

Tachyonic and parametric resonances for massive particle production in an intense scalar plane wave background

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We investigate the stability of an intensive plane wave of a massless or light scalar field ϕ in a trilinear scalar model $g\phi\chi^2$ due to the resonant production of massive χ particles in a perturbatively forbidden regime. We apply two methods. First, we solve the Heisenberg equation for the quantum amplitudes of the field χ in an external plane wave background, generalizing the solution of Arza [A. Arza, Production of massive bosons from the decay of a massless particle beam, *Phys. Rev. D* **105**, 036004 (2022)]. Second, for the light but massive ϕ , we perform the relativistic boost to the rest frame of ϕ , reducing the problem to the stability of the thoroughly investigated massive condensate. It turns out that the stability properties are significantly different for the cases of massless and light fields. In the first case, one should adopt the Heisenberg equation approach, since the alternative method cannot provide a comprehensive outcome. In the second case, the use of the Mathieu equation provides a more accurate solution, while for the massless case, this approach is not applicable.

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I. INTRODUCTION

Particle production in intensive oscillating fields is an interesting phenomenon in quantum field theory that has important applications in cosmology and astrophysics [1–7]. An interesting example of the aforementioned phenomena is the production of particles by an intensive oscillating field enhanced by parametric resonance [8–12]. This process is of great interest in cosmology, since it describes extremely fast production of scalar particles driven by the oscillations of the inflaton field at the preheating stage after inflation [8,13–18], see also [19] for a review. Mathematically, one solves the Heisenberg equation for the amplitudes of a certain bosonic field mode in an external oscillating field, which in the most popular two-scalar model with quartic coupling $g\phi^2\chi^2$ reduces (neglecting the expansion of the Universe for simplicity) to the Mathieu equation [20–22]. Depending on the parameters of the model, one distinguishes the analytical regimes of the “narrow” and “broad” resonance [13,23], which affect the preheating efficiency in different ways.

Besides, the properties of the solution of this equation and the diagram of its stability are well studied mathematically and presented in many works, e.g., in [20,21,24].

In addition to quartic coupling, one also considers trilinear coupling $g\phi\chi^2$ [15] in which the tachyonic instability¹ exists [25,26]. In this model, the Mathieu equation is interpreted as an equation for the amplitudes of the χ field, just as in the previous case. Dufaux *et al.* [10] have shown that in a certain range of parameters this instability implies a very fast tachyonic resonance for the production of particles χ from ϕ condensate.

Preheating is not the only application of such a resonant particle production. Hypothetical compact objects from a Bose condensate of a scalar field (Bose stars) may have an instability due to a similar mechanism [27,28].

Besides the consideration of the condensate instability due to the resonant particle production, one can also consider an intense plane wave in the initial state. If the plane wave is constructed from a massive field, the stability investigation reduces to the previous case of the condensate by a relativistic boost. However, alternative formalisms are also interesting, referring to the fact that such a boost is unavailable in the massless case. One of the recently discussed examples is the instability of a plane electromagnetic wave due to the production of axions in a colinear limit [29].

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¹One can regularize the theory by an extra $\lambda\chi^4$ term.

In scalar quantum electrodynamics, the Klein-Gordon equation for scalar electrons in an intense field of a plane wave like background [30–32] can be reduced to the Mathieu equation. Such conditions can be realized in a medium of refractive index $n > 1$ [30–32] as well as with the intersection of plane waves [33].

The intense plane wave instability in a trilinear model of two scalar fields was considered in [34]. For a massless plane wave field, the Heisenberg equation does not imply the Mathieu equation, so a different method with additional assumptions has been applied. In the work [34], it was shown that with a sufficiently large amplitude, a plane wave of the massless field is unstable, which leads to the formation of massive particles. For this reason, the Heisenberg equation for modes of a massive scalar field was solved. It should be noted that the calculation was carried out under the assumption of low mass of produced particles compared to the frequency of a plane wave.

In this article, we summarize the calculations [34] for the case of arbitrary masses of produced particles and also investigate in detail the connection of the method based on the Heisenberg equation with the Mathieu equation.

The paper is organized as follows. In Sec. II, we consider a toy model $g\phi\chi^2$ and obtain the equations of motion. In Sec. III, we study the solution of the Heisenberg equation for a high-intensity plane wave for the trilinear interaction and obtain solutions both in the case of small mass approximation and for the case of arbitrary mass. In Sec. IV, we describe the intensely oscillating condensate using the Mathieu equation. In this section, we also describe the narrow, broad, and tachyon resonances. In Sec. V, we present a comparison of the two approaches for the condensate. Section VI is devoted to the conclusion.

II. THE MODEL

We consider the theory of two interacting scalar fields ϕ and χ with a trilinear coupling described by a Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{2}m_\chi^2\chi^2 - g\phi\chi^2, \quad (1)$$

where m_ϕ and m_χ are the masses of ϕ and χ fields, and g is the trilinear coupling constant. We are interested in studying the instability of the plane wave of field ϕ ,

$$\phi(t, \vec{x}) = \Phi \cos(\omega_p t - \vec{p} \cdot \vec{x}), \quad (2)$$

related to the production of χ particles. Here, Φ is the amplitude of the plane wave, and ω_p and $\vec{p}$ are its frequency and momentum. Special relativity implies $\omega_p^2 - \vec{p}^2 = m_\phi^2$. We consider both the cases of massless $m_\phi = 0$ and the massive $m_\phi \neq 0$ field ϕ .

The process of χ particle production is a simple perturbative process $\phi \rightarrow \chi\chi$ if $m_\phi > 2m_\chi$. Otherwise,

$m_\phi < 2m_\chi$, and the perturbative process is forbidden due to energy-momentum conservation. However, one should take into account that the solution for a free particle χ modifies in the presence of a ϕ plane wave with a sufficiently large amplitude Φ which can lead to non-perturbative decay channels.

The dynamics of the fields χ and ϕ is governed by equations of motion (EOM),

$$(\square + m_\chi^2)\chi = -2g\phi\chi, \quad (3)$$

$$(\square + m_\phi^2)\phi = -g\chi^2. \quad (4)$$

Equation (3) refers to the production of field χ in an external plane wave of field ϕ and Eq. (4) to the production of ϕ modes in the generated χ field. The last process can be neglected at early times while the amplitude of the produced χ field is significantly less than those of ϕ , and the backreaction is not taken into account. At the initial moment of time, χ particles are completely absent in the system, and the ϕ field can be considered as the background for the χ vacuum.

We perform two approaches. First, we solve Eq. (3) for the amplitudes of χ field following [34] in the approximation ($m_\chi/\omega_p \ll 1$) and beyond it, both for massless and massive ϕ . The second approach is related exclusively to the case of massive ϕ and implies the relativistic boost to the frame in which the plane wave (2) is a condensate,

$$\phi(t) = \Phi \cos(m_\phi t), \quad (5)$$

and the problem reduces to the known Mathieu equation.

III. THE HEISENBERG EQUATION APPROACH: PLANE WAVE

We solve Eq. (3) with the classical field ϕ (2). The field χ can be decomposed in Fourier series as

$$\chi(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_{\vec{k}}}} (\chi_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}} + \chi_{\vec{k}}^*(t) e^{-i\vec{k} \cdot \vec{x}}), \quad (6)$$

where $\Omega_{\vec{k}} = \sqrt{\vec{k}^2 + m_\chi^2}$, and the operators $\chi_{\vec{k}}$ and $\chi_{\vec{k}}^\dagger$ satisfy the commutation relations $[\chi_{\vec{k}}, \chi_{\vec{k}'}] = 0$, $[\chi_{\vec{k}}, \chi_{-\vec{k}}^\dagger] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$.

Substituting the decomposition (6) into the equations of motion (3), encountering the plane wave field ϕ (2), and denoting the Bogolubov transformation $A_{\vec{k}} = \chi_{\vec{k}} + \chi_{-\vec{k}}^\dagger$, one finally obtains (cf. [34]),

$$(\partial_t^2 + \Omega_k^2)A_{\vec{k}} = -\omega_{\vec{p}}^2 \alpha \left(\sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{k}-\vec{p}}}} A_{\vec{k}-\vec{p}} e^{-i\omega_{\vec{p}} t} + \sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{k}+\vec{p}}}} A_{\vec{k}+\vec{p}} e^{i\omega_{\vec{p}} t} \right), \quad (7)$$

where $\alpha \equiv \frac{q\Phi}{\omega_{\vec{p}}^2}$. We highlight the standard oscillation part $\chi_{\vec{k}}^-$, $\chi_{\vec{k}}^- = a_{\vec{k}}^-(t) e^{-i\Omega_{\vec{k}} t}$ and $\chi_{-\vec{k}}^+ = a_{-\vec{k}}^+(t) e^{i\Omega_{-\vec{k}} t}$ (where $\Omega_{\vec{k}} = \Omega_{-\vec{k}}$) and substitute it into Eq. (7). The time evolution for the amplitudes $a_{\vec{k}}^-$, $a_{-\vec{k}}^+$ is governed by the equation,

$$\begin{aligned} e^{-i\Omega_{\vec{k}} t} (\ddot{a}_{\vec{k}}^- - 2i\Omega_{\vec{k}} \dot{a}_{\vec{k}}^-) + e^{i\Omega_{-\vec{k}} t} (\ddot{a}_{-\vec{k}}^+ + 2i\Omega_{-\vec{k}} \dot{a}_{-\vec{k}}^+) \\ = -\omega_{\vec{p}}^2 \alpha \left(\sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{k}+\vec{p}}}} (a_{-\vec{k}+\vec{p}}^+ e^{i(\Omega_{-\vec{k}+\vec{p}} - \omega_{\vec{p}}) t} + a_{\vec{k}+\vec{p}}^- e^{-i(\Omega_{\vec{k}+\vec{p}} - \omega_{\vec{p}}) t}) \right. \\ \left. + \sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{k}-\vec{p}}}} (a_{-\vec{k}-\vec{p}}^+ e^{i(\Omega_{-\vec{k}-\vec{p}} - \omega_{\vec{p}}) t} + a_{\vec{k}-\vec{p}}^- e^{-i(\Omega_{\vec{k}-\vec{p}} - \omega_{\vec{p}}) t}) \right). \quad (8) \end{aligned}$$

This is an infinite system of second order differential linear equations. The right column of Eq. (8) can be obtained from the left column using a Hermitian conjugation and $\vec{k} \rightarrow -\vec{k}$ substitution. We can conditionally separate Eq. (8) into two parts, one part corresponds to $a_{\vec{k}}^-$, and the other part corresponds to $a_{-\vec{k}}^+$. Consequently, the solutions for the amplitudes of each of these parts will contain parameters associated with $a_{\vec{k}}^-$ or $a_{-\vec{k}}^+$, respectively. Terms with $a_{\vec{p}-\vec{k}}^+$ and $a_{-\vec{p}-\vec{k}}^-$ correspond to momentum $\vec{k}$, and terms with $a_{\vec{k}+\vec{p}}^-$ and $a_{-\vec{k}-\vec{p}}^+$ correspond to momentum $-\vec{k}$. The general solution is a linear combination of complex and real exponents. If the solution contains exponential growing with a time term, $a_{\vec{k}}^-(t) = \dots + A_{\vec{k}'} e^{c_{\vec{k}'} t} a_{\vec{k}'}^+(0) + \dots$ for certain $\vec{k}'$, the occupation number for modes of momentum $\vec{k}'$ grows resonantly,

$$f_{\chi, \vec{k}'}(t) = \langle 0 | a_{\vec{k}}^+(t) a_{\vec{k}}^-(t) | 0 \rangle = A_{\vec{k}}^2 e^{2c_{\vec{k}'} t} + \dots$$

In case of several resonant terms, one should sum over all of them.

Note that the terms in the rhs of any line of Eq. (8) are obtained from the corresponding lhs terms of that line. Thus, it is sufficient to search for the resonant solution in the lhs column of Eq. (8) separately. Denoting additionally

$$\sigma_{\vec{p}-\vec{k}} = -\omega_{\vec{p}}^2 \alpha \sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{p}-\vec{k}}}} \quad \text{and} \quad \sigma_{\vec{p}+\vec{k}} = -\omega_{\vec{p}}^2 \alpha \sqrt{\frac{\Omega_{\vec{k}}}{\Omega_{\vec{p}+\vec{k}}}}, \quad (9)$$

we rewrite Eq. (8) as

$$\begin{aligned} e^{-i\Omega_{\vec{k}} t} (\ddot{a}_{\vec{k}}^- - 2i\Omega_{\vec{k}} \dot{a}_{\vec{k}}^-) = \sigma_{\vec{p}-\vec{k}} a_{\vec{p}-\vec{k}}^+ e^{i(\Omega_{\vec{p}-\vec{k}} - \omega_{\vec{p}}) t} \\ + \sigma_{\vec{p}+\vec{k}} a_{-\vec{p}-\vec{k}}^+ e^{i(\Omega_{-\vec{p}-\vec{k}} + \omega_{\vec{p}}) t}. \quad (10) \end{aligned}$$

The term with $a_{\vec{p}-\vec{k}}^+$ is related to the production of χ modes with momentum $\vec{p} - \vec{k}$, $a_{-\vec{p}-\vec{k}}^+$, with momentum $(-\vec{p} - \vec{k})$.

A. The solution using approximation

The equation for the amplitude (10) reads (amplitude $a_{\vec{p}-\vec{k}}^+$ is leading),

$$\ddot{a}_{\vec{k}}^- - 2i\Omega_{\vec{k}} \dot{a}_{\vec{k}}^- = \sigma_{\vec{p}-\vec{k}} a_{\vec{p}-\vec{k}}^+ e^{i(\Omega_{\vec{k}} + \Omega_{\vec{p}-\vec{k}} - \omega_{\vec{p}}) t}. \quad (11)$$

This equation can be simplified if $a_{\vec{k}}^-(t)$ varies slower with time than $\chi_{\vec{k}}^-(t)$ in a way that we can neglect the second derivative in (11),

$$|\ddot{a}_{\vec{k}}^-| \ll |\Omega_{\vec{k}} \dot{a}_{\vec{k}}^-|, \quad (12)$$

where the limit is referred to as the rotating wave approximation (RWA) in [34]. A similar approach is employed in the context of particle production in electromagnetism [35,36]. Thus, in this approximation, the first term in (11) can be neglected,

$$\dot{a}_{\vec{k}}^- = -i \frac{\sigma_{\vec{p}-\vec{k}}}{\Omega_{\vec{k}}} a_{\vec{p}-\vec{k}}^+ e^{i\epsilon_{\vec{k}} t}. \quad (13)$$

The solution [34] reads (in our notations),

$$\begin{aligned} a_{\vec{k}}^-(t) = e^{i\epsilon_{\vec{k}} t/2} \left(a_{\vec{k}}^-(0) \left(\cosh(s_{\vec{k}}^0 t) - i \frac{\epsilon_{\vec{k}}}{2s_{\vec{k}}^0} \sinh(s_{\vec{k}}^0 t) \right) \right. \\ \left. - i \frac{\sigma_{\vec{p}-\vec{k}}}{2s_{\vec{k}}^0 \Omega_{\vec{k}}} a_{\vec{p}-\vec{k}}^+(0) \sinh(s_{\vec{k}}^0 t) \right), \quad (14) \end{aligned}$$

where

$$s_{\vec{k}}^0 = \frac{1}{2} \sqrt{\frac{\sigma_{\vec{p}-\vec{k}}^2}{\Omega_{\vec{k}}^2} - \epsilon_{\vec{k}}^2}, \quad \epsilon_{\vec{k}} = \epsilon_{\vec{p}-\vec{k}} = \Omega_{\vec{k}} + \Omega_{\vec{p}-\vec{k}} - \omega_{\vec{p}}. \quad (15)$$

In order to obtain the resonance, the argument of the hyperbolic sine in the last term of (14) should be real, $\text{Im } s_{\vec{k}}^0 = 0$. Note that there is a symmetry regarding the replacement of $\vec{k}$ by $\vec{p} - \vec{k}$. This is in agreement with the production of two particles in the final state. For fixed α, m_{χ} , the momentum threshold for resonance (14) and (15) reads $k = p/2$.

Now, we return to the limits of applicability of this result. Substituting the solution (14) and (15) into the approximation (12), we obtain

$$\epsilon_k^- \ll \Omega_k^-, \quad s_k^0 \ll \Omega_k^-. \quad (16)$$

The first condition reads $\Omega_{\vec{p}-\vec{k}} \ll \omega_{\vec{p}}$, which restricts us to the cases of small $m_\chi \ll \omega_{\vec{p}}^2$ and not very small $|\vec{k}| \gg \frac{m_\chi^2}{2\omega_{\vec{p}}}$. The second condition reduces to $\alpha \ll 1$.

The occupancy number for a mode of a field χ with momentum $\vec{k}$ grows resonantly [6],

$$f_{\chi, \vec{k}}(t) = \langle 0 | a_{\vec{k}}^\dagger(t) a_{\vec{k}}(t) | 0 \rangle = \frac{\sigma_{\vec{p}-\vec{k}}^2 \sinh^2(s_k^0 t)}{4 (s_k^0)^2 \Omega_k^2}. \quad (17)$$

The total density of all resonantly produced modes reads

$$n_\chi(t) = \int \frac{d^3 k}{(2\pi)^3} f_{\chi, \vec{k}}(t), \quad (18)$$

where the momentum integral $\int d^3 k$ should be taken only over momenta satisfying the resonant condition $(s_k^0)^2 > 0$. Consequently, the condition of the resonance boundary reads

$$s_k^0 = 0. \quad (19)$$

The resonance condition (19) and (15) for the threshold value $k = p/2$ is

$$\frac{4g^2\Phi^2}{\omega_p^2 + 4m_\chi^2 - m_\phi^2} = \left(\sqrt{\omega_p^2 + 4m_\chi^2 - m_\phi^2} - \omega_p \right)^2. \quad (20)$$

In the limit of our approximation $m_\chi \ll \omega_p$, one derives that the resonance condition does not depend on ω_p ,

$$g\Phi = m_\chi^2 - \frac{m_\phi^2}{4}. \quad (21)$$

In the case of massless ϕ (as well as in the case $m_\phi \ll m_\chi$), one divides each side of Eq. (21) by ω_p^2 obtaining a dimensionless bound (cf. [34]),

$$\alpha = \mu_\chi^2, \quad \mu_\chi^2 \equiv \frac{m_\chi^2}{\omega_p^2}, \quad (22)$$

while in the case of nonzero m_ϕ , one may rewrite (21) using

²More accurately, $m_\chi^2 - \frac{m_\phi^2}{4} \ll \omega_p^2$.

relativistic invariant dimensionless parameters,

$$\frac{4g\Phi}{m_\phi^2} = \frac{4m_\chi^2}{m_\phi^2} - 1. \quad (23)$$

B. The solution without approximation

Here, we are looking for a solution which covers arbitrary m_χ (not only $m_\chi \ll \omega_{\vec{p}}$). For this reason, we solve Eq. (11) without an approximation.

After including the phase $\epsilon_k^- t/2$ into the definition of the amplitude operator, we obtain

$$a_{\vec{k}} = b_{\vec{k}} e^{i\epsilon_k^- t/2}. \quad (24)$$

In terms of $b_{\vec{k}}, b_{\vec{k}}^\dagger$ operators, we rewrite Eq. (11) as

$$\ddot{b}_{\vec{k}} - i(2\Omega_{\vec{k}} - \epsilon_{\vec{k}}) \dot{b}_{\vec{k}} + \left(\Omega_{\vec{k}} \epsilon_{\vec{k}} - \frac{\epsilon_{\vec{k}}^2}{4} \right) b_{\vec{k}} = \sigma_{\vec{p}-\vec{k}} b_{\vec{p}-\vec{k}}^\dagger. \quad (25)$$

We determine the solution of Eqs. (25) and (24) using an ansatz of the form of Eq. (14) with different values of coefficients. Precisely, the solution reads

$$a_{\vec{k}}(t) = e^{i\epsilon_k^- t/2} \left[a_{\vec{k}}(0) \left(\cosh(s_k^- t) - i \frac{\epsilon_{\vec{k}}^2/4 - s_k^2 - \Omega_{\vec{k}} \epsilon_{\vec{k}}}{s_k^- (2\Omega_{\vec{k}} - \epsilon_{\vec{k}})} \sinh(s_k^- t) \right) - a_{\vec{p}-\vec{k}}^\dagger(0) \cdot i \frac{\sigma_{\vec{p}-\vec{k}}}{s_k^- (2\Omega_{\vec{k}} - \epsilon_{\vec{k}})} \sinh(s_k^- t) \right], \quad (26)$$

where

$$s_k^- = \sqrt{\Omega_{\vec{k}}^2 (2\Omega_{\vec{k}} - \epsilon_{\vec{k}})^2 + \alpha^2 \omega_p^4 \frac{\Omega_{\vec{k}}}{\Omega_{\vec{p}-\vec{k}}} - \Omega_{\vec{k}} (2\Omega_{\vec{k}} - \epsilon_{\vec{k}}) - \frac{\epsilon_{\vec{k}}^2}{4}}. \quad (27)$$

Expanding expression (27) into the Taylor series over a small parameter $\alpha \ll 1$ and additionally over $\epsilon_{\vec{k}}/\Omega_{\vec{k}} \ll 1$, one derives the approximated result Eq. (15) in the leading order.

The result (27) is not invariant under symmetry $\vec{k} \rightarrow \vec{p} - \vec{k}$ beyond the leading-order perturbative expansion (15). Thus, in general, (26) and (27) is still a solution only if $\vec{k} = \frac{\vec{p}}{2}$. This may indicate that the ansatz we have chosen is incomplete.

Note that if we do not take into account the absence of symmetry, then $k = p/2$ (or $\kappa = 0.5$ in dimensionless case) still remains the lower, and therefore the most general, boundary of instability.

The occupancy number for a mode of a field χ with momentum $\vec{k}$ grows resonantly [6],

$$f_{\chi, \vec{k}}(t) = \langle 0 | a_{\vec{k}}^\dagger(t) a_{\vec{k}}(t) | 0 \rangle = \frac{\sigma_{\vec{p}-\vec{k}}^2}{4} \frac{\sinh^2(s_{\vec{k}} t)}{s_{\vec{k}}^2 \epsilon_{\vec{k}}^2}. \quad (28)$$

The total density of all resonantly produced modes is calculated using the same formula (18) as in the previous case. In physical terms, the exponential increase in the occupancy numbers can be interpreted as an increase in the number of particles with momentum k . However, the resonance condition is determined by $s_{\vec{k}}^2 > 0$.

In order to precisely determine the integration domain, one should solve $s_{\vec{k}}^2 = 0$ for (27) with respect to the momentum for various numbers of parameters.

In addition, it is convenient to introduce dimensionless notation $\vec{k} = \vec{k}/\omega$, $\mu_\chi = m_\chi/\omega$, $\vec{v} = \vec{p}/\omega$. The dimensionless frequencies reads

$$\begin{aligned} \frac{\Omega_{\vec{k}}}{\omega_{\vec{p}}} &\equiv \beta_{\vec{k}} = \sqrt{\kappa^2 + \mu_\chi^2}, \\ \frac{\Omega_{\vec{p}-\vec{k}}}{\omega_{\vec{p}}} &\equiv \beta_{\vec{v}-\vec{k}} = \sqrt{1 - 2\kappa + \kappa^2 + \mu_\chi^2}. \end{aligned} \quad (29)$$

In terms of dimensionless variables (29), Eq. (27) reads (furthermore, we introduce the dimensionless parameter $\eta_{\vec{k}} = s_{\vec{k}}/\omega_{\vec{p}}$)

$$\begin{aligned} \eta_{\vec{k}}^2 &= \sqrt{\beta_{\vec{k}}^2(\beta_{\vec{k}} - \beta_{\vec{v}-\vec{k}} + 1)^2 + \alpha^2 \beta_{\vec{k}} \beta_{\vec{v}-\vec{k}}^{-1} -} \\ &\quad \beta_{\vec{k}}(\beta_{\vec{k}} - \beta_{\vec{v}-\vec{k}} + 1) - \frac{(\beta_{\vec{k}} + \beta_{\vec{v}-\vec{k}} - 1)^2}{4}. \end{aligned} \quad (30)$$

Fixing $\vec{k}$ in Eq. (30), we determine the threshold value of α for fixed $\vec{k}$,

$$\alpha_{\vec{k}} = \frac{1}{4} \sqrt{\frac{\beta_{\vec{v}-\vec{k}}}{\beta_{\vec{k}}} (\beta_{\vec{k}} + \beta_{\vec{v}-\vec{k}} - 1)^2 (-3\beta_{\vec{k}} + \beta_{\vec{v}-\vec{k}} - 1)^2}. \quad (31)$$

This dependence is shown in Fig. 1 for several values of momentum $\vec{k}$.

The resonance condition Eq. (27) at the threshold $k = p/2$ reads directly Eq. (21) without any additional approximation on ω_p . This illustrates the fact that the value of ω_p determines only the frame; the stability boundary does not depend on it due to the Lorentz invariance.

C. Problems of the Heisenberg equation approach

Equations (26) and (27) are not symmetric with respect to the substitution $\vec{k} \rightarrow \vec{p} - \vec{k}$ unless $\vec{k} = \vec{p}/2$. In the approximate case $\alpha \ll 1$, the symmetry is restored.

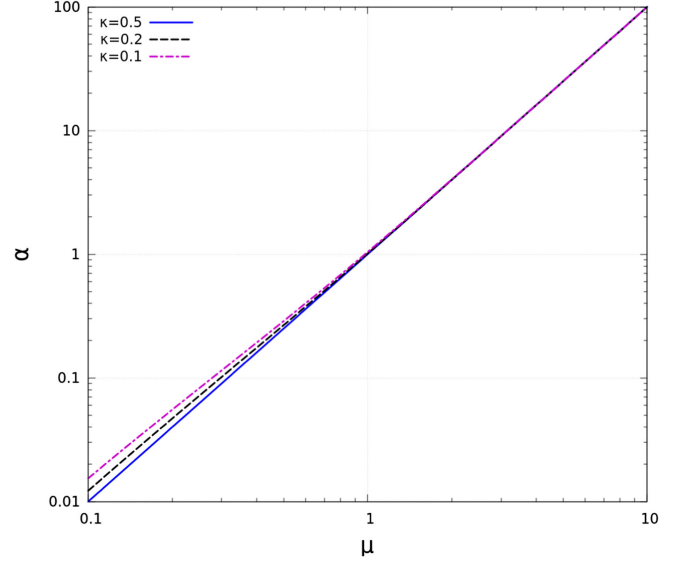


FIG. 1. Dependence $\alpha_{\vec{k}}(\mu_\chi)$ at fixed $\kappa = 0.1, 0.2, 0.5$. The upper region related to the resonance.

This result may indicate that the ansatz (26) is incomplete, and other research methods are required. Hence, we consider an alternative method in the next section.

IV. MATHIEU EQUATION ANALYSIS FOR THE INSTABILITY IN CONDENSATE

In this section, we consider a massive plane wave that has undergone the Lorentz boost (see details in Appendix C) to its rest frame being a condensate (5); $\omega_p = m_\phi$, $\vec{p} = 0$. The amplitude Φ is a Lorentz scalar, so it has not changed after the boost.

The EOM for the mode $\chi_{\vec{k}}$ reads

$$\ddot{\chi}_{\vec{k}} + (k^2 + m_\chi^2 + 2g\Phi \cos(m_\phi t))\chi_{\vec{k}} = 0, \quad (32)$$

which is the Mathieu equation,

$$\chi_{\vec{k}}'' + (A_k + 2q \cos(2z))\chi_{\vec{k}} = 0, \quad (33)$$

where $z = \frac{m_\phi t}{2}$, and

$$A_k = 4 \frac{k^2 + m_\chi^2}{m_\phi^2}, \quad q = 4 \frac{g\Phi}{m_\phi^2}. \quad (34)$$

The number of modes is determined as

$$n_{\vec{k}} = \langle 0 | a_{\vec{k}}(t) a_{\vec{k}}^\dagger(t) | 0 \rangle = |\chi_{\vec{k}}(t)|^2. \quad (35)$$

The Mathieu equation and its solutions are extensively researched in the literature [20,21]. It appears in physical systems which undergo parametric resonance, such as a mathematical pendulum in an external driving force. In the

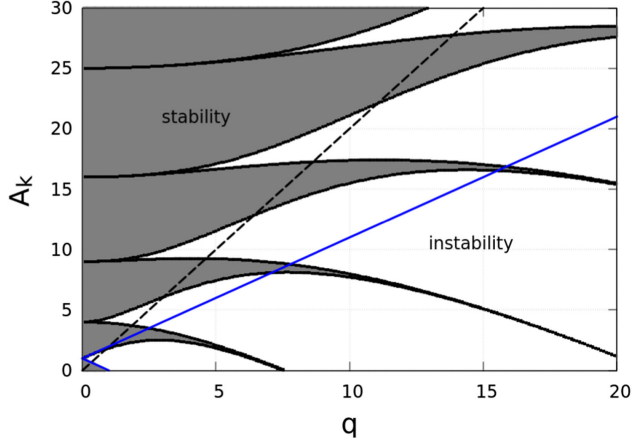


FIG. 2. Stability diagram of the Mathieu equation and stability bounds for the Heisenberg case [Eq. (50)]. Vertical axis: A_k . Horizontal axis: q . Gray area: stability regions of Mathieu equation. Blue solid line: Heisenberg instability bound, see Eq. (50). Black dashed line: line $A_k = 2q$.

parametric plane (A_k, q) one distinguishes the area of stability, where the solutions of the Mathieu equation oscillate, and the area of instability, where one of the solutions exponentially grows. This stability chart is shown in Fig. 2, where the gray area refers to stability and the white area to instability.

Besides the pendulum in classical mechanics, the Mathieu equation for given modes in field theory arises in the context of resonant matter particle production by an inflaton field at the end of inflation (preheating), [8], see [22] for review. The Mathieu equation appears in the most studied model $g\phi^2\chi^2$ which does not have kinematical instability and in the trilinear model $g\phi\chi^2$. In inflation models, the mass of the matter particles is usually significantly less than the mass of the inflaton, $m_\chi \ll m_\phi$, so the m_χ^2 term in (32) is usually neglected, taking $A_k = \frac{4k^2}{m_\phi^2}$. Otherwise, in our task $m_\chi > m_\phi/2$, we cannot neglect it. The case $k = 0$ is related to the threshold $k = p/2$ in the unboosted case.

Depending on the model parameters, one distinguishes the analytically solvable cases of narrow and broad resonances.

A. Narrow resonance

The regime $q \ll 1$ shows a simple solution in terms of elementary functions [37]. The growing solution χ_k^- of Eq. (33) can be considered as an oscillating function modulated by a growing exponent,

$$\chi_k^-(z) = e^{\mu z} \cos(z + \varphi), \quad (36)$$

where φ just defines the initial condition, and the exponent μ reads [37],

$$\mu = \frac{1}{2} \sqrt{q^2 - (A_k - 1)^2}. \quad (37)$$

A graph illustrating a narrow resonant solution is shown in the upper left corner of Fig. 3. The number of χ particles grows exponentially, $n_\chi = |\chi_k^-|^2 \sim e^{2\mu z}$ which determines the so-called narrow resonance [8,9,20]. The condition $\mu = 0$ shows the boundary of the resonance,

$$q = |A_k - 1|. \quad (38)$$

This dependence is shown graphically in the lower left corner of Fig. 2 (blue line). Using notation (34) with $m_\chi > m_\phi/2$ and setting the threshold zero momentum $k = 0$, one obtains the instability bound (23).

There is the following interpretation for this phenomenon. We take one massive particle ϕ from the condensate and consider decay $\phi \rightarrow 2\chi$. Due to the fact that χ particles are bosons, their presence in the final state causes Bose enhancement, which ultimately leads to an exponential increase in the number of χ particles produced. This process is the main resonance effect for the narrow peak [8,9,38].

Several massive particles of ϕ can merge producing χ particles, $N\phi \rightarrow 2\chi$, with subsequent Bose enhancement. The corresponding solutions (characteristic functions of Mathieu [20,21]) of Eq. (33) give additional narrow peaks at Fig. 2 that reach $q = 0$ at $A_k = N^2$, each of those narrower than the previous.

The solution for $N = 2$ is shown in the lower left part of Fig. 3. We see that there is an effect of additional oscillations called beats. In large $N \gg 1$, this supplementary oscillation takes the form seen in Fig. 3 right panel.

Note that the additional narrow peaks ($N > 1$) of the Mathieu equation, after boost transformation, lead to the effect of χ particle production in an interaction of several quanta from a massive plane wave ϕ . In the massless limit, the effect vanishes since two or more quanta of massless field propagate with the same velocity and do not interact with each other. Thus, the instability boundary for the massless case does not include these $N > 1$ peaks.

B. Broad resonance

The narrow resonance analytical solution fails in the regime $q \gtrsim 1$. However, there is another analytical solution called broad resonance in the regime $g \gg 1$ which is even more efficient.

The broad resonance regime is defined so that during one period of ϕ field oscillation the field χ oscillates many times. In this case, the frequency of the χ field [Eq. (32)] is determined as follows:

$$\omega = \sqrt{k^2 + m_\chi^2 + 2g\phi(t)}. \quad (39)$$

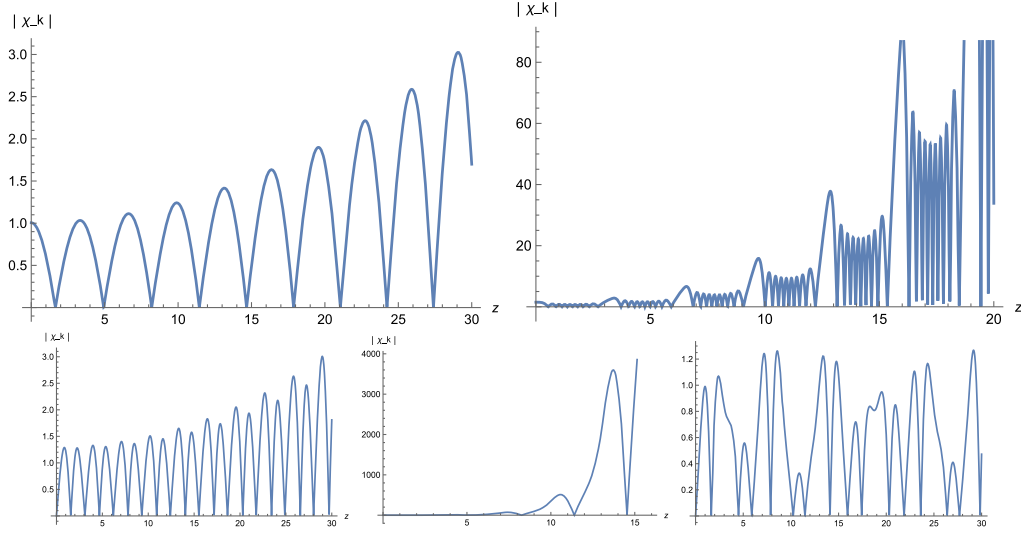


FIG. 3. The resonant solutions. Upper line, left panel: narrow resonance, $A_k = 1$, $q = 0.1$. Right panel: broad resonance. $A_k = 100$, $q = 50$. Lower line, left panel: $N = 2$ band resonance, $A_k = 4$, $q = 1$. Central panel: tachyonic resonance, $A_k = 1$, $q = 1.5$. Right panel: destructive parametrical resonance in naively tachyonic unstable area, $A_k = 3$, $q = 2$.

The graph of the solution corresponding to a broad parametric resonance is shown in the upper right corner of Fig. 3.

In the end of inflation, preheating can start with a broad resonance and continue as a narrow [8]. During a broad resonance, particles are produced by bursts, which are divided in time by $\sim T$, where $T = \frac{2\pi}{m_\phi}$ is the period of oscillation of the condensate. These bursts occur every time when the condition of adiabaticity [9,10],

$$\frac{\dot{\omega}(k, t)}{\omega^2(k, t)} \ll 1, \quad (40)$$

is violated. Inserting (39) into (40) yields the broad resonance condition for the Mathieu equation,

$$2q \sin(2z) \gtrsim (A_k + 2q \cos(2z))^{\frac{2}{3}}. \quad (41)$$

This expression implies two different conditions for the broad resonance [9]

$$A_k < 2|q|, \quad m_\phi t = \pi/2 + \pi N, \quad (42)$$

or

$$A_k - 2|q| \ll |q|^{1/2}, \quad m_\phi t = \pi + 2\pi N. \quad (43)$$

In the laboratory frame, particle production due to the broad resonance occurs by bursts as well, each burst arising at the beginning of the period $T = 2\pi/\omega_p$. However, in the theory with massless ϕ , there is no rest frame. Moreover, both conditions (42), (43) are not satisfied in the limit $m_\phi \rightarrow 0$. Consequently, a broad resonance does not emerge in the massless case.

1. Tachyonic instability

The parameters of the tachyon instability satisfy the condition of broad resonance, but there is an additional requirement. For a trilinear interaction at $q > A_k/2$, Eq. (33) shows that for several ranges of z the effective mass becomes negative (tachyonic) for that range of z . The following solution is expected to exponentially increase in that range, so the instability under the line $A_k = 2q$ in Fig. 2 is called the tachyon instability [9,10]. Tachyon preheating can be much more efficient than a conventional preheating mechanism. It is important to consider this phenomenon in the context of our work, as previously such a case of the ratio of the mass of condensate and the particles produced ($m_\phi < 2m_\chi$) had not been investigated in the literature.

If we consider only positive q , then the conditions of tachyon resonance will look like this,

$$0 < A_k < 2q. \quad (44)$$

The conditions for broad resonance, which are described in the previous section, show that the tachyon resonance region is part of the broad resonance region [9].

Comparative analysis of the conditions of tachyon and broad resonances demonstrated a range of parameters in which broad resonance will be observed, but there will be no tachyon resonance. We examine the second condition for broad resonance and, at the same time, the condition that tachyon resonance does not occur,

$$A_k - 2q \ll q^{1/2} \quad \text{and} \quad A_k > 2q.$$

Consider $A_k = 2q + \varepsilon$, where ε is some small parameter, then $2q + \varepsilon - 2q \ll q^{1/2} \Rightarrow \varepsilon \ll q^{1/2} \Rightarrow \varepsilon \ll \frac{\sqrt{2q\Phi}}{m_\phi}$. From

this, we can conclude that we converge to the parameter space of broad resonance and remain outside the region of tachyon instability when deviating from $A_k = 2q$ by an ϵ value.

On the basis of the results of this paragraph, it can be concluded that tachyon resonance satisfies the conditions of broad resonance, but it has an additional constraint. As a result, it turns out that for a massless plane wave, the solution mainly belongs to the region of tachyon instability. It also turns out that there is only a small range of parameters in which the latter is not observed, but there is an exceptionally broad resonance.

Note that the case of condensate may in some cases be similar to a standing wave. For our solution, the case of a standing wave was considered in paper [39], and we came to the conclusion that it reduces to a combination of plane waves.

V. COMPARISON OF TWO APPROACHES FOR CONDENSATE

In this section, we show that the Heisenberg equation solution for the decay of a massive condensate corresponds exactly to a narrow resonance, and compare it with the exact Mathieu chart.

A. The approach through the Heisenberg equation for condensate

Equation (11) for mass $m_\phi \neq 0$ and momentum $p = 0$ reads

$$e^{-i\Omega_k t} [\ddot{a}_k - 2i\Omega_k \dot{a}_k + 2\alpha\omega^2 \cos(m_\phi t) a_k] + \quad (45)$$

$$+ e^{i\Omega_k t} [\ddot{a}_{-k}^\dagger + 2i\Omega_k \dot{a}_{-k}^\dagger + 2\alpha\omega^2 \cos(m_\phi t) a_{-k}^\dagger] = 0. \quad (46)$$

Including the rotation $a_k(t) = b_k(t)e^{i\epsilon_k t/2}$, we obtain the equation for b_k .

$$\ddot{b}_k + i\dot{b}_k(\epsilon_k - 2\Omega_k) + b_k \left(\epsilon_k \Omega_k - \frac{\epsilon_k^2}{4} \right) + \alpha m_\phi^2 b_{-k}^\dagger = 0. \quad (47)$$

We replace an analogue of the approximate solution type. Naturally, our solution, when substituting parameters that satisfy the approximate result, reduces to the expression in Sec. III B. The solution to Eq. (47) is

$$b_k(t) = b_k(0) \left(\cosh(st) - i \frac{\frac{\epsilon_k^2}{4} - s^2 - \Omega_k \epsilon_k}{s(2\Omega_k - \epsilon_k)} \sinh(st) \right) - \quad (48)$$

$$ib_{-k}^\dagger \frac{\alpha\omega^2}{s(2\Omega_k - \epsilon_k)} \sinh(st),$$

where

$$s = \sqrt{\Omega_k^2 (2\Omega_k - \epsilon_k)^2 + \alpha^2 m_\phi^4 - \Omega_k (2\Omega_k - \epsilon_k) - \frac{\epsilon_k^2}{4}}. \quad (49)$$

In terms of the parameters of the Mathieu equation, we obtain the following relation for the stability boundary $s^2 = 0$ for any k :

$$q = |A_k - 1|. \quad (50)$$

This is explicitly the instability bound for the narrow resonance. This expression is completely consistent with Eq. (38) for the first peak of the Mathieu instability boundary. It is depicted in Fig. 2 by the solid blue line.

These results demonstrate that the ansatz used describes a narrow resonance and only the first peak, even for the massive case. Consequently, in the case $m_\phi \neq 0$, one should use a more complete ansatz or conduct the study using the Mathieu equation.

VI. CONCLUSION

In this article, we examined the process of massive particle production in a plane wave background of an intensive massless or small mass field in a toy scalar model with interaction $g\phi\chi^2$. We applied two methods: the solution of the Heisenberg equation for the amplitudes and reduction to the known case of particle production in a massive condensate, which is mathematically described by the Mathieu equation, and compared them. In addition, we compare both methods for the condensate.

The Heisenberg equation method provides a solution [34] which has the form of the narrow resonance in a massive scalar condensate but works also well in a different physical model, related to the tachyon instability.

The author of [34] presents the method in the case of low mass; we considered the case of an arbitrary mass. We have found an analytical solution to the Heisenberg equation for the amplitude of the χ field for arbitrary model parameters. We also showed that resonance is possible beyond the approximation discussed in [34]. Subsequently, we calculated that in the case of a large mass (here we considered the case when the initial plane wave ϕ is massless or has a mass less than the mass of two χ particles) a higher value of the amplitude is required to achieve resonance. This implies the existence of a threshold value for the plane wave amplitude, which is given by the inequality

$$\Phi \geq \frac{m_\chi^2}{g}.$$

The second solution method is better suited to describe massive plane waves. We reduce the equation of motion for a low-mass condensate to the Mathieu equation and analyze the instability diagram for the resulting parameters. Note that for the massless case, the parameters correspond to the broad resonance regime, but no resonance actually occurs

in the limit $m_\phi \rightarrow 0$. Consequently, this method cannot be applied in such cases.

We obtained a solution to the Heisenberg equation for a light condensate and compared it with the previously obtained solution of the Mathieu equation for the same physical scenario. By analyzing the instability boundary derived from the Heisenberg equation solution and comparing it with the Mathieu stability diagram, we observed that the first solution aligns with the first peak of the stability diagram within a narrow resonant region. Moreover, this comparison showed that the solution of the Heisenberg equation describes the tachyon instability, except for a constrained region of light χ . However, for systems with significant mass, the Heisenberg equation-based approach proves inadequate. This is because it provides only an approximate description of the first peak in a narrow resonance regime and fails to accurately characterize the parameter space associated with a broad resonance. Among other findings, we note that for a heavy χ particle ($m_\chi \gg m_\phi$) the instability threshold is two nonrelativistic particles with zero momentum $k = 0$. The corresponding $A_{k=0} \gg 1$ is related to the resonance, which is not narrow but broad/tachyonic.

On the other hand, when we considered a massless plane wave and its instability, we found that in this case there would be only one line of instability. Based on the particle interpretation ($n\phi \rightarrow 2\chi$), for the second and subsequent peaks, two or more massless waves should decay into two χ particles, but they cannot catch up with each other. So, the remaining peaks for a massless plane wave are not observed.

Moreover, the interpretation through the Mathieu stability diagram cannot be properly applied to the massless case, as resonance does not emerge under these conditions. Furthermore, the ansatz employed in this solution may be incomplete, suggesting that further investigation is required to refine the theoretical framework.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: SOLUTION FOR NARROW RESONANCE REGIME

Consider the Mathieu equation in the form

$$y'' + (a - 2q \cos(2z))y = 0.$$

where the approximation is $a \gg q$, $q > 0$; q is real; a lies in a narrow unstable region where $b_m < a < a_m$; m is the

peak number; a_m , b_m are characteristic numbers of the Mathieu equation.

The solution for the Mathieu equation is in this approximation,

$$\begin{aligned} y_1 &\simeq e^{\mu z} [C_m c e_m(z, q) + S_m s e_m(z, q)], \\ y_2 &\simeq e^{-\mu z} [C_m c e_m(z, q) - S_m s e_m(z, q)], \\ \mu &\simeq \pm \frac{[(a_m - a)(a - a_m)]^{\frac{1}{2}}}{2m}. \end{aligned}$$

For the first resonant band ($a \approx 1$),

$$\begin{aligned} a_1 &\simeq 1 + q - \frac{1}{8}q^2, \\ b_1 &\simeq 1 - q - \frac{1}{8}q^2, \\ \Rightarrow \mu &\simeq \pm \frac{1}{2} [(1 + q - a)(a - (1 - q))]^{\frac{1}{2}} = \frac{1}{2} \sqrt{q^2 - (a - 1)^2}. \end{aligned}$$

In our notations, $a = A_k$,

$$\mu \simeq \frac{1}{2} \sqrt{q^2 - (A_k - 1)^2}.$$

APPENDIX B: CONDITION FOR BROAD RESONANCE

The condition for a broad resonance is a violation of the adiabaticity condition (40). It follows from the fact that the Wentzel–Kramers–Brillouin approximation is used in the solution [9,10] and at the moments when it is violated and a broad resonance occurs. Then, the condition for a broad resonance will be reduced to the expression

$$\frac{d\omega}{dt} \gtrsim \omega^2. \quad (\text{B1})$$

Using expression (39) for the frequency of the chi field, we obtain

$$\begin{aligned} 2g\dot{\phi} &\gtrsim (k^2 + m_\chi^2 + 2g\phi(t))^{2/3}, \\ \phi(t) &= \Phi \cos(m_\phi t). \end{aligned} \quad (\text{B2})$$

It is maximum ϕ when $\cos(m_\phi t) = 1$ and minimum ϕ when $\cos(m_\phi t) = 0$.

$$\begin{aligned} 1) \quad \phi(t) &= 0 \quad \text{for } m_\phi t = \pi/2 + N\pi. \\ 2) \quad \phi &= \Phi \quad \text{for } m_\phi t = 2\pi N. \end{aligned}$$

Decompose the cosine into a Maclaurin series (the series converges for any argument),

$$\cos(x) = 1 - \frac{x^2}{2!} + \dots$$

1) $\dot{\phi} \approx -m_\phi(\frac{\pi}{2} + \pi N)$, where N is an integer. From expression (B1), we get the condition for k^2 ,

$$k^2 \lesssim (-g\Phi m_\phi(\frac{\pi}{2} + \pi N))^{2/3} - m_\chi^2 - 2g\phi(t). \quad (\text{B3})$$

Then, the condition for ϕ near zero will be

$$\phi(t) \lesssim \frac{(-g\Phi m_\phi(\frac{\pi}{2} + \pi N))^{2/3} - m_\chi^2}{2g}. \quad (\text{B4})$$

2) $\dot{\phi} \approx \Phi 2\pi N m_\phi$, where N is an integer. From expression (B1), we get the condition for k^2 ,

$$k^2 \lesssim (-g\Phi m_\phi(2\pi N))^{2/3} - m_\chi^2 - 2g\phi(t). \quad (\text{B5})$$

Then, the condition for ϕ near Φ will be

$$\phi(t) \lesssim \frac{(-g\Phi m_\phi(2\pi N))^{2/3} - m_\chi^2}{2g}.$$

3) $\cos(m_\phi t) = -1$ when $m_\phi t = \pi + 2\pi N$, then $\dot{\phi} \approx \Phi m_\phi(\pi + 2\pi N)$,

$$k^2 \lesssim (-g\Phi m_\phi(\pi + 2\pi N))^{2/3} - m_\chi^2 - 2g\phi(t),$$

and

$$\phi(t) \lesssim \frac{(-g\Phi m_\phi(\pi + 2\pi N))^{2/3} - m_\chi^2}{2g}. \quad (\text{B6})$$

APPENDIX C: BOOST

The classical plane wave of a massive scalar field is a condensate in a boosted coordinate system. Due to Lorentz invariance, the observable physical values such as the stability of the ϕ field configuration should remain the same in any coordinate system. In particular, the stability chart cannot depend on the plane wave frequency $\omega_{\vec{p}}$ or momentum $\vec{p}$ separately, only on the combination $\sqrt{\omega_{\vec{p}}^2 - p^2} = m_\phi$. The boost factor $\omega_{\vec{p}}/m_\phi$ describes the transfer between the laboratory and the center-of-mass systems.

To make a boost at $p \neq 0$, we use Lorentz transformations [40]. Consider

$$\phi' = \Phi \cos(m_\phi t) = \Phi \cos(p_\mu x_\mu),$$

where the ϕ' field is a scalar, respectively, and the amplitude is not transformed during the Lorentz transformation.

We used the notation ' for the values up to the boost at $p = 0$, and the parameters of the Mathieu equation in terms of the Heisenberg equation will look like this,

$$A'_k = 4 \frac{k^2 + m_\phi^2}{\omega'^2}, \quad q = 4 \frac{g\Phi}{\omega'^2}. \quad (\text{C1})$$

The density of the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}(\omega^2 + p^2 + m_\phi^2) \cos^2(p_\mu x_\mu),$$

$\omega^2 - p^2 = m_\phi^2$ $\mathcal{H} = \omega^2 \cos^2(p_\mu x_\mu)$. Before boost, $\mathcal{H}_{m_\phi} = m_\phi^2 \cos^2(m_\phi t)$, after boost, $\mathcal{H}_p = \omega_\phi^2 \cos^2(\omega t - p x)$. We know $\mathcal{H} = T_{00}$; then, when we transition from one frame of reference to another, $T_{00} \rightarrow \frac{\omega^2}{m_\phi^2} T_{00}$.

Therefore, we arrive at the result that the γ factor is equal to

$$\gamma = \frac{\omega}{m_\phi} = \frac{1}{\sqrt{1 - V^2}}.$$

From here, we can find the velocity of the moving frame of reference,

$$V = \frac{\sqrt{\omega^2 - m_\phi^2}}{\omega} = \frac{p}{\omega}.$$

It follows from the Lorentz transformations for momentum that

$$k = (\pm k' + V \mathcal{E}') \gamma = \left(\pm k' + \frac{p m_\phi}{\omega} \right) \frac{\omega}{m_\phi} = \pm k' \frac{\omega}{m_\phi} + \frac{p}{2},$$

where $\mathcal{E}' = \frac{m_\phi}{2}$ is the energy of the resulting particle in the reference frame with momentum $p = 0$, and if we consider the boundary case $k' = 0$, then as a result of the boost, k will be equal to $p/2$. It is this boundary case that we consider in the following paragraphs.

After the boost, we can no longer apply Mathieu's methods to solving the equation. However, we can study how the relationship between the parameters of the Mathieu equation A_k and q will change after the boost $p \neq 0$,

$$A_k = \frac{4k^2}{\omega^2} + \frac{4m_\chi^2}{\omega^2}, \quad q = \frac{4g\Phi}{\omega^2},$$

where $\omega^2 = m_\phi^2 + p^2$.

- [1] V. Ritus and A. Nikishov, Quantum electrodynamics phenomena in the intense field, *Trudy FIAN* **111** (1979), <https://proceedings.lebedev.ru/0111-1979/>.
- [2] W. Dittrich and H. Gies, *Springer Tracts in Modern Physics*, Vol. 166 (Springer, Berlin, Heidelberg, 2000).
- [3] A. Kuznetsov and N. Mikheev, *Springer Tracts in Modern Physics*, Vol. 252 (Springer, New York, 2013).
- [4] E. A. Calzetta and B.-L. B. Hu, *Nonequilibrium Quantum Field Theory*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2023).
- [5] A. Fedotov, A. Ilderton, F. Karbstein, B. King, D. Seipt, H. Taya, and G. Torgrimsson, Advances in QED with intense background fields, *Phys. Rep.* **1010**, 1 (2023).
- [6] A. A. Grib, S. G. Mamaev, and V. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Friedmann Laboratory Pub., 1994).
- [7] A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Friedmann Laboratory Publishing, St. Petersburg, 1994).
- [8] L. Kofman, A. D. Linde, and A. A. Starobinsky, Towards the theory of reheating after inflation, *Phys. Rev. D* **56**, 3258 (1997).
- [9] K. Lozanov, *Reheating After Inflation* (Springer, New York, 2020).
- [10] J. F. Dufaux, G. N. Felder, L. Kofman, M. Peloso, and D. Podolsky, Preheating with trilinear interactions: Tachyonic resonance, *J. Cosmol. Astropart. Phys.* **07** (2006) 006.
- [11] C. Eröncel, Axion-like-particle dark matter beyond the standard paradigm, [arXiv:2501.11717](https://arxiv.org/abs/2501.11717).
- [12] R. Wiley Deal, L. Barrowes, J. T. Giblin, K. Sinha, S. Watson, and F. C. Adams, Cosmological moduli and non-perturbative production of axions, [arXiv:2501.17229](https://arxiv.org/abs/2501.17229).
- [13] L. Kofman, A. D. Linde, and A. A. Starobinsky, Reheating after inflation, *Phys. Rev. Lett.* **73**, 3195 (1994).
- [14] J. H. Traschen and R. H. Brandenberger, Particle production during out-of-equilibrium phase transitions, *Phys. Rev. D* **42**, 2491 (1990).
- [15] Y. Shtanov, J. H. Traschen, and R. H. Brandenberger, Universe reheating after inflation, *Phys. Rev. D* **51**, 5438 (1995).
- [16] S. Y. Khlebnikov and I. I. Tkachev, Classical decay of inflaton, *Phys. Rev. Lett.* **77**, 219 (1996).
- [17] S. Y. Khlebnikov and I. I. Tkachev, The Universe after inflation: The wide resonance case, *Phys. Lett. B* **390**, 80 (1997).
- [18] R. Allahverdi, R. Brandenberger, F.-Y. Cyr-Racine, and A. Mazumdar, Reheating in inflationary cosmology: Theory and applications, *Annu. Rev. Nucl. Part. Sci.* **60**, 27 (2010).
- [19] M. A. Amin, M. P. Hertzberg, D. I. Kaiser, and J. Karouby, Nonperturbative dynamics of reheating after inflation: A review, *Int. J. Mod. Phys. D* **24**, 1530003 (2014).
- [20] N. W. McLachlan, *Theory and Application of Mathieu Functions* (Clarendon Press, Oxford, 1947).
- [21] R. N. Jazar, *Perturbation Methods in Science and Engineering* (Springer, New York, 2021).
- [22] K. Schmitz and G. Vertongen, Reheating and preheating after inflation: An introduction, *Reh* **6**, t2 (2010), https://www.desy.de/~westphal/workshop_seminar_fall_2010/reheating.pdf.
- [23] H. Fujisaki, K. Kumekawa, M. Yoshimura, and M. Yamaguchi, Particle production and gravitino abundance after inflation, *Phys. Rev. D* **54**, 2494 (1996).
- [24] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series (Dover Publications, New York, 1965).
- [25] A. Y. Andreev and D. A. Kirzhnits, Tachyons and the instability of physical systems, *Phys. Usp.* **39**, 1071 (1996).
- [26] G. Feinberg, Possibility of faster-than-light particles, *Phys. Rev.* **159**, 1089 (1967).
- [27] S. Khlebnikov and I. Tkachev, Resonant decay of cosmological Bose condensates, *Phys. Rev. Lett.* **79**, 1607 (1997).
- [28] D. G. Levkov, A. G. Panin, and I. I. Tkachev, Radio-emission of axion stars, *Phys. Rev. D* **102**, 023501 (2020).
- [29] K. A. Beyer, G. Marocco, C. Danson, R. Bingham, and G. Gregori, Parametric co-linear axion photon instability, *Phys. Lett. B* **839**, 137759 (2023).
- [30] C. Cronström and M. Noga, Photon induced relativistic band structure in dielectrics, *Phys. Lett.* **60A**, 137 (1977).
- [31] W. Becker, Relativistic charged particles in the field of an electromagnetic plane wave in a medium, *Physica A (Amsterdam)* **87**, 601 (1977).
- [32] T. Heinzl, A. Ilderton, and B. King, Classical and quantum particle dynamics in univariate background fields, *Phys. Rev. D* **94**, 065039 (2016).
- [33] B. King and H. Hu, Classical and quantum dynamics of a charged scalar particle in a background of two counter-propagating plane waves, *Phys. Rev. D* **94**, 125010 (2016).
- [34] A. Arza, Production of massive bosons from the decay of a massless particle beam, *Phys. Rev. D* **105**, 036004 (2022).
- [35] J. T. Mendonça and A. Serbeto, Volkov solutions for relativistic quantum plasmas, *Phys. Rev. E* **83**, 026406 (2011).
- [36] E. Raicher, S. Eliezer, and A. Zigler, Nonlinear Compton scattering in a strong rotating electric field, *Phys. Rev. A* **94**, 062105 (2016).
- [37] L. D. Landau and E. Lifshitz, *Mechanics: Volume 1* (Butterworth-Heinemann, 1982).
- [38] R. Allahverdi, R. H. A. D. Shaw, and B. A. Campbell, Parametric resonance for complex fields, *Phys. Lett. B* **473**, 246 (2000).
- [39] E. Dmitrieva and P. Satunin, The effects of the birth of particles in the field of waves of high intensity, *Proc. Sci. ICPPCRubakov2023* (2024) 047.
- [40] L. D. Landau and E. Lifshitz, *The Classical Theory of Fields: Volume 2* Course of theoretical physics (Elsevier Science, 1975).