

Shear viscosity of a massless quark-gluon gas in chemical equilibrium in terms of all $2 \leftrightarrow 2$ cross sections

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The analytical expressions of the shear viscosity of both one and two particle species with Boltzmann statistics and $2 \rightarrow 2$ elastic scatterings are known from the Chapman-Enskog method and have been shown to be quite accurate. The expression for a multispecies hadronic gas under $2 \rightarrow 2$ elastic scatterings is also known. Here we use the Chapman-Enskog method to derive the explicit expression of shear viscosity of a massless quark-gluon gas of N_f quark flavors in chemical equilibrium subjected to all $2 \rightarrow 2$ parton scatterings including for the first time inelastic scatterings. We then verify the expression in a general single-species limit, where the shear viscosity of the quark-gluon gas should reduce to the result for a single particle species. In addition, we show the explicit analytical result in terms of the seven independent cross sections for the special case of isotropic and energy-independent cross sections. The analytical expressions derived here can be useful for determining the shear viscosity of parton transport models with any $2 \rightarrow 2$ scattering cross sections. They can also be coupled with finite temperature QCD cross sections to help study the shear viscosity of the quark gluon plasma.

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I. INTRODUCTION

The quark-gluon plasma (QGP) is a state of matter achieved when conditions are sufficient for quarks and gluons to become deconfined from hadrons. The QGP is believed to have been created in heavy ion collisions at the Large Hadron Collider and the Relativistic Heavy Ion Collider [1]. The QGP is also theorized to be the state of matter of the Universe less than 10^{-5} sec after the big bang [2], so understanding its properties will enable a better understanding of the Universe's evolution. Observables from heavy ion collisions including the anisotropic flows indicate that the QGP is a near perfect fluid with a very small shear viscosity to entropy density ratio η/s . This ratio has been conjectured to have a lower bound of $1/(4\pi)$ [3]. Extractions of the shear viscosity from comparisons between theoretical models and the experimental data [4–7] have shown that the QGP η/s within a few times of the QCD phase transition temperature ($T_c \simeq 156$ MeV) is not too far from the conjectured lower bound, although the uncertainty is large and its temperature dependence cannot be yet determined well.

While the QGP shear viscosity including its temperature dependence is an input function in hydrodynamical models of heavy ion collisions, it is not an input in transport models [8–12]. Instead, it is implicitly determined by the interactions among quarks and gluons. Therefore, the ability to compute the shear viscosity within the quasiparticle picture is important for the development and application of the transport model approach [8–13] and the QCD effective kinetic theory approach [14]. These two approaches are both well suited to the description of nonequilibrium dynamics, which is expected to be important for small collision systems as well as early times or heavy flavor observables in large collision systems. An example is the parton escape mechanism for anisotropic flows [15,16], which can efficiently convert spatial anisotropies into momentum anisotropies via a small number of scatterings far from equilibrium. An extensive computation of the shear viscosity of the QGP at high temperatures has been performed in the AMY framework developed by Arnold, Moore and Yafee [17,18], where both $2 \leftrightarrow 2$ and $1 \leftrightarrow 2$ interactions have been taken into account and the $2 \leftrightarrow 2$ perturbative QCD (pQCD) cross sections are screened with thermal self energies. When one extrapolates the AMY results to finite temperatures below several T_c , the η/s values are much higher than $1/(4\pi)$ or the current η/s range extracted from the experimental data. On the other hand, next-to-leading order corrections have been found to be large and significantly decrease the η/s value at physically relevant temperatures [19].

Regarding analytical relations between the shear viscosity and arbitrary scattering cross sections, there are

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several methods to calculate the shear viscosity for a single particle species with Boltzmann statistics and $2 \rightarrow 2$ elastic scatterings [20,21], including the relaxation time approximation and the Chapman-Enskog method. For a massless parton gas at temperature T that scatters with an isotropic and energy-independent (elastic) cross section σ , different methods give essentially the same result $\eta_1^{\text{iso}} = 6T/(5\sigma)$. For anisotropic scatterings, however, only the Chapman-Enskog method is accurate enough based on the comparisons with numerical results obtained using the Green-Kubo relation [13,21]. The calculation of shear viscosity for a hadronic resonance gas with elastic $2 \rightarrow 2$ scatterings [22] provides the analytical result for a multispecies system, where the explicit η expression for a binary mixture was given [22] and has been recently shown to agree well with the Green-Kubo numerical results [12]. Note that another study [23] addressed the special case of isotropic and energy-independent elastic $2 \rightarrow 2$ scatterings for a multispecies system.

The goal of this work is to derive an explicit analytical relationship between the shear viscosity of a massless quark-gluon gas in chemical equilibrium and all the $2 \leftrightarrow 2$ differential cross sections including the inelastic channels. We will use the Boltzmann statistics for the computation. The paper is organized as follows. After the introduction, we will review in Sec. II the method used for this work. Then we will consider all parton $2 \leftrightarrow 2$ scatterings and derive the shear viscosity expression in Sec. III. In Sec. IV we verify the general η expression in the single-species limit, where the shear viscosity should reduce to the single-species result, and we also give a more explicit η expression in the special case of isotropic and energy-independent cross sections. After some discussions in Sec. V, we summarize in Sec. VI.

II. THE CHAPMAN-ENSKOG METHOD FOR SHEAR VISCOSITY

In this section, we closely follow the book by de Groot *et al.* [24] while showing the key steps in the derivations. We begin by defining the following variables:

$$n \equiv \sum_i n_i, \quad x_i \equiv \frac{n_i}{n}, \quad z_i \equiv \frac{m_i}{T}, \quad \pi_i^\mu \equiv \frac{p_i^\mu}{T}, \quad \tau_i \equiv \frac{p_i^\mu U_\mu}{T}. \quad (1)$$

In the above, n_i is the number density for particle species i (from 1 to N), n is the total number density of all N species, m_i is the particle mass that we shall take to zero for this study, p_i^μ is the particle 4-momentum, and U^μ is the 4-velocity of the gas. We also define the projection tensor

$$\Delta^{\mu\nu} \equiv g^{\mu\nu} - U^\mu U^\nu, \quad (2)$$

which generates from any tensor $a^{\mu\nu}$ the following symmetric and traceless tensor:

$$\overset{\circ}{a}^{\mu\nu} \equiv \frac{1}{2}(\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\beta^\mu \Delta_\alpha^\nu) a^{\alpha\beta} - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} a^{\alpha\beta}. \quad (3)$$

Similarly, one can generate from a vector product $b^\mu b^\nu$ a symmetric and traceless $\overset{\circ}{b}^\mu \overset{\circ}{b}^\nu$. We will also make use of the following inner product and collision bracket [24]:

$$\begin{aligned} \langle F_i, G_i \rangle_i &\equiv \frac{T}{n_i} \int \frac{d^3 \vec{p}_i}{p_i^0} F_i(p_i) G_i(p_i) f_i(p_i) \\ &= \frac{1}{8\pi T^2} \int \frac{d^3 \vec{p}_i}{p_i^0} F_i(p_i) G_i(p_i) e^{-\tau_i}, \end{aligned} \quad (4)$$

$$\begin{aligned} [F, G]_{ij \rightarrow kl} &= \frac{1}{128\pi^2 T^6} \int \frac{d^3 \vec{p}_i}{p_i^0} \frac{d^3 \vec{p}_j}{p_j^0} \frac{d^3 \vec{p}_k}{p_k^0} \frac{d^3 \vec{p}_l}{p_l^0} \\ &\quad \times e^{-(\tau_i + \tau_j)} F G W_{ij \rightarrow kl}. \end{aligned} \quad (5)$$

In the above, $n_i = d_i z_i^2 K_2(z_i) T^3 / (2\pi^3)$ is the number density of parton species i with mass $z_i T$, and we have used the relation $z_i^2 K_2(z_i) = 2$ as $z_i \rightarrow 0$ to reduce it for the massless case, where $K_n(x)$ represents the modified Bessel of the second kind. The term $f_i(p_i) \equiv dN_i / (d^3 x d^3 p) = d_i e^{-\tau_i} / (8\pi^3)$ is the equilibrium distribution for Boltzmann statistics with d_i being the degeneracy factor, and $W_{ij \rightarrow kl}$ is the collision kernel defined as

$$W_{ij \rightarrow kl} \equiv s \delta^{(4)}(p_i + p_j - p_k - p_l) \frac{d\sigma^{ij \rightarrow kl}}{d\Omega}, \quad (6)$$

where s is the Mandelstam variable (from now on) and $d\sigma/d\Omega$ is the differential cross section for the scattering $i + j \rightarrow k + l$.

Referring to previous works [24–26], the first Chapman-Enskog approximation allows us to write the shear viscosity as

$$\eta = \frac{T}{10} \sum_i x_i \langle \pi_i^\mu \overset{\circ}{\pi}_i^\nu, C_i(\tau_i) \pi_{i\mu} \overset{\circ}{\pi}_{i\nu} \rangle_i. \quad (7)$$

The $C_i(\tau_i)$ above is the collision term specified by [24]

$$\sum_{j=1}^N x_j R_{ij} [C_i(\tau_i) \pi_i^\mu \overset{\circ}{\pi}_i^\nu] = \pi_i^\mu \overset{\circ}{\pi}_i^\nu, \quad (8)$$

where

$$\begin{aligned} R_{ij}[\phi_i] &= \frac{1}{16\pi T^4} \sum_{k,l} \int \frac{d^3 \vec{p}_j}{p_j^0} \frac{d^3 \vec{p}_k}{p_k^0} \frac{d^3 \vec{p}_l}{p_l^0} \\ &\quad \times e^{-\tau_j} (\phi_i + \phi_j - \phi_k - \phi_l) W_{ij \rightarrow kl}. \end{aligned} \quad (9)$$

Applying the above to Eq. (7), we get

$$\begin{aligned} \eta &= \frac{T}{10} \sum_{i,j} x_i x_j \left\langle R_{ij} \left[C_i(\tau_i) \pi_i^{\mu} \pi_i^{\nu}, C_i(\tau_i) \pi_{i\mu} \pi_{i\nu} \right]_i \right. \\ &= \sum_{i,j,k,l} \frac{x_i x_j}{1280 \pi^2 T^5} \int \frac{d^3 \vec{p}_i}{p_i^0} \frac{d^3 \vec{p}_j}{p_j^0} \frac{d^3 \vec{p}_k}{p_k^0} \frac{d^3 \vec{p}_l}{p_l^0} e^{-(\tau_i + \tau_j)} W_{ij \rightarrow kl} \\ &\quad \times \left[C_i(\tau_i) \pi_i^{\mu} \pi_i^{\nu} + C_j(\tau_j) \pi_j^{\mu} \pi_j^{\nu} - C_k(\tau_k) \pi_k^{\mu} \pi_k^{\nu} \right. \\ &\quad \left. \left. - C_l(\tau_l) \pi_l^{\mu} \pi_l^{\nu} \right] C_i(\tau_i) \pi_{i\mu} \pi_{i\nu}. \end{aligned} \quad (10)$$

Approximating $C_i(\tau_i)$ as a trial function of a finite power series up to power P [24]

$$C_i(\tau_i) \approx C_i^{(P)}(\tau_i) = \sum_{r=0}^P C_{i,r}^{(P)} \tau_i^r, \quad (11)$$

and rearranging the indices to extract the constant term $C_{i,r}^{(P)} C_{j,s}^{(P)}$ [for example, by replacing $C_{i,s}^{(P)}$ with $\sum_m \delta_{im} C_{m,s}^{(P)}$ for the term $C_{i,r}^{(P)} C_{i,s}^{(P)}$], we can write the shear viscosity as

$$\eta = \frac{T}{10} \sum_{r,s} \sum_{i,j} C_{ij}^{rs} C_{i,r}^{(P)} C_{j,s}^{(P)}, \quad (12)$$

where the matrix elements are defined in terms of the collision brackets as

$$\begin{aligned} C_{ij}^{rs} &\equiv \delta_{ij} x_i \sum_{k,l,m} x_m \left[\tau_i^r \pi_i^{\mu} \pi_i^{\nu}, \tau_i^s \pi_{i\mu} \pi_{i\nu} \right]_{im \rightarrow kl} \\ &\quad + x_i x_j \sum_{k,l} \left[\tau_i^r \pi_i^{\mu} \pi_i^{\nu}, \tau_j^s \pi_{j\mu} \pi_{j\nu} \right]_{ij \rightarrow kl} \\ &\quad - x_j \sum_{k,m} x_m \left[\tau_i^r \pi_i^{\mu} \pi_i^{\nu}, \tau_j^s \pi_{j\mu} \pi_{j\nu} \right]_{jm \rightarrow ik} \\ &\quad - x_j \sum_{k,m} x_m \left[\tau_i^r \pi_i^{\mu} \pi_i^{\nu}, \tau_j^s \pi_{j\mu} \pi_{j\nu} \right]_{jm \rightarrow ki}. \end{aligned} \quad (13)$$

Note that the above is somewhat different [27] from Eq. (VI.3.100) of the book [24] in that the last two terms show up in the book as $-2x_j \sum_{k,m} x_m \left[\tau_i^r \pi_i^{\mu} \pi_i^{\nu}, \tau_j^s \pi_{j\mu} \pi_{j\nu} \right]_{jm \rightarrow ik}$; they would be the same if $W_{ij \rightarrow kl} = W_{ij \rightarrow lk}$, but this is not the case for certain pQCD $2 \rightarrow 2$ scatterings including $gq \rightarrow gq$ and $q\bar{q} \rightarrow q\bar{q}$.

Equations (7) and (11) also allow us to write the shear viscosity as

$$\eta = \frac{T}{10} \sum_{i=1}^N \sum_{r=0}^P x_i \gamma_{i,r} C_{i,r}^{(P)}, \quad (14)$$

where $\gamma_{i,r} \equiv \langle \tau_i^r \pi_i^{\mu} \pi_i^{\nu}, \pi_{i\mu} \pi_{i\nu} \rangle_i$. For massless partons, $\gamma_{i,r} = (r+5)!/3$ is independent of the particle species i , e.g., $\gamma_{i,0} \equiv \gamma_0 = 40$ for $r=0$. Next we apply the inner product $\langle F, x_i \tau_i^r \pi_{i\mu} \pi_{i\nu} \rangle_i$ to Eq. (8), where F represents either side of the equation, and we obtain [24] (see Appendix A for details)

$$\sum_{j=1}^N \sum_{s=0}^P C_{ij}^{rs} C_{j,s}^{(P)} = x_i \gamma_{i,r}. \quad (15)$$

In this study, we only derive the first-order ($P=0$) Chapman-Enskog expression, i.e., we assume that the collision coefficient $C_i(\tau_i)$ is constant with respect to energy. Note that it has been shown for a single particle species that adding higher-order terms only changes the result by several percent [20]. For $P=0$, Eq. (15) gives

$$\vec{c} = \mathbf{C}^{-1} \vec{\gamma}, \quad (16)$$

where, for a system of massless gluons and N_f flavors of massless quarks, we write

$$\vec{c} \equiv \begin{pmatrix} C_{g,0}^{(0)} \\ C_{q_1,0}^{(0)} \\ C_{\bar{q}_1,0}^{(0)} \\ \vdots \\ C_{q_{N_f},0}^{(0)} \\ C_{\bar{q}_{N_f},0}^{(0)} \end{pmatrix}, \quad \vec{\gamma} \equiv \gamma_0 \begin{pmatrix} x_g \\ x_{q_1} \\ x_{\bar{q}_1} \\ \vdots \\ x_{q_{N_f}} \\ x_{\bar{q}_{N_f}} \end{pmatrix}, \quad \mathbf{C} \equiv \begin{pmatrix} C_{gg}^{00} & C_{gq_1}^{00} & C_{g\bar{q}_1}^{00} & \cdots & C_{gq_{N_f}}^{00} & C_{g\bar{q}_{N_f}}^{00} \\ C_{q_1g}^{00} & C_{q_1q_1}^{00} & C_{q_1\bar{q}_1}^{00} & \cdots & C_{q_1q_{N_f}}^{00} & C_{q_1\bar{q}_{N_f}}^{00} \\ C_{\bar{q}_1g}^{00} & C_{\bar{q}_1q_1}^{00} & C_{\bar{q}_1\bar{q}_1}^{00} & \cdots & C_{\bar{q}_1q_{N_f}}^{00} & C_{\bar{q}_1\bar{q}_{N_f}}^{00} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{q_{N_f}g}^{00} & C_{q_{N_f}q_1}^{00} & C_{q_{N_f}\bar{q}_1}^{00} & \cdots & C_{q_{N_f}q_{N_f}}^{00} & C_{q_{N_f}\bar{q}_{N_f}}^{00} \\ C_{\bar{q}_{N_f}g}^{00} & C_{\bar{q}_{N_f}q_1}^{00} & C_{\bar{q}_{N_f}\bar{q}_1}^{00} & \cdots & C_{\bar{q}_{N_f}q_{N_f}}^{00} & C_{\bar{q}_{N_f}\bar{q}_{N_f}}^{00} \end{pmatrix}. \quad (17)$$

Equation (14) then gives the shear viscosity for $P=0$ as

$$\eta = \frac{T}{10} \vec{\gamma}^T \cdot \vec{c} = \frac{T}{10} \vec{\gamma}^T \mathbf{C}^{-1} \vec{\gamma}. \quad (18)$$

III. EXPLICIT RESULTS FOR A MASSLESS QUARK-GLUON GAS

We now consider a system of massless gluons and N_f flavors of quarks in chemical equilibrium, where we write $x_{q_i} \equiv x_q$, $x_{\bar{q}_i} \equiv x_{\bar{q}}$ for $i \in [1, N_f]$ with $x_q = x_{\bar{q}}$. For convenience, we write the matrix elements of $\mathbf{C}$ in the following forms [22]:

$$C_{ii}^{00} = x_i^2 \tilde{c}_0[i] + x_i \sum_{j \neq i} x_j \tilde{c}_1[ij] + \tilde{\omega}[ii], \quad C_{ij}^{00} = x_i x_j \tilde{c}_2[ij] + \tilde{\omega}[ij] \quad (\text{for } i \neq j). \quad (19)$$

Following the steps shown in Appendix B, for massless partons we obtain the elastic terms as follows (with $v = \sqrt{s}/T$):

$$\tilde{c}_0[i] = \frac{1}{384} \int_0^\infty dv v^6 [(3v^2 + 4)K_3(v) - 6vK_2(v)] \sigma_{\text{tr}}^{ii \rightarrow ii}(v), \quad (20)$$

$$\tilde{c}_1[ij] = -\frac{1}{192} \int_0^\infty dv v^2 \int_{-v^2 T^2}^0 dt \frac{t}{T^2} \left\{ \left[3v^2 \left(\frac{t}{T^2} + v^2 \right) + 4 \left(\frac{t}{T^2} + 6v^2 \right) \right] K_3(v) - 2v \left(\frac{3t}{T^2} - 2v^2 \right) K_2(v) \right\} \frac{d\sigma^{ij \rightarrow ij}}{dt}, \quad (21)$$

$$\tilde{c}_2[ij] = -\frac{1}{192} \int_0^\infty dv v^2 \int_{-v^2 T^2}^0 dt \frac{t}{T^2} \left\{ \left[3v^2 \left(\frac{t}{T^2} + v^2 \right) + 4 \left(\frac{t}{T^2} - 4v^2 \right) \right] K_3(v) - 2v \left(\frac{3t}{T^2} + 8v^2 \right) K_2(v) \right\} \frac{d\sigma^{ij \rightarrow ij}}{dt}. \quad (22)$$

We also obtain the inelastic terms as

$$\tilde{\omega}[gg] = \frac{x_g^2 N_f}{192} \int_0^\infty dv v^6 [(v^2 + 28)K_3(v) - 2vK_2(v)] \sigma_{\text{tr}}^{gg \rightarrow q\bar{q}}(v), \quad (23)$$

$$\begin{aligned} \tilde{\omega}[gq] &= \frac{x_g^2}{768} \int_0^\infty dv v^6 [(3v^2 + 4)K_3(v) - 6vK_2(v)] \sigma_{\text{tr}}^{gg \rightarrow q\bar{q}}(v) \\ &\quad - \frac{x_g^2}{384} \int_0^\infty dv v^6 [(v^2 + 28)K_3(v) - 2vK_2(v)] \sigma_{\text{tr}}^{gg \rightarrow q\bar{q}}(v), \end{aligned} \quad (24)$$

$$\tilde{\omega}[qq] = \frac{x_q x_{\bar{q}}}{384} \int_0^\infty dv v^6 [(v^2 + 48)K_3(v) + 8vK_2(v)] [\sigma^{q\bar{q} \rightarrow gg}(v) + (N_f - 1) \sigma^{q\bar{q} \rightarrow q'\bar{q}'}(v)], \quad (25)$$

$$\tilde{\omega}[q\bar{q}] = \frac{x_q x_{\bar{q}}}{384} \int_0^\infty dv v^6 [(v^2 + 8)K_3(v) - 12vK_2(v)] [\sigma^{q\bar{q} \rightarrow gg}(v) + (N_f - 1) \sigma^{q\bar{q} \rightarrow q'\bar{q}'}(v)], \quad (26)$$

$$\begin{aligned} \tilde{\omega}[qq'] &= \frac{x_q x_{\bar{q}}}{768} \int_0^\infty dv v^6 [(3v^2 + 4)K_3(v) - 6vK_2(v)] \sigma_{\text{tr}}^{q\bar{q} \rightarrow q'\bar{q}'}(v) \\ &\quad - \frac{x_q x_{\bar{q}}}{384} \int_0^\infty dv v^6 [(v^2 + 28)K_3(v) - 2vK_2(v)] \sigma^{q\bar{q} \rightarrow q'\bar{q}'}(v). \end{aligned} \quad (27)$$

Note that we have made use of the standard symmetries in pQCD, e.g., the facts that $d\sigma^{q\bar{q}_i \rightarrow q\bar{q}_i}/dt = d\sigma^{q\bar{q}_j \rightarrow q\bar{q}_j}/dt \equiv d\sigma^{q\bar{q} \rightarrow q\bar{q}}/dt$ and $d\sigma^{gg \rightarrow q_i \bar{q}_i}/dt \equiv d\sigma^{gg \rightarrow q\bar{q}}/dt$ are independent of quark flavor for massless partons. Also note that the total cross section and the total transport cross section in the above relations are given by

$$\sigma^{ij \rightarrow kl}(v) = \frac{1}{1 + \delta_{kl}} \int_{-s}^0 \frac{d\sigma^{ij \rightarrow kl}}{dt} dt, \quad \sigma_{\text{tr}}^{ij \rightarrow kl}(v) = \frac{1}{1 + \delta_{kl}} \int_{-s}^0 \frac{d\sigma^{ij \rightarrow kl}}{dt} \sin^2 \theta_{\text{cm}} dt, \quad (28)$$

respectively, where θ_{cm} is the scattering angle in the two-parton center of mass frame. In chemical equilibrium, we find $\tilde{\omega}[gg] = \tilde{\omega}[q\bar{q}] = \tilde{\omega}[q\bar{q}] = \tilde{\omega}[\bar{q}q]$, $\tilde{\omega}[qq] = \tilde{\omega}[\bar{q}\bar{q}]$, $\tilde{\omega}[q\bar{q}] = \tilde{\omega}[\bar{q}q]$, and $C_{ij}^{00} = C_{ji}^{00}$.

After inverting the matrix $\mathbf{C}$, we use Eq. (18) to obtain the general expression of the shear viscosity of a multispecies massless quark-gluon gas in chemical equilibrium as

$$\eta = 160 T \frac{x_g^2 [C_{qq}^{00} + C_{q\bar{q}}^{00} + 2(N_f - 1)C_{q\bar{q}'}^{00}] - 4N_f x_g x_q C_{gq}^{00} + 2N_f x_q^2 C_{gg}^{00}}{C_{gg}^{00} [C_{qq}^{00} + C_{q\bar{q}}^{00} + 2(N_f - 1)C_{q\bar{q}'}^{00}] - 2N_f (C_{gq}^{00})^2}. \quad (29)$$

Note that in chemical equilibrium $x_g = 8/(8 + 6N_f)$ and $x_q = x_{\bar{q}} = 3(1 - \delta_{0N_f})/(8 + 6N_f)$. Also note that for the $N_f = 0$ case, Eq. (18) as well as Eq. (29) lead to $\eta = 160T/C_{gg}^{00} = 160T/\tilde{c}_0[g]$, which as expected is the Chapman-Enskog result for a single species of massless parton [13,21]. Interestingly, the explicit expression (i.e., after the inversion of matrix $\mathbf{C}$) of η in Eq. (29) involves the matrix elements C_{ij}^{00} at most to second order, not to the order of the dimension of the matrix ($1 + 2N_f$); this is a result of the symmetry properties of matrix $\mathbf{C}$ and vector $\vec{\gamma}$.

IV. THE SINGLE-SPECIES LIMITS

In certain limits, which we call the single-species limits, the quark-gluon gas should behave in the same way as a single species in terms of momentum transfer and thus

$$\begin{aligned} C_{gg}^{00} &= x_g^2 \tilde{c}_0[g] + x_g(1 - x_g) \tilde{c}_1[gq], & C_{qq}^{00} &= x_q^2 \tilde{c}_0[q] + x_q(1 - x_q) \tilde{c}_1[gq], \\ C_{gq}^{00} &= x_g x_q \tilde{c}_2[gq], & C_{q\bar{q}}^{00} &= x_q^2 \tilde{c}_2[gq], & C_{q\bar{q}'}^{00} &= x_q^2 \tilde{c}_2[gq]. \end{aligned} \quad (31)$$

In the above, we have taken advantage of the facts that all the inelastic $\tilde{\omega}[ij]$ terms are zero and that $\tilde{c}_0[i]$, $\tilde{c}_1[ij]$ and $\tilde{c}_2[ij]$ are all independent of parton species. We have also used the relations $x_q = x_{\bar{q}}$ and $\sum x_i = 1$ that were imposed in deriving Eq. (29). Noting $\tilde{c}_0[i] = \tilde{c}_1[ij] + \tilde{c}_2[ij]$, Eq. (29) then gives

$$\eta = \frac{61440 T}{\int_0^\infty dv v^6 [(3v^2 + 4)K_3(v) - 6vK_2(v)] \sigma_{\text{tr}}(v)}, \quad (32)$$

shear viscosity. So we can use the previously known Chapman-Enskog result for a single particle species to check the shear viscosity result for multiple parton species such as Eq. (29). One such limit occurs when all the parton species scatter elastically (but not inelastically) with the same cross section and angular distribution, i.e., when $d\sigma^{ij \rightarrow ij}/dt = (1 + \delta_{ij})d\sigma/dt$, where $d\sigma/dt$ is a given arbitrary angular distribution. Note that in this case

$$\sigma_{\text{tr}}^{ii \rightarrow ii}(v) = \sigma_{\text{tr}}^{ij \rightarrow ij}(v) = \int_{-s}^0 \sin^2 \theta_{\text{cm}} \frac{d\sigma}{dt} dt \equiv \sigma_{\text{tr}}(v) \quad (30)$$

are independent of parton species (i and j), so we call this the equal-cross-section case of the single-species limit. We now check Eq. (29) in this limit, where we find the following matrix elements:

which is exactly the single-species result [13,21].

For η expressions that are more explicit than Eq. (29) and written in terms of cross sections, let us consider the special case of isotropic and energy(s)-independent (or constant) scatterings, where $d\sigma^{ij \rightarrow kl}/dt$ is a constant that can depend on i, j, k, l but not the Mandelstam variables. For this case, we get the following shear viscosity expression for a massless quark-gluon gas of N_f quark flavors in chemical equilibrium:

$$\begin{aligned} \eta^{\text{iso}} &= \frac{12TA}{5B}, & A &= 240(2N_f \sigma_{gg} + \sigma_4) + 8(112 + 48N_f + 63N_f^2) \sigma_{gq} + 9(48 + 32N_f + 27N_f^2) \sigma_{q\bar{q}}^{gg}, \\ B &= 8[8\sigma_{gq}(28\sigma_{gg} + 27N_f \sigma_{gq}) + 3\sigma_4(20\sigma_{gg} + 21N_f \sigma_{gq})] + 9(96\sigma_{gg} + 176N_f \sigma_{gq} + 27N_f \sigma_4) \sigma_{q\bar{q}}^{gg} + 351N_f (\sigma_{q\bar{q}}^{gg})^2. \end{aligned} \quad (33)$$

For brevity, we have written the elastic cross section $\sigma^{ij \rightarrow ij}$ as σ_{ij} , the two independent inelastic cross sections as $\sigma_{q\bar{q}}^{gg} \equiv \sigma(q\bar{q} \rightarrow gg)$ and $\sigma_{q\bar{q}'}^{q\bar{q}'} \equiv \sigma(q\bar{q} \rightarrow q'\bar{q}')$, respectively, and

$$\sigma_4 \equiv \sigma_{qq} + \sigma_{q\bar{q}} + (N_f - 1)(2\sigma_{q\bar{q}'} + \sigma_{q\bar{q}}^{q\bar{q}'}). \quad (34)$$

Note that $\sigma_{gg}^{q\bar{q}} \equiv \sigma(gg \rightarrow q\bar{q}) = 9\sigma(q\bar{q} \rightarrow gg)/32$; thus it is not independent.

For this case of isotropic and constant cross sections, in the limit that the collision rate of every parton species per parton is the same, the shear viscosity should be given by the single-species result $\eta_1^{\text{iso}} = 6T/(5\sigma)$. For the collision rate r_i of parton species i per parton, we have

$$r_g/n = x_g \sigma_{gg} + 2N_f x_q \sigma_{gq} + N_f x_g \sigma_{gg}^{q\bar{q}}, \quad r_q/n = r_{\bar{q}}/n = x_g \sigma_{gq} + x_q \sigma_4 + x_q \sigma_{q\bar{q}}^{gg}. \quad (35)$$

Thus the equal rate condition $r_g = r_q$ requires the following relation among the seven independent cross sections:

$$\sigma_{gg} = \frac{3}{8} \sigma_4 + \frac{(4 - 3N_f)}{32} (8\sigma_{gq} + 3\sigma_{q\bar{q}}^{gg}), \quad (36)$$

then Eq. (33) reduces to

$$\eta = \frac{6(4 + 3N_f)T}{5(4\sigma_{gg} + 3N_f\sigma_{gq} + 4N_f\sigma_{g\bar{q}})}. \quad (37)$$

On the other hand, the quark-gluon gas in this equal-rate limit behaves, in terms of momentum transfer, as a single parton species with an effective isotropic cross section σ_{eff} that gives the same collision rate per parton. From $n\sigma_{\text{eff}} = r_g$, we get the equivalent effective cross section as

$$\sigma_{\text{eff}} = \frac{4\sigma_{gg} + 3N_f\sigma_{gq} + 4N_f\sigma_{g\bar{q}}}{4 + 3N_f}. \quad (38)$$

The single-species result $\eta_1^{\text{iso}} = 6T/(5\sigma)$ for $\sigma = \sigma_{\text{eff}}$ thus agrees with Eq. (37), i.e., the shear viscosity for isotropic and constant cross sections in Eq. (33) satisfies this single-species limit.

We can extend the above cases to a general single-species limit, where $d\sigma^{ij \rightarrow kl}/dt = f_{ij}^{kl}(1 + \delta_{kl})d\sigma_0/dt$ with $d\sigma_0/dt$ being a given arbitrary angular distribution and f_{ij}^{kl} being constants that depend on i, j, k, l but not on the Mandelstam variables. When the seven independent $\sigma^{ij \rightarrow kl}(v)$ values satisfy Eq. (36), then each parton species has the same collision rate per parton and the general Eq. (29) indeed reduces to the single species result of Eq. (32). We note that the previous two single-species limits in this section are simply special cases of this general single-species limit. The proof of this general equal-rate case of the single-species limit is shown in Appendix C, which serves as a strong check of the general expression of shear viscosity in Eq. (29).

V. DISCUSSIONS

In a follow-up study [28], we applied the general shear viscosity expression of Eq. (29) to given parton cross sections at finite temperature based on perturbative QCD. We find that the Chapman-Enskog results of shear viscosity are qualitatively similar to the corresponding leading-order results from the AMY framework [17,18,29]; quantitatively they are higher partly due to the usage of parton thermal masses (instead of self-energies) to screen the cross sections. It will also be very useful to compare the Chapman-Enskog results for given parton cross sections with numerical results of shear viscosity obtained from transport models using the Green-Kubo relation. In addition, it will be interesting to consider the case of finite parton mass including different masses of different particle species; we plan to carry out this separate study in the near future.

This study includes inelastic two-body cross sections in the Chapman-Enskog calculation of shear viscosity for the first time. The effects of these inelastic processes have been shown in follow-up studies [28,30] for several choices of

the renormalization scale Q used in the strong coupling constant and the scaling factor ξ used for screening the scattering matrix elements. For $N_f = 3$, inelastic two-body scatterings reduce the shear viscosity by $\sim 5\%$ – 12% depending on the temperature as well as the values of Q and ξ . We also note that this study only considers Boltzmann statistics but not quantum statistics. We have compared the η/s values [28] between Boltzmann statistics and quantum statistics from the leading-order AMY framework [29], where the shear viscosity at $N_f = 3$ for two-body scatterings from Boltzmann statistics is higher than that from quantum statistics by $\sim 13\%$ – 15% . Therefore, the effect from quantum statistics needs to be considered in the future for quantitative comparisons between parton transport results and the experimental data.

VI. CONCLUSION

We have calculated the shear viscosity of a massless quark-gluon gas in chemical equilibrium subjected to any $2 \leftrightarrow 2$ interactions, including all elastic and inelastic collisions with arbitrary angular distributions. The general result is an explicit first-order Chapman-Enskog expression for the shear viscosity for a system of gluons and N_f flavors of quarks with the Boltzmann statistics. As expected, the general result for a quark-gluon gas reduces to the previously known result for a massless single particle species in the equal-cross section case, where all the parton species scatter elastically with the same cross section and angular distribution without inelastic scatterings. We then design a general single-species limit, where the shear viscosity of the quark-gluon system is expected to reduce to the single species result. In this general limit, all the parton species scatter with the same angular distribution but can have different cross section magnitudes and scatter both elastically and inelastically, and the seven independent cross sections satisfy one relation so that the collision rate of each parton species is the same. In this limit, which can be called the equal-rate case of the single-species limit, we show that the general expression of the shear viscosity indeed reduces to the single-species result. This provides a strong check of the general expression because the general single-species limit checks not only the elastic terms but also the inelastic terms in the collision matrix. In addition, we give an explicit expression of the shear viscosity in terms of all the $2 \leftrightarrow 2$ cross sections for the case of isotropic and energy-independent scatterings, where it is much easier to see how different processes contribute to the shear viscosity. These analytical results of the shear viscosity should be useful for parton transport models and its extraction of shear viscosity from comparisons with the experimental data. They can also be applied to QCD cross sections to facilitate studies of the shear viscosity of quark-gluon matter at finite temperature with parton transport models or QCD effective kinetic theory.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: DERIVATION OF EQ. (15)

We start from the following Eq. (8):

$$\sum_{j=1}^N x_j R_{ij} \left[C_i(\tau_i) \pi_i^\mu \pi_i^\nu \right] = \pi_i^\mu \pi_i^\nu. \quad (\text{A1})$$

Let us apply the inner product $\langle F, x_i \tau_i^r \pi_i^\mu \pi_i^\nu \rangle_i$ to the above equation, where F represents either side of the equation. When F is replaced with the right-hand side, the above procedure simply gives $x_i \gamma_{i,r}$, where $\gamma_{i,r} \equiv \langle \tau_i^r \pi_i^\mu \pi_i^\nu, \pi_i^\mu \pi_i^\nu \rangle_i$. When F is replaced with the left-hand side, the above procedure gives four terms since $R_{ij}[\phi_i] \propto (\phi_i + \phi_j - \phi_k - \phi_l)$ as shown in Eq. (9). The first term (T_1) is given by

$$T_1 = \frac{1}{128\pi^2 T^6} \sum_{s=0}^P \sum_{j,k,l} \int \frac{d^3 \vec{p}_i}{p_i^0} \frac{d^3 \vec{p}_j}{p_j^0} \frac{d^3 \vec{p}_k}{p_k^0} \frac{d^3 \vec{p}_l}{p_l^0} e^{-\tau_i - \tau_j} x_j C_{i,s}^{(P)} \tau_i^s \pi_i^\mu \pi_i^\nu W_{ij \rightarrow kl} x_i \tau_i^r \pi_{i\mu} \pi_{i\nu}, \quad (\text{A2})$$

where we have made the approximation $C_i(\tau_i) \approx \sum_{s=0}^P C_{i,s}^{(P)} \tau_i^s$. Writing $C_{i,s}^{(P)}$ as $\sum_m \delta_{im} C_{m,s}^{(P)}$ and then making the exchange $m \leftrightarrow j$, we get

$$T_1 = \sum_{s=0}^P \sum_j C_{j,s}^{(P)} \frac{\delta_{ij} x_i}{128\pi^2 T^6} \sum_{k,l,m} x_m \int \frac{d^3 \vec{p}_i}{p_i^0} \frac{d^3 \vec{p}_m}{p_m^0} \frac{d^3 \vec{p}_k}{p_k^0} \frac{d^3 \vec{p}_l}{p_l^0} e^{-\tau_i - \tau_m} \tau_i^s \pi_i^\mu \pi_i^\nu W_{im \rightarrow kl} \tau_i^r \pi_{i\mu} \pi_{i\nu} \quad (\text{A3})$$

$$= \sum_{s=0}^P \sum_j C_{j,s}^{(P)} \delta_{ij} x_i \sum_{k,l,m} x_m \left[\tau_i^r \pi_i^\mu \pi_i^\nu, \tau_i^s \pi_{i\mu} \pi_{i\nu} \right]_{im \rightarrow kl}. \quad (\text{A4})$$

We see that the expression after the term $C_{j,s}^{(P)}$ in the above is just the first term of the matrix element C_{ij}^{rs} in Eq. (13). The second term of the left-hand side from the above procedure is simpler and given by

$$T_2 = \sum_{s=0}^P \sum_j C_{j,s}^{(P)} x_i x_j \sum_{k,l} \left[\tau_i^r \pi_i^\mu \pi_i^\nu, \tau_j^s \pi_{j\mu} \pi_{j\nu} \right]_{ij \rightarrow kl}, \quad (\text{A5})$$

which contains the second term of the matrix element C_{ij}^{rs} .

The third term is given by

$$T_3 = \frac{-1}{128\pi^2 T^6} \sum_{s=0}^P \sum_{j,k,l} \int \frac{d^3 \vec{p}_i}{p_i^0} \frac{d^3 \vec{p}_j}{p_j^0} \frac{d^3 \vec{p}_k}{p_k^0} \frac{d^3 \vec{p}_l}{p_l^0} e^{-\tau_i - \tau_j} x_j C_{k,s}^{(P)} \tau_k^s \pi_k^\mu \pi_k^\nu W_{ij \rightarrow kl} x_i \tau_i^r \pi_{i\mu} \pi_{i\nu}. \quad (\text{A6})$$

Making the exchange $j \leftrightarrow k$ and the replacement $l \rightarrow m$, we get

$$T_3 = \frac{-1}{128\pi^2 T^6} \sum_{s=0}^P \sum_{j,k,m} \int \frac{d^3 \vec{p}_i}{p_i^0} \frac{d^3 \vec{p}_j}{p_j^0} \frac{d^3 \vec{p}_k}{p_k^0} \frac{d^3 \vec{p}_m}{p_m^0} e^{-\tau_i - \tau_k} x_k C_{j,s}^{(P)} \tau_j^s \pi_j^\mu \pi_j^\nu W_{ik \rightarrow jm} x_i \tau_i^r \pi_{i\mu} \pi_{i\nu}. \quad (\text{A7})$$

Because of the detailed balance relation (for massless particles) $d_i d_k W_{ik \rightarrow jm} = d_j d_m W_{jm \rightarrow ik}$ with d_i being the degeneracy factor of parton species i , we have $x_i x_k W_{ik \rightarrow jm} = x_j x_m W_{jm \rightarrow ik}$ in chemical equilibrium. Also noting the energy-momentum conservation relation $\tau_i + \tau_k = \tau_j + \tau_m$, we then get

$$T_3 = \frac{-1}{128\pi^2 T^6} \sum_{s=0}^P \sum_{j,k,m} \int \frac{d^3\vec{p}_i}{p_i^0} \frac{d^3\vec{p}_j}{p_j^0} \frac{d^3\vec{p}_k}{p_k^0} \frac{d^3\vec{p}_m}{p_m^0} e^{-\tau_j - \tau_m} x_j C_{j,s}^{(P)} \tau_j^s \pi_j^\mu \pi_j^\nu W_{jm \rightarrow ik} x_m \tau_i^r \pi_{i\mu} \pi_{i\nu} \quad (\text{A8})$$

$$= \sum_{s=0}^P \sum_j C_{j,s}^{(P)} x_j \sum_{k,m} x_m \left[\tau_i^r \pi_i^\mu \pi_i^\nu, \tau_j^s \pi_{j\mu} \pi_{j\nu} \right]_{jm \rightarrow ik}, \quad (\text{A9})$$

which contains the third term of the matrix element C_{ij}^{rs} . Similarly, one can show that the fourth term reduces to the following (after taking advantage of $W_{ij \rightarrow kl} = W_{ji \rightarrow lk}$):

$$T_4 = \sum_{s=0}^P \sum_j C_{j,s}^{(P)} x_j \sum_{k,m} x_m \left[\tau_i^r \pi_i^\mu \pi_i^\nu, \tau_j^s \pi_{j\mu} \pi_{j\nu} \right]_{jm \rightarrow ki}, \quad (\text{A10})$$

which contains the forth term of the matrix element C_{ij}^{rs} .

Adding all four terms from the left-hand side together, we then obtain Eq. (15) [24],

$$\sum_{j=1}^N \sum_{s=0}^P C_{ij}^{rs} C_{j,s}^{(P)} = x_i \gamma_{i,r}. \quad (\text{A11})$$

APPENDIX B: CALCULATING THE COLLISION BRACKETS IN MATRIX C

For completeness, here we show the derivations of the collision brackets for massless partons, similar to those for massive partons with elastic collisions in the book [24]. For two incoming partons (with momenta p_i and p_j) and two outgoing partons (with momenta p_k and p_l), the total 4-momentum is

$$P^\mu \equiv p_i^\mu + p_j^\mu = p_k^\mu + p_l^\mu \equiv P'^\mu, \quad (\text{B1})$$

and the projected relative 4-momenta are defined as [24]

$$Q^\mu \equiv \Delta_P^{\mu\nu} (p_{i\nu} - p_{j\nu}), \quad Q'^\mu \equiv \Delta_P^{\mu\nu} (p_{k\nu} - p_{l\nu}), \quad \text{with} \quad \Delta_P^{\mu\nu} \equiv g^{\mu\nu} - P^\mu P^\nu / P^2. \quad (\text{B2})$$

Taking advantage of the orthogonality relations $P^\mu Q_\mu = P^\mu Q'_\mu = 0$, we can write the individual momentum of each massless parton as

$$p_i^\mu = (P^\mu + Q^\mu)/2, \quad p_j^\mu = (P^\mu - Q^\mu)/2, \quad p_k^\mu = (P^\mu + Q'^\mu)/2, \quad p_l^\mu = (P^\mu - Q'^\mu)/2. \quad (\text{B3})$$

We can then write a generic collision integral [24], similar in structure to Eq. (5), which can represent all the terms in the elements of matrix C,

$$J_{ij \rightarrow kl}^{(a,b,d,e,f)} = \frac{1}{128\pi^2 T^6} \int \frac{d^3\vec{p}_i}{p_i^0} \frac{d^3\vec{p}_j}{p_j^0} \frac{d^3\vec{p}_k}{p_k^0} \frac{d^3\vec{p}_l}{p_l^0} e^{-\frac{P \cdot U}{T}} \left(\frac{P^2}{T^2} \right)^a \left(\frac{P \cdot U}{T} \right)^b \left(\frac{Q \cdot U}{T} \right)^d \left(\frac{Q' \cdot U}{T} \right)^e \left(\frac{-Q \cdot Q'}{T^2} \right)^f W_{ij \rightarrow kl}, \quad (\text{B4})$$

where all the upper indices (a, b, d, e, f) are integers. Following earlier derivations [23,24], we get the final result for the J integral for massless partons as

$$J_{ij \rightarrow kl}^{(a,b,d,e,f)} = \frac{\pi(-1)^b (d+e+1)!!}{8} \int_0^\infty dv v^{2a+b+d+e+2f+5} \left(\frac{d^b}{dv^b} \frac{K_{(d+e)/2+1}(v)}{v^{(d+e)/2+1}} \right) \sum_{g=0}^{\min(d,e)} K(d, e, g) \sigma_{ij \rightarrow kl}^{(f,g)}(v). \quad (\text{B5})$$

In the above, $K_n(v)$ is the modified Bessel of the second kind,

$$K(d, e, g) = \frac{d!e!}{(d-g)!!(d+g+1)!!(e-g)!!(e+g+1)!!} \quad (\text{B6})$$

for even $(d - g)$ and even $(e - g)$ while $K(d, e, g) = 0$ otherwise, and

$$\sigma_{ij \rightarrow kl}^{(f,g)}(v) = \frac{2g+1}{4\pi} \int d\Omega \cos^f \Theta P_g(\cos \Theta) \frac{d\sigma^{ij \rightarrow kl}}{d\Omega}, \quad (\text{B7})$$

where $v = \sqrt{s}/T$ and $P_g(x)$ represents the Legendre polynomials.

Examining the collision brackets in C_{ij}^s of Eq. (13), we can rewrite the integrands using

$$\begin{aligned} a^\mu \overleftrightarrow{a}^\nu b_\mu \overleftrightarrow{b}_\nu &= (\Delta_{\mu\nu} a^\mu b^\nu)^2 - \frac{1}{3} (\Delta_{\mu\nu} a^\mu a^\nu) (\Delta_{\alpha\beta} b^\alpha b^\beta) \\ &= [(a \cdot b) - (a \cdot U)(b \cdot U)]^2 - \frac{1}{3} [a^2 - (a \cdot U)^2] [b^2 - (b \cdot U)^2]. \end{aligned} \quad (\text{B8})$$

For massless partons, the first term on the right-hand side of Eq. (13) corresponds to the following integrals:

$$\left[\tau_i^r \pi_i^\mu \overleftrightarrow{\pi}_i^\nu, \tau_i^s \pi_{i\mu} \overleftrightarrow{\pi}_{i\nu} \right]_{im \rightarrow kl} = \frac{2}{3} [\tau_i^{r+2}, \tau_i^{s+2}]_{im \rightarrow kl}, \quad (\text{B9})$$

$$[\tau_i^r, \tau_i^s]_{im \rightarrow kl} = \frac{1}{2^{r+s}} \sum_{n=0}^{r+s} \binom{r+s}{n} J_{im \rightarrow kl}^{(0,n,r+s-n,0,0)}. \quad (\text{B10})$$

The second term in Eq. (13) corresponds to the s -channel integrals,

$$\begin{aligned} \left[\tau_i^r \pi_i^\mu \overleftrightarrow{\pi}_i^\nu, \tau_j^s \pi_{j\mu} \overleftrightarrow{\pi}_{j\nu} \right]_{ij \rightarrow kl} &= -2 \left[\tau_i^{r+1} \pi_i^\mu, \tau_j^{s+1} \pi_{j\mu} \right]_{ij \rightarrow kl} - \frac{4}{3} [\tau_i^{r+2}, \tau_j^{s+2}]_{ij \rightarrow kl} \\ &\quad + \frac{1}{2^{r+s+2}} \sum_{n_1=0}^r \sum_{n_2=0}^s (-1)^{s-n_2} \binom{r}{n_1} \binom{s}{n_2} J_{ij \rightarrow kl}^{(2,n_1+n_2,r+s-n_1-n_2,0,0)}, \end{aligned} \quad (\text{B11})$$

$$\left[\tau_i^r \pi_i^\mu, \tau_j^s \pi_{j\mu} \right]_{ij \rightarrow kl} = -[\tau_i^{r+1}, \tau_j^{s+1}]_{ij \rightarrow kl} + \frac{1}{2^{r+s+1}} \sum_{n_1=0}^r \sum_{n_2=0}^s (-1)^{s-n_2} \binom{r}{n_1} \binom{s}{n_2} J_{ij \rightarrow kl}^{(1,n_1+n_2,r+s-n_1-n_2,0,0)}, \quad (\text{B12})$$

$$[\tau_i^r, \tau_j^s]_{ij \rightarrow kl} = \frac{1}{2^{r+s}} \sum_{n_1=0}^r \sum_{n_2=0}^s (-1)^{s-n_2} \binom{r}{n_1} \binom{s}{n_2} J_{ij \rightarrow kl}^{(0,n_1+n_2,r+s-n_1-n_2,0,0)}. \quad (\text{B13})$$

In the above, $\pi_i^{\bar{\mu}} \equiv \Delta_{\nu}^{\mu} \pi_i^{\nu}$, where $\Delta^{\mu\nu}$ has been defined in Eq. (2).

The third term in Eq. (13) corresponds to the t -channel integrals,

$$\begin{aligned} \left[\tau_i^r \pi_i^\mu \overleftrightarrow{\pi}_i^\nu, \tau_j^s \pi_{j\mu} \overleftrightarrow{\pi}_{j\nu} \right]_{jm \rightarrow ik} &= -2 [\tau_i^{r+1} \pi_i^\mu, \tau_j^{s+1} \pi_{j\mu}]_{jm \rightarrow ik} - \frac{4}{3} [\tau_i^{r+2}, \tau_j^{s+2}]_{jm \rightarrow ik} \\ &\quad + \frac{1}{2^{r+s+4}} \sum_{n_1=0}^r \sum_{n_2=0}^s \binom{r}{n_1} \binom{s}{n_2} \left[J_{jm \rightarrow ik}^{(2,n_1+n_2,s-n_2,r-n_1,0)} - 2J_{jm \rightarrow ik}^{(1,n_1+n_2,s-n_2,r-n_1,1)} + J_{jm \rightarrow ik}^{(0,n_1+n_2,s-n_2,r-n_1,2)} \right], \end{aligned} \quad (\text{B14})$$

$$\left[\tau_i^r \pi_i^\mu, \tau_j^s \pi_{j\mu} \right]_{jm \rightarrow ik} = -[\tau_i^{r+1}, \tau_j^{s+1}]_{jm \rightarrow ik} + \frac{1}{2^{r+s+2}} \sum_{n_1=0}^r \sum_{n_2=0}^s \binom{r}{n_1} \binom{s}{n_2} \left[J_{jm \rightarrow ik}^{(1,n_1+n_2,s-n_2,r-n_1,0)} - J_{jm \rightarrow ik}^{(0,n_1+n_2,s-n_2,r-n_1,1)} \right], \quad (\text{B15})$$

$$[\tau_i^r, \tau_j^s]_{jm \rightarrow ik} = \frac{1}{2^{r+s}} \sum_{n_1=0}^r \sum_{n_2=0}^s \binom{r}{n_1} \binom{s}{n_2} J_{jm \rightarrow ik}^{(0,n_1+n_2,s-n_2,r-n_1,0)}. \quad (\text{B16})$$

The fourth term in Eq. (13) corresponds to the u -channel integrals,

$$\begin{aligned} \left[\tau_i^r \pi_i^{\bar{\mu}} \pi_i^{\bar{\nu}}, \tau_j^s \pi_{j\mu}^{\bar{\mu}} \pi_{j\nu}^{\bar{\nu}} \right]_{jm \rightarrow ki} &= -2 \left[\tau_i^{r+1} \pi_i^{\bar{\mu}}, \tau_j^{s+1} \pi_{j\mu}^{\bar{\mu}} \right]_{jm \rightarrow ki} - \frac{4}{3} \left[\tau_i^{r+2}, \tau_j^{s+2} \right]_{jm \rightarrow ki} \\ &+ \frac{1}{2^{r+s+4}} \sum_{n_1=0}^r \sum_{n_2=0}^s (-1)^{r-n_1} \binom{r}{n_1} \binom{s}{n_2} \left[J_{jm \rightarrow ki}^{(2, n_1+n_2, s-n_2, r-n_1, 0)} \right. \\ &\left. + 2J_{jm \rightarrow ki}^{(1, n_1+n_2, s-n_2, r-n_1, 1)} + J_{jm \rightarrow ki}^{(0, n_1+n_2, s-n_2, r-n_1, 2)} \right], \end{aligned} \quad (\text{B17})$$

$$\left[\tau_i^r \pi_i^{\bar{\mu}}, \tau_j^s \pi_{j\mu}^{\bar{\mu}} \right]_{jm \rightarrow ki} = - \left[\tau_i^{r+1}, \tau_j^{s+1} \right]_{jm \rightarrow ki} + \frac{1}{2^{r+s+2}} \sum_{n_1=0}^r \sum_{n_2=0}^s (-1)^{r-n_1} \binom{r}{n_1} \binom{s}{n_2} \left[J_{jm \rightarrow ki}^{(1, n_1+n_2, s-n_2, r-n_1, 0)} + J_{jm \rightarrow ki}^{(0, n_1+n_2, s-n_2, r-n_1, 1)} \right], \quad (\text{B18})$$

$$[\tau_i^r, \tau_j^s]_{jm \rightarrow ki} = \frac{1}{2^{r+s}} \sum_{n_1=0}^r \sum_{n_2=0}^s (-1)^{r-n_1} \binom{r}{n_1} \binom{s}{n_2} J_{jm \rightarrow ki}^{(0, n_1+n_2, s-n_2, r-n_1, 0)}. \quad (\text{B19})$$

APPENDIX C: VERIFICATION OF THE FIRST-ORDER RESULT IN A GENERAL SINGLE-SPECIES LIMIT

Here we prove that the shear viscosity expression in Eq. (29) satisfies the general single-species limit, where one can write

$$\frac{d\sigma^{ij \rightarrow kl}}{dt} = f_{ij}^{kl} (1 + \delta_{kl}) \frac{d\sigma_0}{dt} \quad (\text{C1})$$

and each parton species has the same collision rate per parton. In the above, $d\sigma_0/dt$ is an arbitrary given differential cross section, and f_{ij}^{kl} are constants that depend on

i, j, k, l but not on the Mandelstam variables. As a result of Eq. (28), we then have

$$\sigma_{ij}^{kl} \equiv \sigma^{ij \rightarrow kl}(v) = f_{ij}^{kl} \int_{-s}^0 \frac{d\sigma_0}{dt} dt \equiv f_{ij}^{kl} \sigma_0(v), \quad (\text{C2})$$

where $v = \sqrt{s}/T$, thus the cross section of every $2 \rightarrow 2$ scattering channel has the same energy(s) dependence and is proportional to the f_{ij}^{kl} constant. For elastic scatterings, we shall write $\sigma^{ij \rightarrow ij}(v)$ as σ_{ij} and f_{ij}^{ij} as f_{ij} in the following.

We can write the following matrix elements that Eq. (29) needs:

$$\begin{aligned} C_{gg}^{00} &= x_g^2 \tilde{c}_0[g] + x_g(1-x_g) \tilde{c}_1[gq] + \tilde{\omega}[gg], \\ C_{qq}^{00} &= x_q^2 \tilde{c}_0[q] + x_q[x_g \tilde{c}_1[gq] + x_q \tilde{c}_1[q\bar{q}] + 2(N_f - 1)x_q \tilde{c}_1[qq']] + \tilde{\omega}[qq], \\ C_{gq}^{00} &= x_g x_q \tilde{c}_2[gq] + \tilde{\omega}[gq], \quad C_{q\bar{q}}^{00} = x_q^2 \tilde{c}_2[q\bar{q}] + \tilde{\omega}[q\bar{q}], \quad C_{qq'}^{00} = x_q^2 \tilde{c}_2[qq'] + \tilde{\omega}[qq'], \end{aligned} \quad (\text{C3})$$

where according to Eq. (20) we have

$$\tilde{c}_0[g] = \frac{f_{gg}}{384} \int_0^\infty dv v^6 [(3v^2 + 4)K_3(v) - 6vK_2(v)] \sigma_{0,\text{tr}}(v) \equiv f_{gg} \tilde{c}_0. \quad (\text{C4})$$

In the above, $\sigma_{0,\text{tr}}(v) = \int_{-s}^0 dt \sin^2 \theta_{\text{cm}} d\sigma_0/dt$. We then write the inelastic terms as

$$\begin{aligned} \tilde{\omega}[gg] &= \sigma_{gg}^{q\bar{q}} w_1, \quad \tilde{\omega}[gq] = \sigma_{gg}^{q\bar{q}} w_2, \quad \tilde{\omega}[qq] = \left[\sigma_{qq}^{gg} + (N_f - 1) \sigma_{qq}^{q'\bar{q}'} \right] w_3, \\ \tilde{\omega}[q\bar{q}] &= \left[\sigma_{q\bar{q}}^{gg} + (N_f - 1) \sigma_{q\bar{q}}^{q'\bar{q}'} \right] w_4, \quad \tilde{\omega}[qq'] = \sigma_{qq}^{q'\bar{q}'} w_5, \end{aligned} \quad (\text{C5})$$

where the dimensionless w_1 to w_5 are defined according to Eqs. (23)–(27) as the following:

$$w_1 = \frac{x_g^2 N_f}{192 \sigma_0(v)} \int_0^\infty dv v^6 [(v^2 + 28)K_3(v) - 2vK_2(v)] \sigma_0(v), \quad (C6)$$

$$w_2 = \frac{x_g^2}{768 \sigma_0(v)} \int_0^\infty dv v^6 [(3v^2 + 4)K_3(v) - 6vK_2(v)] \sigma_{0,\text{tr}}(v) - \frac{x_g^2}{384 \sigma_0(v)} \int_0^\infty dv v^6 [(v^2 + 28)K_3(v) - 2vK_2(v)] \sigma_0(v), \quad (C7)$$

$$w_3 = \frac{x_q^2}{384 \sigma_0(v)} \int_0^\infty dv v^6 [(v^2 + 48)K_3(v) + 8vK_2(v)] \sigma_0(v), \quad (C8)$$

$$w_4 = \frac{x_q^2}{384 \sigma_0(v)} \int_0^\infty dv v^6 [(v^2 + 8)K_3(v) - 12vK_2(v)] \sigma_0(v), \quad (C9)$$

$$w_5 = \frac{x_q^2}{768 \sigma_0(v)} \int_0^\infty dv v^6 [(3v^2 + 4)K_3(v) - 6vK_2(v)] \sigma_{0,\text{tr}}(v) - \frac{x_q^2}{384 \sigma_0(v)} \int_0^\infty dv v^6 [(v^2 + 28)K_3(v) - 2vK_2(v)] \sigma_0(v). \quad (C10)$$

Note that although w_k ($k = 1$ to 5) above has a v dependence due to the $\sigma_0(v)$ term in its denominator, $\tilde{\omega}[ij]$ in Eq. (C5) is independent of v because the v dependence of $\sigma_0(v)$ is canceled by that of $\sigma^{ij \rightarrow kl}(v)$.

For the limit under consideration here, we find the following relations:

$$\tilde{c}_0[g] = \frac{\sigma_{gg}}{\sigma_{gq}} (\tilde{c}_1[gq] + \tilde{c}_2[gq]), \quad \frac{\tilde{c}_0[q]}{\tilde{c}_0[g]} = \frac{\sigma_{qq}}{\sigma_{gg}}, \quad \frac{\tilde{c}_i[q\bar{q}]}{\tilde{c}_i[gq]} = \frac{\sigma_{q\bar{q}}}{\sigma_{gq}}, \quad \frac{\tilde{c}_i[qq']}{\tilde{c}_i[gq]} = \frac{\sigma_{qq'}}{\sigma_{gq}}, \quad (C11)$$

where $i = 1$ or 2 . Then we can write the shear viscosity of Eq. (29) in terms of $\tilde{c}_1[gq]$, $\tilde{c}_2[gq]$ and w_k , where w_3 and w_4 always appear as $(w_3 + w_4)$. We also find the following relations:

$$w_1 = \frac{N_f x_g^2}{x_q^2} (w_3 + w_4), \quad w_2 = \frac{x_g^2}{x_q^2} w_5, \quad \frac{\tilde{c}_0[g]}{\sigma_{gg}} = \frac{w_1}{N_f x_g^2} + \frac{2w_2}{x_g^2}. \quad (C12)$$

which we then use to write the shear viscosity of Eq. (29) in terms of $\tilde{c}_1[gq]$, $\tilde{c}_2[gq]$ and w_1 .

For the collision rates per parton, we have

$$r_g/n = x_g \bar{\sigma}_{gg} + 2N_f x_q \bar{\sigma}_{gq} + N_f x_g \bar{\sigma}_{gg}^{q\bar{q}}, \\ r_q/n = r_{\bar{q}}/n = x_g \bar{\sigma}_{gq} + x_q \bar{\sigma}_{qq} + x_q \bar{\sigma}_{q\bar{q}} + 2(N_f - 1)x_q \bar{\sigma}_{qq'} + (N_f - 1)x_q \bar{\sigma}_{q\bar{q}}^{q'\bar{q}'} + x_q \bar{\sigma}_{q\bar{q}}^{gg}. \quad (C13)$$

Here we have use the notation $\bar{\sigma} \equiv \langle \sigma v_{\text{rel}} \rangle$ for the thermal average of a cross section, with the relative velocity between two colliding massless partons given by $v_{\text{rel}} = s/(2E_1 E_2)$. For the general single-species limit, the equal rate condition $r_g = r_q$ requires the seven independent cross sections to satisfy Eq. (36), i.e.,

$$\sigma_{gg} = \frac{3}{8} \left[\sigma_{qq} + \sigma_{q\bar{q}} + (N_f - 1) \left(2\sigma_{qq'} + \sigma_{q\bar{q}}^{q'\bar{q}'} \right) \right] + \frac{(4 - 3N_f)}{32} (8\sigma_{gq} + 3\sigma_{q\bar{q}}^{gg}). \quad (C14)$$

Applying the above relation, Eq. (29) then reduces to the following (without any w_i terms):

$$\eta = \frac{1}{(x_g f_{gg} + 2N_f x_q f_{gq} + N_f x_g f_{gg}^{q\bar{q}})} \frac{160T}{\tilde{c}_0}, \quad (C15)$$

where $\tilde{c}_0$ has been defined in Eq. (C4).

On the other hand, we expect the shear viscosity in this equal-rate limit to be the single-species result with an effective cross section σ_{eff} that gives the same collision rate per parton. From $n\langle\sigma_{\text{eff}}v_{\text{rel}}\rangle = r_g$, we obtain the equivalent effective cross section as

$$\sigma_{\text{eff}} = (x_g f_{gg} + 2N_f x_q f_{gq} + N_f x_g f_{gq}^{qq})\sigma_0(v). \quad (\text{C16})$$

The single-species result in Eq. (32) for $\sigma = \sigma_{\text{eff}}$ thus agrees with Eq. (C15), i.e., the Chapman-Enskog expression of shear viscosity in Eq. (29) satisfies this general single-species limit.

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