}$, the effective vertex is order α^3/m^8 , and so on.

From a modern perspective, the HE Lagrangian is a paradigmatic example of a low-energy effective field theory: the effective Lagrangian is an expansion in terms of operators of increasing mass dimension, each term weighted by a dimensionless coupling constant divided by the appropriate energy or cutoff scale (here the electron mass, m). In condensed notation, this expansion (in $d = 4$) may be written as in (1),

$$\mathcal{L}_{\text{eff}} = \sum_{n,i} \mathcal{L}_{n,i} \equiv \sum_{n,i} c_{n,i} \frac{\alpha^{n/2}}{m^{2n+i-4}} \partial^{(i)} F^n, \quad (6)$$

guaranteeing mass dimension 4 for the Lagrangian with dimensionless low-energy constants $c_{n,i}$. Furry's theorem demands that the number n of field operators F (or its dual) is even. Their total mass dimension is $2n$ to which one has to add an even number i counting the derivatives as well as their mass dimension. The original all-orders HE Lagrangian is the sum of all terms with $n \geq 4$ and no derivatives ($i = 0$), which hence represents the QED effective action in a constant background. The low-energy couplings, $c_{n,i}$, are determined by the QED Lagrangian, the ultraviolet completion of the effective Lagrangian (6). The underlying procedure is referred to as matching: one demands that the low-energy limit of the QED photon scattering amplitude agrees with its pendant calculated in the effective theory given by $\mathcal{L}_{\text{eff}}$.

In this paper, we are interested in the derivative corrections with $i = 2$, the minimum value required by Lorentz invariance. By construction, these corrections extend the validity of the effective Lagrangian to higher energies. At LO in field strength, $n = 2$, there is only one term, the LO vacuum polarization correction, which has been known since the original calculations of Dirac [25] and Heisenberg [26] at the dawn of QED. In our notation, one has

$$\mathcal{L}_{2,2} = \frac{1}{60\pi} \frac{\alpha}{m^2} F_{\mu\nu} \square F^{\mu\nu}. \quad (7)$$

This is often referred to as the (low-energy limit of) the Uehling potential [27] induced by vacuum polarization. As shown in [28], this term is a redundant operator in the effective field theory as it can be removed by a field redefinition. Adding just $\mathcal{L}_{2,2}$ to the Maxwell Lagrangian (with $c_{2,2}$ as a free parameter), yields what is called Bopp-Podolsky electrodynamics [29,30]. It was originally introduced to modify the UV behavior of the theory and to deal with the problem of radiation reaction. This has recently been taken up again in [31,32].

The next correction with two derivatives involves four field strengths ($n = 4$). To the best of our knowledge, it was first determined by Dicus *et al.* [33] who found *three* terms at this order,

$$\begin{aligned}\mathcal{L}_{4,2} = & \frac{1}{945} \frac{\alpha^2}{m^6} [(\partial_\alpha \partial_\beta F_{\mu\nu}) F^{\mu\nu} F_\rho^\alpha F^{\rho\beta} \\ & + 3(\partial^\alpha F_{\mu\nu})(\partial_\alpha F^{\mu\nu}) F_{\rho\sigma} F^{\rho\sigma} \\ & + 11(\partial^\alpha F_{\mu\nu}) F^{\nu\rho} (\partial_\alpha F_{\rho\sigma}) F^{\sigma\mu}].\end{aligned}\quad (8)$$

This correction has been recalculated at least twice, first by Gusynin and Shovkovy, who obtain seven terms altogether [34]. However, they state that on-shell (when $k^2 = 0$, equivalent to using the free equation of motion), their answer reduces to the three terms in (8). More recently, a result with four terms has been reported by Karbstein [35]. It can be straightforwardly shown, however, that two of Karbstein's terms can be combined after an integration by parts. One can check the three couplings in Eq. (8) by leaving them undetermined and matching with the QED amplitudes for photon-photon scattering in the low-energy limit [36]. We performed this check and confirm the couplings in Eq. (8) as given by [33]. For the sake of completeness, we note that Dicus *et al.* have also worked out the next order ($n = i = 4$), which again yields three terms,

$$\begin{aligned}\mathcal{L}_{4,4} = & \frac{1}{9450} \frac{\alpha^2}{m^8} [-33(\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\mu\nu})(\partial_\beta F_{\rho\sigma})(\partial^\beta F^{\rho\sigma}) \\ & + 106(\partial_\alpha F_{\mu\nu})(\partial^\beta F^{\mu\nu})(\partial_\beta F_{\rho\sigma})(\partial^\alpha F^{\rho\sigma}) \\ & - 262(\partial^\alpha \partial_\beta F_{\mu\rho}) F^{\rho\nu} (\partial^\mu \partial_\nu F_{\alpha\sigma}) F^{\sigma\beta}].\end{aligned}\quad (9)$$

The results above imply that at $n = 4$ the number of terms with $i = 0, 2, 4$ derivatives in the effective Lagrangian equals $k = 2, 3, 3$, respectively. We have seen, however, that even for $i = 2$ there are three different answers in the literature, although these could be boiled down to a single Lagrangian with $k = 3$ terms. It would be useful to have a general counting scheme for these operators, in particular when going to higher orders. Fortunately, such a counting scheme exists and goes by the name of *Hilbert series*. Put in simplest terms, a Hilbert series is just the generating function for the number k of operators of a given mass dimension in an effective field theory. As usual, cf. (1) and (6), these operators are ordered by increasing mass dimension. For any given mass dimension, the choice of operators is not unique as there are a number of redundancies such as integration-by-parts (IBP) and equation-of-motion (EOM) ambiguities which allow for field redefinitions. This freedom of choice is accompanied by restrictions due to symmetry. For instance, if one imposes parity symmetry, the number of operators gets reduced as briefly discussed above. Analogous statements hold for other symmetries, both local and global. In any case, the possible equivalent choices of operators yield the same physics (or S-matrix). The existence of a Hilbert series then implies that the *number* of operators (hence the number of couplings) for a given mass dimension is unique. The literature on Hilbert series by now is too extensive for providing a complete list of references. Brief reviews with an

emphasis on effective field theory may be found, e.g., in [37–41] which also list some relevant earlier references.

For the case at hand, i.e., the fourth order field strength operators contributing to $\mathcal{L}_{4,i}$, the Hilbert series may be found in Table 2 of [41] (the case including parity symmetry), and explicitly reads

$$\begin{aligned}H_{F^4}^P(t) &= \frac{2 + 3t^2 + t^4 - t^6}{(1 - t^4)(1 - t^6)} \\ &= 2 + 3t^2 + 3t^4 + 4t^6 + 6t^8 + 5t^{10} + \dots\end{aligned}\quad (10)$$

Looking at the expansion coefficients in the second line we conclude that there are two terms without derivatives (LO HE), cf. (2), three terms with two derivatives, cf. (8), three terms with four derivatives, cf. (9), four terms with six derivatives, and so on.

For later use we also quote the result without parity symmetry from [41],

$$\begin{aligned}H_{F^4}^\not{P}(t) &= \frac{3 + 5t^2 + t^4 - 2t^6}{(1 - t^4)(1 - t^6)} \\ &= 3 + 5t^2 + 4t^4 + 6t^6 + 9t^8 + 7t^{10} + \dots,\end{aligned}\quad (11)$$

which unsurprisingly shows an increase in the number of operators. As expected, at lowest order (no derivatives, $i = 0$), there is one additional term, which we identified earlier as $\mathcal{SP}$, a pseudoscalar that indeed violates parity. QED of course predicts that its coupling is zero, but an axion field, for instance, would couple to this operator. At NLO (two derivatives, $i = 2$) the number of operators increases from three to five. This has recently been confirmed in [42].

There is a rather useful way of addressing the issue of parity by introducing the “Silberstein variables” [43], i.e., the linear combinations,

$$F_\pm := F \pm i\tilde{F}.\quad (12)$$

These are (anti)self-dual and transform under parity according to $F_\pm \rightarrow F_\mp$. For this reason, they are also referred to as chiral fields—the subscripts denoting their chirality ($\pm =$ left/right). The tensors $F_\pm$ have a number of useful properties. For instance, they allow for an elegant classification of electromagnetic field strength tensors according to their Lorentz properties [44]. For our purposes they provide an alternative choice of operator basis which has been employed in a similar context before in [42,45].

To actually perform the change of basis we trade F and $\tilde{F}$ for $F_\pm$ in the invariants (4) and (5),

$$S = \frac{1}{16} (\text{tr} F_+^2 + \text{tr} F_-^2),\quad (13)$$

$$\mathcal{P} = \frac{1}{16i} (\text{tr} F_+^2 - \text{tr} F_-^2).\quad (14)$$

Under parity, the labels $+$ and $-$ get swapped such that $\mathcal{P}$ changes sign as befits a pseudoscalar. Plugging (13) and (14) into (2) and omitting the trace symbols, the LO HE Lagrangian becomes [45]

$$\mathcal{L}_{4,0} = \frac{\alpha^2}{5760m^4} \{-3[(F_+^2)^2 + (F_-^2)^2] + 22F_+^2 F_-^2\}. \quad (15)$$

We note that parity conservation requires that the two terms $(F_\pm^2)^2$ come with the same prefactor, so that the overall number of independent terms remains two. If one allows for parity violation, this is no longer true, and one has three independent terms as stated above, cf. [42].

Recasting $\mathcal{L}_{4,2}$ from (8) into chiral form is a more onerous task. From (11) we know that there are altogether five operators accompanied by two derivatives. These operators are $F_\pm^4$, $F_+^2 F_-^2$ which are already present in $\mathcal{L}_{4,0}$, see (15), plus two additional ones, namely $F_\pm^3 F_\mp$ [42]. In the parity symmetric case (QED), quartic and cubic terms appear in the symmetric combinations $F_+^4 + F_-^4$ and $F_+^3 F_- + F_-^3 F_+$ which reduces the number of independent operators to three as required by the associated Hilbert series (10). A somewhat tedious calculation utilizing IBP and EOM redundancies yields the explicit chiral representation,

$$\mathcal{L}_{4,2} = \frac{\alpha^2}{15120m^6} \{O^{++++} + O^{+---} + O^{++--} + (+ \leftrightarrow -)\}, \quad (16)$$

with the chiral operators given by

$$O^{\pm\pm\pm\pm} \sim -13(F_\pm \partial F_\pm)^2 - 4F_\pm (F_\pm^2 \partial^2) F_\pm, \quad (17)$$

$$O^{\pm\pm\mp\mp} \sim 9(F_\pm \partial F_\pm)(F_\mp \partial F_\mp) + 4F_\mp (F_\pm^2 \partial^2) F_\mp, \quad (18)$$

$$O^{\pm\pm\mp\mp} \sim 4F_\pm (F_\pm F_\mp + F_\mp F_\pm) \partial^2 F_\pm, \quad (19)$$

suppressing the coupling constants throughout. In the above, we have used the abbreviation (omitting $\pm$ subscripts on F),

$$F^2 \partial^2 \equiv F^\alpha_\lambda F^{\lambda\beta} \partial_\alpha \partial_\beta. \quad (20)$$

Once the effective Lagrangian has been written in terms of the chiral fields, one can employ the one-to-one correspondence between the basic operators and the helicity amplitudes for photon-photon scattering. Informally, this can be written as

$$F_{\lambda_1} F_{\lambda_2} F_{\lambda_3} F_{\lambda_4} \sim M^{\lambda_1 \lambda_2 \lambda_3 \lambda_4}, \quad (21)$$

where each $\lambda_i = \pm$ represents a helicity label. The correspondence has some interesting consequences. First note that the LO Lagrangian (15) does not contain terms with an

odd number of $F_\pm$ factors. Hence, the associated helicity amplitudes vanish [45],

$$M^{\pm\mp\mp\mp} = M^{\mp\pm\pm\pm} = M^{\mp\mp\pm\pm} = M^{\mp\pm\mp\mp} = 0. \quad (22)$$

At the root of this property is the fact that the invariants $\mathcal{S}$ and $\mathcal{P}$ contain only even powers of $F_\pm$, cf. (13) and (14). As this is true to all orders in these invariants, all amplitudes with an odd number of either $+$ or $-$ helicities vanish for the complete HE Lagrangian, $\mathcal{L}_{\text{HE}} \equiv \sum_n \mathcal{L}_{n,0}$. In other words, at LO in the derivative expansion ($i = 0$) resummed over all powers n of field strength, F , (22) remains valid [46]. For $n = 4$, $i = 0$, the nonvanishing LO helicity amplitudes are [45,47,48]

$$\begin{aligned} M_{\text{LO}}^{\pm\pm\mp\mp} &= \frac{11\alpha^2}{45m^4} s^2, \\ M_{\text{LO}}^{\pm\mp\pm\mp} &= \frac{11\alpha^2}{45m^4} t^2, \\ M_{\text{LO}}^{\pm\mp\mp\pm} &= \frac{11\alpha^2}{45m^4} u^2, \\ M_{\text{LO}}^{\pm\pm\pm\pm} &= -\frac{\alpha^2}{15m^4} (s^2 + t^2 + u^2), \end{aligned} \quad (23)$$

where all photons are assumed incoming and s, t, u are the usual Mandelstam invariants. The first three lines represent the operator $F_+^2 F_-^2$, the last line the operator $F_+^4 + F_-^4$. We note in passing that these amplitudes have recently been calculated to two-loop order in QED and QCD [48,49] and thus are known to order α^3 and $\alpha^2 \alpha_s$ for arbitrary center-of-mass energy (α_s denoting the strong coupling constant). For our purposes, the first few orders in the low-energy/derivative expansion are sufficient.

At NLO in derivatives ($n = 4$, $i = 2$) one first encounters the new operator $F_+^3 F_- + F_-^3 F_+$. In consequence, the amplitudes (22) no longer vanish [45,47],

$$M_{\text{NLO}}^{\pm\mp\mp\mp} = M_{\text{NLO}}^{\mp\pm\pm\pm} = M_{\text{NLO}}^{\mp\mp\pm\pm} = M_{\text{NLO}}^{\mp\pm\mp\mp} = \frac{-\alpha^2}{315m^6} stu. \quad (24)$$

These are *cubic* in the Mandelstam variables as $stu = -(s^3 + t^3 + u^3)/3$, a consequence of $s + t + u = 0$. The NLO corrections to the amplitudes (23) are found to be [45,47]

$$\begin{aligned} M_{\text{NLO}}^{\pm\pm\mp\mp} &= -\frac{4\alpha^2}{315m^6} s^3, \\ M_{\text{NLO}}^{\pm\mp\pm\mp} &= -\frac{4\alpha^2}{315m^6} t^3, \\ M_{\text{NLO}}^{\pm\mp\mp\pm} &= -\frac{4\alpha^2}{315m^6} u^3, \\ M_{\text{NLO}}^{\pm\pm\pm\pm} &= \frac{2\alpha^2}{63m^6} stu. \end{aligned} \quad (25)$$

These NLO contributions are normally subleading since the Mandelstam variables are small compared to the electron rest mass (squared) in scenarios considered suitable for laser experiments. However, in the case of *circular birefringence*, the LO terms are suppressed and the NLO terms are dominant as will be shown below. Before addressing this issue, we introduce our conventions and some intuitive formalism essentially borrowed from optics.

III. SETUP AND FORMALISM

Let us consider the scattering of a linearly polarized (LP) photon in a CP plane-wave electromagnetic background, characterized by fixed lightlike four-momentum, κ , $\kappa^2 = 0$. The situation is sketched in Fig. 1. Modeling the intense laser background as a plane wave is a reasonable first approximation and allows for a straightforward demonstration of the main phenomenology.

Mathematically, the background is described by a vector potential $a = eA$, rescaled with elementary charge $e > 0$ and decomposed into positive and negative frequency components, $a = a^{(+)} + a^{(-)}$, according to

$$a^{(\pm)} = \frac{m\xi}{\sqrt{2}} \epsilon^{(\mp)} \psi(\varphi) e^{\pm i\varphi}; \quad \epsilon^{(\mp)} = \frac{1}{\sqrt{2}} [\epsilon_1 \mp i\epsilon_2]. \quad (26)$$

The background phase is $\varphi = \kappa \cdot x$, while $\epsilon_{1,2}$ are orthonormal polarization vectors normalized to $(\epsilon_{1,2})^2 = -1$. It follows that $\epsilon^{(\mp)} \cdot \epsilon^{(\mp)*} = -1$ and $a \cdot a = -m^2 \xi^2 \psi^2(\varphi)$ which identifies ξ as the dimensionless laser amplitude and ψ as the phase-dependent profile function. We choose the linear polarization vectors of the probe photon to be

$$\epsilon_j = \epsilon_j - \kappa \frac{\ell \cdot \epsilon_j}{\kappa \cdot \ell}, \quad (27)$$

where $j \in \{1, 2\}$ and $\ell \cdot \epsilon_j = 0$.

A plane-wave background is special because, due to light-front momentum conservation and the on-shell condition, the scattered photon momentum ℓ' is the same as the initial photon momentum ℓ (forward scattering). As there is no momentum transfer, the only quantum number that may change upon scattering is helicity or polarization. We hence

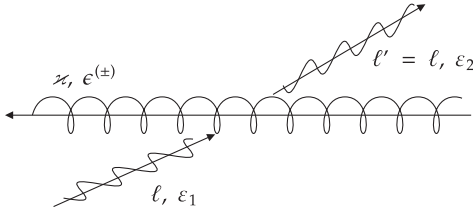


FIG. 1. Microscopic view of the circular birefringence scenario: There is a nonvanishing probability for the *linear* polarization of probe photons to flip upon scattering off the CP plane-wave background (momentum and polarization labels shown).

focus on the probability for polarization flip of an incoming LP photon which we write as

$$P^{(21)} = |\mathfrak{M}^{(21)}|^2; \quad \mathfrak{M}^{(21)} = \langle 2 | S | 1 \rangle \equiv S_{21}, \quad (28)$$

where $|1\rangle$ and $|2\rangle$ represent orthonormal photon linear polarization states corresponding to (27), $\mathfrak{M}$ is the amplitude and S is the scattering matrix. In essence, for a photon colliding with a plane-wave background, the polarization state is described by a two-level system.

We hence employ the simplest possible Hilbert space, $\mathbb{C}^2$, with (polarization) basis vectors $|1\rangle$ and $|2\rangle$ obeying the completeness relation $\mathbb{I} = |1\rangle\langle 1| + |2\rangle\langle 2|$. For illustration purposes, we consider a vacuum with homogeneous refractive index. The associated formalism is the starting point of, and nicely explained in, Baym's text on quantum mechanics [50] where further details (such as the mapping between polarization states and the electromagnetic field) can be found.

The S-matrix is just a unitary two-by-two matrix, its eigenvalues given by phases, $\exp(i\delta_j)$, $j \in \{1, 2\}$. Let us first assume an LP background which represents the standard scenario for vacuum birefringence (as recently reviewed in [11]). In this case, the S-matrix is indeed diagonal,

$$S_{LP} \equiv \sum_{i=1,2} \exp(i\delta_i) |i\rangle\langle i| = \begin{pmatrix} \exp i\delta_1 & 0 \\ 0 & \exp i\delta_2 \end{pmatrix} \equiv \exp iD. \quad (29)$$

The matrix $D \equiv \text{diag}(\delta_1, \delta_2)$ defines the exponential representation of the S-matrix, S_{LP} , a generalization of which has recently been employed in the context of gravitational scattering amplitudes and their classical limit [51,52]. It is useful to introduce half the phase sum and difference,

$$\Sigma \equiv \frac{\delta_1 + \delta_2}{2}, \quad \Delta \equiv \frac{\delta_1 - \delta_2}{2}, \quad (30)$$

and rewrite the S-matrix (29) as

$$S_{LP} = e^{i\Sigma} \begin{pmatrix} \exp i\Delta & 0 \\ 0 & \exp -i\Delta \end{pmatrix} \equiv e^{i\Sigma} \exp i\Delta. \quad (31)$$

This is the polar decomposition of S_{LP} into a $U(1)$ factor $e^{i\Sigma}$ and an $SU(2)$ matrix $\exp i\Delta$ with unit determinant. To elucidate the physics we use the formalism of [50], which relates the phase exponents, δ_i , to the refractive indices, n_i , according to

$$\delta_i = \omega_\ell dn_i \equiv \phi_0 n_i, \quad (32)$$

$$\Sigma = \phi_0 \frac{n_1 + n_2}{2}, \quad (33)$$

$$\Delta = \phi_0 \frac{n_1 - n_2}{2}, \quad (34)$$

where ω_ℓ is the photon energy, d the duration of propagation through the background, and ϕ_0 is the photon phase change without birefringence, i.e., in a classical vacuum. In the weak-field regime, the refractive indices differ only slightly from unity, their vacuum value, such that $\delta n_i \equiv n_i - 1 \ll 1$. We can thus rewrite

$$\Sigma = \phi_0 \left(1 + \frac{\delta n_1 + \delta n_2}{2} \right) \equiv \phi_0 + \sigma, \quad (35)$$

$$\Delta = \phi_0 \frac{\delta n_1 - \delta n_2}{2}, \quad (36)$$

with σ and Δ being small compared to unity, so that they may serve as expansion parameters. The overall phase factor, $e^{i\Sigma} = e^{i\phi_0} e^{i\sigma}$, acts as a straightforward multiplication, shifting all incoming polarizations by the common phase, $\Sigma = \phi_0 + \sigma$.

In the generic case, $\delta_1 \neq \delta_2$ or $n_1 \neq n_2$, the second factor, $\exp i\Delta$, in the S-matrix (31) describes birefringence, as each basis polarization, $|i\rangle$, undergoes its own phase shift, $\pm\Delta$. This may also be inferred by identifying $\mathbf{S}_{\text{LP}}$ with the Jones matrix for a birefringent medium [53,54]. For instance, an initial state polarized linearly at an angle of β_0 from the 1-axis,

$$|\text{in}\rangle = \cos \beta_0 |1\rangle + \sin \beta_0 |2\rangle, \quad (37)$$

will turn into the transformed state,

$$|\text{in}'\rangle = \mathbf{S}_{\text{LP}} |\text{in}\rangle = e^{i\Sigma} (e^{i\Delta} \cos \beta_0 |1\rangle + e^{-i\Delta} \sin \beta_0 |2\rangle). \quad (38)$$

Apart from the overall phase factor, $e^{i\Sigma}$, this is a typical Jones vector for *elliptic* polarization [54], with a phase difference of $2\Delta \equiv \delta_1 - \delta_2$ between the “horizontal” and “vertical” components. The nonflip amplitude measures the “persistence” of the incoming state, hence the overlap

$$\mathfrak{M}_{\text{nf}} = \langle \text{in} | \text{in}' \rangle = e^{i\Sigma} (\cos^2 \beta_0 e^{i\Delta} + \sin^2 \beta_0 e^{-i\Delta}). \quad (39)$$

This result is formally exact, valid to all orders in Σ and in particular Δ , as the S-matrix must have the form (31). However, the phases δ_i will typically be only known through their perturbation series in σ and Δ which in turn will be given in terms of their expansion in the QED coupling α . The LO HE 4-photon vertex implies that both σ and Δ are of order α^2 . For the nonflip amplitude (39), the leading order in σ and Δ corresponds to

$$\mathfrak{M}_{\text{nf}} \simeq e^{i\phi_0} (1 + i\sigma + i\Delta \cos 2\beta_0) = e^{i\phi_0} (1 + i\sigma), \quad (40)$$

where the last equality is valid for the standard choice $\beta_0 = \pi/4$. In this case, the approximate nonflip probability

is thus $P_{\text{nf}} \simeq 1 + \sigma^2$. The fact that this is greater than unity is an artifact of the approximation as the exact result (for $\beta_0 = \pi/4$) is $P_{\text{nf}} = \cos^2 \Delta$. This is a simple example of an apparent unitarity violation that is fixed by “resummation,” i.e., by working out the exact, all-orders answer. Where this is not possible, higher-order corrections may at least be indicative of the remedy [55,56].

Of particular interest is the amplitude for polarization flip into the state

$$|f\rangle = -\sin \beta_0 |1\rangle + \cos \beta_0 |2\rangle, \quad (41)$$

which is orthogonal to $|\text{in}\rangle$. This yields the flip amplitude

$$\begin{aligned} \mathfrak{M}_{\text{f}} &= \langle f | \text{in}' \rangle = -e^{i\Sigma} \sin \beta_0 \cos \beta_0 (e^{i\Delta} - e^{-i\Delta}) \\ &\simeq -ie^{i\phi_0} \Delta \sin 2\beta_0 = -ie^{i\phi_0} \Delta, \end{aligned} \quad (42)$$

with, again, the last identity holding for $\beta_0 = \pi/4$. The exact flip probability is $P_{\text{f}} = \sin^2 \Delta = \Delta^2 + O(\Delta^4)$, such that indeed $P_{\text{nf}} + P_{\text{f}} = 1$. Note that the exact, nonperturbative probabilities are independent of the average σ . In perturbation theory, however, the nonflip and flip amplitudes are governed by both the sum, σ , and difference, Δ , of the vacuum refractive indices. For low-energy QED this translates into a dependence on the sum and difference of the two LO low-energy couplings in the HE Lagrangian, $\mathcal{L}_{4,0}$ [11,57].

In this paper, however, we are interested in a CP background, to be probed by an LP beam, say in state (37). In this case, the S-matrix is diagonal in the *helicity* basis, $|\lambda\rangle = |\pm\rangle$, with states

$$|\pm\rangle \equiv \frac{1}{\sqrt{2}} (|1\rangle \pm i|2\rangle) \quad (43)$$

corresponding to right (+) and left (−) circular polarization [50]. The completeness relation is $\mathbb{I} = |+\rangle\langle+| + |-\rangle\langle-|$. Given the LP initial state (37), we need to write the S-matrix in the polarization basis $|i\rangle$,

$$\mathbf{S}_{\text{CP}} = \sum_{\lambda=\pm} e^{i\delta_\lambda} |\lambda\rangle\langle\lambda| = \sum_{i,j=1}^2 |i\rangle \left(\sum_{\lambda=\pm} e^{i\delta_\lambda} \langle i|\lambda\rangle\langle\lambda|j\rangle \right) \langle j|. \quad (44)$$

The expression in parentheses defines the entries of the S-matrix we are looking for. They take on the explicit form

$$\begin{aligned} \mathbf{S}_{\text{CP}} &= U \begin{pmatrix} e^{i\delta_+} & 0 \\ 0 & e^{i\delta_-} \end{pmatrix} U^\dagger = e^{i\Sigma} \begin{pmatrix} \cos \Delta & \sin \Delta \\ -\sin \Delta & \cos \Delta \end{pmatrix} \\ &\equiv e^{i\Sigma} R(\Delta), \end{aligned} \quad (45)$$

with $\Sigma \equiv (\delta_+ + \delta_-)/2$ and $\Delta \equiv (\delta_+ - \delta_-)/2$. The S-matrix (45) is analogous to (31), though no longer diagonal.

Instead we find a polar representation, given by the product of a global $U(1)$ phase shift, $e^{i\Sigma}$ and a *rotation*, $R(\Delta) \in SO(2)$, by an angle Δ . (We use the same symbols as for LP to avoid cluttered notation.) The first equality in (45) is nothing but the statement that $\mathbf{S}_{\text{CP}}$ is diagonalized by the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad (46)$$

which transforms the linear polarization basis $|i\rangle$ to the circular polarization or helicity basis $|\lambda\rangle$. [Recall that the columns of (46) are the images of the basis vectors.] To study the fate of the LP probe state (37), we act with the CP S-matrix (45) and find

$$|\text{in}'\rangle = \mathbf{S}_{\text{CP}}|\text{in}\rangle = e^{i\Sigma}[\cos(\beta_0 - \Delta)|1\rangle + \sin(\beta_0 - \Delta)|2\rangle]. \quad (47)$$

This state corresponds to photons polarized linearly at angle $\beta_0 - \Delta$. The initial polarization has thus been rotated by an amount of Δ . In other words, the background acts like an optically active *chiral* medium similar to a solution of dextrose or a chiral metamaterial [58]. As the background has preferred handedness (chirality), different helicities (positive or negative) acquire different phases upon traversing the background (or chiral medium),

$$\mathbf{S}_{\text{CP}}|\pm\rangle = e^{i\delta_{\pm}}|\pm\rangle, \quad (48)$$

as is obvious from the diagonal form (45) in the helicity basis. For incoming linear polarization, the nonflip and flip amplitudes become

$$\mathfrak{M}_{\text{nf}} = \langle \text{in} | \text{in}' \rangle = e^{i\Sigma} \cos \Delta \simeq e^{i\phi_0} (1 + i\sigma), \quad (49)$$

$$\mathfrak{M}_{\text{f}} = \langle \text{f} | \text{in}' \rangle = -e^{i\Sigma} \sin \Delta \simeq -\Delta e^{i\phi_0}. \quad (50)$$

Interestingly, the exact nonflip and flip probabilities are the same as in the LP case, given by $\cos^2 \Delta$ and $\sin^2 \Delta$, respectively. Note that the amplitudes are independent of the initial rotation angle, β_0 , which implies that nonflip and flip amplitudes are simply given by the following matrix elements (in the LP basis),

$$\mathfrak{M}_{\text{nf}} = \langle 1 | \mathbf{S}_{\text{CP}} | 1 \rangle, \quad (51)$$

$$\mathfrak{M}_{\text{f}} = \langle 2 | \mathbf{S}_{\text{CP}} | 1 \rangle, \quad (52)$$

irrespective of the initial direction of (linear) polarization.

The different combinations of incoming (= probe), background, and outgoing polarizations are summarized in Table I.

Often, the effect of vacuum polarization on a probe laser beam is studied, which corresponds to replacing in states of

TABLE I. The effects on an incoming probe beam for varying polarizations (LP = linearly polarized, CP = circularly polarized, EP = elliptically polarized). All outputs are phase shifted compared to the input. Circular birefringence, which is the focus of our discussion, corresponds to case (ii) with LP' denoting a rotation of the input LP.

	Input (probe)	Background	Output
(i)	LP	LP	EP
(ii)	LP	CP	LP'
(iii)	CP	LP	EP
(iv)	CP	CP	CP

photons with a classical field comprising positive and negative frequencies [see, e.g., Eq. (26)], i.e., studying the “ $0 \rightarrow 0$ ” process. The connection can be made with the above analysis by adding the out state from a photon of fixed polarization and frequency to its complex conjugate. The results in the “Output” column of Table I can then be understood in terms of classical beams. For example an initially CP probe beam of the form Eq. (26) becomes elliptically polarized when propagating through an LP background, which can be understood as a *rotation* of the initial CP beam into another CP beam with the opposite chirality, by angle Δ , with each beam acquiring an overall phase shift of Σ .

A particularly intuitive graphical overview is given by the Poincaré sphere [59][Ch. XII]² and [61–63], where a generic point represents elliptic polarization, the equator collects all possible linear polarizations while north and south poles correspond to right, $|+\rangle$, and left, $|-\rangle$, circular polarization, respectively. Orthogonal vectors are given by antipodal points. Any change in polarization caused by a background field (and described by the corresponding S-matrix) is represented by a great circle segment on the sphere which results from a rotation around the “fast axis” of background, hence the eigenvector associated with the smaller phase shift, $\min(\delta_1, \delta_2)$ or $\min(\delta_+, \delta_-)$, as the velocities of light are $v_i = 1/n_i \simeq 1 - \delta_i$ and analogously for $v_{\pm}$. Cases (i) and (ii) are depicted in Fig. 2, cases (iii) and (iv) in Fig. 3.

IV. VACUUM CIRCULAR BIREFRINGENCE IN A PLANE WAVE BACKGROUND

In the remainder of the paper we focus on the calculation of the polarization flip and nonflip amplitudes for a CP background, hence the matrix elements of $\mathbf{S}_{\text{CP}}$, in QED and its low-energy approximation. We will compare three approaches based on (i) the derivative corrections to the LCFA, (ii) the derivative corrections (8) to the HE

²Interestingly, Poincaré entitles this chapter as “Circular polarization—theory of M. Mallard” which presumably refers to the second volume of Mallard’s treatise on crystallography [60]. While the latter discusses birefringence at length, we could not find any precursor of the Poincaré sphere there.

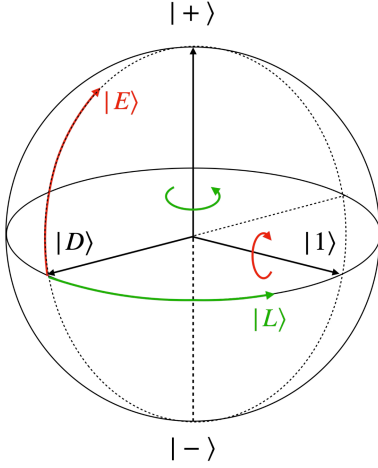


FIG. 2. Poincaré sphere with polarization changes for linear and circular birefringence [cases (i) and (ii) of Table I]. The input state is $|D\rangle = (1/\sqrt{2})(|1\rangle - |2\rangle)$. For linear birefringence, this is mapped to an elliptically polarized state $|E\rangle$ (red arrows), for circular polarization (a chiral background), $|D\rangle$ is mapped to a rotated LP state, $|L\rangle$ (green arrows).

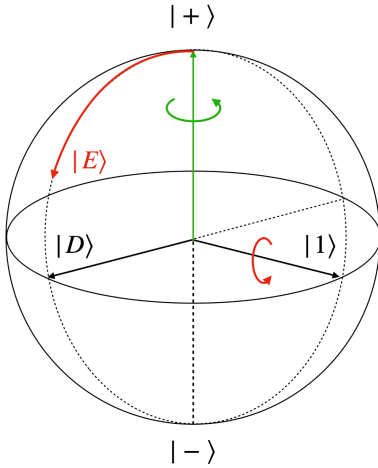


FIG. 3. Poincaré sphere with polarization changes for a CP probe [cases (iii) and (iv) of Table I]. The input state is $|+\rangle$ which is a fixed point, hence mapped to itself, for a CP background (green arrows). For an LP background, $|+\rangle$ is mapped to an elliptically polarized state $|E\rangle$ (red arrows), unless the phase difference is fine tuned to $\pi/2$ (as for a quarter waveplate) for which one reaches the equator, hence an LP configuration.

Lagrangian, and (iii) the low-energy expansion of the QED amplitude for $2 \rightarrow 2$ scattering. All relevant amplitudes are depicted in Fig. 4 in terms of Feynman diagrams stemming from $0 \rightarrow 0$, $1 \rightarrow 1$, and $2 \rightarrow 2$ photon processes, respectively.

A. dLCFA: Derivative corrections to the LCFA

The main advantage of plane-wave backgrounds is their high symmetry leading to superintegrable dynamics [64].

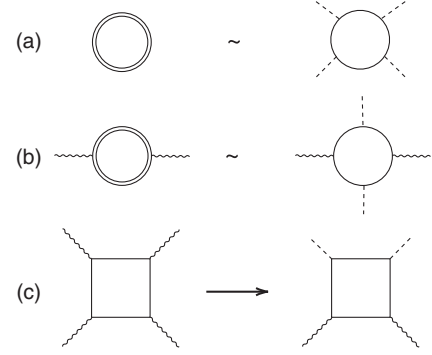


FIG. 4. (a) The LO term in field strength of the HE Lagrangian ($0 \rightarrow 0$ process, i.e., a dressed vacuum loop), with the probe and background as classical fields (dashed lines). (b) The LO expansion in field strength of the $1 \rightarrow 1$ process of photon scattering in a plane-wave background (polarization tensor). (c) $2 \rightarrow 2$ photon-photon scattering with one incoming and one outgoing photon (wavy lines) replaced by a classical plane-wave background field.

As a result, one can derive exact results, valid to all orders in field strength and/or energy. Alternatively, one can view these results as being due to a resummation of the weak-field or low-energy expansions. Plane-wave backgrounds thus provide a useful test bed for assessing the quality of the derivative expansion described in the previous section.

To proceed, we use (52) and express the probability (28) as

$$\mathbf{P}^{(21)} = |\langle 2 | \mathbf{S}_{\text{CP}} | 1 \rangle|^2 = |\epsilon_{2\mu}^* \mathfrak{M}_{\text{CP}}^{\mu\nu} \epsilon_{1\nu}|^2, \quad (53)$$

where we have mapped the states $|1\rangle$ and $|2\rangle$ back to the polarization 4-vectors ϵ_1 and ϵ_2^* for incoming and outgoing photons, respectively. The tensor amplitude $\mathfrak{M}_{\text{CP}}$ represents an integral over the background phase,

$$\mathfrak{M}_{\text{CP}} = \int \mathfrak{M}_{\text{CP}}(\varphi) d\varphi \quad (54)$$

and the background has been specified to be circularly polarized, with Lorentz indices suppressed for convenience. The integrand for arbitrary background polarization, $\mathfrak{M}(\varphi)$, has been found in, e.g., [65–68]. Below, we use the double-integral representation given in [21]:

$$\begin{aligned} \mathfrak{M}^{\mu\nu}(\varphi) = & \frac{\alpha}{8\pi\eta} \int_0^\infty \frac{d\theta}{\theta} \int_0^1 dr \exp\left[\frac{i\theta\mu(\theta)}{2\eta r(1-r)}\right] \\ & \times \left\{ \frac{2(1-2r)^2}{r(1-r)} (\bar{a}^\mu - \langle a^\mu \rangle)(\bar{a}^\nu - \langle a^\nu \rangle) \right. \\ & \left. - \frac{2}{r(1-r)} (a^\mu - \langle a^\mu \rangle)(\bar{a}^\nu - \langle a^\nu \rangle) \right\}. \end{aligned} \quad (55)$$

Here $\eta = \kappa \cdot \ell / m^2$ is the photon energy parameter, while the integration variable r denotes a light-front momentum

fraction of the virtual positron or electron. We have abbreviated $a = a(\varphi + \theta/2)$, $\bar{a} = a(\varphi - \theta/2)$ and introduced the phase-window average

$$\langle f \rangle = \theta^{-1} \int_{\varphi-\theta/2}^{\varphi+\theta/2} f(x) dx \quad (56)$$

such that the Kibble mass squared, $\mu(\theta) = 1 + \langle a \rangle \cdot \langle a \rangle - \langle a \cdot a \rangle$ (in units of m^2). The result (55) is valid for any center-of-mass energy and field intensity, but specific to a plane-wave background. Hence, (55) is an example of an exact result, valid to all orders in η and the background intensity parameter ξ defined in (26).

The LCFA and its derivative corrections are derived from (55) in the standard way (see, e.g., [17]). As we are interested in a CP background, we employ the CP potential from (26). We also assume the background wave has many cycles so that $\psi'(\varphi)/\psi(\varphi) \ll 1$, so that derivatives of the envelope can be ignored. Defining the local intensity parameter $\xi(\varphi) = \xi\psi(\varphi)$ and substituting

$$\theta \rightarrow 2u \left[\frac{r(1-r)\eta}{\xi^2(\varphi)} \right]^{1/3}, \quad (57)$$

in (55), the exponent can be expanded in the small parameter, $1/\xi$. The leading order in this expansion is cubic in u . We depart from the usual LCFA derivation by including all orders of the expansion of the preexponent in θ as well as Taylor expanding the part of the exponent that is higher order in $1/\xi$. For example, when expanded in θ , the preexponent in curly braces in (55) becomes

$$\{\} \rightarrow a'^2(\varphi)\theta^2 + \frac{3a'(\varphi)a^{(3)}(\varphi) - a''^2(\varphi)}{36}\theta^4 + \dots \quad (58)$$

Because the derivative “corrections” will actually dominate in this case, we refer to the resulting approximation as “dLCFA” to differentiate it from the “standard” LCFA. Defining

$$\mathfrak{M}^{(21)}(\varphi) = \varepsilon_{2\mu}^* \mathfrak{M}^{\mu\nu}(\varphi) \varepsilon_{1\nu}$$

and putting everything together, the dLCFA of the amplitude for the CP background from (26) becomes

$$\begin{aligned} \mathfrak{M}_{\text{CP,dLCFA}}^{(21)}(\varphi) &= \frac{\alpha}{8\pi\eta} \int_0^1 dr \int_0^\infty du (1 + ik_5 u^5 + \dots) \\ &\times \sum_{j \geq 1} c_j u^j \exp \left[i \left(zu + \frac{u^3}{3} \right) \right], \end{aligned} \quad (59)$$

where $z = [\chi(\varphi)r(1-r)]^{-2/3}$ with $\chi(\varphi) = \chi\psi(\varphi)$, $\chi = \eta\xi$, and $k_5 = 4/[45z\xi^2(\varphi)]$. The c_j are phase-dependent expansion coefficients, some of which we calculate below. The series with k_5 and higher-order terms originates from

Taylor-expanding the “non-Airy” parts of the exponent, which scale with higher powers of the small parameter $1/\xi$. This procedure is similar to the LCFA + approximation of tree-level processes [69–71].

The u integral can be performed by replacing powers of u in the preexponent with derivatives:

$$\begin{aligned} \mathfrak{M}_{\text{CP,dLCFA}}^{(21)}(\varphi) &= \frac{\alpha}{8\eta} \int_0^1 dr \left(1 + k_5 \frac{d^5}{dz^5} + \dots \right) \\ &\times \sum_{j \geq 1} (-i)^j c_j \frac{d^j}{dz^j} [\text{Ai}(z) + i\text{Gi}(z)], \end{aligned} \quad (60)$$

where Gi is the first Scorer function [72]. Expression (60) exhibits the dLCFA at the amplitude level: the successive higher-order z derivatives of the LCFA kernel are associated with increasing powers of the energy parameter η . Whereas the standard LCFA can be interpreted as integrating the instantaneous probability in a constant crossed field background, the dLCFA includes a description of background inhomogeneities. Depending on the parameter regime of interest and the relative size of χ and η , we could choose to expand the Airy-type functions in χ or use the derivative expansion to include higher powers in η . The LO ($j = 1$) term in (60) is the form of the standard LCFA (with k_5 and $\dots$ terms set to zero). However, as was noted in [21], for the CP background we find $c_1 \propto \sin(2\varphi)$, which is rapidly oscillating and integrates to zero in the long-pulse limit. Thus the NLO ($j = 2$) term is dominant and we see the same disruption of hierarchy as will be seen in the effective field theory approach.

As is well known [65,66,68], a comparison can be made between the plane-wave result and the HE approach by calculating the LCFA of the former and taking the limit of small strong-field parameter, $\chi = \xi\eta \ll 1$. (The small- χ limit is consistent with a low-energy limit, since the LCFA requires $\xi \gg 1$, which needs to be overcompensated by small $\eta \ll 1$.) For now, we concentrate on birefringence at small χ and use the asymptotic result [73]:

$$\text{Gi}(z) \sim \frac{1}{\pi z} \sum_{k=0}^{\infty} \frac{(3k)!}{k!(3z^3)^k}. \quad (61)$$

[We neglect the contribution from $\text{Ai}(z)$, which is exponentially suppressed for small η i.e., large z .] This allows the probability to be expanded in the form

$$\mathcal{P}_{\text{dLCFA}}^{(21)} = \left| \alpha \sum_{i,j=0}^{\infty} C_{i,j} \chi^{2i} \eta^{2j} \right|^2 \quad (62)$$

where $C_{i,j}$ are real coefficient. Expanding in η , say up to $O(\eta^6)$, we find

$$\begin{aligned} \mathcal{P}_{\text{dLCFA}}^{(21)} = & \frac{\alpha^2}{\pi^2} \left| \frac{2\chi^2}{315} \left(1 + \frac{8\eta^2}{55} + \frac{24\eta^4}{1001} \right) \mathcal{I}_2 \right. \\ & \left. + \frac{4\chi^4}{693} \mathcal{I}_4 + \frac{64\chi^6}{3861} \mathcal{I}_6 \right|^2, \end{aligned} \quad (63)$$

where we have reintroduced $\chi = \eta\xi$ to facilitate comparison with the LCFA.³ The $\mathcal{I}_{2n} = \int \psi^{2n}(\varphi) d\varphi$ represent integrals involving the envelope ψ and are independent of η . We note the first term of the amplitude in Eq. (63), which scales with $\sim O(\eta^0 \chi^2)$, is a factor η larger than the LO of the result, for the standard vacuum linear birefringence case of photon helicity flipping (CP) in an LP background, which scales as $\sim O(\chi^2/\eta)$ (see, e.g., [68]). The LO term in Eq. (63) corresponds to the NLO order in the derivative expansion. For the plane-wave setup of this section, the standard Mandelstam variables become

$$s = (\chi + \ell)^2 = 2m^2\eta; \quad t = -s; \quad u = 0. \quad (64)$$

Therefore, the NLO term of the LCFA having one higher power of η corresponds to an amplitude with one higher power of the Mandelstam variables, which is exactly what was found for the NLO terms (25) in the derivative expansion in the previous section.

To confirm our findings, we recalculate the flip amplitude from the effective field theory of Sec. II.

B. Derivative corrections to the HE lagrangian

We use the low-energy limit of the HE Lagrangian together with its NLO derivative correction given by (2) and (8). Following [23] we express the effective Lagrangian in terms of the total electromagnetic field tensor, which for LP photons flipping polarization in a CP plane-wave background is written $F^{\mu\nu} = \mathcal{F}^{\mu\nu} + f^{\mu\nu} + f'^{\mu\nu}$, where $\mathcal{F}$ is the plane-wave background field corresponding to the vector potential given in (26),

$$\begin{aligned} \mathcal{F}_{\mu\nu} = & \frac{m\xi}{\sqrt{2}} \psi(\varphi) \left[(\epsilon_{\mu}^{(+)} \chi_{\nu} - \epsilon_{\nu}^{(+)} \chi_{\mu}) e^{-i\varphi} \right. \\ & \left. + (\epsilon_{\mu}^{(-)} \chi_{\nu} - \epsilon_{\nu}^{(-)} \chi_{\mu}) e^{i\varphi} \right] \end{aligned} \quad (65)$$

while f and f' correspond to the incoming and outgoing LP photons,

$$f^{\mu\nu} = \frac{1}{\sqrt{2V\ell^0}} e^{i\ell \cdot x} (\ell^{\mu} \epsilon^{\nu} - \ell^{\nu} \epsilon^{\mu}), \quad (66)$$

$$f'^{\mu\nu} = \frac{1}{\sqrt{2V\ell'^0}} e^{-i\ell' \cdot x} (\ell'^{\mu} \epsilon'^{\nu*} - \ell'^{\nu} \epsilon'^{\mu*}). \quad (67)$$

³The term $\chi^4 \eta^2$ evaluates to zero when the contribution of k_5 is taken into account.

V is a volume factor, ℓ and ℓ' are the momenta of the incoming and outgoing photons, and ϵ, ϵ'^* their polarization vectors, which will be taken to be in the linear polarization basis. Substituting all field strength terms into the effective Lagrangian, we obtain the schematic derivative expansion

$$\mathcal{L}_{\text{CP,eff}} \simeq \mathcal{L}_{\text{CP,eff}}^{(4,0)} + \mathcal{L}_{\text{CP,eff}}^{(4,2)} + \dots, \quad (68)$$

where the (4,0) and (4,2) terms correspond to replacements of F for the case in hand, in expressions (2) and (8). The field strength of the plane-wave background is assumed to be much larger than that of the incoming and outgoing LP photons, and because we are interested in the birefringence signal, we keep only terms linear in the photon fields, f and f' . Schematically, the contributions are of the form

$$\begin{aligned} \mathcal{L}_{\text{CP,eff}}^{(4,0)} & \sim f \cdot F F \cdot f' \sim \xi^2 (\chi \cdot \ell)^2 \cos \varphi \sin \varphi, \\ \mathcal{L}_{\text{CP,eff}}^{(4,2)} & \sim (\partial f \cdot \partial F) F \cdot f' \sim \xi^2 (\chi \cdot \ell)^3. \end{aligned} \quad (69)$$

The (4,0) term is proportional to a fast oscillation on the timescale of the plane wave and integrates to zero. In contrast, the (4,2) term involves a derivative of the background implying a nonvanishing contribution after integration over the phase. Because the field tensor of the background and of each photon individually square to zero, high powers of derivatives must involve contractions between photons and the background. Hence the (4, 2n) term will scale as $(\chi \cdot \ell)^{2+n} \sim \eta^{2+n}$.

Using first-order perturbation theory, we calculate the probability,

$$\mathcal{P}_{\text{CP,eff}}^{(21)} = V \int \frac{d^3 \ell'}{(2\pi)^3} \left| -i \int d^4 x \mathcal{L}_{\text{CP,eff}} \right|^2, \quad (70)$$

and make the required substitutions in the Lagrangian to find

$$\begin{aligned} \mathcal{L}_{\text{CP,eff}}^{(4,2)} = & -i \frac{\alpha \chi^2}{315\pi V \sqrt{\ell'^0 \ell^0}} \left\{ 2(\ell \cdot \chi) \right. \\ & \left. + 5(e_1 \cdot \ell)(e_2 \cdot \delta \ell) - \frac{\delta \ell^0}{\ell^0} (\ell \cdot \chi) \right\} + O(\delta \ell^2). \end{aligned} \quad (71)$$

We have employed the photon polarization states from (27) and assumed that terms with higher powers of $\delta \ell \equiv \ell' - \ell$ are small compared to the leading-order terms above, hence the expansion in $\delta \ell$. When the space-time integral is performed, a delta function in $\delta \ell$ is clearly formed, which will be evaluated when the scattered momentum is integrated over. Including the (4,0) and (4,2) terms in the expansion, we then obtain for the probability:

$$\mathcal{P}_{\text{CP,eff}}^{(21)} = \frac{\alpha^2}{\pi^2} \left| \frac{2\chi^2}{315} \mathcal{I}_2 \right|^2, \quad (72)$$

i.e., the same as the leading, $O(\chi^2\eta^0)$ term in (63). The NNLO (4,4) term of the derivative expansion (9) can also be included in this analysis; we find for the scenario studied here that it yields terms that are again proportional to a fast oscillation term as for the LO term; therefore the (4,4) term does not contribute. Indeed, this is also consistent with the dLCFA analysis as the (4,4) term would contribute terms of order $\sim\eta\chi^2$, which were integrated over and thus are absent in (63).

So far we have assumed that derivatives of the pulse envelope, ψ , are negligible. If we relax this assumption, we can estimate their contribution. The envelope ψ in the vector potential is a function of φ/Φ , where Φ represents the characteristic phase duration of the pulse. Every power $\mathcal{F}^{\mu\nu}$ of the background then contains a term proportional to the derivative $(1/\Phi)\psi'(\varphi/\Phi)$. The elastic LO contribution is

$$\begin{aligned} \mathcal{L}_{\text{ENV,eff}}^{(4,0)} &\equiv \frac{4\alpha^2 m^2 \chi^2 e^{i(\ell-\ell')\cdot x}}{15V\sqrt{\ell^0\ell'^0}} \left[\psi\left(\frac{\varphi}{\Phi}\right) \psi'\left(\frac{\varphi}{\Phi}\right) \cos(2\varphi) \frac{1}{\Phi} \right. \\ &\quad \left. + \left(\psi'^2\left(\frac{\varphi}{\Phi}\right) \frac{1}{\Phi^2} - \psi^2\left(\frac{\varphi}{\Phi}\right) \right) \frac{\sin(2\varphi)}{2} \right]. \quad (73) \end{aligned}$$

If the pulse profile is symmetric around the origin, $\psi(\varphi/\Phi) = \psi(-\varphi/\Phi)$, the integral of (73) over φ gives exactly zero (using $\ell = \ell'$), despite including the finite pulse duration. However, the NLO (4,2) terms, given by

$$\begin{aligned} \mathcal{L}_{\text{ENV,eff}}^{(4,2)} &= \frac{-2i\alpha\chi^2(\ell \cdot \chi) e^{ix\cdot\delta\ell}}{315\pi V\sqrt{\ell^0\ell'^0}} \\ &\quad \times \left\{ \psi^2\left(\frac{\varphi}{\Phi}\right) + \frac{1}{\Phi^2} \left[2\psi'^2\left(\frac{\varphi}{\Phi}\right) \right. \right. \\ &\quad \left. \left. - \psi\left(\frac{\varphi}{\Phi}\right) \psi''\left(\frac{\varphi}{\Phi}\right) \right] \right\}, \quad (74) \end{aligned}$$

yield a nonzero contribution upon integration, i.e., when calculating the probability according to (70). (The replacement $\delta\ell \rightarrow 0$ has already been made in the preexponent, and we have assumed a symmetric pulse envelope.) The derivative corrections will thus appear at $\sim O(\Phi^{-2})$. (For a laser pulse, typically $\Phi \sim 2\pi N$ where $N \geq 1$ is the number of cycles.)

C. Photon-photon scattering

The general result for $2 \rightarrow 2$ photon-photon scattering can be directly related to $1 \rightarrow 1$ scattering in a classical plane wave by a replacement prescription. It can be shown [13] that the probability can be written as

$$\mathcal{P}_{\text{QED}} = V \int \frac{d^3\ell'}{(2\pi)^3} |T_{\text{fi}}|^2; \quad T_{\text{fi}} = K(\ell' - \ell) \frac{2V}{\chi^0} [\mathfrak{M}^{2 \rightarrow 2}]_{1 \rightarrow 1} \quad (75)$$

where K is a *form factor*. For a weakly focused beam it can be expressed as a Fourier integral over the intensity distribution,

$$K(q) = (m\xi x^0)^2 \int \psi^2(\varphi) e^{iq\cdot x} d^4x. \quad (76)$$

The labels on the invariant amplitude $\mathfrak{M}$ correspond to replacing the momentum and polarization of one incoming and one outgoing photon with the momentum and polarization of a Fourier mode of the plane-wave background. For the CP background (26) one obtains

$$\mathcal{P}_{\text{QED}} = \left| \frac{1}{\eta} [\bar{\mathfrak{M}}^{2 \rightarrow 2}]_{1 \rightarrow 1} \int \frac{1}{2} \xi^2 \psi^2(\varphi) d\varphi \right|^2, \quad (77)$$

where the bar on $\mathfrak{M}$ indicates that the wave function normalization factors of the form $(2V k^0)^{-1/2}$ have been set to unity. We can compare with literature results for helicity/polarization amplitudes through the definition

$$\mathfrak{M}^{2 \rightarrow 2} = \varepsilon_{k_1}^i \varepsilon_{k_2}^j \varepsilon_{k_3}^{i'} \varepsilon_{k_4}^{j'} \mathfrak{M}_{ij i' j'}^{2 \rightarrow 2}, \quad (78)$$

with polarization vectors ε_k and a summation over polarization labels $i, j, i', j' = 1, 2$ implied. We then perform the replacement in (77), employing

$$\mathfrak{M}_{1+ \rightarrow 2}^{2 \rightarrow 2} = \frac{i}{2} (\mathfrak{M}_{1212} - \mathfrak{M}_{1122}). \quad (79)$$

Using the low-energy expansion of the helicity amplitudes available in the literature [36,45,47], we infer that the LO terms are given by the difference of the first two equations in (23) and thus are proportional to $s^2 - t^2$. In a plane-wave background $t = -s$, which is why the LO contributions cancel. (For a general background, the scattered and incident photon momentum may differ, and the leading order may no longer vanish.) For the plane-wave case, the probability depends on linear combinations of the NLO terms listed in (25). Specifically, we find a proportionality $\propto s^3 - t^3$, which no longer vanishes when $t = -s$. Using the generating functions in [47], one can calculate arbitrary high powers in the low-energy expansion with the result. Including the first four orders:

$$|\bar{\mathfrak{M}}_{1+ \rightarrow 2}^{2 \rightarrow 2}|^2 = \left| \frac{4\alpha\eta^3}{315\pi} \left(1 + \frac{8\eta^2}{55} + \frac{24\eta^4}{1001} - \frac{6272\eta^6}{415701} \right) \right|^2. \quad (80)$$

(Recall that we are interested in the flipping of linearly polarized photons in a CP background. For the alternative

scenario of a low-energy expansion in an LP background see, e.g., the recent study [74].)

For our case (LP probe, CP background), the flip probability inferred from (80) becomes

$$P_{\text{QED}}^{(21)} = \frac{\alpha^2}{\pi^2} \left| \frac{2\chi^2}{315} \left(1 + \frac{8\eta^2}{55} + \frac{24\eta^4}{1001} - \frac{6272\eta^6}{415701} \right) \right|, \quad (81)$$

which has the same coefficients at the same orders of $\chi^2\eta^{2n}$ as (63). To NLO, we therefore find the three methods discussed in this section to be equivalent.

The evaluation of the QED amplitude for the box diagram of photon-photon scattering suggests that if circular birefringence is calculated in a setup where $\ell' \neq \ell$, for example if the background is not a plane wave, then the LO contribution will not necessarily disappear, although it may still be strongly suppressed. In the next section, we look at the example of a rotation standing wave background.

V. ROTATING STANDING WAVE BACKGROUND

In a Volkov approach, the electromagnetic background must be close to a plane wave for the calculation to yield a good approximation. Employing the derivative expansion of the QED effective action, however, allows for treating backgrounds that are not restricted to being a plane wave. As an example of this kind we consider a standing-wave background formed by two counter-propagating CP plane waves. We choose a particular configuration where the plane waves collide head on, with opposite helicities and such that the electric fields of each plane wave are parallel (and likewise for the magnetic fields). We call this the “rotating standing wave” setup. The corresponding vector potential takes the form

$$A = a^{(+)}(\varphi_r) + a^{(-)}(\varphi_r) + a^{(+)}(\varphi_l) + a^{(-)}(\varphi_l) \quad (82)$$

where $\varphi_r = \mathbf{x}_r \cdot \mathbf{x} = \mathbf{x}_r^+ x_+$ and $\varphi_l = \mathbf{x}_l \cdot \mathbf{x} = \mathbf{x}_l^- x_-$. Again, we consider the incoming photon to be in linear polarization state $|1\rangle$ corresponding to polarization 4-vector ε_1 . However, as the background now differs from a plane wave, the scattered photon momentum ℓ' is not automatically equal to the incoming momentum ℓ . Instead, photon scattering in the standing wave can be understood as a collision of three plane waves, the kinematics of which have been well studied in the literature [75–78]. One finds the photon may be scattered into three peaks: a central peak with momentum ℓ and two neighboring peaks with momentum $\ell \pm \Delta\ell$ where $\Delta\ell = \mathbf{x}_r - \mathbf{x}_l$. Because the photon polarization is transverse to the momentum, this also means there are three different polarization bases, $\{\varepsilon_1, \varepsilon_2\}$, $\{\varepsilon_{1l}, \varepsilon_{2l}\}$, $\{\varepsilon_{1r}, \varepsilon_{2r}\}$: the Hilbert space of photon polarization is no longer two dimensional. The setup is depicted in Fig. 5. The signature of the NLO terms can be identified by considering the polarization-flipped signal in the central peak, which we find is

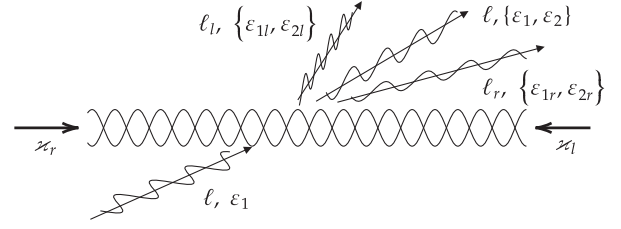


FIG. 5. Sketch of the circular standing-wave scenario: The linear polarization of probe photons flips due to scattering off the standing CP plane-wave background (with momentum and polarization labels). The main signal has unchanged momentum ℓ (forward scattering), while the photons on the side peaks have altered momenta $\ell_{r,l}$ and polarizations $\neq \varepsilon_{1,2}$.

$$\mathfrak{M}^{(21)} \approx \frac{4\alpha}{315\pi} \frac{1}{2V\sqrt{l^0 l'^0}} \times \int d^4x [(\ell \cdot \mathbf{x}_r) \chi_r^2(\varphi_r) + (\ell \cdot \mathbf{x}_l) \chi_l^2(\varphi_l)] e^{i\delta\ell \cdot \mathbf{x}}, \quad (83)$$

where $\delta\ell = \ell' - \ell$. We have already set $\ell' = \ell$ in the preexponent in anticipation of the delta function that will be generated and integrated over. In addition, we have only included terms to zeroth order in $\mathbf{x}_r \cdot \mathbf{x}_l$, assuming that $\mathbf{x}_r \cdot \mathbf{x}_l \ll \ell \cdot \mathbf{x}_r$ and $\mathbf{x}_r \cdot \mathbf{x}_l \ll \ell \cdot \mathbf{x}_l$. In this limit, the polarization flip amplitude is simply the sum of contributions from each of the two plane waves that comprise the standing wave background. However, at the probability level, an interference term is generated, which is absent from the plane-wave treatment. For the probability, we thus find (after mod-squaring and integrating over momentum)

$$P = \left| \frac{2\alpha}{315\pi} \int d\varphi [\chi_r^2(\varphi) + \chi_l^2(\varphi)] \right|^2, \quad (84)$$

where $\chi_{r,l} = \chi_{r,l}(\mathbf{x}_{r,l} \cdot \mathbf{x})$. We note that if $\eta_r \gg \eta_\ell$ or vice versa, the result tends to the plane-wave result [e.g., Eq. (63)] with a wave vector originating from the larger of η_r or η_l . We can also rephrase the result via the derivative-modified LCFA by calculating χ for the full background and dropping fast oscillating factors (which will give only a negligible contribution once the phases are integrated over), leading to $\chi^2 = \chi_r^2 + \chi_l^2$. Hence, for the central peak of the standing-wave background, assuming that $\mathbf{x}_l \cdot \mathbf{x}_r \ll \mathbf{x}_l \cdot \ell$ and $\mathbf{x}_l \cdot \mathbf{x}_r \ll \mathbf{x}_r \cdot \ell$, the result (84) can be recovered by simply using the plane-wave result (63) and calculating χ for the standing-wave background.

For the side peaks (an effect beyond a plane-wave background, which the LCFA cannot replicate), the main contribution to the polarization-flip probability is from the regular, LO terms of the HE Lagrangian. This, however, is not the dominant polarization-flip signal. Instead, the LO terms are proportional to the momentum exchange $\delta\ell$; if

the probe energy is much larger than the central frequency of the standing wave, then $\delta\ell \ll \ell$. Specifically, due to momentum conservation, $\ell \pm \kappa_r = \ell' \pm \kappa_l$, the side peaks correspond to a change in photon momentum given by $\delta\ell = \pm\Delta\ell$ (recalling $\Delta\ell = \kappa_r - \kappa_l$). To understand the magnitude of the LO term compared to the NLO term, we can use the low-energy expansion of the box diagram [36,47]. For the case of a plane-wave background studied earlier, we found the LO terms were proportional to $\mathfrak{M}_{1212} - \mathfrak{M}_{1122} \sim s^2 - t^2 = 0$ and we expect the standing wave LO terms to scale similarly. For the kinematics of the side peaks scattered off the standing wave, one has $s = \pm 2\ell \cdot \kappa_r$ and $t = \mp 2\ell \cdot \kappa_l$, giving $s^2 - t^2 \sim \eta_r^2 - \eta_l^2$. Analogously, the NLO term scales as $\sim s^3 - t^3 \sim \pm(\eta_r^3 + \eta_l^3)$. The NLO contribution thus remains dominant in the side peaks if

$$\eta_r^3 + \eta_l^3 \gg |\eta_r^2 - \eta_l^2|. \quad (85)$$

VI. EXPERIMENTAL OUTLOOK

One way to experimentally test the theory above is to pick a set-up of colliding beams in which the derivative term (NLO) is the main contribution to the experimental signature. For plane-wave backgrounds, corrections to the NLO, Eq. (63), scale as χ^2 . To suppress these (and to avoid pair creation) one would require $\chi \ll 1$. Since measuring photon polarization flip at LO is already an experimental challenge, the NLO signal should ideally not be much smaller than this. The ratio of the derivative term to the standard nonderivative HE term scales as η , and therefore it would lead to a significant NLO signal if $\eta \not\ll 1$. Since $\chi = \eta\xi$ in a plane wave, these two requirements then imply ξ should not be too large. An example combination would be a high-energy photon source with a weakly focused laser. To obtain some approximate numerical values, we model the envelope of the laser pulse as

$$\psi(\varphi) = \sin^2\left(\frac{\varphi}{2N}\right); \quad 0 < \varphi < 2\pi N, \quad (86)$$

and $\psi(\varphi) = 0$ otherwise, such that N measures the phase duration. Some typical example parameters and corresponding polarization flip probabilities, calculated from

TABLE II. Some example parameters and ensuing probabilities for the polarization flip of an LP photon in a CP background (circular birefringence/optical rotation).

	η	ξ	N	$\mathbf{P}$
ELI-beamlines [80]	0.01	20	40	5.2×10^{-8}
LUXE [81,82]	0.1	5	16	1.3×10^{-6}
FCC-ee [79]	1	0.5	32	2.6×10^{-6}

(63), are given in Table II. Alternative scenarios at the electron-positron collider of the Future Circular Collider (FCC-ee) could include an inverse Compton scattering source of high energy photons with energies up to around 100 GeV [79].

VII. CONCLUSION

Vacuum *circular* birefringence (or optical activity) is a prediction of QED that has been much less studied than its linear counterpart. It occurs whenever there is a nonzero probability for the linear polarization of a probe photon to flip upon traversing a circularly polarized background. This endows the vacuum with a preferred “handedness” and thus turns it into a chiral “medium” akin to a solution of dextrose.

Both variants of birefringence can be studied using a two-by-two S-matrix formalism with polarization or helicity states which is basically equivalent to the Jones matrix approach of optics. The outcomes can be intuitively visualized on the Poincaré sphere (Figs. 2 and 3). The flip amplitudes/S-matrix entries, though, must be calculated in QED or its low-energy effective approximation as has been done in this paper.

In this context it was already noted in [21] that the vacuum circular birefringence signal cannot be described by the standard Heisenberg-Euler approach (the leading order in a derivative expansion). In this paper we have shown that the inclusion of derivative corrections reproduces the low-energy limit of the Volkov approach based on the dressed polarization tensor. A third method of using the helicity amplitudes for photon-photon scattering in full QED leads to agreement with these other two methods.

Calculating the derivative corrections to the HE Lagrangian is a nontrivial task; even for the first correction, i.e., the next-to-leading-order Lagrangian, there are three different expressions stated in the literature. We outlined how Hilbert series can be used to identify the (minimal) number of inequivalent operators in the expansion and matched these operators with helicity amplitudes.

We also demonstrated how derivative corrections can be calculated to extend the locally constant field approximation for photon polarization flipping. Although the LCFA usually only depends on the strong-field parameter, χ , the extended LCFA with derivative corrections (dubbed dLCFA) generated extra powers of η , thus doubling the dimensions of parameter space.

A virtue of the effective field theory approach is its capability to tackle more realistic backgrounds with less symmetry than the superintegrable plane wave. We demonstrated this by calculating the polarization flip of a linearly polarized photon scattering off a rotating standing wave background given by the superposition of *two* circularly polarized plane waves.

The observation of vacuum circular birefringence or “vacuum chirality” would be a test of higher-dimensional operators in the low-energy effective field theory for QED and experimentally yield the next-to-leading order effective couplings (collectively denoted $c_{4,2}$ in the Introduction). To realize this goal one has to achieve a sufficiently high center-of-mass energy while maintaining a moderate strong-field parameter.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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