

First order electroweak radiative corrections to the decay of the polarized W boson

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Analytic results for the next-to-leading order electroweak radiative corrections to the decay of the polarized W boson into a pair of heavy quarks are presented. Finite fermion masses are taken into account throughout this calculation, and in performing the collinear limit for the final quark states and all light fermions taking part in this process, the importance of mass effects is demonstrated. This gives hints for the deviation of the total decay rate and the branching ratios into massive fermions between experimental measurements and theoretical predictions.

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I. INTRODUCTION

The W boson as one of the two massive gauge bosons of the electroweak theory has been subject to intense investigations recently. Besides the mass discrepancy with the Standard Model (SM) value, challenged by a new interpretation of old data from 2012 by the CDF collaboration that indicate a deviation of approximately 4 standard deviations in 2022 [1], there are also less profound deviations reported in Ref. [2], for instance, in the measured decay width of the W decay of $\Gamma_W = 2.137 \pm 0.032$ GeV [3,4], contrasted with the SM prediction of $\Gamma_W = 2.0892 \pm 0.0008$ GeV [5,6] that is 1.5σ smaller than the measured result. While measurements of the leptonic branching ratios from ATLAS and CMS are in good agreement with lepton universality [7,8], older, less precise data from LEP 2 indicate a branching ratio for $W \rightarrow \tau + \bar{\nu}_\tau$ that is 2.6σ larger than the electron-muon average [9]. This and the experimentally measured larger decay rate might indicate that the difference is due to mass effects and radiative corrections to the leading order (LO) Born-term result not taken properly into account. In this work we deal with these issues.

Pairs of W bosons can be produced at lepton or hadron colliders (cf., e.g., Refs. [10,11]). However, as the main SM channel for the single W boson is the decay process $t \rightarrow W^+ + X_b$, in Refs. [12–16] first order

quantum chromodynamics (QCD) and electroweak (EW) radiative corrections have been calculated in order to obtain next-to-leading order (NLO) results for the decay processes $t \rightarrow b + W^+(\uparrow)$ and the similar process $b \rightarrow c + W^-(\uparrow)$, flipped to the process at hand. Note that due to the left-handed $V-A$ coupling of the electroweak interaction, the W boson is polarized [17], and the polarization can be analyzed by looking at the angular distribution of the decay [18,19] and can be considered in WW scattering at the LHC [20,21]. NLO EW radiative corrections to the process $W \rightarrow \ell^+ + \nu_\ell$ have been calculated in Ref. [6] without taking into account the polarization of the decaying particle. This gap was closed by a master thesis defended at the University of Tartu in 2009 [22]. Taking quark masses into account and using similarities to the process $Z(\uparrow) \rightarrow q + \bar{q}$ [23–25], NLO QCD radiative corrections for the process $W^+(\uparrow) \rightarrow Q + \bar{q}$ were calculated in a master thesis at the University of Tartu in 2010 [26–29]. In order to discern from W boson physics beyond the SM [30–32] and in rare decays [33,34] it is inevitable to improve the precision of the SM prediction by including NLO EW radiative corrections [35] and by taking care of effects from the masses of the decay products. This is the aim of the present publication.

The central object of interest is the angular dependence of the decay rate for the polarized W boson, as it can be inferred by looking at Refs. [27–29]. It contains the spin-density matrix ρ that describes the polarization of the W boson, the decay functions $H^{mm'}$, named here as helicity bilinears and calculated for the decay channel of the W boson, and the polar angle dependence on θ . The azimuthal angle dependence is found in Refs. [12,14], the decay rate function [29] is

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$$\begin{aligned}
W(\theta) &= \sum_{m=0,\pm 1} \rho_{mm} H^{mm}(\theta) \\
&= \sum_{m,m'=0,\pm 1} \rho_{mm} d_{mm'}^1(\theta) d_{mm'}^1(\theta) H^{m'm'} \\
&= \frac{3}{8} (1 + \cos^2 \theta) ((\rho_{++} + \rho_{--})(H^{++} + H^{--}) + 2\rho_{00} H^{00}) \\
&\quad + \frac{3}{4} \cos \theta ((\rho_{++} - \rho_{--})(H^{++} - H^{--})) \\
&\quad + \frac{3}{4} \sin^2 \theta ((\rho_{++} + \rho_{--}) H^{00} + \rho_{00} (H^{++} + H^{--} - H^{00})),
\end{aligned} \tag{1}$$

using the normalized spin-density matrix elements $\rho_{mm'}$ with $\rho_{++} + \rho_{00} + \rho_{--} = 1$ and the helicity bilinears $H^{mm'}$ normalized in such a way that $H^{++} + H^{00} + H^{--} = 1$ at the Born-term level for massless fermions. The results are given in the rest frame of the W boson with the original direction of flight of the W boson fixed in boosting to this rest frame, while the polar angle θ is given as the angle between this direction and the momentum of the quark. While the spin-density matrix elements $\rho_{00} = 0.687(5)$, $\rho_{++} = 0.0017(1)$, and $\rho_{--} = 0.311(5)$ obtained from the production of the W boson in top quark decay are inputs for our calculation

taken from Ref. [36], here we have to calculate the helicity bilinears.

The work is divided up as follows: In Secs. II and III we give the Born-term results and the tree and loop contributions to the EW radiative corrections to the decay process $W^+ \rightarrow c\bar{b}$ of an on-shell W boson into charm and anti-bottom quarks. Though contributing only with a small branching ratio to the total decay rate due to the smallness of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{cb} , this particular choice is made because for on-shell W bosons the masses of the two quarks are maximal and will most prominently show the mass effects being presented in this paper. The results are incorporated in the helicity bilinears in Sec. IV. In Sec. V we give analytical results for the helicity bilinears, taking also into account detailed counterterms for the cancellation of IR singularities. A discussion of the result in terms of the angular dependence of the decay rate is given in Sec. VI. Our conclusions are found in Sec. VII, while detailed calculations and formulas are given in four Appendixes.

II. TREE CONTRIBUTIONS

For the Born term, the squared absolute value of the matrix element is given by

$$\begin{aligned}
|\mathcal{M}_0|^2 &= \sum_{s_1, s_2} \mathcal{M}_0^* \mathcal{M}_0 \\
&= \frac{e^2}{8s_W^2} |V_{cb}|^2 \text{tr}((\not{p}_2 - m_2)\gamma^\mu(1 - \gamma_5)(\not{p}_1 + m_1)\gamma^\nu(1 - \gamma_5)) \varepsilon_\mu^*(q, \lambda) \varepsilon_\nu(q, \lambda) \\
&= \frac{e^2}{s_W^2} |V_{cb}|^2 [p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu}(p_2 p_1) - p_{2\kappa} p_{1\lambda} i \epsilon^{\kappa\lambda\mu\nu}] \varepsilon_\mu^*(q, \lambda) \varepsilon_\nu(q, \lambda),
\end{aligned} \tag{2}$$

with $s_W = \sin \theta_W$ the sine of the Weinberg angle. p_1 and p_2 are the momenta of the quark and antiquark, respectively, with corresponding masses m_1 and m_2 . In later steps we will be more specific in choosing $m_1 = m_c$ and $m_2 = m_b$. For an unpolarized W boson we sum over the polarizations λ , and choosing the unitary gauge $\xi_W = \infty$, we obtain [37]

$$\sum_\lambda \varepsilon_\mu^*(q, \lambda) \varepsilon_\nu(q, \lambda) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{m_W^2}. \tag{3}$$

On mass shell we can use $q^2 = m_W^2$, where $q = p_1 + p_2$ is the momentum of the W boson. As the W boson is the incoming particle, instead of summing over the (three) polarizations $\lambda = +, -, \text{ and } 0$ we have to calculate the mean value over these. This gives an additional factor $1/3$ to the decay rate, in compliance with Refs. [6,29]. Rearranging $q = p_1 + p_2$ accordingly and squaring, one obtains

$$\begin{aligned}
p_1 p_2 &= \frac{q^2}{2} (1 - \mu_1 - \mu_2), & p_1 q &= \frac{q^2}{2} (1 + \mu_1 - \mu_2), \\
p_2 q &= \frac{q^2}{2} (1 - \mu_1 + \mu_2),
\end{aligned} \tag{4}$$

where $m_1^2 = p_1^2 = \mu_1 q^2$ and $m_2^2 = p_2^2 = \mu_2 q^2$ introduces two dimensionless mass parameters μ_1 and μ_2 . Therefore, summing over the polarizations of the final states one has

$$\begin{aligned}
\overline{|\mathcal{M}_0|^2} &= \frac{e^2 |V_{cb}|^2}{3s_W^2} (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - p_1 p_2 g^{\mu\nu} + i p_{1\kappa} p_{2\lambda} \epsilon^{\kappa\lambda\mu\nu}) \\
&\quad \times \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_W^2} \right) \\
&= \frac{e^2 |V_{cb}|^2 m_W^2}{6s_W^2} (2 - \mu_1 - \mu_2 - (\mu_1 - \mu_2)^2).
\end{aligned} \tag{5}$$

In our approach, the Weinberg angle θ_W is fixed by the boson masses, $\cos \theta_W = m_W/m_Z$. The first line of Eq. (5) can be written as

$$|\overline{\mathcal{M}}_0|^2 = \frac{e^2 |V_{cb}|^2 q^2}{3s_W^2} H^{\mu\nu}(\text{Born}) \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{m_W^2} \right), \quad (6)$$

where $H^{\mu\nu}(\text{Born})$ is the Born-term hadron tensor. In calculating the helicity bilinears $H^{\lambda_1 \lambda_2}$, instead of Eq. (3) one contracts with the product $\varepsilon_\mu^*(q, \lambda_1) \varepsilon_\nu(q, \lambda_2)$ and normalizes the result by dividing by the sum of the unnormalized results for $(\lambda_1, \lambda_2) = (+, +)$, $(-, -)$, and $(0, 0)$. This will be exemplified in Sec. IV.

A. The two body decay

For the decay rate, one has to integrate the absolute square of the matrix element over the phase space. In order to calculate the phase space, we have to make explicit the kinematics of the decay in the rest frame of the W boson. In the two-body decay of the W boson into up-type quark $Q(p_1)$ and down-type antiquark $\bar{q}(p_2)$, the two quarks are produced back to back. Here we can choose two frames. In the *quark frame* we take the positive z axis to be the direction of flight of the charm quark. One obtains

$$p_1 = (E_1; 0, 0, |\vec{p}|), \quad p_2 = (E_2; 0, 0, -|\vec{p}|), \quad (7)$$

where $p_1 + p_2 = q = (\sqrt{q^2}; 0, 0, 0)$ and

$$E_1 = \frac{1}{2}(1 + \mu_1 - \mu_2)\sqrt{q^2}, \quad E_2 = \frac{1}{2}(1 - \mu_1 + \mu_2)\sqrt{q^2},$$

$$|\vec{p}| = \frac{1}{2}\sqrt{\lambda(1, \mu_1, \mu_2)}\sqrt{q^2}, \quad (8)$$

with the Källén function given by $\lambda(a, b, c) := a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$. Taking into account that the axis of flight of the W boson produced in, e.g., the dominant process $t \rightarrow W^+ + b$, this defines another axis. Considering the polarization of the W boson as an important parameter for our analysis, the polar angle θ between the quark frame and the axis of flight of the W is of importance here. The rest frame of the W boson with the

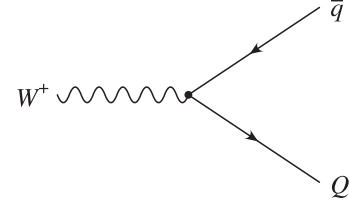


FIG. 1. Born-term contribution.

direction of flight pointing in the positive z direction is called the *W frame* in the following. The two-particle phase space depends on the polar angle and is given by

$$dPS_2 = \frac{1}{16\pi q^2} \sqrt{\lambda(q^2, m_1^2, m_2^2)} d(\cos \theta)$$

$$= \frac{1}{16\pi} \sqrt{\lambda(1, \mu_1, \mu_2)} d(\cos \theta). \quad (9)$$

According to Fermi's golden rule, for the calculation of the decay rate we have to combine the squared absolute value of the matrix element and the phase space factor to obtain

$$d\Gamma = \frac{1}{2m_W} |\mathcal{M}|^2 dPS. \quad (10)$$

After integration over the polar angle, the full unpolarized Born-term decay rate for the process shown in Fig. 1 with an on-shell W boson ($q^2 = m_W^2$) is given by

$$\Gamma_0 = \frac{e^2 m_W}{96\pi s_W^2} N_c |V_{cb}|^2 \sqrt{\lambda(1, \mu_1, \mu_2)} (2 - \mu_1 - \mu_2 - (\mu_1 - \mu_2)^2). \quad (11)$$

B. The three-body decay

In this work we are dealing with NLO electroweak corrections. Taking into account the tree corrections with a real photon emitted, we end up with the three-body decays shown in Fig. 2, with the kinematics shown in Fig. 3. For the phase space integration it would be appropriate to have the denominator factors in the easiest shape. For this we define dimensionless quantities y_1 and y_2 by

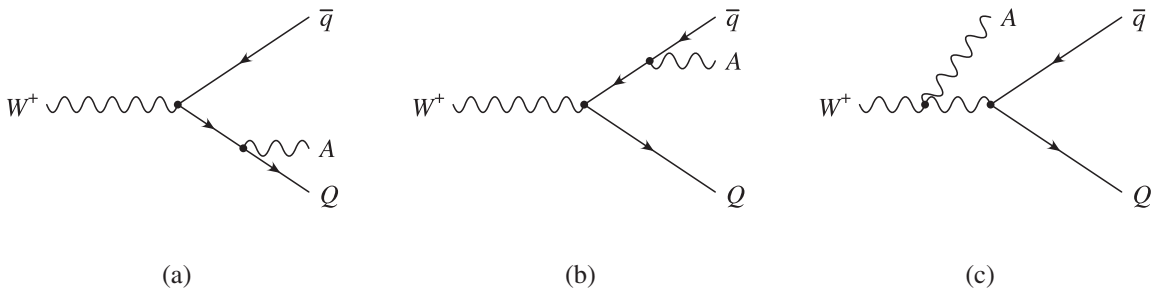


FIG. 2. First order electroweak tree corrections off (a) the quark, (b) the antiquark, and (c) the W boson.

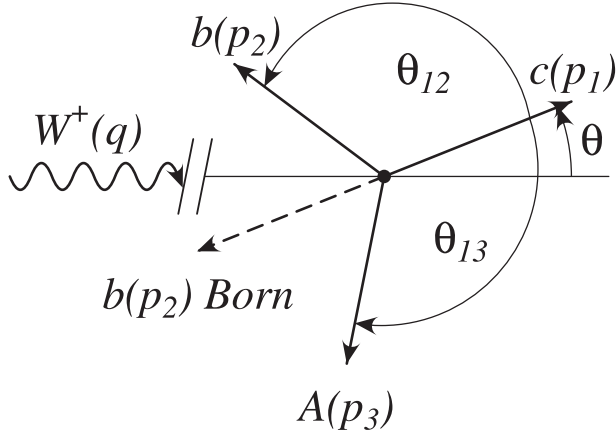


FIG. 3. Kinematics of the three (and two) particle decay.

$$\begin{aligned}
 y_1 q^2 &:= (p_1 + p_3)^2 - m_1^2 = p_1^2 + 2p_1 p_3 + p_3^2 - m_1^2 \\
 &= 2p_1 p_3 + p_3^2, \\
 y_2 q^2 &:= (p_2 + p_3)^2 - m_2^2 = p_2^2 + 2p_2 p_3 + p_3^2 - m_2^2 \\
 &= 2p_2 p_3 + p_3^2
 \end{aligned} \tag{12}$$

with $q^2 = (p_1 + p_2 + p_3)^2 = p_1^2 + 2p_1 p_2 + 2p_1 p_3 + p_2^2 + 2p_2 p_3 + p_3^2$. As the integration over the phase space will result in IR divergencies, we use mass regularization by giving a small mass $m_A^2 = p_3^2 = \Lambda q^2$ to the emitted photon with momentum p_3 . The scalar products read

$$\begin{aligned}
 p_1 p_2 &= \frac{1}{2}(1 - (\mu_1 + y_1) - (\mu_2 + y_2) + \Lambda)q^2, \\
 p_1 p_3 &= \frac{1}{2}(y_1 - \Lambda)q^2, \quad p_2 p_3 = \frac{1}{2}(y_2 - \Lambda)q^2.
 \end{aligned} \tag{13}$$

The general ansatz for the kinematics in the quark frame is given by $p_1 = (E_1; 0, 0, |\vec{p}_1|)$,

$$\begin{aligned}
 p_2 &= (E_2; |\vec{p}_2| \sin \theta_{12}, 0, |\vec{p}_2| \cos \theta_{12}), \\
 p_3 &= (E_3; |\vec{p}_3| \sin \theta_{13}, 0, |\vec{p}_3| \cos \theta_{13})
 \end{aligned} \tag{14}$$

with

$$\begin{aligned}
 E_1 &= \frac{1}{2}(1 + \mu_1 - (\mu_2 + y_2))\sqrt{q^2} & |\vec{p}_1| &= \frac{1}{2}\sqrt{\lambda(1, \mu_1, \mu_2 + y_2)}\sqrt{q^2} \\
 E_2 &= \frac{1}{2}(1 - (\mu_1 + y_1) + \mu_2)\sqrt{q^2} & |\vec{p}_2| &= \frac{1}{2}\sqrt{\lambda(1, \mu_1 + y_1, \mu_2)}\sqrt{q^2} \\
 E_3 &= \frac{1}{2}(y_1 + y_2)\sqrt{q^2} & |\vec{p}_3| &= \frac{1}{2}\sqrt{(y_1 + y_2)^2 - 4\Lambda}\sqrt{q^2}
 \end{aligned} \tag{15}$$

and the cosines and sines of the relative angles are given by

$$\begin{aligned}
 \cos \theta_{12} &= \frac{y_1 y_2 + (1 - \mu_1 + \mu_2)y_1 + (1 + \mu_1 - \mu_2)y_2 - \lambda - 2\Lambda}{\sqrt{\lambda(1, \mu_1, \mu_2 + y_2)}\sqrt{\lambda(1, \mu_1 + y_1, \mu_2)}}, \\
 \sin \theta_{12} &= \frac{2\sqrt{N(y_1, y_2)}}{\sqrt{\lambda(1, \mu_1, \mu_2 + y_2)}\sqrt{\lambda(1, \mu_1 + y_1, \mu_2)}}, \\
 \cos \theta_{13} &= \frac{-y_1(1 - \mu_1 + \mu_2 + y_2) + (1 + \mu_1 - \mu_2 - y_2)y_2 + 2\Lambda}{\sqrt{\lambda(1, \mu_1, \mu_2 + y_2)}\sqrt{(y_1 + y_2)^2 - 4\Lambda}}, \\
 \sin \theta_{13} &= \frac{-2\sqrt{N(y_1, y_2)}}{\sqrt{\lambda(1, \mu_1, \mu_2 + y_2)}\sqrt{(y_1 + y_2)^2 - 4\Lambda}},
 \end{aligned} \tag{16}$$

with

$$\begin{aligned}
 N(y_1, y_2) &= (1 - \mu_1 - \mu_2)y_1 y_2 - y_1^2(\mu_2 + y_2) - (\mu_1 + y_1)y_2^2 \\
 &\quad + \Lambda((1 - \mu_1 + \mu_2)y_1 + (1 + \mu_1 - \mu_2)y_2 + y_1 y_2 - \lambda) - \Lambda^2.
 \end{aligned} \tag{17}$$

The three-body phase space is given by

$$dPS_3 = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - q) \prod_{i=1}^3 \frac{d^4 p_i}{(2\pi)^4} (2\pi) \delta(p_i^2 - m_i^2) \theta(p_i^0), \tag{18}$$

which in the quark frame simplifies to

$$dPS_3 = dPS_2 \times \frac{q^2}{(4\pi)^2 \sqrt{\lambda(1, \mu_1, \mu_2)}} dy_1 dy_2. \quad (19)$$

The phase limits in terms of y_1 and y_2 read $y_{1-}(y_2) \leq y_1 \leq y_{1+}(y_2)$ and

$$y_{20} := \Lambda + 2\sqrt{\Lambda\mu_2} \leq y_2 \leq (1 - \sqrt{\mu_1})^2 - \mu_2 := y_{2-}, \quad (20)$$

with

$$y_{1\pm}(y_2) = \frac{1}{2(\mu_2 + y_2)} (y_2(1 - \mu_1 - (\mu_2 + y_2)) + \Lambda(1 - \mu_1 + (\mu_2 + y_2)) \pm \sqrt{(y_2 - \Lambda)^2 - 4\Lambda\mu_2} \sqrt{\lambda(1, \mu_1, \mu_2 + y_2)}). \quad (21)$$

The IR singularity resides in the phase space at $y_1 = y_2 = 0$ and is regularized by the parameter Λ . Details about the calculation of the phase space integrals and the basic terms contained in the tree contributions are found in Appendix A.

III. LOOP CONTRIBUTIONS

The tree contributions are not the only first order corrections. In addition, we have to take into account a large number of loop corrections to the vertex displayed in Fig. 4. The results we obtain from the loop corrections are both infrared and ultraviolet singular. For the latter, we have to apply the renormalization procedure. The NLO result of the decay rate is contributed by a matrix element $\mathcal{M}(q, \lambda) = \mathcal{M}^\mu \varepsilon_\mu(q, \lambda)$, where $\varepsilon(q, \lambda)$ is the polarization vector of the W boson, and

$$\mathcal{M}^\mu = \frac{ie}{\sqrt{2}s_W} \left\{ \gamma^\mu \Lambda_- \left[V_{cb} \left(1 + \delta Z_e - \frac{\delta s_W}{s_W} + \delta Z_{WW} \right) + \delta V_{cb} + \sum_k (\delta Z_{ck}^{L*} V_{kb} + V_{ck} \delta Z_{kb}^L) \right] + V_{cb} V_b^\mu \right\}, \quad (22)$$

where the sum runs over corresponding quark flavors. The result is renormalized by the counterterms δZ_e for the charge, $\delta s_W/s_W$ for the sine of the Weinberg angle, and

counterterms related to the particles involved in the decay process. For different renormalization schemes, these counterterms are defined in Appendix B. The result for the bare vertex correction V_b^μ can, in principle, be split up into six contributions according to

$$V_b^\mu = V_-^0 \bar{u}(p_1) \gamma^\mu \Lambda_- v(p_2) + V_+^0 \bar{u}(p_1) \gamma^\mu \Lambda_+ v(p_2) + V_-^1 \bar{u}(p_1) p_1^\mu \Lambda_- v(p_2) + V_+^1 \bar{u}(p_1) p_1^\mu \Lambda_+ v(p_2) + V_-^2 \bar{u}(p_1) p_2^\mu \Lambda_- v(p_2) + V_+^2 \bar{u}(p_1) p_2^\mu \Lambda_+ v(p_2) \quad (23)$$

with $\Lambda_\pm = (1 \pm \gamma_5)/2$. However, considering only on-shell W boson decays, the renormalized loop corrections are divided up into four form factors V_- , V_+ , V_1 , and V_2 only, defined by

$$\begin{aligned} \frac{\alpha}{4\pi} V_- &= \text{Re} \left(V_-^0 + \delta Z_{\text{CKM}} + \delta Z_e - \frac{\delta s_W}{s_W} + \delta Z_{WW} + \delta Z_{cc}^L + \delta Z_{bb}^L \right), \\ \frac{\alpha}{4\pi} \mu_1 \mu_2 V_+ q^2 &= m_1 m_2 \text{Re} V_+^0, \\ \frac{\alpha}{4\pi} \mu_1 V_1 &= m_1 \text{Re}(V_-^1 - V_-^2), \\ \frac{\alpha}{4\pi} \mu_2 V_2 &= m_2 \text{Re}(V_+^1 - V_+^2), \end{aligned} \quad (24)$$

where δZ_{CKM} is the counterterm for the CKM matrix, as dealt with in the following subsection. As will be discussed in Sec. VA, IR singularities will be subtracted from V_- to obtain V_-^* . The form factors V_-^* , V_+ , V_1 , and V_2 are cross-checked against the form factors in Ref. [5]. In doing so, we found agreement with our result.

A. Renormalization of the CKM matrix

As the CKM matrix element V_{cb} dominating the decay $W^+ \rightarrow c\bar{b}$ is not close to the unit, we have to take into account the mixing of quark states and, related to this, the renormalization of the mixing matrix [5,38–45]. Using

$$\delta V_{cb} = \frac{1}{2} \sum_k ((\delta Z_{ck}^L - \delta Z_{ck}^{L*}) V_{kb} - V_{ck} (\delta Z_{kb}^L - \delta Z_{kb}^{L*})), \quad (25)$$

one obtains

$$\begin{aligned} \mathcal{M}^\mu &= \frac{ie}{\sqrt{2}s_W} \left\{ \gamma^\mu \Lambda_- \left[V_{cb} \left(1 + \delta Z_e - \frac{\delta s_W}{s_W} + \delta Z_{WW} + \delta Z_{cc}^L + \delta Z_{bb}^L \right) + \text{Re} \delta Z_{cu}^L V_{ub} + \text{Re} \delta Z_{ct}^L V_{tb} + V_{cd} \text{Re} \delta Z_{db}^L + V_{cs} \text{Re} \delta Z_{sb}^L \right] + V_{cb} V_b^\mu \right\}. \end{aligned} \quad (26)$$

Defining $\delta Z_{\text{CKM}} := (\text{Re} \delta Z_{cu}^L V_{ub} + \text{Re} \delta Z_{ct}^L V_{tb} + V_{cd} \text{Re} \delta Z_{db}^L + V_{cs} \text{Re} \delta Z_{sb}^L) / V_{cb}$, one ends up with V_- as given in Eq. (24).

In the $\alpha(0)$ scheme, the UV singular part within V_b^μ is given by $V_s^\mu = \gamma^\mu \Lambda_- V_s^0$ with

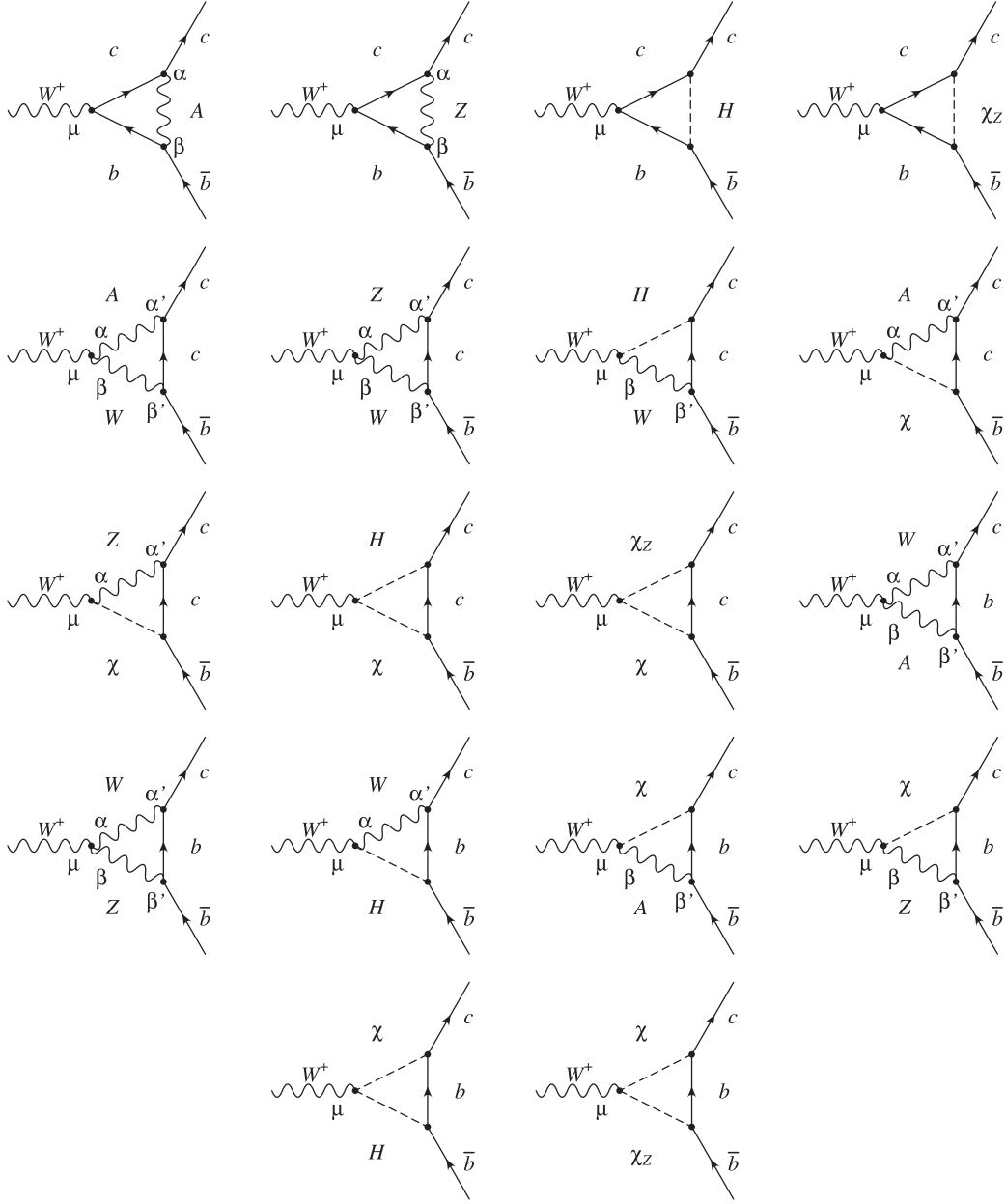


FIG. 4. Loop corrections.

$$V_s^0 = \frac{e^2}{4m_W^2 s_W^2 \epsilon} [m_c^2 + m_b^2 + 12m_W^2 - m_Z^2(1 - 2(Q_c - Q_b + 2Q_c Q_b)s_W^2)] \quad (27)$$

with space-time dimension $D = 4 - 2\epsilon$, and $Q_c = 2/3$ and $Q_b = -1/3$ for the electric charges of the quarks, while $\delta Z_e^s + \delta Z_{WW}^s - \delta s_W^s/s_W = -2e^2/(s_W^2 \epsilon)$ and [46]

$$\begin{aligned} \delta Z_{bb}^{Ls} &= -\frac{e^2}{8m_W^2 s_W^2 \epsilon} \left[m_b^2 + \sum_k |V_{kb}|^2 (m_k^2 + 2m_W^2) + (1 + 4Q_c Q_b s_W^2) m_Z^2 \right], \\ \delta Z_{cc}^{Ls} &= -\frac{e^2}{8m_W^2 s_W^2 \epsilon} \left[m_c^2 + \sum_k |V_{ck}|^2 (m_k^2 + 2m_W^2) + (1 + 4Q_c Q_b s_W^2) m_Z^2 \right], \\ \delta Z_{ij}^{Ls} &= \frac{e^2}{4m_W^2 s_W^2 \epsilon} \sum_k V_{ik} V_{kj} \frac{m_i^2 m_j^2 - 2m_i^2 m_k^2 - m_j^2 m_k^2 + 2m_j^2 m_W^2}{m_i^2 - m_j^2}. \end{aligned} \quad (28)$$

Note that according to Eq. (4.5.27) in Ref. [46], one has $\delta Z_{ij}^{Ls*} = \delta Z_{ij}^{Ls}|_{m_i^2 \leftrightarrow m_j^2}$. Therefore,

$$\text{Re} \delta Z_{ij}^{Ls} = \frac{-e^2}{8m_W^2 s_W^2 \epsilon} \sum_k V_{ik} V_{kj} (m_k^2 + 2m_W^2), \quad V_{kj} = V_{jk}^*. \quad (29)$$

Without the mixed contributions (and without the general factor V_{cb}), one first obtains

$$\begin{aligned} V_s^0 + \delta Z_e^s - \frac{\delta s_W^s}{s_W} + \delta Z_{WW}^s + \delta Z_{cc}^{Ls} + \delta Z_{bb}^{Ls} \\ = \frac{e^2}{8m_W^2 s_W^2 \epsilon} \left[m_c^2 + m_b^2 + 4m_W^2 - \sum_k |V_{ck}|^2 (m_k^2 + 2m_W^2) \right. \\ \left. - \sum_k |V_{kb}|^2 (m_k^2 + 2m_W^2) \right]. \end{aligned} \quad (30)$$

On the other hand, the mixed part gives

$$\begin{aligned} \text{Re} \delta Z_{cu}^{Ls} V_{ub} + \text{Re} \delta Z_{ct}^{Ls} V_{tb} + V_{cd} \text{Re} \delta Z_{db}^{Ls} + V_{cs} \text{Re} \delta Z_{sb}^{Ls} \\ = \frac{-e^2}{8m_W^2 s_W^2 \epsilon} \sum_k (V_{ck} V_{ku} V_{ub} + V_{ck} V_{kt} V_{tb} + V_{cd} V_{dk} V_{kb} \\ + V_{cs} V_{sk} V_{kb}) (m_k^2 + 2m_W^2) \\ = \frac{-e^2 V_{cb}}{8m_W^2 s_W^2 \epsilon} \left[m_b^2 + 2m_W^2 - \sum_k |V_{ck}|^2 (m_k^2 + 2m_W^2) + m_c^2 \right. \\ \left. + 2m_W^2 - \sum_k |V_{kb}|^2 (m_k^2 + 2m_W^2) \right]. \end{aligned} \quad (31)$$

This cancels exactly against the previous contribution, gaining an UV finite form factor.

IV. HELICITY BILINEARS

The helicity bilinears are obtained by contracting the hadron tensor with the polarization vectors $\epsilon(q, \lambda)$ which for the Born-term diagram is given in Eq. (5). First we use the helicity basis in the W frame, i.e., the rest frame of the decaying W boson with the z axis as the initial direction of flight of the W boson, given by

$$\begin{aligned} \epsilon(q, \pm) &= \frac{1}{\sqrt{2}} (0; \mp 1, -i, 0), \quad \epsilon(q, 0) = (0; 0, 0, 1), \\ \epsilon(q, t) &= (1; 0, 0, 0). \end{aligned} \quad (32)$$

Note that $\lambda = \pm, 0$ describe the different magnetic quantum numbers for the total angular momentum quantum number $j = 1$ while $\lambda = t$ indicates the magnetic quantum number zero for $j = 0$. This timelike component does not appear for on-shell W bosons and will be skipped in the following. At the Born-term level, up to a general factor $e^2 q^2 |V_{cb}|^2 / s_W^2$ the helicity bilinears are given by

$$\begin{aligned} q^2 H^{\lambda_1 \lambda_2} &= [p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - p_1 p_2 g^{\mu\nu} + i p_{1\kappa} p_{2\lambda} \epsilon^{\kappa\lambda\mu\nu}] \\ &\times \epsilon_\mu(q, \lambda_1) \epsilon_\nu^*(q, \lambda_2). \end{aligned} \quad (33)$$

This results in the W frame read

$$\begin{aligned} H''(\theta) &= \frac{1}{2} (\mu_1 + \mu_2 - (\mu_1 - \mu_2)^2), \\ H^{00}(\theta) &= H^{0t}(\theta) = \frac{1}{2} (\mu_1 - \mu_2) \sqrt{\lambda} \cos \theta, \\ H^{t\pm}(\theta) &= H^{\pm t}(\theta) = \mp \frac{1}{2\sqrt{2}} (\mu_1 - \mu_2) \sqrt{\lambda} \sin \theta, \\ H^{00}(\theta) &= \frac{1}{2} (1 - \mu_1 - \mu_2 - \lambda \cos^2 \theta), \\ H^{0\pm}(\theta) &= H^{\pm 0}(\theta) = -\frac{1}{2\sqrt{2}} \sqrt{\lambda} (1 \mp \sqrt{\lambda} \cos \theta) \sin \theta, \\ H^{\pm\pm}(\theta) &= \frac{1}{4} (1 + \mu_1 - \mu_2 \mp \sqrt{\lambda} \cos \theta) (1 - \mu_1 + \mu_2 \mp \sqrt{\lambda} \cos \theta), \\ H^{\pm\mp}(\theta) &= \frac{1}{4} \lambda \sin^2 \theta. \end{aligned} \quad (34)$$

Changing to the quark frame as the frame of reference for the angular distribution (1), one has to take $\theta = 0$. Only the helicity bilinears $H^{00} = H^{00}(\theta = 0)$, $H^{\pm\pm} = H^{\pm\pm}(\theta = 0)$ necessary for the angular distribution and $H^{\pm\mp} = H^{\pm\mp}(\theta = 0)$ (for completeness) will be considered in the following.

A. Helicity bilinears from the tree corrections

The hadron tensor for the NLO tree correction to be contracted with the polarization vectors is obtained by replacing $|\mathcal{M}_0|^2$ in Eq. (5) by $|\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3|^2$, where

the matrix elements $\mathcal{M}_i$ ($i = 1, 2, 3$) correspond to the three Feynman diagrams in Fig. 2. We have calculate the helicity bilinears both for Feynman and unitary gauge and found agreement. Up to a general factor $e^2 q^2 |V_{cb}|^2 / s_W^2 \times \alpha / (4\pi\sqrt{\lambda})$, the NLO tree contributions to the helicity bilinears read

$$\begin{aligned}
H^{00}(\text{tree}) = & \frac{1}{2} [Q_c^2 + Q_b^2 + Q_W^2] \sqrt{\lambda} (-\mu_1 + \mu_1^2 - \mu_2 - 2\mu_1\mu_2 + \mu_2^2) \ell_\zeta \\
& + \frac{1}{2} [(1 - \mu_2) Q_c^2 - \mu_1 (Q_b^2 - Q_W^2)] (-\mu_1 + \mu_1^2 - \mu_2 - 2\mu_1\mu_2 + \mu_2^2) (t_\zeta^\ell - 2t_{zW}^{\ell+}) \\
& - \frac{1}{2} [Q_c^2 - Q_b^2 + (\mu_1 - \mu_2) Q_W^2] (-\mu_1 + \mu_1^2 - \mu_2 - 2\mu_1\mu_2 + \mu_2^2) (t_{\zeta W}^\ell + 2t_{zW}^{\ell+}) \\
& - 2\mu_1 [(1 + 5\mu_1 - \mu_2) Q_c^2 + 2\mu_1 (Q_b^2 - Q_W^2)] (t_z^\ell + t_z^{-\ell} - t_z^{+\ell}) \\
& - \sqrt{\mu_1} [(1 - 10\mu_1 - 3\mu_1^2 - 2\mu_2 + 2\mu_1\mu_2 + \mu_2^2) Q_c^2 - 2\mu_1 (1 + \mu_1 - \mu_2) (Q_b^2 - Q_W^2)] (t_z^{-\ell} + t_z^{+\ell}) \\
& + 2[(1 + \mu_1 - \mu_2) (Q_c^2 - Q_b^2) - (3 + 2\mu_1 + \mu_1^2 - 4\mu_2 - 2\mu_1\mu_2 + \mu_2^2) Q_W^2] (t_{zW}^\ell + t_{zW}^{-\ell} - t_{zW}^{+\ell}) \\
& - \frac{1}{\sqrt{\mu_1}} (1 + \mu_1 - \mu_2) [(1 + \mu_1 - \mu_2) (Q_c^2 - Q_b^2) - 2(1 + 2\mu_1 - \mu_2) Q_W^2] (t_{zW}^{-\ell} + t_{zW}^{+\ell}) \\
& + \frac{1}{4} [(4 - 25\mu_1 - 12\mu_1^2 - 9\mu_2 + 26\mu_1\mu_2 - 2\mu_1^2\mu_2 + 6\mu_2^2 - 3\mu_1\mu_2^2 - \mu_2^3) Q_c^2 \\
& - (4 + 6\mu_1 + 6\mu_1^2 - \mu_1^3 - 6\mu_2 - 2\mu_1\mu_2 + 9\mu_1^2\mu_2 - 4\mu_1\mu_2^2 + 2\mu_2^3) Q_b^2 \\
& - 2(4 + 9\mu_1 + 3\mu_1^2 + \mu_1^3 - 7\mu_2 - 7\mu_1\mu_2 - 2\mu_1^2\mu_2 + 2\mu_2^2 - 2\mu_1\mu_2^2 + \mu_2^3) Q_W^2] \ell_1 \\
& - \frac{1}{4} [(4 - 23\mu_1 - 16\mu_1^2 + 2\mu_1^3 - 7\mu_2 + 12\mu_1\mu_2 - 6\mu_1^2\mu_2 + 4\mu_2^2 + 6\mu_1\mu_2^2 - 2\mu_2^3) Q_c^2 \\
& - (4 + 5\mu_1 + 8\mu_1^2 - 2\mu_1^3 - 3\mu_2 - 4\mu_1\mu_2 + 6\mu_1^2\mu_2 - 4\mu_2^2 - 6\mu_1\mu_2^2 + 2\mu_2^3) Q_b^2 \\
& - 2(4 + 8\mu_1 + 5\mu_1^2 - 12\mu_2 - 12\mu_1\mu_2 + 7\mu_2^2) Q_W^2] \ell_{1W} \\
& + \frac{1}{8} [(16 - 67\mu_1 - 15\mu_1^2 - 3\mu_2 + 38\mu_1\mu_2 - 11\mu_2^2) Q_c^2 \\
& - (8 + 11\mu_1 + 11\mu_1^2 - 21\mu_2 - 38\mu_1\mu_2 + 15\mu_2^2) Q_b^2 \\
& - 2(24 + \mu_1 + 9\mu_1^2 - 31\mu_2 - 14\mu_1\mu_2 + 9\mu_2^2) Q_W^2] \sqrt{\lambda}, \tag{35}
\end{aligned}$$

$$\begin{aligned}
H^{++}(\text{tree}) = & -\frac{1}{2} [Q_c^2 + Q_b^2 + Q_W^2] \sqrt{\lambda} ((1 - \mu_1 - \mu_2 - \sqrt{\lambda}) \ell_\zeta + 2\sqrt{\lambda} \ell_+) \\
& - \frac{1}{2} [(1 - \mu_2) Q_c^2 - \mu_1 (Q_b^2 - Q_W^2)] ((1 - \mu_1 - \mu_2) (t_\zeta^\ell - 2t_{zW}^{\ell+}) - \sqrt{\lambda} (t_\zeta^\ell - 2t_{zW}^{\ell-})) \\
& + \frac{1}{2} [Q_c^2 - Q_b^2 + (\mu_1 - \mu_2) Q_W^2] ((1 - \mu_1 - \mu_2) (t_{\zeta W}^\ell + 2t_{zW}^{\ell+}) - \sqrt{\lambda} (t_{\zeta W}^\ell + 2t_{zW}^{\ell-})) \\
& + [(1 - 2\mu_1 - 2\mu_2 - \mu_1\mu_2 + \mu_2^2) Q_c^2 - \mu_1 (1 + \mu_1 - \mu_2) (Q_b^2 - Q_W^2)] t_z^\ell \\
& + \mu_1 [(1 + 5\mu_1 - \mu_2) Q_c^2 + 2\mu_1 (Q_b^2 - Q_W^2)] (t_z^\ell + t_z^{-\ell} - t_z^{+\ell}) \\
& + \frac{1}{2} \sqrt{\mu_1} [(1 - 10\mu_1 - 3\mu_1^2 - 2\mu_2 + 2\mu_1\mu_2 + \mu_2^2) Q_c^2 - 2\mu_1 (1 + \mu_1 - \mu_2) (Q_b^2 - Q_W^2)] (t_z^{-\ell} + t_z^{+\ell}) \\
& - [(1 + \mu_1 - \mu_2) (Q_c^2 - Q_b^2) - (3 + 2\mu_1 + \mu_1^2 - 4\mu_2 - 2\mu_1\mu_2 + \mu_2^2) Q_W^2] (2t_{zW}^\ell + t_{zW}^{-\ell} - t_{zW}^{+\ell}) \\
& + \frac{1}{2\sqrt{\mu_1}} (1 + \mu_1 - \mu_2) [(1 + \mu_1 - \mu_2) (Q_c^2 - Q_b^2) - 2(1 + 2\mu_1 - \mu_2) Q_W^2] (t_{zW}^{-\ell} + t_{zW}^{+\ell}) \\
& + \frac{1}{2} [(1 - \mu_1) (5 - 3\mu_1 + 4\mu_2) Q_c^2 - (9 - 10\mu_1 + \mu_1^2 + 6\mu_2 - 2\mu_1\mu_2) Q_b^2 + \\
& - 2(1 - \mu_1) (5 - \mu_1 + \mu_2) Q_W^2] \ell_0
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}[(3-8\mu_1-4\mu_1^2-6\mu_2+10\mu_1\mu_2+3\mu_2^2+(1+5\mu_1-\mu_2)\sqrt{\lambda})Q_c^2 \\
& -\mu_1(4+7\mu_1-4\mu_2)Q_b^2-2(1-\mu_2)(1+5\mu_1-\mu_2)Q_W^2]\ell_1 \\
& +\frac{1}{4}[(1-4\mu_1-6\mu_1^2-4\mu_2+6\mu_2^2+8\sqrt{\lambda})Q_c^2-(1+2\mu_1+8\mu_1^2+6\mu_2-8\mu_1\mu_2+8\sqrt{\lambda})Q_b^2 \\
& -2(2+3\mu_1+\mu_1^2-5\mu_2-6\mu_1\mu_2+5\mu_2^2+4\sqrt{\lambda})Q_W^2]\ell_{1W} \\
& -\frac{1}{4}[(17+7\mu_1-8\mu_2)Q_c^2-(13-3\mu_1+2\mu_2)Q_b^2-6(3+\mu_1-\mu_2)Q_W^2]\lambda_- \\
& +\frac{1}{24}[9(1+5\mu_1+\mu_2)Q_c^2+3(15+7\mu_1-29\mu_2)Q_b^2 \\
& +4(29-10\mu_1-\mu_1^2-34\mu_2+2\mu_1\mu_2-\mu_2^2)Q_W^2]\sqrt{\lambda}+\frac{1}{8}[Q_c^2-9Q_b^2-12Q_W^2]\lambda,
\end{aligned} \tag{36}$$

$$\begin{aligned}
H^{+-}(\text{tree}) &= H^{-+}(\text{tree}) = -\mu_1[(1+5\mu_1-\mu_2)Q_c^2+2\mu_1(Q_b^2-Q_W^2)](t_z^\ell+t_z^{-\ell}-t_z^{+\ell}) \\
& -\frac{1}{2}\sqrt{\mu_1}[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2-2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)](t_z^{-\ell}+t_z^{+\ell}) \\
& +[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2](t_{zW}^\ell+t_{zW}^{-\ell}-t_{zW}^{+\ell}) \\
& -\frac{1}{2\sqrt{\mu_1}}(1+\mu_1-\mu_2)[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-2(1+2\mu_1-\mu_2)Q_W^2](t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
& +\frac{1}{2}[(1-6\mu_1-3\mu_1^2-2\mu_2+6\mu_1\mu_2+\mu_2^2)Q_c^2-(1+\mu_1+2\mu_1^2-2\mu_2-\mu_1\mu_2+\mu_2^2)Q_b^2 \\
& -2(1+\mu_1-\mu_2)^2Q_W^2]\ell_1 \\
& -\frac{1}{2}[(1-6\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2-(1+\mu_1+2\mu_1^2-\mu_2-2\mu_1\mu_2)Q_b^2 \\
& -2(1+\mu_1-2\mu_2)(1+\mu_1-\mu_2)Q_W^2]\ell_{1W} \\
& +\frac{1}{2}[2(1-5\mu_1-\mu_2)Q_c^2-(1+3\mu_1-\mu_2)Q_b^2-2(3+\mu_1-3\mu_2)Q_W^2]\sqrt{\lambda},
\end{aligned} \tag{37}$$

$$\begin{aligned}
H^{--}(\text{tree}) &= \frac{1}{2}[Q_c^2+Q_b^2+Q_W^2]\sqrt{\lambda}((1-\mu_1-\mu_2+\sqrt{\lambda})\ell_\xi-2\sqrt{\lambda}\ell_+) \\
& -\frac{1}{2}[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)]((1-\mu_1-\mu_2)(t_\xi^\ell-2t_z^{\ell+})+\sqrt{\lambda}(t_\xi^\ell-2t_z^{\ell-})) \\
& +\frac{1}{2}[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2]((1-\mu_1-\mu_2)(t_{\xi W}^\ell+2t_{zW}^{\ell+})+\sqrt{\lambda}(t_{\xi W}^\ell+2t_{zW}^{\ell-})) \\
& -[(1-2\mu_1-2\mu_2-\mu_1\mu_2+\mu_2^2)Q_c^2-\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)]t_z^\ell \\
& +\mu_1[(1+5\mu_1-\mu_2)Q_c^2+2\mu_1(Q_b^2-Q_W^2)](t_z^\ell+t_z^{-\ell}-t_z^{+\ell}) \\
& +\frac{1}{2}\sqrt{\mu_1}[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2-2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)](t_z^{-\ell}+t_z^{+\ell}) \\
& -[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2](t_{zW}^{-\ell}-t_{zW}^{+\ell}) \\
& +\frac{1}{2\sqrt{\mu_1}}(1+\mu_1-\mu_2)[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-2(1+2\mu_1-\mu_2)Q_W^2](t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
& -\frac{1}{2}[(1-\mu_1)(5-3\mu_1+4\mu_2)Q_c^2-(9-10\mu_1+\mu_1^2+6\mu_2-2\mu_1\mu_2)Q_b^2 \\
& -2(1-\mu_1)(5-\mu_1+\mu_2)Q_W^2]\ell_0
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}[(3-8\mu_1-4\mu_1^2-6\mu_2+10\mu_1\mu_2+3\mu_2^2-(1+5\mu_1-\mu_2)\sqrt{\lambda})Q_c^2 \\
& -\mu_1(4+7\mu_1-4\mu_2)Q_b^2-2(1-\mu_2)(1+5\mu_1-\mu_2)Q_W^2]\ell_1 \\
& +\frac{1}{4}[(1-4\mu_1-6\mu_1^2-4\mu_2+6\mu_2^2-8\sqrt{\lambda})Q_c^2-(1+2\mu_1+8\mu_1^2+6\mu_2-8\mu_1\mu_2-8\sqrt{\lambda})Q_b^2 \\
& -2(2+3\mu_1+\mu_1^2-5\mu_2-6\mu_1\mu_2+5\mu_2^2-4\sqrt{\lambda})Q_W^2]\ell_{1W} \\
& +\frac{1}{4}[(17+7\mu_1-8\mu_2)Q_c^2-(13-3\mu_1+2\mu_2)Q_b^2-6(3+\mu_1-\mu_2)Q_W^2]\lambda_- \\
& +\frac{1}{24}[9(1+5\mu_1+\mu_2)Q_c^2+3(15+7\mu_1-29\mu_2)Q_b^2 \\
& +4(29-10\mu_1-\mu_1^2-34\mu_2+2\mu_1\mu_2-\mu_2^2)Q_W^2]\sqrt{\lambda}-\frac{1}{8}[Q_c^2-9Q_b^2-12Q_W^2]\lambda
\end{aligned} \tag{38}$$

with $\lambda_{\pm} := (1 \pm \sqrt{\mu_1})^2 - \mu_2$ and $\lambda_- \lambda_+ = \lambda := \lambda(1, \mu_1, \mu_2)$. The dilogarithmic terms contained in these expressions are given by

$$\begin{aligned}
t_{\zeta}^{\ell} &= \int_{\zeta_-}^1 \left(\frac{1}{\zeta} - \frac{2}{1+\zeta} \right) \ln \left(\frac{\zeta + \zeta_0}{1 + \zeta_0 \zeta} \right) d\zeta, \quad \zeta_0 = \frac{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}, \\
t_{\zeta W}^{\ell} &= \int_{\zeta_-}^1 \left(\frac{1}{\zeta} - \frac{2}{1+\zeta} \right) \ln \left(\frac{\zeta + \zeta_{0W}}{1 + \zeta_{0W} \zeta} \right) d\zeta, \quad \zeta_{0W} = \frac{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}, \\
t_z^{\ell-} &= \int_{z_-}^1 \left[\left(\frac{1}{z-z_-} - \frac{1}{z-z_+} \right) L_z(z) - \frac{L_z(z_-)}{z-z_+} \right] dz, \\
t_z^{\ell+} &= \int_{z_-}^1 \left[\left(\frac{1}{z-z_-} + \frac{1}{z-z_+} - \frac{1}{z} \right) L_z(z) - \frac{L_z(z_-)}{z-z_+} \right] dz, \\
t_{zW}^{\ell-} &= \int_{z_-}^1 \left[\left(\frac{1}{z-z_-} - \frac{1}{z-z_+} \right) L_{zW}(z) - \frac{L_{zW}(z_-)}{z-z_+} \right] dz, \\
t_{zW}^{\ell+} &= \int_{z_-}^1 \left[\left(\frac{1}{z-z_-} + \frac{1}{z-z_+} - \frac{1}{z} \right) L_{zW}(z) - \frac{L_{zW}(z_-)}{z-z_+} \right] dz, \\
t_z^{\pm\ell} &= \int_{z_-}^1 \frac{L_z(z)}{1 \pm z} dz, \quad t_z^{\ell} = \int_{z_-}^1 \frac{L_z(z)}{z} dz, \\
t_{zW}^{\pm\ell} &= \int_{z_-}^1 \frac{L_{zW}(z)}{1 \pm z} dz, \quad t_{zW}^{\ell} = \int_{z_-}^1 \frac{L_{zW}(z)}{z} dz,
\end{aligned} \tag{39}$$

with

$$L_z(z) = \ln \left(\frac{1 - z\sqrt{\mu_1}}{(z - \sqrt{\mu_1})z} \right), \quad L_{zW}(z) = \ln \left(\frac{z - \sqrt{\mu_1}}{z(1 - z\sqrt{\mu_1})} \right) \tag{40}$$

and

$$\zeta_- = \frac{\Lambda\mu_2}{((1 - \sqrt{\mu_1})^2 - \mu_2)^2}, \quad z_{\pm} = \frac{1}{2\sqrt{\mu_1}}(1 + \mu_1 - \mu_2 \pm \sqrt{\lambda}), \quad \sqrt{\lambda} = \sqrt{\lambda(1, \mu_1, \mu_2)}. \tag{41}$$

The logarithmic terms ℓ_{ζ} , ℓ_0 , ℓ_+ , ℓ_1 and ℓ_{1W} are listed in Eq. (A2).

B. Helicity bilinears from the loop corrections

Again up to a general factor $e^2 q^2 |V_{cb}|^2 / s_W^2 \times \alpha / (4\pi\sqrt{\lambda})$, the NLO loop contributions read

$$\begin{aligned} H^{00}(\text{loop}) &= -\frac{1}{2}(-\mu_1 + \mu_1^2 - \mu_2 - 2\mu_1\mu_2 + \mu_2^2)\sqrt{\lambda}V_- \\ &\quad + \mu_1\mu_2\sqrt{\lambda}V_+ - \frac{1}{4}\lambda\sqrt{\lambda}(\mu_1V_1 + \mu_2V_2), \\ H^{++}(\text{loop}) &= \frac{1}{2}(1 - \mu_1 - \mu_2 - \sqrt{\lambda})\sqrt{\lambda}V_- + \mu_1\mu_2\sqrt{\lambda}V_+, \\ H^{+-}(\text{loop}) &= H^{-+}(\text{loop}) = 0, \\ H^{--}(\text{loop}) &= \frac{1}{2}(1 - \mu_1 - \mu_2 + \sqrt{\lambda})\sqrt{\lambda}V_- + \mu_1\mu_2\sqrt{\lambda}V_+. \end{aligned} \quad (42)$$

V. FULL RESULTS FOR THE HELICITY BILINEARS

Having used the Feynman gauge up to now, it is worth mentioning here that we have performed the calculation of the helicity bilinears for the tree corrections also in the unitary gauge. The unitary gauge used for the W propagator in Fig. 2.3 allows us to drop contributions from the charged Goldstone boson which is not indicated but implicitly assumed in this diagram. We found that the results for the helicity bilinears $H^{00}(\text{tree})$, $H^{\pm\pm}(\text{tree})$, and $H^{\pm\mp}(\text{tree})$ turn

out to be exactly the same. Both tree and loop contributions, however, contain IR singularities which have to cancel according to the Lee-Nauenberg theorem.

A. Counterterms for the IR singularities

In order to deal with the IR singularities in a consistent way, we need a convenient method to extract them. For the first order tree contributions we have seen that the IR singular parts are contained in ℓ_ζ , t_ζ^ℓ , and $t_{\zeta W}^\ell$. While ℓ_ζ itself, as defined in Eq. (A2), can be used as a counterterm, for the other two expressions in the first order tree contributions we use

$$t_\zeta^\ell = t_\zeta^{\ell*} + \ell_1 \ell_\zeta, \quad t_{\zeta W}^\ell = t_{\zeta W}^{\ell*} + \ell_{1W} \ell_\zeta, \quad (43)$$

where the starred quantities are found in Appendix A. On the other hand, the form factors contain IR singularities only in the three-point functions with the photon on one of the internal lines. The scalar three-point functions are given by

$$C(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3) = \frac{i}{(4\pi)^2} C_f(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3) \quad (44)$$

with [5]

$$C_f(p_1^2, p_2^2, p_3^2; m_1, m_2, m_3) = \frac{-1}{\sqrt{\lambda(p_1^2, p_2^2, p_3^2)}} \sum_{i=1}^3 \sum_{\pm} \left[\text{Li}_2\left(\frac{1 - y_{i0}}{y_{i\pm} \pm i\epsilon - y_{i0}}\right) - \text{Li}_2\left(\frac{-y_{i0}}{y_{i\pm} \pm i\epsilon - y_{i0}}\right) \right], \quad (45)$$

where

$$\begin{aligned} y_{10} &= \frac{(p_2^2 + m_2^2 - m_3^2)(\sqrt{\lambda'} + p_1^2 - p_2^2 - p_3^2) + 2p_2^2(p_3^2 - m_1^2 + m_2^2)}{2p_2^2\sqrt{\lambda'}}, \\ y_{20} &= \frac{(p_3^2 + m_1^2 - m_2^2)(\sqrt{\lambda'} - p_1^2 + p_2^2 - p_3^2) + 2p_3^2(p_1^2 - m_3^2 + m_1^2)}{2p_3^2\sqrt{\lambda'}}, \\ y_{30} &= \frac{(p_1^2 + m_3^2 - m_1^2)(\sqrt{\lambda'} - p_1^2 - p_2^2 + p_3^2) + 2p_1^2(p_2^2 - m_2^2 + m_3^2)}{2p_1^2\sqrt{\lambda'}} \end{aligned} \quad (46)$$

and

$$\begin{aligned} y_{1\pm} &= \frac{p_2^2 + m_2^2 - m_3^2 \pm \sqrt{\lambda(p_2^2, m_2^2, m_3^2)}}{2p_2^2}, \\ y_{2\pm} &= \frac{p_3^2 + m_1^2 - m_2^2 \pm \sqrt{\lambda(p_3^2, m_1^2, m_2^2)}}{2p_3^2}, \\ y_{3\pm} &= \frac{p_1^2 + m_3^2 - m_1^2 \pm \sqrt{\lambda(p_1^2, m_3^2, m_1^2)}}{2p_1^2}. \end{aligned} \quad (47)$$

Therefore, we again define starred quantities by

$$\begin{aligned}
C_f(m_1^2, m_2^2, m_W^2; m_A, m_W, m_1) &= C_f^*(m_1^2, m_2^2, m_W^2; m_A, m_W, m_1) + \frac{\ell_1 - \ell_{1W}}{2m_W^2\sqrt{\lambda}}\ell_\zeta, \\
C_f(m_1^2, m_2^2, m_W^2; m_W, m_A, m_2) &= C_f^*(m_1^2, m_2^2, m_W^2; m_W, m_A, m_2) + \frac{\ell_{1W}}{2m_W^2\sqrt{\lambda}}\ell_\zeta, \\
C_f(m_1^2, m_2^2, m_W^2; m_1, m_2, m_A) &= C_f^*(m_1^2, m_2^2, m_W^2; m_1, m_2, m_A) - \frac{\ell_1 + 2\pi i}{2m_W^2\sqrt{\lambda}}\ell_\zeta
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
C_f^*(m_1^2, m_2^2, m_W^2; m_A, m_W, m_1) &= -\frac{1}{m_W^2\sqrt{\lambda}} \left[\text{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) - \text{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \frac{1}{4}\ell_1^2 + \frac{1}{4}\ell_{1W}^2 - \frac{1}{2}\ln\left(\frac{\mu_1}{\lambda}\right)(\ell_1 - \ell_{1W}) \right], \\
C_f^*(m_1^2, m_2^2, m_W^2; m_W, m_A, m_2) &= -\frac{1}{m_W^2\sqrt{\lambda}} \left[\text{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) - \text{Li}_2\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \frac{1}{4}\ell_1^2 + \frac{1}{4}(\ell_1 - \ell_{1W})^2 - \frac{1}{2}\ln\left(\frac{\mu_2}{\lambda}\right)\ell_{1W} \right], \\
C_f^*(m_1^2, m_2^2, m_W^2; m_1, m_2, m_A) &= -\frac{1}{m_W^2\sqrt{\lambda}} \left[\text{Li}_2\left(\frac{1-\mu_1+\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) + \text{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) + \frac{2\pi^2}{3} + \frac{1}{4}(\ell_1 - \ell_{1W})^2 + \frac{1}{4}\ell_{1W}^2 + \frac{1}{2}\ln\left(\frac{\mu_1\mu_2}{\lambda}\right)(\ell_1 + 2\pi i) \right],
\end{aligned} \tag{49}$$

and inserted into the form factor V_- , one has

$$\begin{aligned}
\sqrt{\lambda}V_- &= \sqrt{\lambda}V_-^* + (Q_c^2 + Q_b^2 + Q_W^2)\sqrt{\lambda}\ell_\zeta + (1 + \mu_1 - \mu_2)Q_cQ_W(\ell_1 - \ell_{1W})\ell_\zeta \\
&\quad - (1 - \mu_1 + \mu_2)Q_bQ_W\ell_{1W}\ell_\zeta + (1 - \mu_1 - \mu_2)Q_cQ_b(\ell_1 + 2\pi i)\ell_\zeta,
\end{aligned} \tag{50}$$

where the starred form factor V_-^* contains the starred three-point functions while the other form factors remain unchanged. The term $(Q_c^2 + Q_b^2 + Q_W^2)\sqrt{\lambda}\ell_\zeta$ originates from the counterterms of the renormalization. Using the starred instead of the usual form factors, one has

$$\begin{aligned}
H^{00}(\text{loop}) &= H^{00*}(\text{loop}) - \frac{1}{2}(-\mu_1 + \mu_1^2 - \mu_2 - 2\mu_1\mu_2 + \mu_2^2)[(Q_c^2 + Q_b^2 + Q_W^2)\sqrt{\lambda} \\
&\quad + ((1 - \mu_2)Q_c^2 - \mu_1(Q_b^2 - Q_W^2))\ell_1 - (Q_c^2 - Q_b^2 + (\mu_1 - \mu_2)Q_W^2)\ell_{1W}]\ell_\zeta.
\end{aligned} \tag{51}$$

B. Analytic results for the $O(\alpha)$ helicity bilinears

Adding up first order tree and loop corrections for all five helicity bilinears under review, the IR singularities cancel and we obtain

$$\begin{aligned}
H^{00}(\alpha) &= -\frac{1}{2}(-\mu_1 + \mu_1^2 - \mu_2 - 2\mu_1\mu_2 + \mu_2^2)\sqrt{\lambda}V_-^* + \mu_1\mu_2\sqrt{\lambda}V_+ - \frac{1}{2}\lambda\sqrt{\lambda}(\mu_1V_1 + \mu_2V_2) \\
&\quad + \frac{1}{2}[(1 - \mu_2)Q_c^2 - \mu_1(Q_b^2 - Q_W^2)](-\mu_1 + \mu_1^2 - \mu_2 - 2\mu_1\mu_2 + \mu_2^2)(t_\zeta^{\ell*} - 2t_z^{\ell+}) \\
&\quad - \frac{1}{2}[Q_c^2 - Q_b^2 + (\mu_1 - \mu_2)Q_W^2](-\mu_1 + \mu_1^2 - \mu_2 - 2\mu_1\mu_2 + \mu_2^2)(t_{\zeta W}^{\ell*} + 2t_{z W}^{\ell+}) \\
&\quad - 2\mu_1[(1 + 5\mu_1 - \mu_2)Q_c^2 + 2\mu_1(Q_b^2 - Q_W^2)](t_z^{-\ell} - t_z^{+\ell} + t_z^{\ell})
\end{aligned}$$

$$\begin{aligned}
& -\sqrt{\mu_1}[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)](t_z^{-\ell}+t_z^{+\ell}) \\
& +2[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2) \\
& -(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2](t_{zW}^{-\ell}-t_{zW}^{+\ell}+t_{zW}^{\ell}) \\
& -\frac{1+\mu_1-\mu_2}{\sqrt{\mu_1}}[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-2(1+2\mu_1-\mu_2)Q_W^2](t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
& +\frac{1}{4}[(4-\mu_2)\lambda-\mu_1(17+16\mu_1-32\mu_2+\mu_1\mu_2+5\mu_2^2))Q_c^2 \\
& -((4-\mu_1+2\mu_2)\lambda+\mu_1(15+8\mu_2+5\mu_1\mu_2+\mu_2^2))Q_b^2 \\
& -2((4+\mu_1+\mu_2)\lambda+\mu_1(16+\mu_1+5\mu_2-\mu_1\mu_2-\mu_2^2))Q_W^2]\ell_1 \\
& -\frac{1}{4}[(2(2+\mu_1-\mu_2)\lambda-17\mu_1-16\mu_1^2+3\mu_2+20\mu_1\mu_2-4\mu_2^2)Q_c^2 \\
& -(2(2-\mu_1+\mu_2)\lambda+15\mu_1+3\mu_2+4\mu_1\mu_2-4\mu_2^2)Q_b^2 \\
& -2(4+8\mu_1+5\mu_1^2-12\mu_2-12\mu_1\mu_2+7\mu_2^2)Q_W^2]\ell_{1W} \\
& +\frac{1}{8}[(16-67\mu_1-15\mu_1^2-3\mu_2+38\mu_1\mu_2-11\mu_2^2)Q_c^2 \\
& -(8+11\mu_1+11\mu_1^2-21\mu_2-38\mu_1\mu_2+15\mu_2^2)Q_b^2 \\
& -2(24+\mu_1+9\mu_1^2-31\mu_2-14\mu_1\mu_2+9\mu_2^2)Q_W^2]\sqrt{\lambda}, \\
H^{++}(\alpha) = & \frac{1}{2}(1-\mu_1-\mu_2-\sqrt{\lambda})\sqrt{\lambda}V_-^*+\mu_1\mu_2\sqrt{\lambda}V_+-(Q_c^2+Q_b^2+Q_W^2)\lambda\ell_+ \\
& -\frac{1}{2}[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)]((1-\mu_1-\mu_2)(t_\zeta^{\ell*}-2t_z^{\ell+})-\sqrt{\lambda}(t_\zeta^{\ell*}-2t_z^{\ell-})) \\
& +\frac{1}{2}[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2]((1-\mu_1-\mu_2)(t_{\zeta W}^{\ell*}+2t_{zW}^{\ell+})-\sqrt{\lambda}(t_{\zeta W}^{\ell*}+2t_{zW}^{\ell-})) \\
& +[(1-2\mu_1-2\mu_2-\mu_1\mu_2+\mu_2^2)Q_c^2-\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)]t_z^{\ell} \\
& +\mu_1[(1+5\mu_1-\mu_2)Q_c^2+2\mu_1(Q_b^2-Q_W^2)](t_z^{-\ell}-t_z^{+\ell}+t_z^{\ell}) \\
& +\frac{1}{2}\sqrt{\mu_1}[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)](t_z^{-\ell}+t_z^{+\ell}) \\
& -[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2) \\
& -(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2](t_{zW}^{-\ell}-t_{zW}^{+\ell}+2t_{zW}^{\ell}) \\
& +\frac{1+\mu_1-\mu_2}{2\sqrt{\mu_1}}[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-2(1+2\mu_1-\mu_2)Q_W^2](t_{zW}^{-\ell}+t_{zW}^{+\ell}) \\
& +\frac{1}{2}[(1-\mu_1)(5-3\mu_1+4\mu_2)Q_c^2-(9-10\mu_1+\mu_1^2+6\mu_2-2\mu_1\mu_2)Q_b^2 \\
& -2(1-\mu_1)(5-\mu_1+\mu_2)Q_W^2]\ell_0 \\
& -\frac{1}{4}[(3-8\mu_1-4\mu_1^2-6\mu_2+10\mu_1\mu_2+3\mu_2^2+(1+5\mu_1-\mu_2)\sqrt{\lambda})Q_c^2 \\
& -\mu_1(4+7\mu_1-4\mu_2)Q_b^2-2(1+5\mu_1-\mu_2)(1-\mu_2)Q_W^2]\ell_1 \\
& +\frac{1}{4}[(1-4\mu_1-6\mu_1^2-4\mu_2+6\mu_2^2+8\sqrt{\lambda})Q_c^2 \\
& -(1+2\mu_1+8\mu_1^2+6\mu_2-8\mu_1\mu_2+8\sqrt{\lambda})Q_b^2 \\
& -2(2+3\mu_1+\mu_1^2-5\mu_2-6\mu_1\mu_2+5\mu_2^2+4\sqrt{\lambda})Q_W^2]\ell_{1W}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}[(17+7\mu_1-8\mu_2)Q_c^2-(13-3\mu_1+2\mu_2)Q_b^2-6(3+\mu_1-\mu_2)Q_W^2]\lambda_- \\
& +\frac{1}{24}[3(3+15\mu_1+3\mu_2+\sqrt{\lambda})Q_c^2+3(15+7\mu_1-29\mu_2-9\sqrt{\lambda})Q_b^2 \\
& +4(29-10\mu_1-\mu_1^2-34\mu_2+2\mu_1\mu_2-\mu_2^2-9\sqrt{\lambda})Q_W^2]\sqrt{\lambda}, \\
H^{+-}(\alpha) = H^{-+}(\alpha) = & -\mu_1[(1+5\mu_1-\mu_2)Q_c^2+2\mu_1(Q_b^2-Q_W^2)](t_z^{-\ell}-t_z^{+\ell}+t_z^{\ell}) \\
& -\frac{1}{2}\sqrt{\mu_1}[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)](t_z^{-\ell}+t_z^{+\ell}) \\
& +[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2) \\
& -(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2](t_z^{-\ell}-t_z^{+\ell}+t_z^{\ell}) \\
& -\frac{1+\mu_1-\mu_2}{2\sqrt{\mu_1}}[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-2(1+2\mu_1-\mu_2)Q_W^2](t_z^{-\ell}+t_z^{+\ell}) \\
& +\frac{1}{2}[(1-6\mu_1-3\mu_1^2-2\mu_2+6\mu_1\mu_2+\mu_2^2)Q_c^2+ \\
& -(1+\mu_1+2\mu_1^2-2\mu_2-\mu_1\mu_2+\mu_2^2)Q_b^2-2(1+\mu_1-\mu_2)^2Q_W^2]\ell_1 \\
& -\frac{1}{2}[(1-6\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -(1+\mu_1+2\mu_1^2-\mu_2-2\mu_1\mu_2)Q_b^2-2(1+\mu_1-2\mu_2)(1+\mu_1-\mu_2)Q_W^2]\ell_{1W} \\
& +\frac{1}{2}[2(1-5\mu_1-\mu_2)Q_c^2-(1+3\mu_1-\mu_2)Q_b^2-2(3+\mu_1-3\mu_2)Q_W^2]\sqrt{\lambda}, \\
H^{--}(\alpha) = & \frac{1}{2}(1-\mu_1-\mu_2+\sqrt{\lambda})\sqrt{\lambda}V_-^*+\mu_1\mu_2\sqrt{\lambda}V_++(Q_c^2+Q_b^2+Q_W^2)\lambda\ell_+ \\
& -\frac{1}{2}[(1-\mu_2)Q_c^2-\mu_1(Q_b^2-Q_W^2)]((1-\mu_1-\mu_2)(t_z^{\ell*}-2t_z^{\ell+})+\sqrt{\lambda}(t_z^{\ell*}-2t_z^{\ell-})) \\
& +\frac{1}{2}[Q_c^2-Q_b^2+(\mu_1-\mu_2)Q_W^2]((1-\mu_1-\mu_2)(t_{zW}^{\ell*}+2t_{zW}^{\ell+})+\sqrt{\lambda}(t_{zW}^{\ell*}+2t_{zW}^{\ell-})) \\
& -[(1-2\mu_1-2\mu_2-\mu_1\mu_2+\mu_2^2)Q_c^2-\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)]t_z^{\ell} \\
& +\mu_1[(1+5\mu_1-\mu_2)Q_c^2+2\mu_1(Q_b^2-Q_W^2)](t_z^{-\ell}-t_z^{+\ell}+t_z^{\ell}) \\
& +\frac{1}{2}\sqrt{\mu_1}[(1-10\mu_1-3\mu_1^2-2\mu_2+2\mu_1\mu_2+\mu_2^2)Q_c^2 \\
& -2\mu_1(1+\mu_1-\mu_2)(Q_b^2-Q_W^2)](t_z^{-\ell}+t_z^{+\ell}) \\
& -[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2) \\
& -(3+2\mu_1+\mu_1^2-4\mu_2-2\mu_1\mu_2+\mu_2^2)Q_W^2](t_z^{-\ell}-t_z^{+\ell}) \\
& +\frac{1+\mu_1-\mu_2}{2\sqrt{\mu_1}}[(1+\mu_1-\mu_2)(Q_c^2-Q_b^2)-2(1+2\mu_1-\mu_2)Q_W^2](t_z^{-\ell}+t_z^{+\ell}) \\
& -\frac{1}{2}[(1-\mu_1)(5-3\mu_1+4\mu_2)Q_c^2-(9-10\mu_1+\mu_1^2+6\mu_2-2\mu_1\mu_2)Q_b^2 \\
& -2(1-\mu_1)(5-\mu_1+\mu_2)Q_W^2]\ell_0 \\
& -\frac{1}{4}[(3-8\mu_1-4\mu_1^2-6\mu_2+10\mu_1\mu_2+3\mu_2^2-(1+5\mu_1-\mu_2)\sqrt{\lambda})Q_c^2 \\
& -\mu_1(4+7\mu_1-4\mu_2)Q_b^2-2(1+5\mu_1-\mu_2)(1-\mu_2)Q_W^2]\ell_1 \\
& +\frac{1}{4}[(1-4\mu_1-6\mu_1^2-4\mu_2+6\mu_2^2-8\sqrt{\lambda})Q_c^2
\end{aligned}$$

$$\begin{aligned}
& - (1 + 2\mu_1 + 8\mu_1^2 + 6\mu_2 - 8\mu_1\mu_2 - 8\sqrt{\lambda})Q_b^2 \\
& - 2(2 + 3\mu_1 + \mu_1^2 - 5\mu_2 - 6\mu_1\mu_2 + 5\mu_2^2 - 4\sqrt{\lambda})Q_W^2\ell_{1W} \\
& + \frac{1}{4}[(17 + 7\mu_1 - 8\mu_2)Q_c^2 - (13 - 3\mu_1 + 2\mu_2)Q_b^2 - 6(3 + \mu_1 - \mu_2)Q_W^2]\lambda_- \\
& + \frac{1}{24}[3(3 + 15\mu_1 + 3\mu_2 - \sqrt{\lambda})Q_c^2 + 3(15 + 7\mu_1 - 29\mu_2 + 9\sqrt{\lambda})Q_b^2 \\
& + 4(29 - 10\mu_1 - \mu_1^2 - 34\mu_2 + 2\mu_1\mu_2 - \mu_2^2 + 9\sqrt{\lambda})Q_W^2]\sqrt{\lambda}.
\end{aligned} \tag{52}$$

VI. DISCUSSION

In general, electroweak corrections are assumed to be small. However, the size of the correction depends strongly on the subtraction scheme used. Using the $\alpha(0)$ scheme as the canonical choice from the perturbation series is not an option for the decay of a massive W boson, as the scale of the problem is far from being zero. Instead, one might consider the $\alpha(m_W^2)$ scheme with the renormalization scale fixed to the mass of the W boson. As argued in Ref. [45], however, the best choice is the G_μ scheme (traditionally known as the G_F scheme related to the Fermi constant as the best measured quantity in the process). This scheme and the consequences of its application are discussed in Appendix B.1. Frankly speaking, using the latter two schemes, large logarithms from light fermion loops are removed from the result which then is indeed a small correction to the Born-term result.

The effect of the different schemes is displayed in the angular distributions shown in Fig. 5. The numerical analysis is done by employing the values $m_W = 80.3962(133)$ GeV, $m_Z = 91.1880(20)$ GeV, $m_H = 125.20(11)$ GeV, $m_c = 1.2730(46)$ GeV, $m_b = 4.183(7)$ GeV, $m_t = 172.57(29)$ GeV, together with values for the lighter leptons taken from Ref. [47]. For the light quarks we use the values $m_u \approx m_d = 0.06$ GeV and $m_s = 0.17$ GeV obtained by fitting these masses to the hadronic vacuum polarization (cf. Appendix D). As it turns out, in the $\alpha(m_W^2)$ scheme the electroweak correction is of the order of 1.9% while for the other schemes the corrections are larger, 4.7% and 5.2% for the G_μ and $\alpha(0)$ schemes, respectively.¹ In the second column of Table I we give the helicity bilinears for the different schemes.

On the other hand, we can investigate the dependence on the masses. For this, the collinear limit $m_c, m_b \rightarrow 0$ can be taken, as the collinear singularities between tree and loop contributions were not only implicitly canceled but explicitly truncated on both sides in our approach.

¹Note that, different from Refs. [27–29], we have chosen the positive z axis in the quark frame as the direction of motion of the quark. A different convention means an interchange $\theta \leftrightarrow -\theta$, i.e., the diagram for the angular distribution will be mirrored at the vertical axis.

The last column shows the results in the collinear limit where all fermion masses except for the top quark mass are set to zero. The collinear limit is used also for the observables in Table II. Finally, in Fig. 6 we have compared the angular distribution for the LO Born-term result and for the NLO result in the G_μ scheme with the results in the collinear limit in order to study mass effects. While mass corrections for the Born-term results are difficult to discern and are of the order -0.2% , mass effects in the G_μ scheme amount to -2.3% , with the mass effects for the $\alpha(m_W^2)$ and $\alpha(0)$ schemes given by -2.4% and -2.2% , respectively. At the same time, we see that in the collinear limit the electroweak correction is larger and amounts to 7.0% , 4.2% , and 7.4% for the G_μ , $\alpha(m_W^2)$ and $\alpha(0)$ schemes, respectively. This is also echoed in Table II. Vice versa, it seems that taking exact fermion masses, the differences due to the choice of the scheme diminish.

Up to a general factor of $e^2 q^2 |V_{cb}|^2 / s_W^2$, the Born-term results in the collinear limit are given by $H^{00}(\text{Born}) = H^{++}(\text{Born}) = H^{+-}(\text{Born}) = H^{-+}(\text{Born}) = 0$, while the only nonvanishing amplitude is $H^{--}(\text{Born}) = 1$. On the other hand, up to an additional general factor of $\alpha/(4\pi)$, the analytic expressions for the NLO EW corrections to the helicity bilinears in the collinear limit for the G_μ scheme read

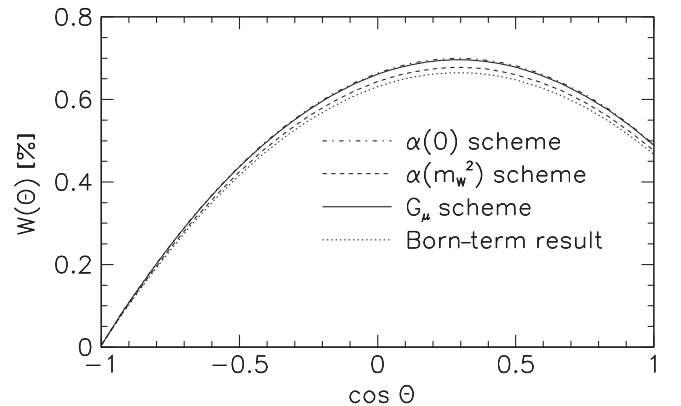


FIG. 5. Angular distribution in dependence on the subtraction scheme employed.

TABLE I. Helicity bilinears for the Born-term contribution and the electroweak radiative corrections. Shown are numerical values for the exact fermion masses, and for the fermion masses except for the top quark mass set to zero, i.e., in the collinear limit.

Scheme	Bilinear	m_f	$m_f = 0, f \neq t$
LO	H^{00}	0.00148	0
	H^{++}	7×10^{-7}	0
	H^{+-}	0	0
	H^{--}	0.99704	1
$\alpha(0)$	H^{00}	+0.00019	+0.00206
	H^{++}	+0.00003	-0.00094
	H^{+-}	+0.00005	+0.00103
	H^{--}	+0.05195	+0.07333
$\alpha(m_W^2)$	H^{00}	+0.00015	+0.00222
	H^{++}	+0.00003	-0.00101
	H^{+-}	+0.00006	+0.00111
	H^{--}	+0.01911	+0.04208
G_μ	H^{00}	+0.00018	+0.00214
	H^{++}	+0.00003	-0.00097
	H^{+-}	+0.00006	+0.00107
	H^{--}	+0.04701	+0.06916

$$\begin{aligned}
H^{00}(\alpha) &= 2Q_c^2 - Q_b^2 - 6Q_W^2 + \frac{2\pi^2}{3} Q_c Q_W - \frac{4\pi^2}{3} Q_b Q_W, \\
H^{++}(\alpha) &= -\frac{15}{4} Q_c^2 + 4Q_b^2 + \frac{47}{6} Q_W^2 - \frac{\pi^2}{2} Q_c Q_W + \frac{3\pi^2}{2} Q_b Q_W, \\
H^{+-}(\alpha) &= H^{-+}(\alpha) = Q_c^2 - \frac{1}{2} Q_b^2 - 3Q_W^2 + \frac{\pi^2}{3} Q_c Q_W - \frac{2\pi^2}{3} Q_b Q_W, \\
H^{--}(\alpha) &= \frac{5}{2} Q_c^2 - \frac{9}{4} Q_b^2 + \frac{11}{6} Q_W^2 - \frac{5\pi^2}{6} Q_c Q_W + \frac{\pi^2}{2} Q_b Q_W \\
&\quad - 2g_c^- g_b^- \left(\frac{m_W^2 + m_Z^2}{m_W^2} \right)^2 \left[\text{Li}_2 \left(-\frac{m_W^2}{m_Z^2} \right) + \ln \left(1 + \frac{m_W^2}{m_Z^2} \right) \ln \left(-\frac{m_W^2}{m_Z^2} \right) \right] \\
&\quad + 2g_c^- \frac{m_W^2 + 2m_Z^2}{m_Z m_W s_W} \left[\text{Li}_2 \left(\frac{2}{1 - \sqrt{1 - 4m_W^2/m_Z^2}} \right) + \text{Li}_2 \left(\frac{2}{1 + \sqrt{1 - 4m_W^2/m_Z^2}} \right) \right. \\
&\quad \left. - \text{Li}_2 \left(\frac{2(1 - m_W^2/m_Z^2)}{1 - \sqrt{1 - 4m_W^2/m_Z^2}} \right) - \text{Li}_2 \left(\frac{2(1 - m_W^2/m_Z^2)}{1 + \sqrt{1 - 4m_W^2/m_Z^2}} \right) \right] \\
&\quad - 2g_b^- \frac{m_W^2 + 2m_Z^2}{m_Z m_W s_W} \left[\text{Li}_2 \left(\frac{2m_W^2/m_Z^2}{1 - \sqrt{1 - 4m_W^2/m_Z^2}} \right) + \text{Li}_2 \left(\frac{2m_W^2/m_Z^2}{1 + \sqrt{1 - 4m_W^2/m_Z^2}} \right) \right] \\
&\quad + \frac{m_Z^2}{2m_W^4} (1 - 2Q_c s_W^2 + Q_b s_W^2 - 4Q_c Q_b s_W^2) \left[(2m_W^2 + m_Z^2) \left(\ln \left(\frac{m_Z^2}{m_W^2} \right) + 1 \right) \right] \\
&\quad + \frac{4m_W^2 - m_Z^2}{2m_W^2 s_W^2} + \frac{1}{4s_W^2} + \frac{g_c^{-2} + g_b^{-2}}{2} \left(\ln \left(\frac{m_Z^2}{m_W^2} \right) + \frac{1}{2} \right) \\
&\quad - \frac{m_t^2 |V_{tb}|^2}{16(m_t^2 - m_W^2)^2 m_W^2 s_W^2} \left[3(m_t^4 - m_W^4) - 2m_t^2(m_t^2 + 2m_W^2) \ln \left(\frac{m_t^2}{m_W^2} \right) \right] \\
&\quad + \frac{48m_W^{10} + 20m_W^8 m_Z^2 - 320m_W^6 m_Z^4 + 249m_W^4 m_Z^6 - 30m_W^2 m_Z^8 - 3m_Z^{10}}{48m_Z^4 m_W^6 s_W^4} \ln \left(\frac{m_Z^2}{m_W^2} \right)
\end{aligned}$$

TABLE II. Numerical values for the three observables c_f , A_{FB} , and $\cos \theta_{\text{extr}}$ to leading order and for three next-to-leading order schemes with exact fermion masses and in the collinear limit.

Scheme	Case	c_f	A_{FB}	$\cos \theta_{\text{extr}}$
LO	$m_f \neq 0$	-0.791269	+0.231288	+0.292300
	$m_f = 0$	-0.795975	+0.231975	+0.291435
$\alpha(0)$	$m_f \neq 0$	-0.832342	+0.243333	+0.292347
	$m_f = 0$	-0.850489	+0.249258	+0.293076
$\alpha(m_W^2)$	$m_f \neq 0$	-0.806268	+0.235714	+0.292352
	$m_f = 0$	-0.825215	+0.242002	+0.293260
G_μ	$m_f \neq 0$	-0.828419	+0.242187	+0.292349
	$m_f = 0$	-0.847012	+0.248296	+0.293143

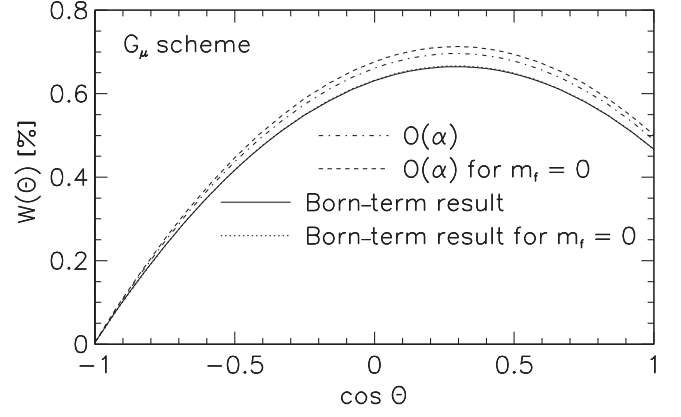


FIG. 6. Mass dependence of the angular distribution in the G_μ scheme.

$$\begin{aligned}
& + \frac{m_t^2(5m_t^4 - 9m_W^4)}{12m_W^6 s_W^2} \ln\left(\frac{m_t^2}{m_W^2}\right) - \frac{(m_t^2 - m_W^2)(5m_t^4 + 5m_t^2 m_W^2 - 4m_W^4)}{12m_W^6 s_W^2} \ln\left(\frac{m_t^2}{m_W^2} - 1\right) \\
& - \frac{3m_H^8 - 18m_H^6 m_W^2 + 51m_H^4 m_W^4 - 64m_H^2 m_W^6 + 12m_W^8}{48(m_H^2 - m_W^2)m_W^6 s_W^2} \ln\left(\frac{m_H^2}{m_W^2}\right) \\
& + \frac{28m_W^6 + 52m_W^4 m_Z^2 - 13m_W^2 m_Z^4 - m_Z^6}{4m_Z^2 m_W^6 s_W^2} \sqrt{4m_W^2 m_Z^2 - m_Z^4} \arctan\left(\sqrt{\frac{2m_W - m_Z}{2m_W + m_Z}}\right) \\
& + \frac{m_H^6 - 7m_H^4 m_W^2 + 20m_H^2 m_W^4 - 28m_W^6}{4(4m_W^2 - m_H^2)m_W^6 s_W^2} \sqrt{4m_H^2 m_W^2 - m_H^4} \arctan\left(\sqrt{\frac{2m_W - m_H}{2m_W + m_H}}\right) \\
& + \frac{3m_H^4 - 10m_t^4 - 10m_H^2 m_W^2 - 8m_t^2 m_W^2 - 184m_W^4 + 38m_W^2 m_Z^2 + 3m_Z^4}{24m_W^4 s_W^2} \\
& + \frac{3V_{tb}^2 m_t^2}{16m_W^2 s_W^2} \left[\frac{m_t^2 + m_W^2}{m_t^2 - m_W^2} - \frac{2m_t^2(m_t^2 + 2m_W^2)}{3(m_t^2 - m_W^2)^2} \ln\left(\frac{m_t^2}{m_W^2}\right) \right]. \tag{53}
\end{aligned}$$

Note that only $H^{--}(\alpha)$ depends on the loop corrections and the corrections due to the renormalization factors and, therefore, on the chosen subtraction scheme.

The considerations done so far are instructive to compare the effect of first order radiative corrections and final state mass effects with the Born-term result in the collinear limit. Note, however, that neither $W(\theta)$ nor the helicity bilinears are observables. Observables are based on the differential decay rate given by

$$\begin{aligned}
\frac{d\Gamma}{d\cos\theta} &= \Gamma_0^0 \sum_{m=0,\pm 1} \rho_{mm} H^{mm}(\theta) = \Gamma_0^0 \sum_{m,m'=0,\pm 1} \rho_{mm} d_{mm'}^1(\theta) d_{mm'}^1(\theta) H^{m'm'} \\
&= \Gamma_0^0 \left[\frac{3}{8} (1 + \cos^2\theta) ((\rho_{++} + \rho_{--})(H^{++} + H^{--}) + 2\rho_{00}H^{00}) \right. \\
&\quad + \frac{3}{4} \cos\theta ((\rho_{++} - \rho_{--})(H^{++} - H^{--})) \\
&\quad \left. + \frac{3}{4} \sin^2\theta ((\rho_{++} + \rho_{--})H^{00} + \rho_{00}(H^{++} + H^{--} - H^{00})) \right], \tag{54}
\end{aligned}$$

where

$$\Gamma_0^0 = \frac{G_\mu m_W^3}{6\pi\sqrt{2}} N_c |V_{ij}|^2 = \frac{\alpha m_W}{12s_W^2} N_c |V_{ij}|^2 \tag{55}$$

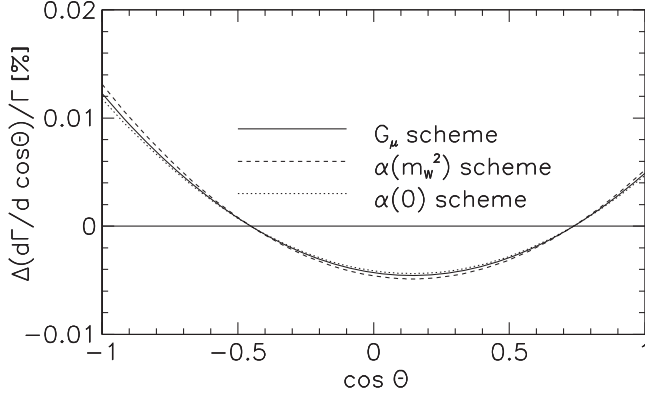


FIG. 7. Differences between the normalized differential decay rate $\hat{\Gamma} = (d\Gamma/d\cos\theta)/\Gamma$ to first order for the different schemes and the Born result.

is the Born-term rate (11) in the collinear limit into quark flavours i and j . In Fig. 7 we show the differences between the normalized decay rates $\hat{\Gamma} = (d\Gamma/d\cos\theta)/\Gamma$ to first order for the different schemes and the Born-term result for the same quantity.

Three observables characterizing the angular distribution are considered. These observables are the maximal curvature of the angular distribution, called the convexity parameter, given by

$$c_f = \frac{d^2\hat{\Gamma}}{d(\cos\theta)^2} = \frac{3}{4}(\rho_{++} - 2\rho_{00} + \rho_{--}) \times \left(\frac{H^{++} - 2H^{00} + H^{--}}{H^{++} + H^{00} + H^{--}} \right), \quad (56)$$

and the forward-backward asymmetry of the decay distribution defined by

$$A_{FB} = \frac{\hat{\Gamma}(F) - \hat{\Gamma}(B)}{\hat{\Gamma}(F) + \hat{\Gamma}(B)} = \frac{3}{4}(\rho_{++} - \rho_{--}) \left(\frac{H^{++} - H^{--}}{H^{++} + H^{00} + H^{--}} \right) \quad (57)$$

with $\hat{\Gamma}(F) = \hat{\Gamma}(0 \leq \theta \leq \pi/2)$ and $\hat{\Gamma}(B) = \hat{\Gamma}(\pi/2 \leq \theta \leq \pi)$, and the position of the extremum

$$\begin{aligned} \cos\theta|_{\text{extr}} &= -\frac{A_{FB}}{c_f} \\ &= -\frac{(\rho_{++} - \rho_{--})}{(\rho_{++} - 2\rho_{00} + \rho_{--})} \left(\frac{H^{++} - H^{--}}{H^{++} - 2H^{00} + H^{--}} \right). \end{aligned} \quad (58)$$

The values for these observables are found in the first lines in the scheme blocks of Table II. The percentage values for the radiatively corrected to the Born-term normalized decay rate are smaller, for $\alpha(0)$, $\alpha(m_W^2)$, and G_μ schemes given by

+0.006%, -0.007%, and -0.006%, respectively, while the corresponding values in the collinear limit are an order of magnitude higher, -0.09%, -0.07%, and -0.10%, respectively. The mass correction in the Born-term case amount of -0.08%, while the mass corrections at first order are -0.0003%, +0.003%, and +0.003%, and thus invisible.

VII. CONCLUSIONS

In this publication we have given analytic results for the NLO electroweak corrections to the decay process $W^+(\uparrow) \rightarrow c\bar{b}$ of a polarized W boson into quarks, taking into account all fermion and boson masses contributing to this process. We have estimated the size of the corrections and shown the dependence on the subtraction scheme of the renormalization procedure. Finally, we have performed the collinear limit in order to estimate the influence of mass effects to this decay process.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: ANALYTICAL EXPRESSIONS FOR THE TREE CORRECTIONS

The integrals necessary for the tree corrections are of the form

$$I(n_1, n_2; \Lambda) = \int_{y_{20}}^{y_{2-}} dy_2 \int_{y_{1-}}^{y_{1+}} dy_1 \frac{y_1^{n_1} y_2^{n_2}}{(y_1 + y_2)^2}. \quad (A1)$$

While the integrals are regular for $n_1 + n_2 > 0$ and can be calculated taking $\Lambda = 0$, the integrals at the border line $n_1 + n_2 = 0$ are IR singular. After performing the integration over y_1 , the integration over y_2 cannot be calculated analytically for a general parameter Λ in a closed form. Instead, we split the integral up into a divergent part $D(n_1, n_2; \Lambda)$ and a convergent part $C(n_1, n_2)$, where the divergent part is given by $I(n_1, n_2; \Lambda)$ with the integrand approximated for small values of y_2 , while the convergent part is given by the limit $\Lambda \rightarrow 0$ of the difference $I(n_1, n_2; \Lambda) - D(n_1, n_2; \Lambda)$ that is finite by construction. In technical terms, for the calculation of the divergent part in terms of a couple of elementary ζ integrals, we use the substitution $y_2 = \Lambda + \sqrt{\Lambda\mu_2}(1 + \zeta)/\sqrt{\zeta}$ to perform the integral over ζ between $\zeta_- = \Lambda\mu_2/((1 - \sqrt{\mu_1})^2 - \mu_2)^2$ to 1. After having performed this substitution, all occurrences of Λ in the (simplified) integrand can be neglected and the

integral can be performed. The simplified integrand is then transformed back to y_2 and subtracted from the full integrand and integrated in the limit $\Lambda \rightarrow 0$ by using the substitution $y_2 = 1 + \mu_1 - \mu_2 - \sqrt{\mu_1}(z + 1/z)$ to obtain the convergent part again in terms of a couple of z integrals. The corresponding limits are given by $z_- = (1 + \mu_1 - \mu_2 - \sqrt{\lambda(1, \mu_1, \mu_2)})/(2\sqrt{\mu_1})$ and 1. A large portion of the ζ and z integrals can be expressed in terms of rational functions and logarithms, containing

$$\begin{aligned} \ell_\zeta &= \ln\left(\frac{\lambda^2}{\Lambda\mu_1\mu_2}\right), & \ell_0 &= \ln\left(\frac{1 - \sqrt{\mu_1}}{\sqrt{\mu_2}}\right), & \ell_+ &= \ln\left(\frac{(1 + \sqrt{\mu_1})^2 - \mu_2}{\sqrt{\mu_1}}\right), \\ \ell_1 &= \ln\left(\frac{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}\right), & \ell_{1W} &= \ln\left(\frac{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}\right). \end{aligned} \quad (\text{A2})$$

Exceptional are integrals containing a logarithm together with ζ or z to the power of -1 . These integrals contain dilogarithms and are kept as closed form terms t_ζ and t_z , the analytic expressions for these found in the following. The ζ -terms

$$\begin{aligned} t_\zeta^{\ell*} &= \text{Li}_2\left(-\frac{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}\right) - \text{Li}_2\left(-\frac{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}\right) \\ &\quad - 2\text{Li}_2\left(\frac{\sqrt{\lambda}}{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}\right) + 2\text{Li}_2\left(\frac{-\sqrt{\lambda}}{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}\right) \\ &\quad + 2\text{Li}_2\left(\frac{2\sqrt{\lambda}}{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}\right) - 2\text{Li}_2\left(\frac{-2\sqrt{\lambda}}{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}\right) \\ &\quad + \ell_1 \ln\left(\frac{4\mu_1((1 - \sqrt{\mu_1})^2 - \mu_2)^2}{\lambda^2}\right), \\ t_{\zeta W}^{\ell*} &= \text{Li}_2\left(-\frac{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}\right) - \text{Li}_2\left(-\frac{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}\right) \\ &\quad - 2\text{Li}_2\left(\frac{\sqrt{\lambda}}{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}\right) + 2\text{Li}_2\left(\frac{-\sqrt{\lambda}}{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}\right) \\ &\quad + 2\text{Li}_2\left(\frac{2\sqrt{\lambda}}{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}\right) - 2\text{Li}_2\left(\frac{-2\sqrt{\lambda}}{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}\right) \\ &\quad + \ell_{1W} \ln\left(\frac{4\mu_1((1 - \sqrt{\mu_1})^2 - \mu_2)^2}{\lambda^2}\right) \end{aligned} \quad (\text{A3})$$

are IR subtracted, cf. the discussion in Sec. [VA](#). The remaining z -terms read

$$\begin{aligned} t_z^{\ell-} &= \text{Li}_2\left(-\frac{(1 - \sqrt{\mu_1})^2 - \mu_2 + \sqrt{\lambda}}{2\sqrt{\mu_1}}\right) - \text{Li}_2\left(-\frac{(1 - \sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{2\sqrt{\mu_1}}\right) \\ &\quad + \text{Li}_2\left(\frac{2\sqrt{\lambda}}{1 + \mu_1 - \mu_2 + \sqrt{\lambda}}\right) + \text{Li}_2\left(\frac{2\sqrt{\lambda}}{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}\right) - \text{Li}_2\left(\frac{-2\sqrt{\lambda}}{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}\right) \\ &\quad + \text{Li}_2\left(\frac{(1 - \sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}\right) - \text{Li}_2\left(\frac{(1 - \sqrt{\mu_1})^2 - \mu_2 + \sqrt{\lambda}}{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}\right) \\ &\quad + \text{Li}_2\left(-\frac{(1 - \sqrt{\mu_1})^2 - \mu_2 + \sqrt{\lambda}}{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}\right) - \text{Li}_2\left(-\frac{(1 - \sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}\right) \\ &\quad + \ell_1 \ln\left(\frac{(1 + \sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{(1 + \sqrt{\mu_1})^2 - \mu_2 + \sqrt{\lambda}}\right), \end{aligned} \quad (\text{A4})$$

$$\begin{aligned}
t_z^{\ell+} = & \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right) + \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{2\sqrt{\mu_1}}\right) + 2\text{Li}_2(\sqrt{\mu_1}) \\
& + \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right) + \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
& - \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) - \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) \\
& - \text{Li}_2\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \text{Li}_2\left(\frac{-2\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) - \text{Li}_2\left(\frac{2\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
& - \text{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2}\right) - \text{Li}_2\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{2}\right) \\
& - \ell_1 \ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{\sqrt{\mu_1}}\right) - \ln^2\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{(1+\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}\right), \tag{A5}
\end{aligned}$$

$$\begin{aligned}
t_z^{\ell-} = & \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right) - \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right) \\
& + \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{2(1-\sqrt{\mu_1})\sqrt{\mu_1}}\right), \tag{A6}
\end{aligned}$$

$$\begin{aligned}
t_z^{+\ell} = & -\text{Li}_2\left(\frac{1-\mu_1+\mu_2+\sqrt{\lambda}}{2(1+\sqrt{\mu_1})}\right) + \text{Li}_2\left(-\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}(1+\sqrt{\mu_1})}\right) + \text{Li}_2\left(-\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2\sqrt{\mu_1}}\right) \\
& - \text{Li}_2\left(-\frac{1-\sqrt{\mu_1}}{1+\sqrt{\mu_1}}\right) + \text{Li}_2\left(\frac{1-\sqrt{\mu_1}}{1+\sqrt{\mu_1}}\right) - \text{Li}_2(-1) \\
& + \frac{1}{2}\ell_1 \ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2}{\sqrt{\mu_1}(1+\sqrt{\mu_1})}\right) - \frac{1}{2}\ell_{1W} \ln\left(\frac{1+\sqrt{\mu_1}}{\sqrt{\mu_1}}\right) \\
& + \frac{1}{4}\ln\left(\frac{1-\mu_1-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) \ln\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) + \ln(\sqrt{\mu_1}) \ln\left(\frac{1-\sqrt{\mu_1}}{\sqrt{\mu_2}}\right), \tag{A7}
\end{aligned}$$

$$\begin{aligned}
t_z^{\ell} = & \text{Li}_2\left(\frac{1+\mu_1-\mu_2+\sqrt{\lambda}}{2}\right) + \text{Li}_2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{2}\right) - 2\text{Li}_2(\sqrt{\mu_1}) \\
& + \frac{1}{4}\ln^2\left(\frac{1+\mu_1-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right), \tag{A8}
\end{aligned}$$

$$\begin{aligned}
t_{zW}^{\ell-} = & \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) - \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1-\mu_2-\sqrt{\lambda}}\right) \\
& - \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) + \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1+\mu_1-\mu_2-\sqrt{\lambda}}\right) \\
& + \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{1-\mu_1+\mu_2+\sqrt{\lambda}}\right) - \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) \\
& - \text{Li}_2\left(\frac{2\sqrt{\lambda}}{1-\mu_1-\mu_2+\sqrt{\lambda}}\right) + \text{Li}_2\left(\frac{-2\sqrt{\lambda}}{1-\mu_1+\mu_2-\sqrt{\lambda}}\right) + \text{Li}_2\left(\frac{2\sqrt{\lambda}}{1+\mu_1-\mu_2+\sqrt{\lambda}}\right) \\
& - \ell_{1W} \ln\left(\frac{(1+\sqrt{\mu_1})^2-\mu_2-\sqrt{\lambda}}{(1+\sqrt{\mu_1})^2-\mu_2+\sqrt{\lambda}}\right), \tag{A9}
\end{aligned}$$

$$\begin{aligned}
t_{zW}^{\ell+} = & \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{1 + \mu_1 - \mu_2 - \sqrt{\lambda}}\right) + \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2 - \mu_2 + \sqrt{\lambda}}{1 + \mu_1 - \mu_2 + \sqrt{\lambda}}\right) - 2\text{Li}_2(\sqrt{\mu_1}) \\
& - \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}\right) - \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2 - \mu_2 + \sqrt{\lambda}}{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}\right) \\
& + \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}\right) + \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2 - \mu_2 + \sqrt{\lambda}}{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}\right) \\
& + \text{Li}_2\left(\frac{2\sqrt{\lambda}}{1 - \mu_1 - \mu_2 + \sqrt{\lambda}}\right) - \text{Li}_2\left(\frac{-2\sqrt{\lambda}}{1 - \mu_1 + \mu_2 - \sqrt{\lambda}}\right) - \text{Li}_2\left(\frac{2\sqrt{\lambda}}{1 + \mu_1 - \mu_2 + \sqrt{\lambda}}\right) \\
& + \text{Li}_2\left(\frac{1 + \mu_1 - \mu_2 - \sqrt{\lambda}}{2}\right) + \text{Li}_2\left(\frac{1 + \mu_1 - \mu_2 + \sqrt{\lambda}}{2}\right) + \ell_{1W} \ln\left(\frac{(1 + \sqrt{\mu_1})^2 - \mu_2}{\sqrt{\mu_1}}\right), \tag{A10}
\end{aligned}$$

$$\begin{aligned}
t_{zW}^{\ell-} = & \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{2\sqrt{\mu_1}}\right) + \text{Li}_2\left(\frac{(1-\sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{2(1-\sqrt{\mu_1})}\right) \\
& + \text{Li}_2\left(-\frac{(1-\sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{2\sqrt{\mu_1}(1-\sqrt{\mu_1})}\right), \tag{A11}
\end{aligned}$$

$$\begin{aligned}
t_{zW}^{\ell+} = & \text{Li}_2\left(-\frac{1 + \mu_1 - \mu_2 - \sqrt{\lambda}}{2\sqrt{\mu_1}}\right) + \text{Li}_2\left(\frac{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}{2(1 + \sqrt{\mu_1})}\right) - \text{Li}_2\left(-\frac{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}{2\sqrt{\mu_1}(1 + \sqrt{\mu_1})}\right) \\
& + \text{Li}_2\left(-\frac{1 - \sqrt{\mu_1}}{1 + \sqrt{\mu_1}}\right) - \text{Li}_2\left(\frac{1 - \sqrt{\mu_1}}{1 + \sqrt{\mu_1}}\right) - \text{Li}_2(-1) + \ln(1 + \sqrt{\mu_1}) \ln\left(\frac{1 - \mu_1 - \mu_2 - \sqrt{\lambda}}{2\sqrt{\mu_1}(1 - \sqrt{\mu_1})}\right) \\
& - \ln\left(1 + \frac{1}{\sqrt{\mu_1}}\right) \ln\left(\frac{1 - \mu_1 + \mu_2 + \sqrt{\lambda}}{2(1 - \sqrt{\mu_1})}\right) - \ell_{1W} \ln\left(\frac{(1 + \sqrt{\mu_1})^2 - \mu_2 - \sqrt{\lambda}}{2\sqrt{\mu_1}}\right), \tag{A12}
\end{aligned}$$

$$t_{zW}^{\ell-} = 2\text{Li}_2(\sqrt{\mu_1}) - \text{Li}_2\left(\frac{1 + \mu_1 - \mu_2 + \sqrt{\lambda}}{2}\right) - \text{Li}_2\left(\frac{1 + \mu_1 - \mu_2 - \sqrt{\lambda}}{2}\right). \tag{A13}$$

APPENDIX B: RENORMALIZATION COUNTERTERMS

In the electroweak theory with its multiple parameters depending on each other, it is essential to decide which of the parameters are independent. In the so-called $\alpha(0)$ scheme, the coupling and the masses of the particles are used as such. However, depending on the energy of the process, more and more fermion loops from the photon and Z boson vacuum polarizations can be resummed to the coupling, so that $\alpha(0)$ is changed to, e.g., $\alpha(m_W^2)$. This is shown in Sec. 8.2.1 of Ref. [5]. The renormalization counterterms are defined by

$$\begin{aligned}
\delta Z_e = & \frac{1}{2} \Pi_{AA}^T(m_A^2) - \frac{s_W}{c_W} \frac{\Sigma_{AZ}^T(m_A^2)}{m_Z^2} - \frac{1}{2} \Delta r, \quad \Pi_{AA}^T(m_A^2) = \left. \frac{\partial \Sigma_{AA}^T(k^2)}{\partial k^2} \right|_{k^2=m_A^2} \\
\frac{\delta s_W}{s_W} = & -\frac{c_W^2}{2s_W^2} \left(\frac{\Sigma_{WW}^T(m_W^2)}{m_W^2} - \frac{\Sigma_{ZZ}^T(m_Z^2)}{m_Z^2} \right), \\
\delta Z_{WW} = & -\frac{1}{2} \left. \frac{\partial \Sigma_{WW}^T(k^2)}{\partial k^2} \right|_{k^2=m_W^2}, \\
\delta Z_{ii}^L = & -\frac{1}{2} \Sigma_{ii}^{bL}(m_i^2) - m_i (m_i \Sigma_{ii}^{bL'}(m_i^2) + \Sigma_{ii}^{br'}(m_i^2)), \\
\delta Z_{ij}^L = & \frac{1}{m_i^2 - m_j^2} (m_j^2 \Sigma_{ij}^{bL}(m_j^2) + m_i m_j \Sigma_{ij}^{bR}(m_j^2) + m_i \Sigma_{ij}^{bL}(m_j^2) + m_j \Sigma_{ij}^{br}(m_j^2)), \tag{B1}
\end{aligned}$$

where the choice of Δr that is absorbed into the coupling $\alpha = \alpha(0)/(1 - \Delta r)$ determines the scheme. While $\Delta r = 0$ for the $\alpha(0)$ scheme, in the $\alpha(m_W^2)$ scheme Δr contains the light fermion loops.

1. Applying the G_μ scheme

In the quite detailed review article in Ref. [45] is argued that for processes involving the electroweak vector bosons, the G_F (or G_μ) scheme based on the Fermi constant (in muon decay) and discussed in Ref. [48] is preferable. Therefore, for our work we will deal with this scheme. From Eqs. (426) and (423) of Ref. [45] we deduce that

$$\delta Z_e|_{G_\mu} = \frac{1}{2} \Pi_{AA}^T(m_A^2) - \frac{s_W}{c_W} \frac{\Sigma_{AZ}^T(m_A^2)}{m_Z^2} - \frac{1}{2} \Delta r|_{G_\mu} \quad (\text{B2})$$

[cf. also Eq. (4.6.11) in Ref. [46]], where

$$\begin{aligned} \Delta r|_{G_\mu} = & \Pi_{AA}^T(m_A^2) + \frac{2c_W}{s_W} \frac{\Sigma_{AZ}^T(m_A^2)}{m_Z^2} + \frac{\Sigma_{WW}^T(0) - \Sigma_{WW}^T(m_W^2)}{m_W^2} \\ & - \frac{c_W^2}{s_W^2} \left(\frac{\Sigma_{ZZ}^T(m_Z^2)}{m_Z^2} - \frac{\Sigma_{WW}^T(m_W^2)}{m_W^2} \right) \\ & + \frac{\alpha}{4\pi s_W^2} \left(6 + \frac{7 - 4s_W^2}{2s_W^2} \ln c_W^2 \right). \end{aligned} \quad (\text{B3})$$

APPENDIX C: CROSS-CHECK FOR THE UNPOLARIZED DECAY RATE

By integrating Eq. (54) over the polar angle θ , we cross-check our result with the unpolarised result given in Refs. [5,6]. One has

$$\Gamma = \Gamma_0^0(\rho_{++} + \rho_{00} + \rho_{--})(H^{++} + H^{00} + H^{--}) =: \Gamma_0^0 H, \quad (\text{C1})$$

where

$$\begin{aligned} H(\text{tree}) = & -\frac{1}{2} [Q_1^2 + Q_2^2 + Q_W^2] \sqrt{\lambda} (2 - \mu_1 - \mu_1^2 - \mu_2 + 2\mu_1\mu_2 - \mu_2^2) \ell_\zeta \\ & - \frac{1}{2} [(1 - \mu_2)Q_1^2 - \mu_1(Q_2^2 - Q_W^2)] (2 - \mu_1 - \mu_1^2 - \mu_2 + 2\mu_1\mu_2 - \mu_2^2) (t_\zeta^\ell - 2t_z^{\ell+}) \\ & + \frac{1}{2} [Q_1^2 - Q_2^2 + (\mu_1 - \mu_2)Q_W^2] (2 - \mu_1 - \mu_1^2 - \mu_2 + 2\mu_1\mu_2 - \mu_2^2) (t_{\zeta W}^\ell + 2t_{zW}^{\ell+}) \\ & - \frac{1}{4} [(2 + 9\mu_1 + 4\mu_1^2 - 3\mu_2 - 6\mu_1\mu_2 + 2\mu_1^2\mu_2 + 3\mu_1\mu_2^2 + \mu_2^3)Q_1^2 \\ & + (4 - 2\mu_1 - 8\mu_1^2 - \mu_1^3 - 6\mu_2 + 6\mu_1\mu_2 + 9\mu_1^2\mu_2 - 4\mu_1\mu_2^2 + 2\mu_2^3)Q_2^2 \\ & + 2(2 - \mu_1 + 3\mu_1^2 + \mu_1^3 - 3\mu_2 + 3\mu_1\mu_2 - 2\mu_1^2\mu_2 - 2\mu_1\mu_2^2 + \mu_2^3)Q_W^2] \ell_1 \\ & - \frac{1}{4} [(2 - 15\mu_1 - 4\mu_1^2 + 2\mu_1^3 + \mu_2 + 12\mu_1\mu_2 - 6\mu_1^2\mu_2 - 8\mu_2^2 + 6\mu_1\mu_2^2 - 2\mu_2^3)Q_1^2 \\ & - (2 + \mu_1 - 8\mu_1^2 - 2\mu_1^3 - 15\mu_2 + 12\mu_1\mu_2 + 6\mu_1^2\mu_2 - 4\mu_2^2 - 6\mu_1\mu_2^2 + 2\mu_2^3)Q_2^2 \\ & - 2(\mu_1 - \mu_2)(2 + 3\mu_1 + 3\mu_2)Q_W^2] \ell_{1W} \\ & + \frac{1}{24} [3(22 - 37\mu_1 - 15\mu_1^2 + 3\mu_2 + 38\mu_1\mu_2 - 11\mu_2^2)Q_1^2 \\ & + 3(22 + 3\mu_1 - 11\mu_1^2 - 37\mu_2 + 38\mu_1\mu_2 - 15\mu_2^2)Q_2^2 \\ & + 2(44 - 43\mu_1 - 31\mu_1^2 - 43\mu_2 + 50\mu_1\mu_2 - 31\mu_2^2)Q_W^2] \sqrt{\lambda}, \\ H(\text{loop}) = & \frac{1}{2} (2 - \mu_1 - \mu_1^2 - \mu_2 + 2\mu_1\mu_2 - \mu_2^2) \sqrt{\lambda} V_- + 3\mu_1\mu_2 \sqrt{\lambda} V_+ - \frac{1}{4} \sqrt{\lambda^3} (\mu_1 V_1 + \mu_2 V_2), \\ H(\alpha) = & \frac{1}{2} (2 - \mu_1 - \mu_1^2 - \mu_2 + 2\mu_1\mu_2 - \mu_2^2) \sqrt{\lambda} V_-^* + 3\mu_1\mu_2 \sqrt{\lambda} V_+ - \frac{1}{4} \sqrt{\lambda^3} (\mu_1 V_1 + \mu_2 V_2) \\ & - \frac{1}{2} [(1 - \mu_2)Q_1^2 - \mu_1(Q_2^2 - Q_W^2)] (2 - \mu_1 - \mu_1^2 - \mu_2 + 2\mu_1\mu_2 - \mu_2^2) (t_\zeta^{\ell*} - 2t_z^{\ell+}) \\ & + \frac{1}{2} [Q_1^2 - Q_2^2 + (\mu_1 - \mu_2)Q_W^2] (2 - \mu_1 - \mu_1^2 - \mu_2 + 2\mu_1\mu_2 - \mu_2^2) (t_{\zeta W}^{\ell*} + 2t_{zW}^{\ell+}) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4}[(2+9\mu_1+4\mu_1^2-3\mu_2-6\mu_1\mu_2+2\mu_1^2\mu_2+3\mu_1\mu_2^2+\mu_2^3)Q_1^2 \\
& + (4-2\mu_1-8\mu_1^2-\mu_1^3-6\mu_2+6\mu_1\mu_2+9\mu_1^2\mu_2-4\mu_1\mu_2^2+2\mu_2^3)Q_2^2 \\
& + 2(2-\mu_1+3\mu_1^2+\mu_1^3-3\mu_2+3\mu_1\mu_2-2\mu_1^2\mu_2-2\mu_1\mu_2^2+\mu_2^3)Q_W^2]\ell_1 \\
& -\frac{1}{4}[(2-15\mu_1-4\mu_1^2+2\mu_1^3+\mu_2+12\mu_1\mu_2-6\mu_1^2\mu_2-8\mu_2^2+6\mu_1\mu_2^2-2\mu_2^3)Q_1^2 \\
& - (2+\mu_1-8\mu_1^2-2\mu_1^3-15\mu_2+12\mu_1\mu_2+6\mu_1^2\mu_2-4\mu_2^2-6\mu_1\mu_2^2+2\mu_2^3)Q_2^2 \\
& - 2(\mu_1-\mu_2)(2+3\mu_1+3\mu_2)Q_W^2]\ell_{1W} \\
& +\frac{1}{24}[3(22-37\mu_1-15\mu_1^2+3\mu_2+38\mu_1\mu_2-11\mu_2^2)Q_1^2 \\
& + 3(22+3\mu_1-11\mu_1^2-37\mu_2+38\mu_1\mu_2-15\mu_2^2)Q_2^2 \\
& + 2(44-43\mu_1-31\mu_1^2-43\mu_2+50\mu_1\mu_2-31\mu_2^2)Q_W^2]\sqrt{\lambda}.
\end{aligned} \tag{C2}$$

We find full agreement, where $\ln w_1 + \ln w_\mu = \ell_{1W}$, $\ln w_1 - \ln w_\mu = \ell_1 - \ell_{1W}$, and

$$8m_W^2(I_{01} + I_{02}) = t_\zeta^\ell - 2t_z^{\ell+}, \quad 8m_W^2 I_{02} = t_{\zeta W}^\ell + 2t_{zW}^{\ell+}. \tag{C3}$$

The result for $H(\text{tree})$ turns out to coincide with Eq. (35) in Ref. [6] and with Eq. (9.16) in Ref. [5]. As the vertex corrections are already checked against Eqs. (9.7)–(9.9) in Ref. [5], we are able to claim that also $H(\text{loop})$ and our formulas in general are consistent with the results for the decay of the unpolarized W decay as given in the literature. In addition, also numerically we are consistent with previous results, as we could reproduce the corresponding line of Table 9.2 in Ref. [5] for the parameters given in this reference.

APPENDIX D: ADAPTION OF THE LIGHT QUARK MASSES

According to Ref. [6] and using techniques from Ref. [49], the light quark masses are adapted to the hadronic contribution to the vacuum polarization. As the old value 0.0286 ± 0.0007 [49] is updated by a more recent one, in Sec. 10 of Ref. [47] cited as

$$\Delta\alpha_{\text{had}}^{(5)}(m_Z) = 0.02783 \pm 0.00006, \tag{D1}$$

a reiteration of the adaption procedure is in order at this point. According to Appendix A of Ref. [49], written in a more compact form, one has

$$\Delta\alpha_{\text{had}}^{(5)}(m_Z) = -e^2 N_c \sum_{i=1}^5 Q_i^2 s \Delta\pi_V^{q_i q_i}(m_Z^2) \tag{D2}$$

with

$$\begin{aligned}
\Delta\pi_V^{q_1 q_2}(q^2) &:= \pi_V^{q_1 q_2}(q^2) - \pi_V^{q_1 q_2}(0) \\
&= q^2 \int_0^\infty \frac{\rho_V^{q_1 q_2}(s) ds}{s(s - q^2 - i\epsilon)},
\end{aligned} \tag{D3}$$

where the spectral density for $s \geq (m_1 + m_2)^2$ is given by

$$\begin{aligned}
\rho_V^{q_1 q_2}(s) &= \frac{\sqrt{\lambda(s, m_1^2, m_2^2)}}{12\pi^2 s} \\
&\times \left[1 - \frac{m_1^2 + m_2^2}{2s} + 3 \frac{m_1 m_2}{s} - \frac{(m_1^2 - m_2^2)^2}{2s^2} \right].
\end{aligned} \tag{D4}$$

For $q_1 = q_2$ the integration results in

$$\begin{aligned}
\Delta\pi_V^{q_i q_i}(q^2) &= \frac{1}{36\pi^2 q^2} \\
&\times \left(5q^2 + 12m_i^2 - 3 \frac{(q^2 + 2m_i^2)(q^2 - 4m_i^2)}{\sqrt{q^2(q^2 - 4m_i^2)}} \right. \\
&\times \ln \left(\frac{\sqrt{q^2} + \sqrt{q^2 - 4m_i^2}}{\sqrt{q^2} - \sqrt{q^2 - 4m_i^2}} \right) \Big).
\end{aligned} \tag{D5}$$

Using this formula, for the mass settings of Ref. [6] given by $m_u = m_d = 0.041$ GeV, $m_s = 0.15$ GeV, $m_c = 1.5$ GeV, and $m_b = 4.5$ GeV, the result of $\Delta\alpha_{\text{had}}^{(5)}(m_Z) = 0.0285$ is reproduced within the error bars. Using instead $m_u = m_d = 0.06$ GeV and $m_s = 0.17$ GeV and the values for the heavy quarks from PDG [47], one obtains $\Delta\alpha_{\text{had}}^{(5)}(m_Z) = 0.027827$ that fits to the actual value. Therefore, in the main text we use these adapted light quark mass values.

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