

QED corrections in unstable vacuum

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Self-energy and vacuum polarization effects in quantum electrodynamics (QED) are calculated for the supercritical Coulomb field, where Dirac energy levels become embedded in the negative-energy continuum. In this regime, the quantum vacuum becomes unstable, resulting in spontaneous electron-positron pair creation. By calculating the imaginary part of the QED correction, we gain access to an unexplored channel of vacuum instability: radiative spontaneous pair creation. Our results show that this radiative channel is greatly enhanced in the vicinity of the threshold of the supercritical regime, providing evidence for nonperturbative effects with respect to the fine-structure constant α . We therefore conjecture that the total probability of spontaneous pair creation could differ significantly from the predictions of Dirac theory, especially near the supercritical threshold.

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I. INTRODUCTION

It has long been realized that for sufficiently large nuclear charges the energies of the lowest bound states of the Dirac equation may fall below the negative-continuum threshold, $E_D < -mc^2$, where m is the electron mass, and c the speed of light. This phenomenon emerges for nuclear charge numbers $Z > Z_{\text{cr}} \approx 173$ and is known as the *diving* of the Dirac bound states into the negative continuum (see Fig. 1). Once a bound state has undergone such diving, it is referred to as a *submerged* state. The wave functions of these states are no longer square-integrable and their energies have an imaginary component, indicating that these originally bound states have become resonance states. This peculiar behavior is a manifestation of the instability of the quantum vacuum in the presence of a sufficiently strong field, known as supercritical. The corresponding physical process is the creation of an electron-positron pair, in which the electron fills the vacancy in the submerged Dirac state and the positron is emitted into the continuum. This effect, known as spontaneous pair creation (SPC), was predicted long ago [1,2] but has not yet been observed. The SPC effect is conceptually similar to the famous Schwinger pair production in an intense electric field [3–5], see also reviews [6–8]. The key difference is that in SPC, the electron is produced in a

bound state and only the positron escapes into the continuum. Furthermore, the Schwinger mechanism allows creation of an indefinite number of electron-positron pairs, whereas in the SPC the number of pairs is limited by the available vacancies in dived states.

Even after the Dirac states dive into the negative-energy continuum, their energies and widths can still be calculated with high precision [9–14]. The resulting widths are typically interpreted in terms of the probability of the SPC process [15]. This interpretation, however, neglects the presence of quantum electrodynamical (QED) effects, assuming they are suppressed by the fine-structure constant α and therefore do not significantly alter the overall physical picture.

QED is among the most rigorously tested theories in modern physics. Owing to the small value of the expansion parameter $\alpha \approx 1/137$, incorporating just the first few terms of the QED perturbation series yields predictions of

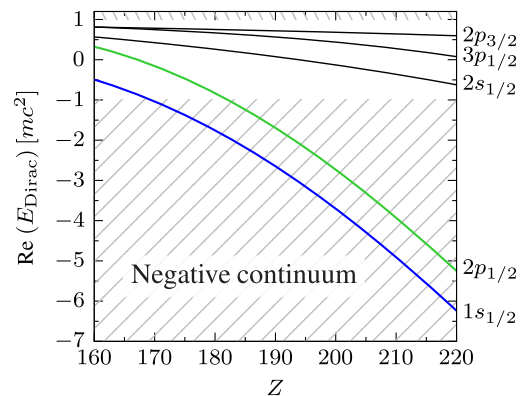


FIG. 1. Real part of the Dirac bound-state energies as a function of the nuclear charge number Z .

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outstanding precision [16]. Although formally the QED perturbation series is most likely asymptotic, its divergence would become significant only at very high orders (so, Dyson [17] argued that this occurs around order $\approx 1/\alpha \approx 137$). In practical contexts, QED perturbation theory is often referred to as “effectively converging”, meaning that the asymptotic nature of the series does not affect its utility at low orders. However, in the supercritical regime, this assumption might need to be revisited. It was argued by Ritus [18] and Narozhny [19] that for sufficiently strong constant crossed fields, QED perturbation theory in α itself may break down, making QED a fully non-perturbative and strongly coupled theory (see also reviews [20,21]).

Although the SPC process has never been observed so far, there are vast experimental efforts directed to the creation and investigation of supercritical fields and unstable vacuum in two main directions. The first one involves high-power optical lasers [22–25] and attempts to reach supercritical fields in laser-electron and electron-electron collisions [26–29]. The theoretical investigations of such scenarios [30–32] have not found confirmations of the Ritus-Narozhny conjecture so far. The second direction of experimental investigations of supercritical fields is connected to low-energetic collisions of two heavy nuclei. In a slow adiabatic collision, a quasimolecule can be formed by two colliding nuclei, with an enhanced nuclear Coulomb field that may reach the supercritical regime. Such experimental studies are planned at number of different facilities around the globe [33–35].

Detection of SPC in the supercritical regime formed during heavy-ion collisions presents a significant experimental challenge. One of the main difficulties arises due to the presence of the so-called dynamical pair creation induced by the time variation of the potential, which can occur even without the formation of the supercritical regime. Since spontaneous and dynamical pair creation is impossible to separate out, one cannot measure directly the presence of SPC contribution, and, thus, cannot unambiguously confirm the submerging of the bound state to the negative continuum. The comparative significance of dynamical and spontaneous channels has been previously discussed, e.g., in Ref. [36] together with possible approaches for identifying traces of SPC [37–39]. Here, we calculate the QED corrections in the field of a static supercritical nucleus, where the dynamic pair production is not present.

In this paper, we explore the convergence of the QED perturbation theory in a supercritical Coulomb field. We address the QED effects of first order in α , evaluating them to all orders in the electron-nucleus coupling strength parameter $Z\alpha$. In the subcritical regime, QED corrections are typically suppressed by a factor of α relative to the Dirac energy. A nonperturbative character of QED in the supercritical regime would manifest itself as a breakdown

of the expansion in α , with QED corrections becoming comparable in magnitude to the Dirac energy. In order to test this conjecture, we calculate the first-order in α corrections to the Dirac energies in a supercritical Coulomb field, namely the electron self-energy and vacuum polarization. Moreover, knowledge of the imaginary part of these QED corrections gives us access to a new, radiative channel of spontaneous pair creation.

II. FORMALISM

There are two main problems in calculations of the QED corrections in a supercritical Coulomb field. The first difficulty is that the electron-nucleus coupling parameter $Z\alpha$ is greater than 1, which requires a nonperturbative in $Z\alpha$ treatment of the electron propagators. The second difficulty is that the reference-state wave function is no longer square integrable. This is the first calculation of radiative QED effects where both of these two problems must be solved simultaneously. The nonperturbative treatment of the electron-nucleus coupling is based on usage of the Dirac-Coulomb Green’s function, for which no closed analytical expression is known. As a consequence, the calculations need to be performed numerically, and no analytical factorization occur, contrary to the Schwinger mechanism.

The starting point of our consideration is the stationary one-electron Dirac Hamiltonian:

$$h_D = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m + V(r), \quad (1)$$

where $\boldsymbol{\alpha}$ and β are the Dirac matrices, $\mathbf{p}$ is the momentum operator, and $V(r)$ a static nuclear-charge potential. We are interested here in the supercritical regime of the nuclear charge potential with $Z > Z_{\text{cr}}$. In the scenario when this regime is achieved in a collision of two nuclei, such potential corresponds to the so-called monopole approximation of the two-center nuclear potential, i.e., keeping the lowest-order term of its multipole expansion. Explicit calculations of two-center QED effects in the sub-critical regime [40,41] demonstrated that the monopole approximation gives a reasonable description of two-center QED effects at small internuclear distances, where the supercritical regime is reached.

Solving the Dirac Hamiltonian for the bound states embedded into the negative-energy continuum is a non-trivial task since these states decay via spontaneous electron-positron pair creation and their wave functions are not square-integrable. A method of complex-scaling (CS) allows a systematic treatment of resonance states, see reviews [42–47] and recent works [48,49]. The method is based on dilating the Dirac Hamiltonian into the complex plane of the radial variable r , e.g., by the uniform complex rotation

$$\mathbf{r} \rightarrow \mathbf{r}e^{i\theta}, \quad (2)$$

leading to the following CS-transformed Hamiltonian

$$h_{D,\theta} = e^{-i\theta} \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m + V(re^{i\theta}). \quad (3)$$

The nuclear-charge potential is approximated by the Gaussian model, which is continued in the complex plane as follows

$$V(re^{i\theta}) = -\frac{\alpha Z}{re^{i\theta}} \operatorname{erf}\left(\sqrt{\frac{3}{2}} \frac{re^{i\theta}}{R_{\text{nuc}}}\right), \quad (4)$$

where erf is the error function and R_{nuc} is the nuclear charge root-mean-square radius. The choice of the Gaussian model is convenient since it can be dilated into the complex plane for a wide range of θ .

After the complex rotation, the Hamiltonian becomes a non-hermitian symmetric operator with complex eigenvalues. The energies of the true bound eigenstates of the rotated Hamiltonian remain unchanged. The energies of the continuum and the submerged states, however, acquire different imaginary parts and thus become separated from each other. The eigenvalues of the submerged states have the form $E = E_{\text{res}} + i\Gamma/2$, where the real part E_{res} is the energy and the imaginary part Γ is interpreted as the decay rate due to the spontaneous positron-bound-electron pair creation. The corresponding calculations were carried out in Refs. [9–11].

The crucial advantage of the CS method is that after the complex rotation the wave functions of the submerged states become square-integrable. This opens a way not only for solving the Dirac equation but also for generalizing this method to calculations of the radiative QED corrections, which is accomplished in the present work.

There are two QED corrections of the first order in α , the electron self-energy and vacuum polarization, each of which will be addressed here.

III. SELF-ENERGY

The self-energy correction of the state a with energy ε_a and wave-function ψ_a is expressed as

$$\begin{aligned} \Delta E_{\text{SE}} = 2i\alpha \int_{C_F} d\omega \int d\mathbf{r}_1 d\mathbf{r}_2 \psi_a^\dagger(\mathbf{r}_1) \alpha_\mu G(\varepsilon_a - \omega, \mathbf{r}_1, \mathbf{r}_2) \\ \times \alpha_\nu \psi_a(\mathbf{r}_2) D^{\mu\nu}(\omega, \mathbf{r}_{12}) - \delta m \int d\mathbf{r} \psi_a^\dagger(\mathbf{r}) \beta \psi_a(\mathbf{r}), \end{aligned} \quad (5)$$

where $\alpha_\mu = (1, \boldsymbol{\alpha})$, δm is the mass counterterm, G the Dirac-Coulomb Green's function, and $D^{\mu\nu}(\omega, \mathbf{r}_{12})$ denotes the photon propagator with $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$, and C_F the Feynman integration contour.

Equation (5) is only a formal expression since it contains ultraviolet divergences that need to be covariantly regularized and eliminated. This is achieved by isolating two first

terms of the expansion of the Dirac Green's function in powers of interaction with the binding potential [50,51]:

$$\Delta E_{\text{SE}} = \Delta E^{(0)} + \Delta E^{(1)} + \Delta E^{(2+)}. \quad (6)$$

The first two terms, referred to as the zero- $\Delta E^{(0)}$ and one-potential $\Delta E^{(1)}$ terms, are calculated in momentum space within the dimensional regularization, whereas the remaining contribution, referred to as the many-potential term $\Delta E^{(2+)}$, is calculated in coordinate space with the partial-wave expansion of the Dirac Green's function. We would like to stress that in contrast to continuum-continuum processes considered in [18–21,52], the diving of the Dirac state corresponds to a bound-continuum transition and, as a consequence, it does not suffer from infrared divergencies.

When the energy of the reference state enters the supercritical regime, $\varepsilon_a < -mc^2$, the reference-state wave function ceases to be square-integrable and Eq. (6) cannot be calculated by techniques developed in previous investigations. We overcome these difficulties by applying the complex rotation of the radial variable (2) and the corresponding rotation of the momentum, $p \rightarrow pe^{-i\theta}$. The complex-rotated Dirac Green's function is obtained by solving the CS Dirac equation (3).

One of the major difficulties in numerical evaluation of the self-energy correction comes from the fact that the complex rotation (2) greatly complicates the analytical structure of the integrand in Eq. (5) as a function of the complex ω , which entails the necessity of using elaborated integration contours in the complex ω plane. Figure 2 shows the poles and branch cuts of the Dirac Green's function, branch cuts of the photon propagator, and the integration contours used for the numerical evaluation of the self-energy correction in the supercritical regime.

We now present expressions for the individual terms in Eq. (6) before and after the CS transformation. Before the CS transformation, the zero-potential and one-potential terms are given by [50]

$$\Delta E_a^{(0)} = \int \frac{d\mathbf{p}}{(2\pi)^3} \bar{\psi}_a(\mathbf{p}) \Sigma_R^{(0)}(\varepsilon_a; \mathbf{p}) \psi_a(\mathbf{p}) \quad (7)$$

and

$$\Delta E_a^{(1)} = \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\mathbf{p}'}{(2\pi)^3} \bar{\psi}_a(\mathbf{p}') V(\mathbf{q}) \Gamma_R^0(\varepsilon_a; \mathbf{p}', \mathbf{p}) \psi_a(\mathbf{p}), \quad (8)$$

where $\bar{\psi}_a = \psi_a^\dagger \gamma^0$, $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ and the Fourier transform of the coordinate-space wave function ψ_a is defined as

$$\psi_a(\mathbf{p}) = \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \psi_a(\mathbf{r}). \quad (9)$$

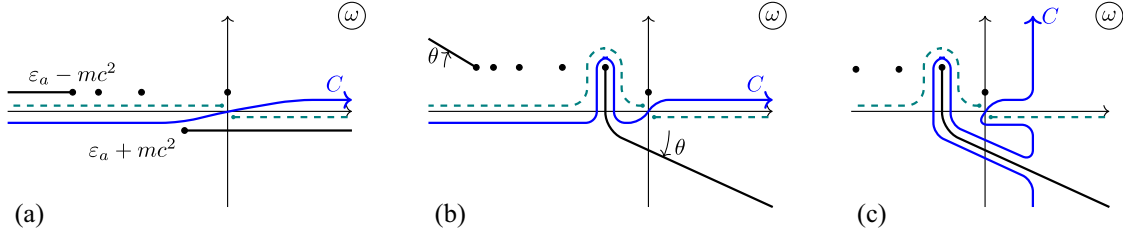


FIG. 2. The analytical structure of the integrand of the self-energy operator in the complex ω plane, for $a = 1s$ reference state, in the supercritical regime $\varepsilon_a < -mc^2$: (a) before the complex rotation, (b) after the complex rotation (with the reference-state energy ε_a acquiring a positive imaginary part), (c) after the deformation of the integration contour for the numerical integration. The dashed turquoise line represents the branch cuts of the photon propagator. Dots and the solid black line show the poles and branch cuts of the electron propagator, respectively. The solid blue line denotes the contour of ω integration.

The explicit expressions for the free self-energy and vertex operators $\Sigma_R^{(0)}$ and Γ_R^0 can be found in Refs. [50,51].

After the CS transformation, the zero- and one-potential terms become

$$\Delta E_a^{(0)} = e^{-3i\theta} \int \frac{d\mathbf{p}}{(2\pi)^3} \bar{\psi}_a(\mathbf{p}e^{-i\theta}) \Sigma_R^{(0)}(\varepsilon_a; \mathbf{p}e^{-i\theta}) \psi_a(\mathbf{p}e^{-i\theta}) \quad (10)$$

and

$$\Delta E_a^{(1)} = e^{-6i\theta} \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\mathbf{p}'}{(2\pi)^3} \bar{\psi}_a(\mathbf{p}'e^{-i\theta}) V(\mathbf{q}e^{-i\theta}) \times \Gamma_R^0(\varepsilon_a; \mathbf{p}'e^{-i\theta}, \mathbf{p}e^{-i\theta}) \psi_a(\mathbf{p}e^{-i\theta}), \quad (11)$$

where

$$\psi_a(\mathbf{p}e^{-i\theta}) = e^{3i\theta} \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \psi_a(\mathbf{r}e^{i\theta}). \quad (12)$$

Note that when the uniform complex rotation is applied, one needs also to modify Eq. (9) by transforming the radial coordinate in accordance with the rule (2). After this transformation, however, the integral will diverge exponentially. To overcome this issue, we perform the uniform complex rotation in the momentum space at the opposite angle, $p \rightarrow pe^{-i\theta}$.

The regularized expression for the zero-potential term, after integration over angular variables, is obtained as

$$\Delta E_a^{(0)} = e^{-3i\theta} \frac{\alpha}{4\pi} \int_0^\infty \frac{p^2 dp}{(2\pi)^3} \{a(\rho_a)[\tilde{g}_a^2 - \tilde{f}_a^2] + b(\rho_a)[\varepsilon_a(\tilde{g}_a^2 + \tilde{f}_a^2) + 2pe^{-i\theta}\tilde{g}_a\tilde{f}_a]\}, \quad (13)$$

where

$$\rho_a \equiv \rho_a(pe^{-i\theta}) = 1 - \varepsilon_a^2 + p^2 e^{-2i\theta} \quad (14)$$

and the functions $a(\rho)$ and $b(\rho)$ are defined as in Refs. [50,51]. The momentum representation of the radial

components of the wave function are, respectively, defined as

$$\tilde{g}_a \equiv \tilde{g}_a(pe^{-i\theta}) = e^{3i\theta} \sqrt{\frac{2}{\pi}} \int_0^\infty dr r^2 j_l(pr) g_a(re^{i\theta}), \quad (15)$$

$$\tilde{f}_a \equiv \tilde{f}_a(pe^{-i\theta}) = -e^{3i\theta} \frac{\kappa}{|\kappa|} \sqrt{\frac{2}{\pi}} \int_0^\infty dr r^2 j_l(pr) f_a(re^{i\theta}). \quad (16)$$

Here $\kappa = (-1)^{j+l+1/2}(j+1/2)$ is the Dirac quantum number with j and l standing for the total and orbital angular momenta, respectively, $\bar{l} = 2j - l$, and j_l is the spherical Bessel function.

The many-potential term is expressed as

$$\Delta E_a^{(2+)} = 2i\alpha e^{2i\theta} \int_{C_F} d\omega \int d\mathbf{r}_1 \int d\mathbf{r}_2 \psi_a^\dagger(\mathbf{r}_1 e^{i\theta}) \alpha_\mu \times G^{2+}(\varepsilon_a - \omega, \mathbf{r}_1 e^{i\theta}, \mathbf{r}_2 e^{i\theta}) \alpha_\nu \psi_a(\mathbf{r}_2 e^{i\theta}) \times D^{\mu\nu}(\omega, \mathbf{r}_{12} e^{i\theta}), \quad (17)$$

where $G^{(2+)}(\omega) \equiv G(\omega) - G^{(0)}(\omega) - G^{(1)}(\omega)$ is the Dirac Green's function containing two or more interactions with the binding potential. Explicit expressions for the CS-transformed free-Dirac and Dirac-Coulomb Green's functions are given in the Appendix.

IV. VACUUM POLARIZATION

The vacuum polarization (VP) potential is usually divided into the Uehling part, containing one interaction with the binding potential inside the lepton loop, and the Wichmann-Kroll part, containing three or more interactions with the binding potential inside the loop. For supercritical fields, both these terms are of the same order and need to be accounted for on the same footing.

In the supercritical regime, $\varepsilon_a < -mc^2$, we need to perform the complex rotation in order to make the reference-state wave function squarely integrable. The Uehling potential then becomes

$$\begin{aligned}
re^{i\theta}V_{\text{Ue}}(re^{i\theta}) &= -\alpha(\alpha Z)\frac{2e^{2i\theta}}{3}\int_0^\infty r'\rho_{\text{nuc}}(r'e^{i\theta})dr' \\
&\times \int_1^\infty dt\left(1+\frac{1}{2t^2}\right)\frac{\sqrt{t^2-1}}{t^3} \\
&\times [e^{-2e^{i\theta}|r-r'|t} - e^{-2e^{i\theta}(r+r')t}], \quad (18)
\end{aligned}$$

where $\rho_{\text{nuc}}(r)$ is the nuclear-charge density. This expression contains only analytic functions in the integrand and thus can be calculated without problems.

The Wichmann-Kroll contribution contains the Dirac Green's function and is much more difficult to compute. After the complex rotation, we write the corresponding potential as

$$\begin{aligned}
V_{\text{WK}}(re^{i\theta}) &= \frac{\alpha}{2\pi i} \int d\mathbf{r}' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \int_{C_F} d\omega \\
&\times \text{Tr} \int e^{i\theta} d\mathbf{x} G^{(0)}(\omega, \mathbf{r}'e^{i\theta}, \mathbf{x}e^{i\theta}) V(\mathbf{x}e^{i\theta}) \\
&\times [G(\omega, \mathbf{x}e^{i\theta}, \mathbf{r}'e^{i\theta}) - G^{(0)}(\omega, \mathbf{x}e^{i\theta}, \mathbf{r}'e^{i\theta})]. \quad (19)
\end{aligned}$$

In order to evaluate the Wichmann-Kroll contribution, we apply the Wick rotation of the ω integration contour, $\omega \rightarrow i\omega$ and evaluate separately the pole contributions due to the most deeply bound states.

V. NUMERICAL DETAILS

Calculations of the first-order QED corrections in strong but sub-critical Coulomb field are presently well established, see, e.g., Refs. [50,53]. Therefore, we here concentrate on new features due to the supercritical regime and the use of the CS-transformed Hamiltonian.

To calculate the SE and VP corrections, one needs the complex-scaled wave functions of the electron in the external central potential (4). We obtain them by modifying the FORTRAN code by Salvat and co-authors [54], introducing a prefactor of $e^{i\theta}$ in front of the radial variable r . In this way we solve the CS-transformed Hamiltonian (3) and

get the electron wave functions that are used for the computation of the SE and VP correction.

The calculation of the zero- and one-potential SE terms is performed in the momentum representation. The radial components g_a and f_a are obtained by numerical Fourier transform according to Eqs. (15) and (16). The calculation of the many-potential term is based on the complex-scaled Green's function, whose computation is described in the Appendix. The remaining computational steps follow the previously developed approach, see Refs. [50,53].

The VP contributions are evaluated by computing expectation values of the Uehling and Wichmann-Kroll potentials with the complex-scaled electron wave functions and propagators, according to Eqs. (18) and (19).

The value of the complex rotation angle θ should not affect results for any observables. In practical computations, however, dependence of the results on θ is present and indicates the level of the numerical accuracy. Actual calculations were performed for θ in the range from 4 to 12 degrees. Variation of the results obtained for θ within this range appeared to be at least 2 orders of magnitude smaller than the uncertainty originating from the uncertainty of numerical integrations, and are therefore negligible.

VI. RESULTS

We now present our numerical results for the QED corrections to the energies of the two lowest-lying states in the supercritical regime. We compute both the real part of the energies and their imaginary parts, which gives us access to the decay probability of these states.

The left panel of Fig. 3 presents our results obtained for the real part of the QED correction to the $1s_{1/2}$ and $2p_{1/2}$ energies, given by the sum of the self-energy and vacuum-polarization contributions. We observe that the QED correction is negative and therefore speeds up the diving of $1s_{1/2}$ and $2p_{1/2}$ levels into the negative continuum.

Of particular importance is the imaginary part of the self-energy correction depicted in the right panel of Fig. 3. For the $1s_{1/2}$ state, the imaginary part is zero before diving ($Z < Z_{\text{cr}} \approx 173$). After the diving, the $1s_{1/2}$ -state

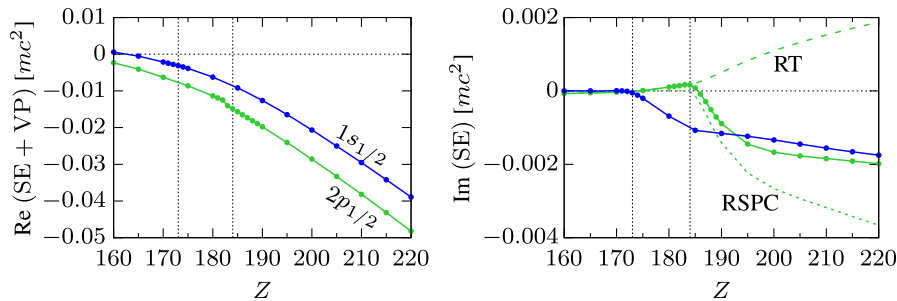


FIG. 3. QED correction to the $1s$ and $2p_{1/2}$ energy levels as a function of the nuclear charge number Z , real part (left), and imaginary part (right).

self-energy acquires the imaginary part, which corresponds to the probability of the spontaneous pair creation accompanied by emission of a photon. We name this process radiative spontaneous pair creation (RSPC). Note that RSPC and SPC channels do not interfere with each other because they produce different final states: the RSPC channel involves an emission of an additional photon compared to SPC. One may view it as the process where an electron from the negative continuum with energy higher than the submerged state decays into it, emitting a photon and creating a positron. The RSPC is a next-order process as compared to the SPC and thus should be parametrically suppressed by an additional factor of α . Our calculations, however, show that it is not always the case.

For the $2p_{1/2}$ state, the situation is getting more complex. There are two diving points, the one of the $1s_{1/2}$ state at $Z \approx 173$ and the one for the $2p_{1/2}$ state at $Z \approx 184$. In the subcritical regime $Z < 173$, the imaginary part is nonzero due to the radiative decay of $2p_{1/2}$ state into the $1s_{1/2}$ state.

In the region between the two diving points $173 < Z < 184$, the imaginary part starts to grow and becomes positive. The reason for this can be explained as follows. After diving of the $1s_{1/2}$ state the electron-positron pair creation can occur, creating the vacancy in the negative-energy continuum. This vacancy, corresponding to the positron emission, can be filled with the electron from $2p_{1/2}$ state. This channel corresponds to the single-quantum electron-positron annihilation process and can be named the radiative transition (RT) decay into the negative continuum.

After diving of the $2p_{1/2}$ state at $Z > 184$, the two decay channels compete with each other. The first is the RSPC process, which is analogous to that for the $1s_{1/2}$ state. Numerical results for this channel are depicted by the dotted green line in the right panel of Fig. 3 and labeled as “RSPC.” The second channel is the radiative transition decay to the negative continuum (with $1s_{1/2}$ state as a part of it), depicted by the dashed green line in the right panel of Fig. 3 and labeled as “RT.” The radiative channel is of course absent in the scenario where the $1s_{1/2}$ state is already filled.

VII. DISCUSSION

Now we turn to testing the Ritus-Narozhny conjecture about the possibility of nonperturbative in α effects in the supercritical regime. To this end, we introduce the enhancement factor $\mathcal{R}$ as the normalized ratio of the probabilities of the radiative and nonradiative SPC processes, which is proportional to the ratio of the imaginary parts of the self-energy correction $\text{Im } E_{\text{SE}}$ and the Dirac energy $\text{Im } E_{\text{D}}$,

$$\mathcal{R} = \frac{1}{\alpha} \left| \frac{\text{Im } E_{\text{SE}}}{\text{Im } E_{\text{D}}} \right|. \quad (20)$$

TABLE I. Imaginary part of the Dirac energy corresponding to SPC (in keV), imaginary part of the SE correction corresponding to RSPC (in keV), and the enhancement factor of the probabilities of the radiative and nonradiative SPC processes $\mathcal{R}$ defined by Eq. (20), for the $1s$ state as a function of the nuclear charge number Z .

Z	$\text{Im } E_{\text{D}}(1s)$	$\text{Im } E_{\text{SE}}(1s)$	$\mathcal{R}_{1s}$
175	0.211	−0.103	66.3
180	1.96	−0.352	24.6
185	5.52	−0.547	13.6
190	10.6	−0.593	7.7
195	16.8	−0.633	5.1
200	24.1	−0.685	3.9

Taking into account that the radiative channel is suppressed by the fine structure constant $\alpha \approx 1/137$ with respect to the nonradiative SPC channel, the enhancement factor $\mathcal{R}$ is expected to be of the order of one in the perturbative regime.

In Table I we present the ratio $\mathcal{R}$ as a function of Z in the supercritical regime for the $1s_{1/2}$ state. We observe that the enhancement factor of the radiative SPC process in the supercritical region is much larger than 1. Moreover, it rapidly increases when Z approaches the diving point in the supercritical regime. This is an evidence that the non-perturbative regime with respect to α could emerge in the vicinity of the diving point.

One may argue that the situation near the diving point is to some extent analogous to the well-known infrared catastrophe [55–57], where the probability of emission of arbitrary numbers of soft photons is comparable and an expansion in the number of emitted photons does not converge. The important difference, however, is that in our case the nonperturbative behavior emerges without any associated infrared divergencies. This is explained by the fact that the diving of the Dirac state is a bound-continuum and not a continuum-continuum transition. The consequence of the nonperturbative regime in our case will be that the total probability of the spontaneous pair creation may differ significantly from the predictions based solely on Dirac theory, especially near the supercritical threshold.

VIII. CONCLUSIONS

We performed calculations of the complete set of QED effects of first order in α in the supercritical Coulomb field, in which the reference-state Dirac energy level is embedded in the negative-energy continuum. Our approach is fully nonperturbative in the parameter $Z\alpha$, which is not small in the supercritical regime, while remaining perturbative in α . We demonstrated that the real part of the QED effects is negative and monotonically increases in magnitude with increase of the nuclear

charge. This confirms the statement expressed in previous studies [58–60] that QED effects do not prevent Dirac levels from diving into the negative continuum.

By calculating the imaginary part of the QED corrections, we obtained the probability of radiative spontaneous pair creation, a process not yet addressed in the literature. Furthermore, an additional channel of the radiative decay of the $2p_{1/2}$ state to the negative continuum was identified. We found that the channel of the one-photon radiative spontaneous pair creation is greatly enhanced in the vicinity of the diving point, as compared to the nonradiative spontaneous pair creation.

Previously, QED effects were studied only in the subcritical regime ($Z < Z_{\text{cr}}$) [40,41,58,61–63]. In particular, the imaginary part of the self-energy and, therefore, the radiative spontaneous pair creation was never considered and generally neglected. Our full-QED calculations demonstrated that the channel of the photon emission with pair creation is important, in contrast to previous assumptions.

For more detailed study of a potentially nonperturbative behavior of QED effects near diving threshold, a calculation of two-loop QED effects in the supercritical regime would be desirable. Such calculations have been accomplished in the subcritical regime [64,65] and their extension to the supercritical fields, although possible, would be extremely difficult.

APPENDIX A: COMPLEX-SCALED FREE-ELECTRON DIRAC GREEN'S FUNCTION

The complex-scaled free-electron Dirac Green's function can be expressed via the solutions of the Dirac equation that are regular at the origin and at the infinity, as follows:

$$G_{\theta}^{(0)}(E, \mathbf{r}_1, \mathbf{r}_2) = -\sum_{\kappa m} [\Phi_{E\kappa m, \theta}^{\infty}(\mathbf{r}_1) \Phi_{E\kappa m, \theta}^{0\dagger}(\mathbf{r}_2) \theta(r_1 - r_2) + \Phi_{E\kappa m, \theta}^0(\mathbf{r}_1) \Phi_{E\kappa m, \theta}^{\infty\dagger}(\mathbf{r}_2) \theta(r_2 - r_1)], \quad (\text{A1})$$

$$G_{\theta}(E, \mathbf{r}_1, \mathbf{r}_2) = -\sum_{\kappa m} [\Phi_{E\kappa m, \theta}^{\infty}(\mathbf{r}_1) \Phi_{E\kappa m, \theta}^{0\dagger}(\mathbf{r}_2) \theta(r_1 - r_2) + \Phi_{E\kappa m, \theta}^0(\mathbf{r}_1) \Phi_{E\kappa m, \theta}^{\infty\dagger}(\mathbf{r}_2) \theta(r_2 - r_1)], \quad (\text{B1})$$

where Φ stands for the solutions of the complex-scaled Dirac equation in the external potential

$$\begin{pmatrix} \frac{d}{dr} + \frac{\kappa}{r} & -e^{i\theta}[1 + E - V(re^{i\theta})] \\ -e^{i\theta}[1 - E + V(re^{i\theta})] & \frac{d}{dr} - \frac{\kappa}{r} \end{pmatrix} \begin{pmatrix} G_{E\kappa, \theta} \\ F_{E\kappa, \theta} \end{pmatrix} = 0. \quad (\text{B2})$$

To solve these equations we use the numerical approach described in Ref. [53]. The solution regular at the origin $\Phi_{E\kappa m, \theta}^0$ is constructed by direct propagation from $r = 0$. To obtain $\Phi_{E\kappa m, \theta}^{\infty}$ we propagate from $r > R$ backwards to $r = 0$. Here R is defined by the requirement that at $r > R$ the potential turns to the pure Coulomb one. For that region, the explicit form of the regular at infinity solution is

where $\Phi_{E\kappa m, \theta}^0$ and $\Phi_{E\kappa m, \theta}^{\infty}$ have the following form:

$$\Phi_{E\kappa m, \theta}(\mathbf{r}) = \frac{e^{-i\theta}}{r} \begin{pmatrix} G_{E\kappa}(re^{i\theta}) \Omega_{\kappa m}(\hat{\mathbf{r}}) \\ i F_{E\kappa}(re^{i\theta}) \Omega_{-\kappa m}(\hat{\mathbf{r}}) \end{pmatrix} \quad (\text{A2})$$

and correspond to the solutions of the complex-scaled Dirac equation

$$\begin{pmatrix} \frac{d}{dr} + \frac{\kappa}{r} & -e^{i\theta}(1 + E) \\ -e^{i\theta}(1 - E) & \frac{d}{dr} - \frac{\kappa}{r} \end{pmatrix} \begin{pmatrix} G_{E\kappa, \theta} \\ F_{E\kappa, \theta} \end{pmatrix} = 0 \quad (\text{A3})$$

that are regular at the origin and the infinity, respectively. The explicit form of these solutions is

$$\begin{pmatrix} G_{E\kappa, \theta}^0 \\ F_{E\kappa, \theta}^0 \end{pmatrix} = re^{i\theta} \begin{pmatrix} j_{|\kappa+1/2|-1/2}(ipr) \\ i \frac{pe^{-i\theta}}{M+E} \frac{\kappa}{|\kappa|} j_{|\kappa-1/2|-1/2}(ipr) \end{pmatrix}, \quad (\text{A4})$$

and

$$\begin{pmatrix} G_{E\kappa, \theta}^{\infty} \\ F_{E\kappa, \theta}^{\infty} \end{pmatrix} = re^{i\theta} \begin{pmatrix} h_{|\kappa+1/2|-1/2}^{(1)}(ipr) \\ i \frac{pe^{-i\theta}}{1+E} \frac{\kappa}{|\kappa|} h_{|\kappa-1/2|-1/2}^{(1)}(ipr) \end{pmatrix} \quad (\text{A5})$$

with $p = \sqrt{e^{2i\theta}(1 - E^2)}$ and $\text{Re}(p) > 0$. The Wronskian is given by

$$W = F_{E\kappa, \theta}^0 G_{E\kappa, \theta}^{\infty} - G_{E\kappa, \theta}^0 F_{E\kappa, \theta}^{\infty} = -[pe^{-i\theta}(1 + E)]^{-1}. \quad (\text{A6})$$

APPENDIX B: COMPLEX-SCALED EXTERNAL-POTENTIAL DIRAC GREEN'S FUNCTION

As in the case of the free Dirac Green's function, the external-potential Dirac Green's function can be expressed as follows:

$$\begin{pmatrix} G_{E\kappa,\theta}^\infty \\ F_{E\kappa,\theta}^\infty \end{pmatrix} = \frac{1}{\sqrt{2pr}} \begin{pmatrix} \left(\kappa + \nu \frac{1}{E}\right) W_{\nu-1/2,\gamma}(2pr) + W_{\nu+1/2,\gamma}(2pr) \\ \frac{pe^{-i\theta}}{1+E} \left[\left(\kappa + \nu \frac{1}{E}\right) W_{\nu-1/2,\gamma}(2pr) - W_{\nu+1/2,\gamma}(2pr) \right] \end{pmatrix}, \quad (\text{B3})$$

where $p = \sqrt{e^{2i\theta}(1-E^2)}$ with $\text{Re}(p) > 0$ and

$$\nu = \frac{\alpha Z E e^{i\theta}}{p} \quad (\text{B4})$$

is the Sommerfeldt parameter. By normalizing the obtained solutions to the unit Wronskian, one obtains the complex-scaled external-potential Green's function.

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