

Recurrence relations and dispersive techniques for precision multiloop calculations

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Ab initio predictions of two-loop electroweak contributions to observables are increasingly essential for precision collider experiments, yet their evaluation remains very challenging. We connect recurrence techniques and dispersive methods in order to evaluate complex multiloop Feynman diagrams. By expressing multipoint Passarino-Veltman functions in a two-point basis and using shifted space-time dimensions with recurrence relations, we minimize the number of required dispersive integrals. This approach reduces computation time and enables a precise and efficient analysis of one- and two-loop diagrams.

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I. INTRODUCTION

Modern collider and low-energy precision programs are driving subpercent uncertainties, demanding *ab initio*, two-loop electroweak predictions for multiscale, multileg processes. To meet this precision frontier, theoretical methods must evolve in parallel with experiments. Flagship measurements include MOLLER [1] at Jefferson Lab (JLab) (low- Q^2 determination of $\sin^2\theta_W$ from parity-violating Møller scattering), P2 [2] at MESA (proton weak charge), and Belle II at KEK SuperKEKB (precision flavor and CP -violation studies). Upcoming programs such as SoLID-PVDIS [3] at JLab and the Electron-Ion Collider (EIC) [4] at Brookhaven National Laboratory will further require comprehensive higher-order theory predictions across multiple scattering channels. The accurate theoretical description of electroweak processes has long been a cornerstone of precision tests of the Standard Model. Over the past four decades, the field has progressed from the first analytical one-loop formalisms to modern semianalytical and numerical two-loop frameworks capable of supporting subpercent experimental precision. This evolution reflects both major theoretical advances in multiloop quantum-field-theory techniques and the rising experimental demands of collider and low-energy programs. The systematic treatment of loop corrections in the electroweak theory was established in the late 1970s and 1980s, culminating in the Passarino-Veltman (PV) tensor-reduction method. The PV algorithm provided a general prescription for expressing one-loop tensor integrals

in terms of scalar functions, establishing the algebraic foundation for all subsequent higher-order calculations [5].

During the following decade, Refs. [6,7] produced comprehensive reviews and explicit one-loop calculations relevant to CERN's Large Electron-Positron collider precision physics, codifying gauge-invariant renormalization schemes and parameter definitions. These works became standard references for both theoretical and computational approaches to radiative corrections. The extension to two-loop order demanded a deeper algebraic understanding of Feynman integrals. References [8,9] introduced dimension-recurrence and propagator-power-reduction identities that relate integrals in D and $D \pm 2$ dimensions. These relations enable systematic reduction of higher-rank and higher-dimensional integrals to a minimal set of master integrals. This formalism remains foundational for modern two-loop reduction algorithms and underpins most symbolic-manipulation packages used today. Concurrently, in [10], a complementary algebraic framework was developed for reducing tensor Feynman integrals to scalar ones using dimension shifting recurrence relations (see also [11–13] and references therein).

Analytic evaluation of increasingly complex diagrams soon became infeasible due to the proliferation of mass scales, external invariants, and threshold singularities. This challenge prompted the development of semianalytical and numerical approaches such as sector-decomposition methods, differential-equation systems for master integrals, and dispersion-relation techniques [14–18]. A major step was the differential-equation approach for master integrals, in which systems of linear equations in kinematic invariants are solved either analytically in canonical (ϵ -form) basis or numerically using series expansions. In [19,20], this framework was refined by introducing canonical-basis

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and uniform-weight formulations, improving both analytic transparency and numerical stability.

A comprehensive two-loop framework was developed in [21–25], specifically for polarized Møller scattering: an essential channel for future parity-violation experiments. The group systematically advanced from reducible two-loop and quadratic one-loop contributions to two-loop irreducible self-energy, vertex, and box calculations. These results quantified higher-order electroweak effects in parity-violating asymmetries, establishing reliable theoretical uncertainties at the sub-percent level which are critical for the forthcoming MOLLER experiment.

Beyond differential-equation and sector-decomposition strategies, a distinct line of development has been the dispersive approach introduced in [26–29]. This methodology expresses multipoint Passarino-Veltman functions in a two-point-function basis, replacing subloop insertions with effective propagators represented by dispersion integrals. The result is a semianalytical bridge between purely analytic amplitude reductions and purely numerical integration techniques. The dispersive framework preserves key analytic properties—threshold behavior, unitarity cuts, and gauge invariance—while reducing computational demands. It is especially attractive for low-energy observables where delicate cancellations between diagrams require high numerical precision. Furthermore, real-experiment implementation often involves acceptance and energy-threshold cuts that significantly affect radiative corrections; the dispersive formulation allows these to be incorporated naturally at the numerical integration stage. An example of an application of the dispersive approach to multiloop “banana” integrals can be found in Ref. [30]. Another example is the precision studies of the Higgs boson at future colliders, which requires the inclusion of next-to-next-to-leading-order (NNLO) electroweak corrections. In [31], the planar and nonplanar double-box topologies with multiple massive propagators in the loops are computed numerically by transforming one of the subloops using Feynman parametrization and a dispersion representation. This was followed by [32], which provides a complete calculation of the NNLO electroweak corrections involving closed fermion loops.

Parallel developments in the phenomenological sector culminated in partial two-loop electroweak predictions for key observables. In Refs. [33,34], the full set of fermionic and bosonic two-loop corrections to Z-boson observables was produced, while [35] addressed hadronic effects in Møller scattering at NNLO. More recently, [36] achieved the analytic evaluation of electroweak double-box integrals relevant for Møller processes. Other important developments can be found in Refs. [37–53].

These advances collectively establish a robust infrastructure for high-precision electroweak phenomenology. The comprehensive review on updated measurements and higher-order theoretical corrections are available in [54,55].

Despite impressive progress, complete NNLO electroweak results remain available only for selected processes due to the technical difficulty of two-loop calculations. Closed-fermion-loop NNLO corrections have been achieved for several key observables, but the general problem of fully automated two-loop amplitude generation, reduction, and evaluation remains open. Seminumerical strategies, such as the dispersive and differential-equation methods, have proven especially effective, balancing analytical control with numerical tractability. The growing complexity of precision calculations arises from the need to handle diagrams with both more external legs and additional loop orders. Each loop introduces new mass and momentum scales, overlapping ultraviolet and infrared divergences, and complicated threshold structures, dramatically increasing algebraic and numerical challenges. At one loop, PV-style analytic reductions allow tensor integrals to be expressed through a small set of scalar functions, often in closed form. At two loops and beyond, however, the explosion of topologies and kinematic invariants typically renders complete analytic evaluation impractical. To address these challenges, the dispersive method reformulates a multiloop integral as a sequence of nested one-loop integrals via spectral representations. This effectively transforms a two-loop problem into integrals over well-behaved spectral densities, isolating singular behavior in analytically controlled functions that can be integrated numerically with high stability. By combining dimensional shifting, recurrence relations, and dispersion theory, the approach achieves algebraic reduction to a minimal set of independent integrals while maintaining analytic transparency.

The present study extends our previous dispersive framework by incorporating shifted-dimension tensor decomposition and dimension-lowering recurrence relations directly into the dispersive representation. These relations, originally formulated within the Tarasov dimension-recursion approach, are adapted here to operate on spectral integrals, leading to a minimal set of independent dispersive building blocks. This algebraic reduction significantly decreases computational time while maintaining high numerical precision. We demonstrate the applicability of this formalism to one-loop self-energy, triangle, and box diagrams, as well as a two-loop example in which a one-loop sub-block is represented through corresponding dispersive integral. The results confirm that the dispersive-recurrence combination provides a stable and robust framework for the precision electroweak calculations. This methodology therefore represents a major step toward an automated, high-precision, *ab initio* framework for two-loop quantum-field-theory calculations capable of supporting the next generation of parity-violation and flavor-physics experiments, including MOLLER, P2, Belle II, and the future programs at EIC.

The paper is organized as follows. In Sec. II, we describe the methods involved in our study, including the tensor

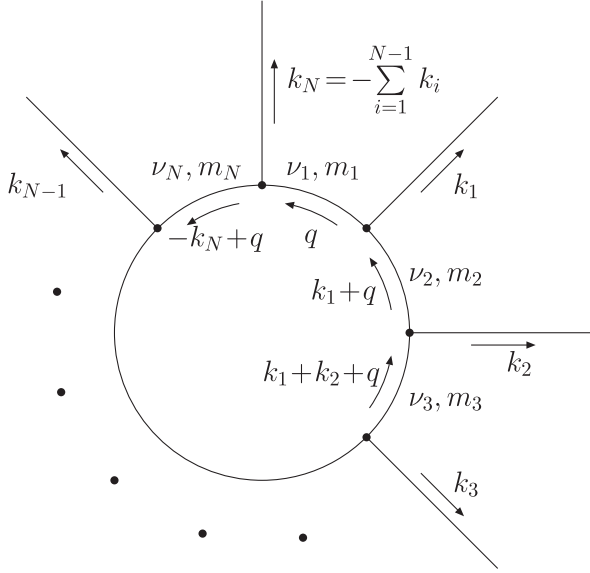


FIG. 1. The one-loop N -point diagram in the notation corresponding to tensors $T_{\mu_1 \dots \mu_M}^{(N)}$.

decomposition, recurrence relations, the momentum expansion, and the dispersion approach. In Sec. III, we present a numerical comparison of the one-loop three- and four-point functions calculated using the described techniques against

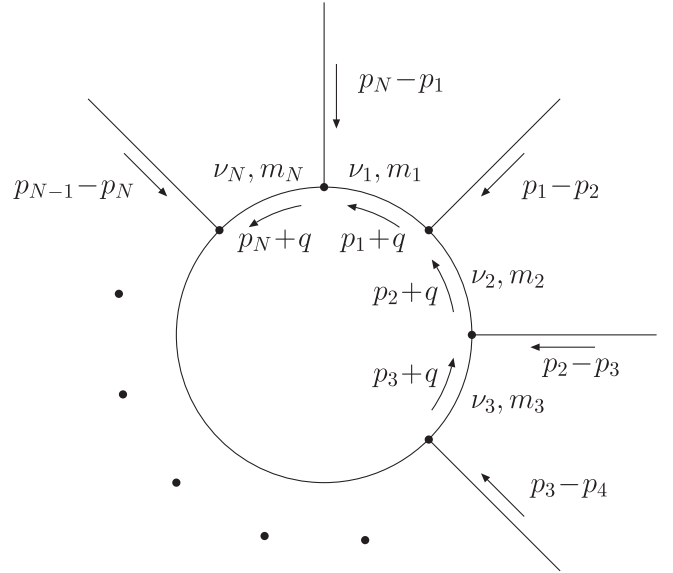


FIG. 2. The one-loop N -point diagram in the notation corresponding to tensors $J_{\mu_1 \dots \mu_M}^{(N)}$.

results from the Collier numerical library. In Sec. IV, we discuss the application of the presented techniques to two-loop calculations and compare our numerical results with those obtained using other available approaches. Finally, in Sec. V, we summarize the main ideas and results.

II. METHODOLOGY

A. Tensor decomposition

The tensor N -point function of an arbitrary rank M is defined as (see Fig. 1)

$$T_{\mu_1 \dots \mu_M}^{(N)} = \frac{\mu^{4-D} e^{\gamma_E(4-D)/2}}{i\pi^{D/2}} \int d^D q \times \frac{q_{\mu_1} \dots q_{\mu_M}}{[q^2 - m_1^2][(k_1 + q)^2 - m_2^2][(k_1 + k_2 + q)^2 - m_3^2] \dots [(k_1 + \dots + k_{N-1} + q)^2 - m_N^2]}, \quad (1)$$

where $D = 4 - 2\epsilon$ is the space-time dimension (ϵ is the standard dimensional regularization parameter) and μ is the mass scale. $T_{\mu_1 \dots \mu_M}^{(N)}$ can be decomposed in terms of the scalar Passarino-Veltman functions $Z_{0 \dots 01 \dots 12 \dots 2 \dots (N-1) \dots (N-1)}$ as (see, e.g., in Ref. [56])

$$T_{\mu_1 \dots \mu_M}^{(N)} = \sum_{\substack{l, n_1, n_2 \\ 2l + n_1 + n_2 + \dots + n_{N-1} = M}} \{[g]^l [k_1]^{n_1} [k_1 + k_2]^{n_2} \dots [k_1 + k_2 + \dots + k_{N-1}]^{n_{N-1}}\}_{\mu_1 \dots \mu_M} \times Z_{\underbrace{0 \dots 0}_{2l} \underbrace{1 \dots 1}_{n_1} \underbrace{2 \dots 2}_{n_2} \dots \underbrace{(N-1) \dots (N-1)}_{n_{N-1}}}, \quad (2)$$

where $\{[g]^l [k_1]^{n_1} [k_1 + k_2]^{n_2} \dots [k_1 + k_2 + \dots + k_{N-1}]^{n_{N-1}}\}_{\mu_1 \dots \mu_M}$ is the symmetrized tensor structure containing l metric tensors g , n_1 vectors k_1 , n_2 vectors $k_1 + k_2, \dots$, and n_{N-1} vectors $k_1 + \dots + k_{N-1}$ ($2l + n_1 + n_2 + \dots + n_{N-1} = M$).

To be able to employ the tensor-decomposition approach developed in Ref. [10], let us define the scalar N -point integral as (see Fig. 2)

$$J^{(N)}(D; \nu_1, \nu_2, \dots, \nu_N) = \int \frac{d^D q}{[(p_1 + q)^2 - m_1^2]^{\nu_1} [(p_2 + q)^2 - m_2^2]^{\nu_2} \dots [(p_N + q)^2 - m_N^2]^{\nu_N}} \quad (3)$$

[it is clear that this scalar integral depends only on the squared momenta $(p_i - p_j)^2$ with $i < j < N$]. Similarly, the N -point tensor integral of an arbitrary rank M is defined as

$$J_{\mu_1 \dots \mu_M}^{(N)}(D; \nu_1, \nu_2, \dots, \nu_N) = \int d^D q \frac{q_{\mu_1} \dots q_{\mu_M}}{[(p_1 + q)^2 - m_1^2]^{\nu_1} [(p_2 + q)^2 - m_2^2]^{\nu_2} \dots [(p_N + q)^2 - m_N^2]^{\nu_N}}. \quad (4)$$

Then the general tensor-decomposition formula [see Eq. (11) of [10]] yields

$$J_{\mu_1 \dots \mu_M}^{(N)}(D; \nu_1, \nu_2, \dots, \nu_N) = \sum_{\substack{\lambda, \kappa_1, \dots, \kappa_N \\ 2\lambda + \kappa_1 + \dots + \kappa_N = M}} \left(-\frac{1}{2}\right)^\lambda \left(\prod_{i=1}^N (\nu_i)_{\kappa_i}\right) \{[g]^\lambda [p_1]^{\kappa_1} [p_2]^{\kappa_2} \dots [p_N]^{\kappa_N}\}_{\mu_1 \dots \mu_M} \\ \times \pi^{\lambda-M} J^{(N)}(D + 2(M - \lambda); \nu_1 + \kappa_1, \nu_2 + \kappa_2, \dots, \nu_N + \kappa_N), \quad (5)$$

where $(\nu)_\kappa \equiv \Gamma(\nu + \kappa)/\Gamma(\nu)$ is the Pochhammer symbol, and the scalar integrals $J^{(N)}$ occurring on the rhs have shifted space-time dimension value $D + 2(M - \lambda)$.

Comparing the definitions of $T_{\mu_1 \dots \mu_M}^{(N)}$ [given in Eq. (1)] and $J_{\mu_1 \dots \mu_M}^{(N)}$ [given in Eq. (4)], we get the following connection formula:

$$T_{\mu_1 \dots \mu_M}^{(N)} = \frac{\mu^{4-D} e^{\gamma_E(4-D)/2}}{i\pi^{D/2}} J_{\mu_1 \dots \mu_M}^{(N)}(D; 1, \dots, 1) \Big|_{p_1=0, p_2=k_1, \dots, p_N=k_1+\dots+k_{N-1}}. \quad (6)$$

Considering Eq. (5) in the case $p_1 = 0, p_2 = k_1, \dots, p_N = k_1 + \dots + k_{N-1}$, we see that only the term with $\kappa_1 = 0$ contributes,

$$J_{\mu_1 \dots \mu_M}^{(N)}(D; \nu_1, \nu_2, \dots, \nu_N) \Big|_{p_1=0, p_2=k_1, \dots, p_N=k_1+\dots+k_{N-1}} \\ = \sum_{\substack{\lambda, \kappa_2, \dots, \kappa_N \\ 2\lambda + \kappa_2 + \dots + \kappa_N = M}} \left(-\frac{1}{2}\right)^\lambda \left(\prod_{i=2}^N (\nu_i)_{\kappa_i}\right) \{[g]^\lambda [k_1]^{\kappa_2} [k_1 + k_2]^{\kappa_3} \dots [k_1 + \dots + k_{N-1}]^{\kappa_N}\}_{\mu_1 \dots \mu_M} \\ \times \pi^{\lambda-M} J^{(N)}(D + 2(M - \lambda); \nu_1, \nu_2 + \kappa_2, \dots, \nu_N + \kappa_N) \Big|_{p_1=0, p_2=k_1, \dots, p_N=k_1+\dots+k_{N-1}}. \quad (7)$$

To compare with $T_{\mu_1 \dots \mu_M}^{(N)}$, we need to put $\nu_1 = \nu_2 = \dots = \nu_N = 1$, $\lambda = l$, $\kappa_2 = n_1$, $\kappa_3 = n_2$, $\dots$, $\kappa_N = n_{N-1}$,

$$J_{\mu_1 \dots \mu_M}^{(N)}(D; 1, \dots, 1) \Big|_{p_1=0, p_2=k_1, \dots, p_N=k_1+\dots+k_{N-1}} \\ = \sum_{\substack{l, n_1, \dots, n_{N-1} \\ 2l + n_1 + \dots + n_{N-1} = M}} \left(-\frac{1}{2}\right)^l \left(\prod_{i=1}^{N-1} n_i!\right) \{[g]^l [k_1]^{n_1} [k_1 + k_2]^{n_2} \dots [k_1 + \dots + k_{N-1}]^{n_{N-1}}\}_{\mu_1 \dots \mu_M} \\ \times \pi^{-l-n_1-\dots-n_{N-1}} J^{(N)}(D + 2l + 2n_1 + \dots + 2n_{N-1}; 1, 1 + n_1, \dots, 1 + n_{N-1}) \Big|_{p_1=0, p_2=k_1, \dots, p_N=k_1+\dots+k_{N-1}}. \quad (8)$$

In this way, we arrive at

$$Z_{\underbrace{0\dots 0}_{2l} \underbrace{1\dots 1}_{n_1} \underbrace{2\dots 2}_{n_2} \dots \underbrace{(N-1)\dots(N-1)}_{n_{N-1}}} = \frac{\mu^{4-D} e^{\gamma_E(4-D)/2}}{i\pi^{l+n_1+\dots+n_{N-1}+D/2}} \left(\prod_{i=1}^{N-1} n_i! \right) \left(-\frac{1}{2} \right)^l \times J^{(N)}(D+2l+2n_1+\dots+2n_{N-1}; 1, 1+n_1, \dots, 1+n_{N-1}) \Big|_{p_1=0, p_2=k_1, \dots, p_N=k_1+\dots+k_{N-1}}. \quad (9)$$

To be consistent with the standard Passarino-Veltman notations, for $N = 2, 3, 4, 5, \dots$ the notation Z should be replaced by B, C, D, E , etc.

For the two-point case ($N = 2$) we get

$$B_{\underbrace{0\dots 0}_{2l} \underbrace{1\dots 1}_n} = \frac{\mu^{4-D} e^{\gamma_E(4-D)/2} n!}{i\pi^{l+n+D/2}} \left(-\frac{1}{2} \right)^l J^{(2)}(D+2l+2n; 1, 1+n) \Big|_{p_1=0, p_2=k_1}. \quad (10)$$

If we have only one external momentum we will usually suppress its index, $k_1 = k$.

For the three-point case ($N = 3$), we get

$$C_{\underbrace{0\dots 0}_{2l} \underbrace{1\dots 1}_{n_1} \underbrace{2\dots 2}_{n_2}} = \frac{\mu^{4-D} e^{\gamma_E(4-D)/2} n_1! n_2!}{i\pi^{l+n_1+n_2+D/2}} \left(-\frac{1}{2} \right)^l \times J^{(3)}(D+2l+2n_1+2n_2; 1, 1+n_1, 1+n_2) \Big|_{p_1=0, p_2=k_1, p_3=k_1+k_2}, \quad (11)$$

where the scalar integral $J^{(3)}$ on the rhs depends on the following momentum invariants: $(p_1 - p_2)^2 = k_1^2$ (for the incoming momentum opposite to the line with m_3), $(p_2 - p_3)^2 = k_2^2$ (for the incoming momentum opposite to the line with m_1), and $(p_3 - p_1)^2 = (k_1 + k_2)^2$ (for the incoming momentum opposite to the line with m_2).

In the occurring three-point integral, we can combine any pair of denominators by using the Feynman parametrization trick, e.g.,

$$\begin{aligned} & J^{(3)}(D+2l+2n_1+2n_2; 1, 1+n_1, 1+n_2) \Big|_{p_1=0, p_2=k_1, p_3=k_1+k_2} \\ &= \int \frac{d^{D+2l+2n_1+2n_2} q}{[q^2 - m_1^2][(k_1 + q)^2 - m_2^2]^{1+n_1}[(k_1 + k_2 + q)^2 - m_3^2]^{1+n_2}} \\ &= \frac{(n_1 + n_2 + 1)!}{n_1! n_2!} \int_0^1 dx x^{n_1} \bar{x}^{n_2} \int \frac{d^{D+2l+2n_1+2n_2} q}{[q^2 - m_1^2][(k_1 + \bar{x}k_2 + q)^2 - x m_2^2 - \bar{x} m_3^2 + x \bar{x} k_2^2]^{2+n_1+n_2}} \\ &= \frac{(n_1 + n_2 + 1)!}{n_1! n_2!} \int_0^1 dx x^{n_1} \bar{x}^{n_2} J^{(2)}(D+2l+2n_1+2n_2; 1, 2+n_1+n_2) \Big|_{\substack{p_1=0, p_2=k_1+\bar{x}k_2, \\ m_2^2 \leftrightarrow x m_2^2 + \bar{x} m_3^2 - x \bar{x} k_2^2}} \end{aligned} \quad (12)$$

where $\bar{x} = 1 - x$.

For the four-point case ($N = 4$), we get

$$D_{\underbrace{0\dots 0}_{2l} \underbrace{1\dots 1}_{n_1} \underbrace{2\dots 2}_{n_2} \underbrace{3\dots 3}_{n_3}} = \frac{\mu^{4-D} e^{\gamma_E(4-D)/2} n_1! n_2! n_3!}{i\pi^{l+n_1+n_2+n_3+D/2}} \left(-\frac{1}{2}\right)^l \times J^{(4)}(D+2l+2n_1+2n_2+2n_3; 1, 1+n_1, 1+n_2, 1+n_3) \Big|_{p_1=0, p_2=k_1, p_3=k_1+k_2, p_4=k_1+k_2+k_3}. \quad (13)$$

In the occurring four-point integral, we can combine any triple of denominators by using the Feynman parametrization trick, e.g.,

$$\begin{aligned} & J^{(4)}(D+2l+2n_1+2n_2+2n_3; 1, 1+n_1, 1+n_2, 1+n_3) \Big|_{p_1=0, p_2=k_1, p_3=k_1+k_2, p_4=k_1+k_2+k_3} \\ &= \int \frac{d^{D+2l+2n_1+2n_2+2n_3} q}{[q^2 - m_1^2][(k_1 + q)^2 - m_2^2]^{1+n_1}[(k_1 + k_2 + q)^2 - m_3^2]^{1+n_2}[(k_1 + k_2 + k_3 + q)^2 - m_4^2]^{1+n_3}} \\ &= \frac{(n_1 + n_2 + n_3 + 2)!}{n_1! n_2! n_3!} \int_0^1 \int_0^1 dx dy x^{n_1} \bar{x}^{n_2+n_3+1} y^{n_2} \bar{y}^{n_3} \\ &\quad \times \int \frac{d^{D+2l+2n_1+2n_2+2n_3} q}{[q^2 - m_1^2][(k_1 + \bar{x}k_2 + \bar{x}\bar{y}k_3 + q)^2 - xm_2^2 - \bar{x}ym_3^2 - \bar{x}\bar{y}m_4^2 + x\bar{x}(k_2 + \bar{y}k_3)^2 + \bar{x}y\bar{y}k_3^2]^{3+n_1+n_2+n_3}} \\ &= \frac{(n_1 + n_2 + n_3 + 2)!}{n_1! n_2! n_3!} \int_0^1 \int_0^1 dx dy x^{n_1} \bar{x}^{n_2+n_3+1} y^{n_2} \bar{y}^{n_3} \\ &\quad \times J^{(2)}(D+2l+2n_1+2n_2+2n_3; 1, 3+n_1+n_2+n_3) \Big|_{\substack{p_1=0, p_2=k_1+\bar{x}k_2+\bar{x}\bar{y}k_3, \\ m_2^2 \leftrightarrow xm_2^2+\bar{x}ym_3^2+\bar{x}\bar{y}m_4^2-x\bar{x}(k_2+\bar{y}k_3)^2-\bar{x}y\bar{y}k_3^2}}, \end{aligned} \quad (14)$$

where $\bar{x} = 1 - x$ and $\bar{y} = 1 - y$.

B. Recurrence relations and the momentum expansion

In the two-point case, according to Eq. (10), we need to calculate the integrals $J^{(2)}(D+2l+2n; 1, 1+n)$ with $l \geq 0$ and $n \geq 0$. Using Feynman parameters, we can express higher functions in terms of the integrals $J^{(2)}$. In the three- and four-point cases, according to Eqs. (12) and (14), we need to calculate the integrals $J^{(2)}(D+2l+2n_1+2n_2; 1, 2+n_1+n_2)$ and $J^{(2)}(D+2l+2n_1+2n_2+2n_3; 1, 3+n_1+n_2+n_3)$, respectively. In general, for an N -point function, we would need to deal with the integrals

$$J^{(2)}(D+2l+2n; 1, N-1+n), \quad \text{with } n = n_1 + \dots n_{N-1}. \quad (15)$$

To decrease the index (power of propagator) $N-1+n$, we can use the recurrence relation (A13) for the case $\nu_1 = 1$,

$$J^{(2)}(D+2; 1, \nu_2+1) = -\frac{\pi}{2\nu_2 k^2} [(k^2 + m_1^2 - m_2^2)J^{(2)}(D; 1, \nu_2) - J^{(2)}(D; 1, \nu_2-1) + J^{(2)}(D; 0, \nu_2)]. \quad (16)$$

Whenever one of the indices on the rhs becomes zero [like, e.g., in $J^{(2)}(D; 0, \nu_2)$], this is a tadpole integral which can be expressed in terms of $J^{(2)}(D; 0, 1)$ or $J^{(2)}(D; 0, 1)$ using Eqs. (A9) and (A10). After that, for the remaining integrals $J^{(2)}(D+2l; 1, 1)$ we can use the recurrence relation (A8) for the case $\nu_1 = \nu_2 = 1$,

$$J^{(2)}(D+2; 1, 1) = -\frac{\pi}{2k^2(D-1)} [\Delta J^{(2)}(D; 1, 1) + (k^2 + m_1^2 - m_2^2)J^{(2)}(D; 1, 0) + (k^2 - m_1^2 + m_2^2)J^{(2)}(D; 0, 1)], \quad (17)$$

with

$$\begin{aligned}\Delta &\equiv \Delta(m_1^2, m_2^2, k^2) = -\lambda(m_1^2, m_2^2, k^2) \\ &= 4m_1^2 m_2^2 - (k^2 - m_1^2 - m_2^2)^2,\end{aligned}\quad (18)$$

where $\lambda(m_1^2, m_2^2, k^2)$ is the standard notation for the Källén function [other representations of Δ are collected in Eq. (A4)].

Another way to deal with the integrals (15) is to use recurrence relations (A2) and (A3). In this way, we can bring them to the integrals $J^{(2)}(D + 2l + 2n; 1, 1)$ (with the same value of the space-time dimension as the original ones) plus tadpoles, and then apply Eq. (17) as many times as needed. However, this option would involve more steps, and it would produce very cumbersome intermediate expressions because of the presence of Δ in the denominators of (A2) and (A3). We found the way based on Eqs. (16) and (17) to be more efficient.

Let us first consider the two- and three-point cases (we will discuss the higher cases later). In the two-point case

($N = 2$), using n times Eq. (16) and l times Eq. (17), we reduce $J^{(2)}(D + 2l + 2n; 1, 1 + n)$ to $J^{(2)}(D; 1, 1)$ plus tadpoles. In the three-point case ($N = 3$), using $n + 1$ times Eq. (16) and l times Eq. (17), we reduce $J^{(2)}(D + 2l + 2n; 1, 2 + n)$ to $J^{(2)}(D - 2; 1, 1)$ plus tadpoles. If we want to use the same basis for the two-point integrals, we need to apply Eq. (17) one more time (shifting $D \rightarrow D - 2$).

In this way, in the two- and three-point cases, starting from the integrals (15) with $D = 4 - 2\varepsilon$ we bring them to the basis of $(2 - 2\varepsilon)$ -dimensional integrals. Namely, by using recurrence relations with respect to the powers of propagators ν_i and the space-time dimension D , we can express all the relevant integrals $J^{(2)}(4 - 2\varepsilon + 2l + 2n; 1, N - 1 + n)$ (with $N = 2$ and $N = 3$) in terms of the master integral $J^{(2)}(2 - 2\varepsilon; 1, 1)$, as well as the tadpoles

$$J^{(2)}(2 - 2\varepsilon; 1, 0) = -i\pi^{1-\varepsilon}\Gamma(\varepsilon)(m_1^2)^{-\varepsilon} \quad \text{and} \quad J^{(2)}(2 - 2\varepsilon; 0, 1) = -i\pi^{1-\varepsilon}\Gamma(\varepsilon)(m_2^2)^{-\varepsilon} \quad (19)$$

(see Appendix A for more details). Analytical results for the integral $J^{(2)}(2 - 2\varepsilon; 1, 1)$ (including the relevant terms of the ε expansion) are collected in Appendix B. This procedure provides analytical results for all required Passarino-Veltman functions (10).

When using the recurrence relations (16) and (17) we are getting powers of k^2 in the denominator. In particular, when reducing $J^{(2)}(4 - 2\varepsilon + 2l + 2n; 1, N - 1 + n)$ (with $N = 2$ or $N = 3$) to $J^{(2)}(2 - 2\varepsilon; 1, 1)$ and tadpoles (19) the maximal power is $(k^2)^{l+n+1}$, i.e.,

$$\begin{aligned}J^{(2)}(4 - 2\varepsilon + 2l + 2n; 1, N - 1 + n) &= \frac{\pi^{l+n+1}}{(k^2)^{l+n+1}} \left[R_{N,l,n}^{(1,1)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2 - 2\varepsilon; 1, 1) \right. \\ &\quad + R_{N,l,n}^{(1,0)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2 - 2\varepsilon; 1, 0) \\ &\quad \left. + R_{N,l,n}^{(0,1)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2 - 2\varepsilon; 0, 1) \right],\end{aligned}\quad (20)$$

where $R_{N,n,l}^{(1,1)}$, $R_{N,n,l}^{(1,0)}$, and $R_{N,n,l}^{(0,1)}$ are algebraic coefficients that are polynomial in k^2 .

To make sure that the resulting expression (20) is not singular as $k^2 \rightarrow 0$, let us employ the small momentum expansion of the integral $J^{(2)}(2 - 2\varepsilon; 1, 1)$. According to Eq. (C9), the terms of the small- k^2 expansion of $J^{(2)}(2 - 2\varepsilon; 1, 1)$ up to $(k^2)^{j_0}$ can be presented as

$$\begin{aligned}J_{[j_0]}^{(2)}(2 - 2\varepsilon; 1, 1) &= \sum_{j=0}^{j_0} (k^2)^j \frac{(1 + \varepsilon)_j}{(m_2^2 - m_1^2)^{1+2j}} \left\{ J^{(2)}(2 - 2\varepsilon; 0, 1) \sum_{l=0}^j \frac{j!}{l!(j-l)!} \frac{(m_2^2)^l (m_1^2)^{j-l}}{(1 - \varepsilon)_l (1 + \varepsilon)_{j-l}} \right. \\ &\quad \left. - J^{(2)}(2 - 2\varepsilon; 1, 0) \sum_{l=0}^j \frac{j!}{l!(j-l)!} \frac{(m_1^2)^l (m_2^2)^{j-l}}{(1 - \varepsilon)_l (1 + \varepsilon)_{j-l}} \right\},\end{aligned}\quad (21)$$

so that

$$J_{[\infty]}^{(2)}(2 - 2\varepsilon; 1, 1) = J^{(2)}(2 - 2\varepsilon; 1, 1). \quad (22)$$

If we subtract the expansion (21) from $J^{(2)}(2 - 2\varepsilon; 1, 1)$, the difference will be of the order $(k^2)^{j_0+1}$, and it can be presented as

$$J^{(2)}(2 - 2\varepsilon; 1, 1) - J_{[j_0]}^{(2)}(2 - 2\varepsilon; 1, 1) = (k^2)^{j_0+1} \bar{J}_{j_0+1}^{(2)}(2 - 2\varepsilon; 1, 1). \quad (23)$$

In our case, we need to put $j_0 = n + l$. Using Eq. (23), we get

$$J^{(2)}(2 - 2\varepsilon; 1, 1) = J_{[n+l]}^{(2)}(2 - 2\varepsilon; 1, 1) + (k^2)^{n+l+1} \bar{J}_{n+l+1}^{(2)}(2 - 2\varepsilon; 1, 1). \quad (24)$$

Combining Eqs. (20), (21), and (24) we get

$$\begin{aligned} J^{(2)}(4 - 2\varepsilon + 2l + 2n; 1, N - 1 + n) &= \frac{\pi^{l+n+1}}{(k^2)^{l+n+1}} \left[R_{N,l,n}^{(1,1)}(m_1, m_2, k^2, \varepsilon) (k^2)^{n+l+1} \bar{J}_{n+l+1}^{(2)}(2 - 2\varepsilon; 1, 1) \right. \\ &\quad + \bar{R}_{N,l,n}^{(1,0)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2 - 2\varepsilon; 1, 0) \\ &\quad \left. + \bar{R}_{N,l,n}^{(0,1)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2 - 2\varepsilon; 0, 1) \right], \end{aligned} \quad (25)$$

where $\bar{R}_{N,l,n}^{(1,0)}$ and $\bar{R}_{N,l,n}^{(0,1)}$ include $R_{N,l,n}^{(1,0)}$ and $R_{N,l,n}^{(0,1)}$ plus the polynomial (in k^2) contributions coming from $R_{N,l,n}^{(1,1)} J_{[n+l]}^{(2)}(2 - 2\varepsilon; 1, 1)$ [see Eq. (21)]. The absence of singularities in k^2 means that in $\bar{R}_{N,l,n}^{(1,0)}$ and $\bar{R}_{N,l,n}^{(0,1)}$ all the powers of k^2 less than $n + l + 1$ should cancel, so that

$$\begin{aligned} \bar{R}_{N,l,n}^{(1,0)}(m_1, m_2, k^2, \varepsilon) &= (k^2)^{l+n+1} \bar{R}_{N,l,n}^{(1,0)}(m_1, m_2, k^2, \varepsilon), \\ \bar{R}_{N,l,n}^{(0,1)}(m_1, m_2, k^2, \varepsilon) &= (k^2)^{l+n+1} \bar{R}_{N,l,n}^{(0,1)}(m_1, m_2, k^2, \varepsilon), \end{aligned} \quad (26)$$

where $\bar{R}_{N,l,n}^{(1,0)}$ and $\bar{R}_{N,l,n}^{(0,1)}$ are also polynomial in k^2 . In this way, we arrive at

$$\begin{aligned} J^{(2)}(4 - 2\varepsilon + 2l + 2n; 1, N - 1 + n) &= \pi^{l+n+1} \left[R_{N,l,n}^{(1,1)}(m_1, m_2, k^2, \varepsilon) \bar{J}_{l+n+1}^{(2)}(2 - 2\varepsilon; 1, 1) \right. \\ &\quad + \bar{R}_{N,l,n}^{(1,0)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2 - 2\varepsilon; 1, 0) \\ &\quad \left. + \bar{R}_{N,l,n}^{(0,1)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2 - 2\varepsilon; 0, 1) \right]. \end{aligned} \quad (27)$$

In particular, this yields the following result for the function (10):

$$\begin{aligned} B_{\underbrace{0 \dots 0}_{2l} \underbrace{1 \dots 1}_n} &= \frac{\mu^{2\varepsilon} e^{\gamma_E \varepsilon} n!}{i\pi^{1-\varepsilon}} \left(-\frac{1}{2} \right)^l \left[R_{2,l,n}^{(1,1)}(m_1, m_2, k^2, \varepsilon) \bar{J}_{l+n+1}^{(2)}(2 - 2\varepsilon; 1, 1) \right. \\ &\quad + \bar{R}_{2,l,n}^{(1,0)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2 - 2\varepsilon; 1, 0) \\ &\quad \left. + \bar{R}_{2,l,n}^{(0,1)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2 - 2\varepsilon; 0, 1) \right]. \end{aligned} \quad (28)$$

Note that the coefficient functions $R_{N,l,n}^{(1,1)}$, $\bar{R}_{N,l,n}^{(1,0)}$, and $\bar{R}_{N,l,n}^{(0,1)}$ do not have poles in ε because in the recurrence relations (16) and (17) the only D -dependent factor in the denominator is $(D - 1)$ which would never produce ε for even dimensions.

We can split the function (28) into two parts, the first one containing the one-point (tadpolelike) integrals $J^{(2)}(2 - 2\varepsilon; 1, 0)$ and $J^{(2)}(2 - 2\varepsilon; 0, 1)$, and the second one involving the genuine (subtracted) two-point integral $\bar{J}_{n+l+1}^{(2)}(2 - 2\varepsilon; 1, 1)$,

$$B_{\underbrace{0\dots 0}_{2l}} \underbrace{1\dots 1}_n \equiv B_{\{2l,n\}}(k^2, m_1^2, m_2^2) = B_{\{2l,n\}}^{1\text{-point}}(k^2, m_1^2, m_2^2) + B_{\{2l,n\}}^{2\text{-point}}(k^2, m_1^2, m_2^2), \quad (29)$$

with

$$B_{\{2l,n\}}^{1\text{-point}}(k^2, m_1^2, m_2^2) = \frac{\mu^{2\varepsilon} e^{\gamma_E \varepsilon} n!}{i\pi^{1-\varepsilon}} \left(-\frac{1}{2} \right)^l \left[\bar{R}_{2,l,n}^{(1,0)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2-2\varepsilon; 1, 0) + \bar{R}_{2,l,n}^{(0,1)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2-2\varepsilon; 0, 1) \right], \quad (30)$$

$$B_{\{2l,n\}}^{2\text{-point}}(k^2, m_1^2, m_2^2) = \frac{\mu^{2\varepsilon} e^{\gamma_E \varepsilon} n!}{i\pi^{1-\varepsilon}} \left(-\frac{1}{2} \right)^l R_{2,l,n}^{(1,1)}(m_1, m_2, k^2, \varepsilon) \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1). \quad (31)$$

Note that all UV singularities are in $B_{\{2l,n\}}^{1\text{-point}}(k^2, m_1^2, m_2^2)$, namely, in the tadpole integrals (19), whereas the term $B_{\{2l,n\}}^{2\text{-point}}(k^2, m_1^2, m_2^2)$ is UV finite.

For the four-point function, the situation is a bit more complicated. Let us start from the integral (15), $J^{(2)}(D+2l+2n; 1, 3+n)$, and use recurrence relations (16) and (17). If we stop the recurrence procedure when the space-time dimension becomes $D-2$ (i.e., $2-2\varepsilon$), then among the remaining integrals we may have not only $J^{(2)}(2-2\varepsilon; 1, 1)$, $J^{(2)}(2-2\varepsilon; 1, 0)$, and $J^{(2)}(2-2\varepsilon; 0, 1)$, but also $J^{(2)}(2-2\varepsilon; 1, 2)$ (this happens at $l=0$). The integral $J^{(2)}(2-2\varepsilon; 1, 2)$ is not independent: using the relation (A3) it can be expressed as

$$J^{(2)}(D; 1, 2) = \frac{1}{\Delta} \left[(D-3)(k^2 + m_1^2 - m_2^2) J^{(2)}(D; 1, 1) - (D-2) J^{(2)}(D; 1, 0) - \frac{(D-2)(k^2 - m_1^2 - m_2^2)}{2m_1^2} J^{(2)}(D; 0, 1) \right]. \quad (32)$$

If we use Eq. (32) for $J^{(2)}(2-2\varepsilon; 1, 2)$ we would get for $N=4$ a representation similar to (20), but the occurring coefficient functions $R_{4,l,n}^{(1,1)}$, etc., will not be polynomial in k^2 , because of the presence of Δ [see Eq. (18)] in their denominators. In this way, we would get rather cumbersome expressions for the higher-order Passarino-Veltman functions.

Another way is to keep the $J^{(2)}(2-2\varepsilon; 1, 2)$ contributions as an extra term

$$R_{N,l,n}^{(1,2)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2-2\varepsilon; 1, 2) \quad (33)$$

in Eq. (20), as well as in Eqs. (25) and (27). For the small- k^2 expansion we can use the derivative of Eq. (21) with respect to m^2 ,

$$J_{[j_0]}^{(2)}(2-2\varepsilon; 1, 2) = \frac{\partial}{\partial m_2^2} J_{[j_0]}^{(2)}(2-2\varepsilon; 1, 1), \quad (34)$$

which can be calculated automatically. In this way, for the four-point case we get the following decomposition:

$$J^{(2)}(4-2\varepsilon+2l+2n; 1, 3+n) = \pi^{l+n+1} \left[R_{4,l,n}^{(1,1)}(m_1, m_2, k^2, \varepsilon) \bar{J}_{l+n+1}^{(2)}(2-2\varepsilon; 1, 1) + R_{4,l,n}^{(1,2)}(m_1, m_2, k^2, \varepsilon) \bar{J}_{l+n+1}^{(2)}(2-2\varepsilon; 1, 2) + \bar{R}_{4,l,n}^{(1,0)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2-2\varepsilon; 1, 0) + \bar{R}_{4,l,n}^{(0,1)}(m_1, m_2, k^2, \varepsilon) J^{(2)}(2-2\varepsilon; 0, 1) \right]. \quad (35)$$

In the same way, for the five-point function we would get in Eq. (20) an extra term involving $J^{(2)}(2-2\varepsilon; 1, 3)$, etc.

C. The dispersion approach

The subtracted integral $\bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1)$ is represented using the dispersive approach. We chose this approach because it allows us to replace the one-loop insertion in the two-loop topology with an effective propagator $1/(s-k^2-i0)$

integrated over an s parameter. Subtracting dispersively represented series expansion of $J_{[n+l]}^{(2)}(2-2\varepsilon; 1, 1)$ around $k^2 \rightarrow 0$ from $J^{(2)}(2-2\varepsilon; 1, 1)$ renders the dispersive one-loop insertion multiply subtracted and convergent. Following Eq. (24), we can write dispersively

$$\begin{aligned} (k^2)^{n+l+1} \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1) &= \frac{i}{\pi} \int_{(m_1+m_2)^2}^{\infty} ds \operatorname{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s \left[\frac{1}{s-k^2-i0} - \sum_{j=0}^{n+l} \frac{(k^2)^j}{s^{j+1}} \right] \\ &= \frac{i}{\pi} (k^2)^{n+l+1} \int_{(m_1+m_2)^2}^{\infty} ds \frac{\operatorname{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s}{s^{n+l+1}(s-k^2-i0)}. \end{aligned}$$

As a result, for the subtracted $\bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1)$ we get

$$\bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1) = \frac{i}{\pi} \int_{(m_1+m_2)^2}^{\infty} ds \frac{\operatorname{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s}{s^{n+l+1}(s-k^2-i0)}, \quad (36)$$

where [see Eq. (B19)]

$$\operatorname{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s = 2\pi^{1-\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{\pi}{\sqrt{-\Delta_s}} \left(\frac{s}{-\Delta_s} \right)^{\varepsilon} \quad (37)$$

(the subscript s means that we substitute $k^2 \rightarrow s$). The first two terms (ε^0 and ε^1) of the ε expansion of $\operatorname{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]$ are given in Eq. (B18). Note that the appearance of the factor $1/s^{n+l}$ in the integrand of Eq. (36) provides better convergence of the dispersive integral. This is another advantage of subtracting the first terms of the Taylor expansion in k^2 .

To deal with the four-point function [see Eq. (35)], we also need the dispersive integral representation for $\bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 2)$. To derive it, let us differentiate Eq. (36) with respect to m_2^2 ,

$$\bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 2) = \frac{\partial}{\partial m_2^2} \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1). \quad (38)$$

For the function in the integrand, we get

$$\begin{aligned} \frac{\partial}{\partial m_2^2} \operatorname{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s &= \operatorname{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 2)]_s \\ &= -\frac{(1+2\varepsilon)(s+m_1^2-m_2^2)}{\Delta_s} \operatorname{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s. \end{aligned} \quad (39)$$

The same result can be obtained by using Eq. (32) and taking into account that the tadpoles do not contribute to the imaginary part. Note that the limit of integration in (36) also depends on m_2 .

Taking into account that separate terms may have singularities as $s \rightarrow (m_1 + m_2)^2$, let us shift the lower limit by a small positive δ ,

$$s_{\delta} = (m_1 + m_2)^2 + \delta, \quad (40)$$

and at the end consider the limit $\delta \rightarrow 0$. Differentiating Eq. (36) with respect to m_2^2 , we get

$$\begin{aligned} \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 2) &= \frac{i}{\pi} \int_{s_\delta}^{\infty} \frac{ds}{s^{n+l+1}(s-k^2-i0)} \frac{\partial}{\partial m_2^2} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s \\ &\quad - \frac{i}{\pi} \frac{m_1 + m_2}{m_2} \frac{1}{s_\delta^{n+l+1}(s_\delta - k^2 - i0)} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_{s_\delta}. \end{aligned} \quad (41)$$

To proceed, let us use the analytic result given in Eq. (37) to calculate the derivatives

$$\begin{aligned} \frac{\partial}{\partial s} \{s^{-\varepsilon} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s\} &= 2\pi^{1-\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \pi \left(\frac{\partial \Delta_s}{\partial s} \right) \left(\frac{\partial}{\partial \Delta_s} (-\Delta_s)^{-1/2-\varepsilon} \right), \\ \frac{\partial}{\partial m_2^2} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s &= 2\pi^{1-\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \pi s^\varepsilon \left(\frac{\partial \Delta_s}{\partial m_2^2} \right) \left(\frac{\partial}{\partial \Delta_s} (-\Delta_s)^{-1/2-\varepsilon} \right). \end{aligned}$$

Combining these equations, we get

$$\frac{\partial}{\partial m_2^2} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s = s^\varepsilon \frac{(\partial \Delta_s / \partial m_2^2)}{(\partial \Delta_s / \partial s)} \frac{\partial}{\partial s} \{s^{-\varepsilon} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s\}. \quad (42)$$

Taking into account that

$$\frac{\partial \Delta_s}{\partial m_2^2} = 2(s + m_1^2 - m_2^2) \quad \text{and} \quad \frac{\partial \Delta_s}{\partial s} = -2(s - m_1^2 - m_2^2),$$

we arrive at

$$\frac{\partial}{\partial m_2^2} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s = -s^\varepsilon \frac{s + m_1^2 - m_2^2}{s - m_1^2 - m_2^2} \frac{\partial}{\partial s} \{s^{-\varepsilon} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s\}. \quad (43)$$

After transforming the derivative with respect to m_2^2 into the derivative with respect to s , we can apply integration by parts to the integral on the rhs of Eq. (41),

$$\begin{aligned} &\frac{i}{\pi} \int_{s_\delta}^{\infty} \frac{ds}{s^{n+l+1}(s-k^2-i0)} \frac{\partial}{\partial m_2^2} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s \\ &= -\frac{i}{\pi} \int_{s_\delta}^{\infty} \frac{s^\varepsilon ds}{s^{n+l+1}(s-k^2-i0)} \frac{s + m_1^2 - m_2^2}{s - m_1^2 - m_2^2} \frac{\partial}{\partial s} \{s^{-\varepsilon} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s\} \\ &= \frac{i}{\pi} \int_{s_\delta}^{\infty} ds s^{-\varepsilon} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s \frac{\partial}{\partial s} \left[\frac{s^\varepsilon}{s^{n+l+1}(s-k^2-i0)} \frac{s + m_1^2 - m_2^2}{s - m_1^2 - m_2^2} \right] \\ &\quad + \frac{i}{\pi} \frac{s_\delta + m_1^2 - m_2^2}{s_\delta - m_1^2 - m_2^2} \frac{1}{s_\delta^{n+l+1}(s_\delta - k^2 - i0)} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_{s_\delta}. \end{aligned} \quad (44)$$

Recalling that $s_\delta = (m_1 + m_2)^2 + \delta$, we can see that in the limit $\delta \rightarrow 0$ the last term on the rhs of Eq. (44) exactly cancels the nonintegral term in Eq. (41). Since the first (integral) term on the rhs of Eq. (44) is finite as $\delta \rightarrow 0$, we can put $\delta = 0$. In this way, we get

$$\begin{aligned} \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 2) &= \frac{i}{\pi} \int_{(m_1+m_2)^2}^{\infty} ds s^{-\varepsilon} \text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s \\ &\quad \times \frac{\partial}{\partial s} \left[\frac{s^\varepsilon}{s^{n+l+1}(s-k^2-i0)} \frac{s + m_1^2 - m_2^2}{s - m_1^2 - m_2^2} \right]. \end{aligned} \quad (45)$$

Using Eq. (45), we can also get another representation,

$$\begin{aligned} \bar{J}_{n+l+1}^{(2)}(2-2\epsilon; 1, 2) &= \frac{i}{\pi} \int_{(m_1+m_2)^2}^{\infty} ds \left\{ s^{-\epsilon} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s - (k^2)^{-\epsilon} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_{k^2} \right\} \\ &\times \frac{\partial}{\partial s} \left[\frac{s^{\epsilon}}{s^{n+l+1}(s-k^2-i0)} \frac{s+m_1^2-m_2^2}{s-m_1^2-m_2^2} \right] \\ &- \frac{i}{\pi} \frac{m_1+m_2}{m_2} \frac{[(m_1+m_2)^2]^{2\epsilon-n-l-1}}{(m_1+m_2)^2-k^2-i0} (k^2)^{-\epsilon} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_{k^2}, \end{aligned} \quad (46)$$

where $k^2 \leftrightarrow k^2 + i0$, in the same way as in the prescription $1/(s-k^2-i0)$.

Using Eq. (46) may lead to numerical issues (slow convergence, etc.), because after taking the derivative with respect to s in the integrand we obtain the term $1/(s-k^2-i0)^2$. To get an alternative representation, we start from Eq. (41) and substitute the result of Eq. (39) for the derivative with respect to m_2^2 ,

$$\begin{aligned} \bar{J}_{n+l+1}^{(2)}(2-2\epsilon; 1, 2) &= -\frac{i}{\pi} (1+2\epsilon) \int_{s_{\delta}}^{\infty} ds \frac{s+m_1^2-m_2^2}{s^{n+l+1}(s-k^2-i0)\Delta_s} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s \\ &- \frac{i}{\pi} \frac{m_1+m_2}{m_2} \frac{1}{s_{\delta}^{n+l+1}(s_{\delta}-k^2-i0)} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_{s_{\delta}}. \end{aligned} \quad (47)$$

Note that the integral in Eq. (47) is singular as $\delta \rightarrow 0$, because

$$\Delta_s = -[s-(m_1+m_2)^2][s-(m_1-m_2)^2] = -(s-s_0)(s-s_1), \quad (48)$$

with $s_0 \equiv (m_1+m_2)^2$, $s_1 \equiv (m_1-m_2)^2$. To separate the finite and the divergent contributions, let us employ the identity

$$\frac{1}{s-k^2-i0} = -\frac{s-s_{\delta}}{(s-k^2-i0)(s_{\delta}-k^2-i0)} + \frac{1}{s_{\delta}-k^2-i0}. \quad (49)$$

In this way, we get

$$\begin{aligned} \bar{J}_{n+l+1}^{(2)}(2-2\epsilon; 1, 2) &= \frac{i}{\pi} \frac{1+2\epsilon}{s_{\delta}-k^2-i0} \int_{s_{\delta}}^{\infty} ds \frac{(s+m_1^2-m_2^2)(s-s_{\delta})}{s^{n+l+1}(s-k^2-i0)\Delta_s} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s \\ &- \frac{i}{\pi} \frac{1+2\epsilon}{s_{\delta}-k^2-i0} \int_{s_{\delta}}^{\infty} ds \frac{s+m_1^2-m_2^2}{s^{n+l+1}\Delta_s} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s \\ &- \frac{i}{\pi} \frac{m_1+m_2}{m_2} \frac{1}{s_{\delta}^{n+l+1}(s_{\delta}-k^2-i0)} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_{s_{\delta}}. \end{aligned} \quad (50)$$

The first integral on the rhs of Eq. (50) is finite as $\delta \rightarrow 0$, so that we can put $\delta=0$ and substitute $(s-s_{\delta})/\Delta_s = (s-s_0)/\Delta_s = -1/(s-s_1)$. To deal with the second integral let us employ the analytic result given in Eq. (37) to calculate the derivative

$$\frac{\partial}{\partial s} \{s^{-\epsilon} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s\} = \frac{(1+2\epsilon)(s-m_1^2-m_2^2)}{\Delta_s} s^{-\epsilon} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s.$$

Therefore,

$$\frac{1+2\epsilon}{\Delta_s} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s = \frac{s^\epsilon}{s-m_1^2-m_2^2} \frac{\partial}{\partial s} \{s^{-\epsilon} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s\}.$$

Integrating by parts, we can transform the second integral in Eq. (50) as

$$\begin{aligned} & -\frac{i}{\pi} \frac{1+2\epsilon}{s_\delta - k^2 - i0} \int_{s_\delta}^{\infty} ds \frac{s+m_1^2-m_2^2}{s^{n+l+1} \Delta_s} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s \\ &= -\frac{i}{\pi} \frac{1}{s_\delta - k^2 - i0} \int_{s_\delta}^{\infty} ds \frac{s^\epsilon (s+m_1^2-m_2^2)}{s^{n+l+1} (s-m_1^2-m_2^2)} \frac{\partial}{\partial s} \{s^{-\epsilon} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s\} \\ &= \frac{i}{\pi} \frac{1}{s_\delta - k^2 - i0} \frac{s_\delta + m_1^2 - m_2^2}{s_\delta^{n+l+1} (s_\delta - m_1^2 - m_2^2)} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_{s_\delta} \\ &+ \frac{i}{\pi} \frac{1}{s_\delta - k^2 - i0} \int_{s_\delta}^{\infty} ds \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s s^{-\epsilon} \frac{\partial}{\partial s} \left[\frac{s^\epsilon (s+m_1^2-m_2^2)}{s^{n+l+1} (s-m_1^2-m_2^2)} \right]. \end{aligned} \quad (51)$$

We can see that in the limit $\delta \rightarrow 0$ the first (nonintegral) term on the rhs of Eq. (51) exactly cancels the nonintegral term in Eq. (50). The remaining integrals contributing to $\bar{J}_{n+l+1}^{(2)}(2-2\epsilon; 1, 2)$ are finite as $\delta \rightarrow 0$. In this way, putting $\delta = 0$ we arrive at the following alternative representation:

$$\begin{aligned} \bar{J}_{n+l+1}^{(2)}(2-2\epsilon; 1, 2) &= -\frac{i}{\pi} \frac{1+2\epsilon}{s_0 - k^2 - i0} \int_{s_0}^{\infty} ds \frac{s+m_1^2-m_2^2}{s^{n+l+1} (s-s_1)(s-k^2-i0)} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s \\ &+ \frac{i}{\pi} \frac{1}{s_0 - k^2 - i0} \int_{s_0}^{\infty} ds \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s s^{-\epsilon} \frac{\partial}{\partial s} \left[\frac{s^\epsilon (s+m_1^2-m_2^2)}{s^{n+l+1} (s-m_1^2-m_2^2)} \right]. \end{aligned} \quad (52)$$

Using partial fractioning in the denominator of the first integral in Eq. (52) and identifying some contributions as specific cases of Eq. (36), we can get the following representation:

$$\begin{aligned} \bar{J}_{n+l+1}^{(2)}(2-2\epsilon; 1, 2) &= -(1+2\epsilon) \frac{k^2 + m_1^2 - m_2^2}{\Delta} \bar{J}_{n+l+1}^{(2)}(2-2\epsilon; 1, 1) \\ &+ (1+2\epsilon) \frac{2m_1(m_1-m_2)}{\Delta} \bar{J}_{n+l+1}^{(2)}(2-2\epsilon; 1, 1) \Big|_{k^2=s_1} \\ &+ \frac{i}{\pi} \frac{1}{s_0 - k^2 - i0} \int_{s_0}^{\infty} ds \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s s^{-\epsilon} \frac{\partial}{\partial s} \left[\frac{s^\epsilon (s+m_1^2-m_2^2)}{s^{n+l+1} (s-m_1^2-m_2^2)} \right]. \end{aligned} \quad (53)$$

Furthermore, using integration by parts we can evaluate the remaining integral in Eq. (53) as

$$\begin{aligned} & \frac{i}{\pi} \int_{s_0}^{\infty} ds \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s s^{-\epsilon} \frac{\partial}{\partial s} \left[\frac{s^\epsilon (s+m_1^2-m_2^2)}{s^{n+l+1} (s-m_1^2-m_2^2)} \right] \\ &= \frac{i}{\pi} \frac{\partial}{\partial m_2^2} \int_{s_0}^{\infty} \frac{ds}{s^{n+l+1}} \text{Im} [i^{-1} J^{(2)}(2-2\epsilon; 1, 1)]_s \\ &= \frac{\partial}{\partial m_2^2} \bar{J}_{n+l}^{(2)}(2-2\epsilon; 1, 1) \Big|_{k^2=0}, \end{aligned} \quad (54)$$

where $\bar{J}_{n+l}^{(2)}(2-2\varepsilon; 1, 1)|_{k^2=0}$ is the $(n+l)$ th coefficient of the small- k^2 expansion of Eq. (21), namely,

$$\begin{aligned} \bar{J}_{n+l}^{(2)}(2-2\varepsilon; 1, 1)|_{k^2=0} = & \frac{(1+\varepsilon)_{n+l}}{(m_2^2 - m_1^2)^{1+2n+2l}} \left\{ J^{(2)}(2-2\varepsilon; 0, 1) \sum_{r=0}^{n+l} \frac{(n+l)!}{r!(n+l-r)!} \frac{(m_2^2)^r (m_1^2)^{n+l-r}}{(1-\varepsilon)_r (1+\varepsilon)_{n+l-r}} \right. \\ & \left. - J^{(2)}(2-2\varepsilon; 1, 0) \sum_{r=0}^j \frac{(n+l)!}{r!(n+l-r)!} \frac{(m_1^2)^r (m_2^2)^{n+l-r}}{(1-\varepsilon)_r (1+\varepsilon)_{n+l-r}} \right\}. \end{aligned} \quad (55)$$

In this way, we get

$$\begin{aligned} \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 2) = & -(1+2\varepsilon) \frac{k^2 + m_1^2 - m_2^2}{\Delta} \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1) \\ & + (1+2\varepsilon) \frac{2m_1(m_1 - m_2)}{\Delta} \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1) \Big|_{k^2=s_1} \\ & + \frac{1}{s_0 - k^2 - i0} \frac{\partial}{\partial m_2^2} \bar{J}_{n+l}^{(2)}(2-2\varepsilon; 1, 1) \Big|_{k^2=0}. \end{aligned} \quad (56)$$

or

$$\begin{aligned} \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 2) = & \frac{1}{s_0 - k^2 - i0} \left\{ -(1+2\varepsilon) \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1) \right. \\ & - (1+2\varepsilon) \frac{2m_1(m_1 - m_2)}{k^2 - s_1} \left[\bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1) - \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1) \Big|_{k^2=s_1} \right] \\ & \left. + \frac{\partial}{\partial m_2^2} \bar{J}_{n+l}^{(2)}(2-2\varepsilon; 1, 1) \Big|_{k^2=0} \right\}. \end{aligned} \quad (57)$$

The dispersion integral representation for $\bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1)$ is given in Eq. (36), and the combination of integrals in the second line of Eq. (57) can be presented as

$$\frac{1}{k^2 - s_1} \left[\bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1) - \bar{J}_{n+l+1}^{(2)}(2-2\varepsilon; 1, 1) \Big|_{k^2=s_1} \right] = \frac{i}{\pi} \int_{s_0}^{\infty} ds \frac{\text{Im} [i^{-1} J^{(2)}(2-2\varepsilon; 1, 1)]_s}{s^{n+l+1} (s - k^2 - i0) (s - s_1 - i0)}.$$

III. NUMERICAL EXAMPLES

In this section, we provide a numerical comparison of three- and four-point functions calculated using techniques outlined in the last section and the Collier [57–60] numerical library.

For numerical integration over the dispersive and Feynman parameters, we have used the *Mathematica* GlobalAdaptive method. As it can be seen from Fig. 3, results are in excellent agreement with Collier. Numerical comparison for four-point functions is given in Fig. 4. As in the case of three-point functions, the four-point example shows that we have rather good consistency with Collier. At this point, we are ready to apply derived many-point

functions in dispersive representation to the evaluation of two-loop diagrams.

IV. ROADMAP TO TWO-LOOP CALCULATIONS

As we can see from the previous sections, we have successfully represented one-loop (up to multiplicity four) integrals with an arbitrary tensor rank using recurrence and dispersive methods. In addition, we were able to reduce higher multiplicity PV functions to the two-point result. Finally, we have adopted the dispersive technique introduced in [26–28] to subtracted two-point functions. Now we have analytical results for PV functions with polynomial terms in k^2 and the dispersive term carrying a

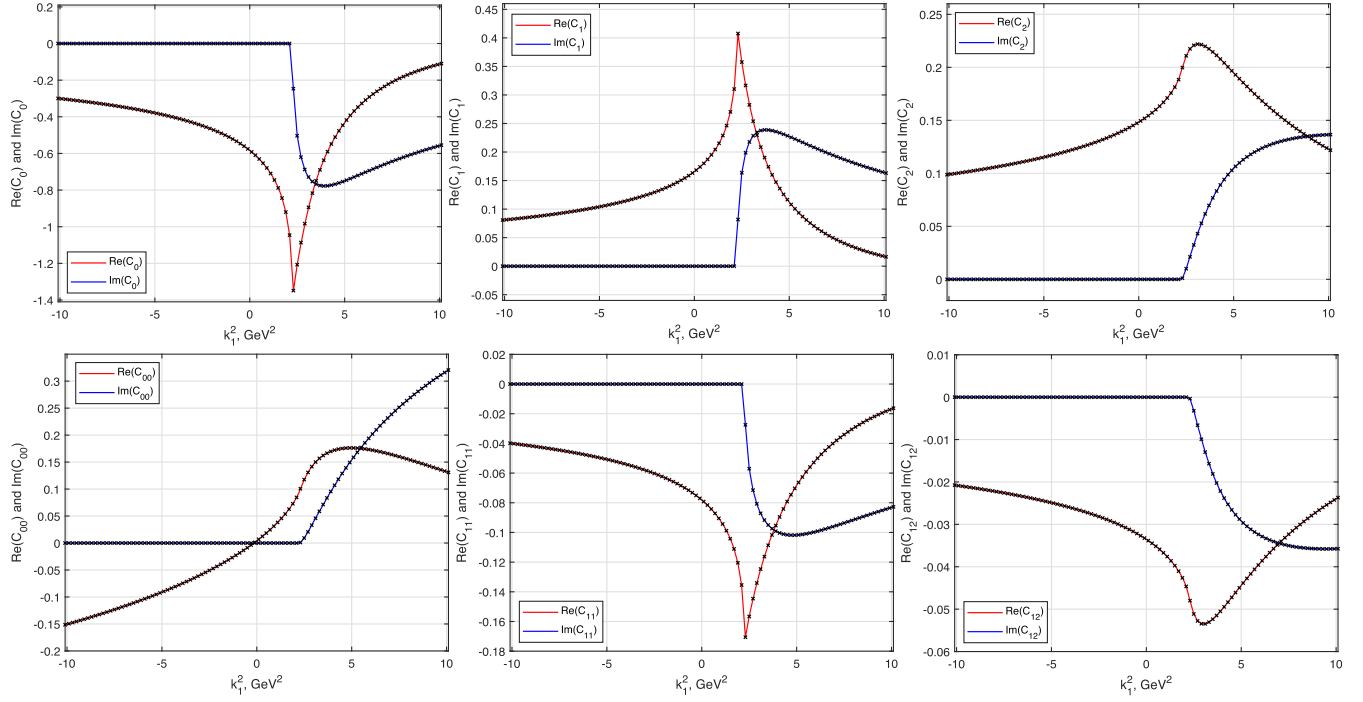


FIG. 3. Numerical results for the three-point functions C_0 , C_1 , C_2 , C_{00} , C_{11} , and C_{12} [$k_2^2 = -1.5 \text{ GeV}^2$, $(k_1 + k_2)^2 = m_3^2$, $m_1 = 0.5 \text{ GeV}$, $m_2 = 1.0 \text{ GeV}$, $m_3 = 1.5 \text{ GeV}$]. The functions C_{2l,n_1,n_2} are defined in Eq. (11). Crossed dots are results based on this work and solid lines are produced from the Collier library.

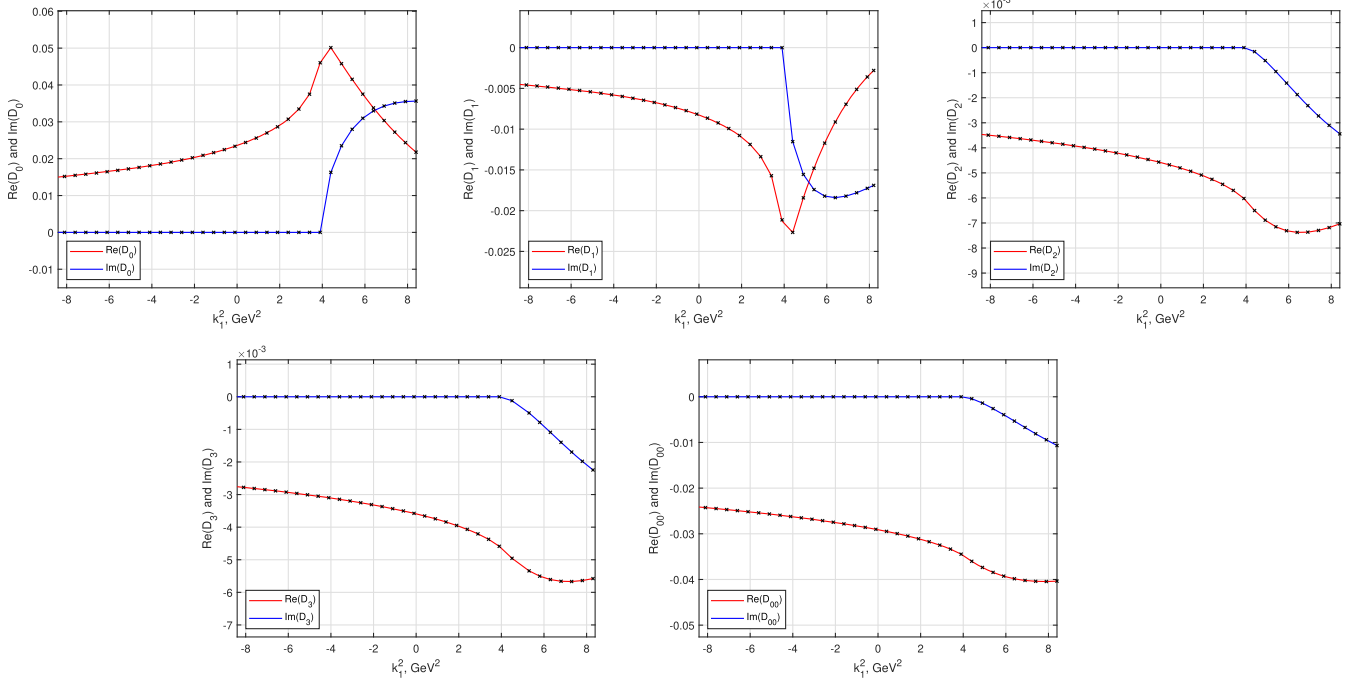


FIG. 4. Numerical results for the four-point functions D_{2l,n_1,n_2,n_3} [$k_2^2 = -1.5 \text{ GeV}^2$, $k_3^2 = -2.5 \text{ GeV}^2$, $k_4^2 = m_4^2$, $(k_1 \cdot k_3) = 4.0 \text{ GeV}^2$, $(k_2 \cdot k_3) = -1.0 \text{ GeV}^2$, $m_1 = 1.5 \text{ GeV}$, $m_2 = 0.5 \text{ GeV}$, $m_3 = 2.0 \text{ GeV}$, $m_4 = 2.5 \text{ GeV}$]. The functions D_{2l,n_1,n_2,n_3} are defined in Eq. (13). Crossed dots are results based on this work and solid lines are produced from the Collier library.

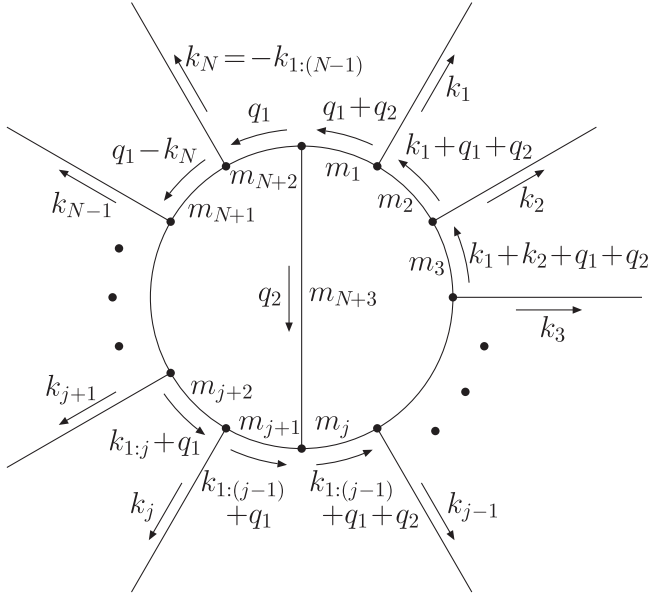


FIG. 5. The two-loop planar N -point diagram ($k_{1:j} \equiv k_1 + \dots + k_j$).

propagatorlike structure $\propto \frac{1}{(s-k^2-i0)}$. This particular representation is most valuable for applications in two-loop calculations for any possible particle physics models. It is

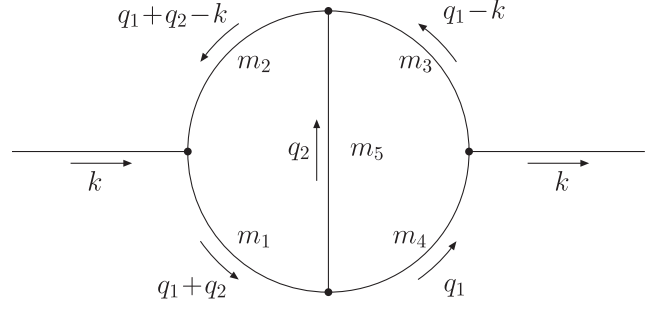


FIG. 6. Two-loop scalar example.

worth mentioning that, in this section, we will consider only planar topology for the two-loop graphs. In the case of nonplanar graphs, we can rearrange momenta in the loop integral and reduce the result to the dispersive representation considered in [27]. First, if we consider a $(j+2)$ -point Feynman graph as an insertion [index $(j+2)$ means that we have j number of external and two internal legs in the insertion] into the two-loop planar topology (see Fig. 5), then polynomial terms in external or second-loop momenta will be a part of the numerator algebra and the dispersive contribution will be treated as an additional propagator in the second-loop integral. Let us define the two-loop integral in a similar way as it was done in Eq. (1),

$$\begin{aligned}
 U_{\mu_1 \dots \mu_{M+L}}^{(N)} &= \left(\frac{\mu^{4-D} e^{\gamma_E(4-D)/2}}{i\pi^{D/2}} \right)^2 \int \int d^D q_1 d^D q_2 \\
 &\times \frac{q_{1\mu_1} \dots q_{1\mu_M} q_{2\mu_{M+1}} \dots q_{2\mu_{M+L}}}{[q_1^2 - m_{N+2}^2][q_2^2 - m_{N+3}^2][(q_1 + q_2)^2 - m_1^2][(k_1 + q_1 + q_2)^2 - m_2^2]} \\
 &\times \frac{1}{[(k_1 + k_2 + q_1 + q_2)^2 - m_3^2] \dots [(k_1 + \dots + k_{j-1} + q_1 + q_2)^2 - m_j^2]} \\
 &\times \frac{1}{[(k_1 + \dots + k_{j-1} + q_1)^2 - m_{j+1}^2] \dots [(k_1 + \dots + k_{N-1} + q_1)^2 - m_{N+1}^2]}. \quad (58)
 \end{aligned}$$

We start our evaluation with the integration over one of the loop momentum. The next step is to apply tensor decomposition and reduce one-loop insertion of Eq. (58) to $J^{(j+2)}(D+2l+2n_1+\dots+2n_{j+1}; 1, 1+n_1, \dots, 1+n_{j+1})$. After that, we can apply the Feynman trick to reduce the number of the propagators in the first-loop integration to two, which will result in the two-point function with one of the propagators in the power of $(1+j+n_1+\dots+n_{j+1})$: $J^{(2)}(D+2l+2n_1+\dots+2n_{j+1}; 1, 1+j+n_1+\dots+n_{j+1})$. Using the recurrence approach, we can reduce the number of the dimensions to $D=2-2\epsilon$ and the propagator's power to one, resulting in the insertion expressed as a subtracted UV-finite $\bar{J}_{l+n_1+\dots+n_{j+1}}^{(2)}(2-2\epsilon; 1, 1)$ two-point function and two UV-divergent tadpoles, $J^{(2)}(2-2\epsilon; 1, 0)$

and $J^{(2)}(2-2\epsilon; 0, 1)$, multiplied by the polynomials in external and second-loop momenta. Subtracted $\bar{J}_{l+n_1+\dots+n_{j+1}}^{(2)}(2-2\epsilon; 1, 1)$ can be expressed dispersively [see Eq. (36)], and in the second-loop integral, we will receive an additional propagator and terms in the numerator expressed as polynomials in the momenta. Second-loop integration is now reduced to a one-loop integral where we can apply well-tested packages, such as x [61,62], FeynCalc [63,64], FormCalc [56], and FORM [65] to complete two-loop evaluation.

In order to demonstrate how the outlined roadmap can be applied to two-loop calculations, we chose to consider the well-known example originally introduced in [66]. We start our example (see Fig. 6) with the two-loop integral

$$U^{(2)} = \left(\frac{\mu^{4-D} e^{\gamma_E(4-D)/2}}{i\pi^{D/2}} \right)^2 \int \int \frac{d^D q_1 d^D q_2}{[q_1^2 - m_4^2][q_2^2 - m_5^2][(q_1 + q_2)^2 - m_1^2][(q_1 + q_2 - k)^2 - m_2^2][(q_1 - k)^2 - m_3^2]}, \quad (59)$$

and first perform integration over the loop momentum q_2 . That results in a three-point integral

$$J^{(3)}(D; 1, 1, 1) \Big|_{\substack{p_1=0, p_2=q_1, p_3=q_1-k \\ m_1^2 \leftrightarrow m_5^2, m_2^2 \leftrightarrow m_1^2, m_3^2 \leftrightarrow m_2^2}} = \int \frac{d^D q_2}{[q_2^2 - m_5^2][(q_1 + q_2)^2 - m_1^2][(q_1 + q_2 - k)^2 - m_2^2]}. \quad (60)$$

Using Eq. (12) (with $\bar{x} = 1 - x$), we get

$$\begin{aligned} J^{(3)}(D; 1, 1, 1) &= \int_0^1 dx J^{(2)}(D; 1, 2) \Big|_{\substack{p_1=0, p_2=q_1-\bar{x}k \\ m_1^2 \leftrightarrow m_5^2, m_2^2 \leftrightarrow xm_1^2 + \bar{x}m_2^2 - x\bar{x}k^2}} \\ &= \int_0^1 dx \int \frac{d^D q_2}{[q_2^2 - m_5^2][(q_1 - \bar{x}k + q_2)^2 - xm_1^2 - \bar{x}m_2^2 + x\bar{x}k^2]^2}, \end{aligned} \quad (61)$$

which effectively gives us a reduction of the three-point integral to a two-point one. Here it is assumed that $D = 4 - 2\varepsilon$. Applying the recursive approach outlined in Sec. II, we can lower the second power of the last propagator in Eq. (61) and arrive at the subtracted $\bar{J}_1^{(2)}(2 - 2\varepsilon; 1, 1)$ plus tadpole terms containing $J^{(2)}(2 - 2\varepsilon; 1, 0)$ and $J^{(2)}(2 - 2\varepsilon; 0, 1)$ [see Eq. (27)]. In this way, Eq. (60) can be written in the following form:

$$\begin{aligned} J^{(3)}(4 - 2\varepsilon; 1, 1, 1) &= \frac{1}{2} \int_0^1 dx \left\{ i\pi^{2-\varepsilon} \Gamma(\varepsilon) \frac{(m_5^2)^{-\varepsilon} - (m_{12x}^2)^{-\varepsilon}}{m_5^2 - m_{12x}^2} \right. \\ &\quad \left. - [(q_1 - \bar{x}k)^2 + m_5^2 - m_{12x}^2] \pi \bar{J}_1^{(2)}(2 - 2\varepsilon; 1, 1) \right\}, \end{aligned} \quad (62)$$

with $m_{12x}^2 = xm_1^2 + \bar{x}m_2^2 - x\bar{x}k^2$. For the subtracted $\bar{J}_1^{(2)}(2 - 2\varepsilon; 1, 1)$, we can apply dispersive representation stemming from Eqs. (36) and (37),

$$\bar{J}_1^{(2)}(2 - 2\varepsilon; 1, 1) = \frac{i}{\pi} \int_{(m_5 + m_{12x})^2}^{\infty} ds \frac{\text{Im} [i^{-1} J^{(2)}(2 - 2\varepsilon; 1, 1)]_s}{s[s - (q_1 - \bar{x}k)^2 - i0]}.$$

Since the insertion $J^{(3)}(D; 1, 1, 1)$ and the entire two-loop integral are UV finite, we can consider the limit $\varepsilon \rightarrow 0$,

$$J^{(3)}(4; 1, 1, 1) = \frac{1}{2} \int_0^1 dx \left\{ \frac{i\pi^2}{m_5^2 - m_{12x}^2} \ln \frac{m_{12x}^2}{m_5^2} - [(q_1 - \bar{x}k)^2 + m_5^2 - m_{12x}^2] \pi \bar{J}_1^{(2)}(2; 1, 1) \right\}. \quad (63)$$

Note that at $\varepsilon = 0$ the imaginary part of $i^{-1} J^{(2)}(2 - 2\varepsilon; 1, 1)$ has a simple structure,

$$\text{Im} [i^{-1} J^{(2)}(2; 1, 1)]_s = \frac{2\pi^2}{\sqrt{(s - m_{12x}^2 - m_5^2)^2 - 4m_{12x}^2 m_5^2}}.$$

TABLE I. Numerical results for $U^{(2)}$ (Fig. 6) calculated from Eq. (65) and compared to [26,66]. The masses are $m_1 = 2.0$, $m_2 = 1.0$, $m_3 = 4.0$, $m_4 = 5.0$, and $m_5 = 3.0$ GeV.

k^2 (GeV ²)	This work	Δt (sec)	[26]	Δt (sec)	[66] (Table I)	Δt (sec)
-50.0	-0.08295	1.0	-0.08296	75.0	...	...
-10.0	-0.18399	0.7	-0.18399	22.0	...	...
-5.0	-0.22180	0.7	-0.22178	17.0	...	...
-1.0	-0.26923	0.7	-0.26919	8.0	...	...
-0.5	-0.27704	0.7	-0.27712	9.0	...	...
-0.1	-0.28372	0.7	-0.28360	9.0	...	...
0.1	-0.28723	0.7	-0.28714	9.0	-0.28724	0.6
0.5	-0.29458	0.7	-0.29443	9.0	-0.29459	0.7
1.0	-0.30451	0.7	-0.30449	10.0	-0.30452	0.7
5.0	-0.45250	0.8	-0.45230	14.0	-0.45252	0.7
10.0	-0.48807 - 0.35309i	4.0	-0.48810 - 0.35318i	30.0	-0.48815 - 0.35322i	0.7
50.0	0.17390 - 0.11804i	76.0	0.17335 - 0.11781i	1120.0	0.17390 - 0.11808i	1.4

At this point, we are ready to complete integration over second-loop momentum q_1 ,

$$U^{(2)} = \frac{1}{2} \int_0^1 dx \left\{ \frac{1}{m_5^2 - m_{12x}^2} \ln \frac{m_{12x}^2}{m_5^2} B_0 + \frac{1}{\pi^2} \int_{(m_5 + m_{12x})^2}^{\infty} ds \frac{\text{Im} [i^{-1} J^{(2)}(2; 1, 1)]_s}{s} [(s + m_5^2 - m_{12x}^2) C_0 + B_0] \right\}, \quad (64)$$

where in the second-loop integration we have used the usual PV functions without dispersive representation. In Eq. (64), the three-point function has the following arguments: $C_0 \equiv C_0(k^2, x^2 k^2, \bar{x}^2 k^2, m_4^2, m_3^2, s)$ [here we have used the following mapping of arguments for $C_0 \equiv C_0(k_1^2, k_2^2, (k_1 + k_2)^2, m_1^2, m_2^2, m_3^2)$] and the two-point function $B_0 \equiv B_0(k^2, m_4^2, m_3^2)$. Since $B_0(k^2, m_4^2, m_3^2)$ does

not depend on either dispersive or Feynman parameters, we can evaluate the dispersive integral multiplied by $B_0(k^2, m_4^2, m_3^2)$ analytically. As a result, the first term in Eq. (64) cancels out with dispersive integration times $B_0(k^2, m_4^2, m_3^2)$. The final two-loop result has a rather simple form,

$$U^{(2)} = \frac{1}{2\pi^2} \int_0^1 dx \int_{(m_5 + m_{12x})^2}^{\infty} \frac{ds}{s} (s + m_5^2 - m_{12x}^2) \text{Im} [i^{-1} J^{(2)}(2; 1, 1)]_s C_0. \quad (65)$$

The three-point function can also be written analytically,

$$C_0 = \frac{1}{s - m_{43x}^2} [x \text{Disc}(x^2 k^2, m_3^2, s) + \bar{x} \text{Disc}(\bar{x}^2 k^2, m_3^2, s) - \text{Disc}(k^2, m_3^2, m_4^2)] + \frac{1}{2x\bar{x}k^2} \left(\ln \frac{m_3^2}{s} - x \ln \frac{m_3^2}{m_4^2} \right). \quad (66)$$

Here, $m_{43x}^2 = xm_4^2 + \bar{x}m_3^2 - x\bar{x}k^2$ and $\text{Disc}(k^2, m_1^2, m_2^2)$ is a discontinuity of the two-point function, which contains a branch cut from $(m_1 + m_2)^2$ to infinity and has the following structure:

$$\text{Disc}(k^2, m_1^2, m_2^2) = \frac{\sqrt{-\Delta(k^2, m_1^2, m_2^2)}}{k^2} \ln \left(\frac{m_1^2 + m_2^2 - k^2 + \sqrt{-\Delta(k^2, m_1^2, m_2^2)}}{2m_1 m_2} \right). \quad (67)$$

At this point, using Eqs. (65)–(67), we can reproduce numerical results for the two-loop graph $U^{(2)}$ in Fig. 6. To make a comparison to the earlier works [26,66], we will use $m_1 = 2.0$, $m_2 = 1.0$, $m_3 = 4.0$, $m_4 = 5.0$, and $m_5 = 3.0$ GeV. For the numerical integration, we shifted k^2 and all the masses by $i \times 10^{-16}$ to remove singular behavior at the poles in Eq. (65). Numerical results in Table I are in very good agreement if compared to previously obtained values in [26,66]. It is worth

noting that, in this work, as well as in Ref. [26], *Mathematica* was used to complete numerical integration using the GlobalAdaptive method. In Ref. [66], the QUADPACK routine was applied for numerical integration.

V. CONCLUSION

In this paper, we continue the development of the dispersive approach for the calculation of multiloop Feynman diagrams. This study builds upon our previous work, where we introduced a general framework based on representing multipoint Passarino-Veltman functions in a two-point-function basis, thereby allowing the replacement of subloop diagrams by effective propagators. In the present work, we extend this framework by employing shifted space-time dimensions in the tensor decomposition of the subloops, together with recurrence relations that systematically lower both the dimensionality and the powers of propagators. These relations algebraically minimize the number of basic dispersive integrals required for numerical evaluation. Furthermore, the complexity of the resulting expressions can be reduced by subtracting a finite number of terms from the small-momentum expansion, which significantly improves the convergence of the dispersion integrals. Compared to the differentiation-based approach with respect to internal masses used in our earlier study, this algebraic reduction scheme proves substantially more efficient numerically, reducing computation time and enabling the treatment of more complex topologies. Our method complements recent advances in two-loop electroweak calculations by providing a semianalytical pathway that combines the dispersive representation of subloops, dimension-recurrence identities, and the two-point-basis decomposition. Since obtaining fully analytic results for general two-loop, multileg electroweak amplitudes remains exceptionally challenging, our approach offers a scalable and robust alternative. Instead of pursuing closed-form

solutions for each diagram, we transform the problem into a compact set of well-behaved dispersive integrals amenable to stable numerical evaluation, enabling precision predictions directly applicable to current and upcoming experiments such as MOLLER, P2, and Belle II. Looking ahead, this framework establishes a solid foundation for the automation of multiloop calculations in a dispersive representation. The next steps will involve implementing the reduction and integration algorithms into a numerical library optimized for precision electroweak observables and extending the method to full two-loop amplitudes with multiple mass scales. Such developments will enable comprehensive, *ab initio* predictions for a broad class of processes relevant to the ongoing and future precision programs at JLab, MESA, and KEK, bridging the gap between analytical theory and phenomenological applications. Ultimately, the goal is not merely the refinement of individual calculations but the construction of a predictive, *ab initio* framework capable of interpreting deviations in upcoming experiments as definitive signals of new physics. In this broader context, the ongoing development of the dispersive approach, along with canonical two-loop methods and numerical integration tools, represents an essential component of the global precision-physics enterprise.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: RECURRENCE RELATIONS

According to the general notation (1) and (3), we define the scalar two-point integrals as

$$J^{(2)}(D; \nu_1, \nu_2) = \int \frac{d^D q}{[q^2 - m_1^2]^{\nu_1} [(k+q)^2 - m_2^2]^{\nu_2}}. \quad (\text{A1})$$

Using the integration by parts technique [67,68], we get the following recurrence relations for these integrals [69]:

$$\begin{aligned} \mathbf{1}^+ J^{(2)}(D; \nu_1, \nu_2) &= \frac{1}{\nu_1 \Delta} \{ [(k^2 - m_1^2)(D - \nu_1 - 2\nu_2) + m_2^2(D - 3\nu_1)] \\ &\quad - 2\nu_2 m_2^2 \mathbf{1}^- \mathbf{2}^+ - \nu_1 (k^2 - m_1^2 - m_2^2) \mathbf{1}^+ \mathbf{2}^- \} J^{(2)}(D; \nu_1, \nu_2), \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \mathbf{2}^+ J^{(2)}(D; \nu_1, \nu_2) &= \frac{1}{\nu_2 \Delta} \{ [(k^2 - m_2^2)(D - 2\nu_1 - \nu_2) + m_1^2(D - 3\nu_2)] \\ &\quad - 2\nu_1 m_1^2 \mathbf{1}^+ \mathbf{2}^- - \nu_2 (k^2 - m_1^2 - m_2^2) \mathbf{1}^- \mathbf{2}^+ \} J^{(2)}(D; \nu_1, \nu_2), \end{aligned} \quad (\text{A3})$$

with $\mathbf{1}^\pm J^{(2)}(D; \nu_1, \nu_2) = J^{(2)}(D; \nu_1 \pm 1, \nu_2)$, $\mathbf{2}^\pm J^{(2)}(D; \nu_1, \nu_2) = J^{(2)}(D; \nu_1, \nu_2 \pm 1)$, and

$$\begin{aligned}
\Delta &\equiv \Delta(m_1^2, m_2^2, k^2) = 2k^2 m_1^2 + 2k^2 m_2^2 + 2m_1^2 m_2^2 - (k^2)^2 - m_1^4 - m_2^4 \\
&= 4m_1^2 m_2^2 - (k^2 - m_1^2 - m_2^2)^2 \\
&= -[k^2 - (m_1 + m_2)^2][k^2 - (m_1 - m_2)^2] \\
&= -\lambda(m_1^2, m_2^2, k^2),
\end{aligned} \tag{A4}$$

where $\lambda(m_1^2, m_2^2, k^2)$ is the standard notation for the Källén function. Note that the sum of the indices ν_1 and ν_2 on the rhs of Eqs. (A2) and (A3) is less by 1 than their sum on the lhs. Therefore, by using these relations, all the integrals with higher integer ν 's can be expressed in terms of the integral $J^{(2)}(D; 1, 1)$ and the massive tadpoles

$$J^{(2)}(D; \nu_1, 0) = i^{1-2\nu_1} \pi^{D/2} \frac{\Gamma(\nu_1 - D/2)}{\Gamma(\nu_1)} (m_1^2)^{D/2-\nu_1} = (-m_1^2)^{1-\nu_1} \frac{\Gamma(\nu_1 - D/2)}{\Gamma(\nu_1)\Gamma(1-D/2)} J^{(2)}(D; 1, 0), \tag{A5}$$

$$J^{(2)}(D; 0, \nu_2) = i^{1-2\nu_2} \pi^{D/2} \frac{\Gamma(\nu_2 - D/2)}{\Gamma(\nu_2)} (m_2^2)^{D/2-\nu_2} = (-m_2^2)^{1-\nu_2} \frac{\Gamma(\nu_2 - D/2)}{\Gamma(\nu_2)\Gamma(1-D/2)} J^{(2)}(D; 0, 1). \tag{A6}$$

Furthermore, to bring the shifted values of the space-time dimension D back to $4 - 2\varepsilon$ (or $2 - 2\varepsilon$) we can use the following relation, which can be obtained by using the geometrical approach [70,71] or the functional relations [9]:

$$\begin{aligned}
J^{(2)}(D+2; \nu_1, \nu_2) &= -\frac{\pi}{2k^2(D-\nu_1-\nu_2+1)} [\Delta J^{(2)}(D; \nu_1, \nu_2) + (k^2 + m_1^2 - m_2^2) J^{(2)}(D; \nu_1, \nu_2 - 1) \\
&\quad + (k^2 - m_1^2 + m_2^2) J^{(2)}(D; \nu_1 - 1, \nu_2)].
\end{aligned} \tag{A7}$$

In particular, for $\nu_1 = \nu_2 = 1$, Eq. (A7) yields

$$J^{(2)}(D+2; 1, 1) = -\frac{\pi}{2k^2(D-1)} [\Delta J^{(2)}(D; 1, 1) + (k^2 + m_1^2 - m_2^2) J^{(2)}(D; 1, 0) + (k^2 - m_1^2 + m_2^2) J^{(2)}(D; 0, 1)]. \tag{A8}$$

To deal with the occurring tadpole integrals, we can use the following formulas [which follow from Eqs. (A5) and (A6)]:

$$J^{(2)}(D+2j; 1, 0) = \pi^j (m_1^2)^j \frac{\Gamma(1-D/2-j)}{\Gamma(1-D/2)} J^{(2)}(D; 1, 0), \tag{A9}$$

$$J^{(2)}(D+2j; 0, 1) = \pi^j (m_2^2)^j \frac{\Gamma(1-D/2-j)}{\Gamma(1-D/2)} J^{(2)}(D; 0, 1). \tag{A10}$$

Using these relations, we can express any integral $J^{(2)}(D+2j; \nu_1, \nu_2)$ (with non-negative integers j , ν_1 , and ν_2) in terms of three integrals, $J^{(2)}(D; 1, 1)$, $J^{(2)}(D; 1, 0)$, and $J^{(2)}(D; 0, 1)$.

Usually the recurrence with respect to the space-time dimension D stops at $D = 4 - 2\varepsilon$. However, as an option, we can also use Eq. (A8) one more time, to reduce $J^{(2)}(4 - 2\varepsilon; 1, 1)$ to $J^{(2)}(2 - 2\varepsilon; 1, 1)$ (which is UV finite as $\varepsilon \rightarrow 0$),

$$\begin{aligned}
J^{(2)}(4 - 2\varepsilon; 1, 1) &= -\frac{\pi}{2k^2(1-2\varepsilon)} [\Delta J^{(2)}(2 - 2\varepsilon; 1, 1) + (k^2 + m_1^2 - m_2^2) J^{(2)}(2 - 2\varepsilon; 1, 0) \\
&\quad + (k^2 - m_1^2 + m_2^2) J^{(2)}(2 - 2\varepsilon; 0, 1)],
\end{aligned} \tag{A11}$$

and then use $J^{(2)}(2 - 2\varepsilon; 1, 1)$, $J^{(2)}(2 - 2\varepsilon; 1, 0)$, and $J^{(2)}(2 - 2\varepsilon; 0, 1)$ as the master integrals.

Combining Eqs. (A2), (A3), and (A7), one can get another pair of useful relations [9],

$$\begin{aligned}
J^{(2)}(D+2; \nu_1+1, \nu_2) &= -\frac{\pi}{2\nu_1 k^2} [(k^2 - m_1^2 + m_2^2) J^{(2)}(D; \nu_1, \nu_2) \\
&\quad + J^{(2)}(D; \nu_1, \nu_2 - 1) - J^{(2)}(D; \nu_1 - 1, \nu_2)],
\end{aligned} \tag{A12}$$

$$J^{(2)}(D+2; \nu_1, \nu_2+1) = -\frac{\pi}{2\nu_2 k^2} [(k^2 + m_1^2 - m_2^2)J^{(2)}(D; \nu_1, \nu_2) - J^{(2)}(D; \nu_1, \nu_2-1) + J^{(2)}(D; \nu_1-1, \nu_2)]. \quad (\text{A13})$$

They can be used to simultaneously reduce one of the indices (ν_1 or ν_2) and the space-time dimension D . A nice property of Eqs. (A12) and (A13) is the absence of Δ in the denominators.

APPENDIX B: ε EXPANSION OF THE MASTER INTEGRAL

In general, the expansion of the master integral $J^{(2)}(4-2\varepsilon; 1, 1)$ is known to an arbitrary order in ε [72–74]. Keeping terms up to the order ε , we get

$$J^{(2)}(4-2\varepsilon; 1, 1) = \frac{i\pi^{2-\varepsilon}\Gamma(1+\varepsilon)}{2(1-2\varepsilon)} \left\{ \frac{(m_1^2)^{-\varepsilon} + (m_2^2)^{-\varepsilon}}{\varepsilon} + \frac{m_1^2 - m_2^2}{\varepsilon k^2} [(m_1^2)^{-\varepsilon} - (m_2^2)^{-\varepsilon}] + \left[1 + \varepsilon \ln\left(\frac{k^2}{\Delta}\right) \right] F_1 + 2\varepsilon F_2 + \mathcal{O}(\varepsilon^2) \right\}, \quad (\text{B1})$$

where $\Delta \equiv \Delta(m_1^2, m_2^2, k^2)$ is defined in Eq. (A4). For the integral in $2-2\varepsilon$ dimensions, we get [e.g., using Eq. (A11)]

$$J^{(2)}(2-2\varepsilon; 1, 1) = -i\pi^{1-\varepsilon}\Gamma(1+\varepsilon) \frac{k^2}{\Delta} \left\{ \left[1 + \varepsilon \ln\left(\frac{k^2}{\Delta}\right) \right] F_1 + 2\varepsilon F_2 + \mathcal{O}(\varepsilon^2) \right\}. \quad (\text{B2})$$

Between the pseudothreshold and the threshold, when $(m_1 - m_2)^2 \leq k^2 \leq (m_1 + m_2)^2$ and $\Delta \geq 0$, the functions F_i can be presented as

$$F_i = \frac{\sqrt{\Delta}}{k^2} \sum_{i=1}^2 [\text{Ls}_i(\pi) - \text{Ls}_i(2\tau'_{0i})], \quad (\text{B3})$$

where

$$\cos \tau'_{01} = \frac{m_1^2 - m_2^2 + k^2}{2m_1 \sqrt{k^2}}, \quad \cos \tau'_{02} = \frac{m_2^2 - m_1^2 + k^2}{2m_2 \sqrt{k^2}}, \quad (\text{B4})$$

and the log-sine integrals are defined as

$$\text{Ls}_j(\theta) \equiv -\int_0^\theta d\theta' \ln^{j-1} \left| 2 \sin \frac{\theta'}{2} \right|. \quad (\text{B5})$$

In particular, $\text{Ls}_1(\theta) = -\theta$ and $\text{Ls}_2(\theta) = \text{Cl}_2(\theta)$, where

$$\text{Cl}_2(\theta) = \frac{1}{2i} [\text{Li}_2(e^{i\theta}) - \text{Li}_2(e^{-i\theta})] \quad (\text{B6})$$

is the Clausen function. Therefore,

$$F_1 = \frac{\sqrt{\Delta}}{k^2} \sum_{i=1}^2 [\text{Ls}_1(\pi) - \text{Ls}_1(2\tau'_{0i})] = -2 \frac{\sqrt{\Delta}}{k^2} \arccos\left(\frac{m_1^2 + m_2^2 - k^2}{2m_1 m_2}\right), \quad (\text{B7})$$

$$F_2 = \frac{\sqrt{\Delta}}{k^2} \sum_{i=1}^2 [\text{Ls}_2(\pi) - \text{Ls}_2(2\tau'_{0i})] = -\frac{\sqrt{\Delta}}{k^2} [\text{Cl}_2(2\tau'_{01}) + \text{Cl}_2(2\tau'_{02})]. \quad (\text{B8})$$

In other regions (where $\Delta < 0$), one can use analytic continuation. This process was described in Ref. [74], at any order in ε . Introducing variables $z_i = e^{i\sigma\theta_i}$, such that $\theta_i = 2\tau'_{0i}$, $\sigma = \pm 1$, we get

$$i\sigma[\text{Ls}_1(\pi) - \text{Ls}_1(\theta_i)] = \ln(-z_i), \quad (\text{B9})$$

$$i\sigma[\text{Ls}_2(\pi) - \text{Ls}_2(\theta_i)] = -\frac{1}{2}[\text{Li}_2(z_i) - \text{Li}_2(1/z_i)]. \quad (\text{B10})$$

In our case, the variables z_1 and z_2 can be presented as

$$z_1 = \frac{k^2 + m_1^2 - m_2^2 + \sqrt{-\Delta}}{k^2 + m_1^2 - m_2^2 - \sqrt{-\Delta}}, \quad z_2 = \frac{k^2 - m_1^2 + m_2^2 + \sqrt{-\Delta}}{k^2 - m_1^2 + m_2^2 - \sqrt{-\Delta}}. \quad (\text{B11})$$

In this way, we get

$$F_1 = -\frac{\sqrt{-\Delta}}{k^2} [\ln(-z_1) + \ln(-z_2)], \quad (\text{B12})$$

$$F_2 = -\frac{\sqrt{-\Delta}}{2k^2} \left[\text{Li}_2\left(\frac{1}{z_1}\right) - \text{Li}_2(z_1) + \text{Li}_2\left(\frac{1}{z_2}\right) - \text{Li}_2(z_2) \right]. \quad (\text{B13})$$

In particular, above the threshold [for $k^2 > (m_1 + m_2)^2$, where $z_1 > 1$ and $z_2 > 1$], we can explicitly separate the real and imaginary parts,

$$F_1 = -\frac{\sqrt{-\Delta}}{k^2} [\ln(z_1 z_2) - 2i\pi], \quad (\text{B14})$$

$$F_2 = -\frac{\sqrt{-\Delta}}{2k^2} \left[2\text{Li}_2\left(\frac{1}{z_1}\right) + 2\text{Li}_2\left(\frac{1}{z_2}\right) - \frac{2}{3}\pi^2 + \frac{1}{2}\ln^2 z_1 + \frac{1}{2}\ln^2 z_2 - i\pi \ln(z_1 z_2) \right]. \quad (\text{B15})$$

Note that in Eq. (B1) we also need to take care of the term

$$\ln\left(\frac{k^2}{\Delta}\right) F_1 \Rightarrow \left[\ln\left(\frac{k^2}{-\Delta}\right) + i\pi \right] F_1. \quad (\text{B16})$$

In this way, we get the following result for the imaginary part of $i^{-1}J^{(2)}(4 - 2\varepsilon; 1, 1)$ above the threshold:

$$\begin{aligned} \text{Im}[i^{-1}J^{(2)}(4 - 2\varepsilon; 1, 1)] &= \frac{\pi^{2-\varepsilon}\Gamma(1+\varepsilon)}{(1-2\varepsilon)} \frac{\pi\sqrt{-\Delta}}{k^2} \left\{ 1 + \varepsilon \ln\left(\frac{k^2}{-\Delta}\right) + \mathcal{O}(\varepsilon^2) \right\} \\ &= \pi^{2-\varepsilon} e^{-\gamma\varepsilon} \frac{\pi\sqrt{-\Delta}}{k^2} \left\{ 1 + 2\varepsilon + \varepsilon \ln\left(\frac{k^2}{-\Delta}\right) + \mathcal{O}(\varepsilon^2) \right\}. \end{aligned} \quad (\text{B17})$$

For $J^{(2)}(2 - 2\varepsilon; 1, 1)$, we get [e.g., using Eq. (A11)]

$$\begin{aligned} \text{Im}[i^{-1}J^{(2)}(2 - 2\varepsilon; 1, 1)] &= 2\pi^{1-\varepsilon}\Gamma(1+\varepsilon) \frac{\pi}{\sqrt{-\Delta}} \left\{ 1 + \varepsilon \ln\left(\frac{k^2}{-\Delta}\right) + \mathcal{O}(\varepsilon^2) \right\} \\ &= 2\pi^{1-\varepsilon} e^{-\gamma\varepsilon} \frac{\pi}{\sqrt{-\Delta}} \left\{ 1 + \varepsilon \ln\left(\frac{k^2}{-\Delta}\right) + \mathcal{O}(\varepsilon^2) \right\}. \end{aligned} \quad (\text{B18})$$

One can also obtain the result for an arbitrary ε (see [75,76]),

$$\text{Im}[i^{-1}J^{(2)}(2 - 2\varepsilon; 1, 1)] = 2\pi^{1-\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{\pi}{\sqrt{-\Delta}} \left(\frac{k^2}{-\Delta}\right)^\varepsilon. \quad (\text{B19})$$

APPENDIX C: SMALL- k^2 EXPANSION OF THE TWO-POINT FUNCTION

For small k^2 and arbitrary m_1 and m_2 , using Eq. (20) of Ref. [77], we get (for unit powers of propagators)

$$J^{(2)}(D; 1, 1) = -i\pi^{D/2}(m_2^2)^{D/2-2}\Gamma(1-D/2)\left\{F_4\left(1, 2-D/2; D/2, 2-D/2\left|\frac{k^2}{m_2^2}, \frac{m_1^2}{m_2^2}\right.\right) - \left(\frac{m_1^2}{m_2^2}\right)^{D/2-1}F_4\left(1, 2-D/2; D/2, 2-D/2\left|\frac{k^2}{m_2^2}, \frac{m_1^2}{m_2^2}\right.\right),\right. \quad (C1)$$

where

$$F_4(a, b; c, d|x, y) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \frac{(a)_{j_1+j_2} (b)_{j_1+j_2} x^{j_1} y^{j_2}}{(c)_{j_1} (d)_{j_2} j_1! j_2!} \quad (C2)$$

is Appell's hypergeometric function of two variables. The sum over j_2 produces the Gauss hypergeometric function,

$$F_4(a, b; c, d|x, y) = \sum_{j=0}^{\infty} \frac{x^j (a)_j (b)_j}{j! (c)_j} {}_2F_1\left(\begin{matrix} a+j, b+j \\ d \end{matrix} \middle| y\right). \quad (C3)$$

Therefore, we can express the integral (C1) as

$$J^{(2)}(D; 1, 1) = -\frac{i\pi^{D/2}}{m_2^2} \sum_{j=0}^{\infty} \left(\frac{k^2}{m_2^2}\right)^j \frac{1}{(D/2)_j} \left\{ (m_2^2)^{D/2-1} (2-D/2)_j {}_2F_1\left(\begin{matrix} 1+j, 2-D/2+j \\ 2-D/2 \end{matrix} \middle| \frac{m_1^2}{m_2^2}\right) - (m_1^2)^{D/2-1} (D/2)_j {}_2F_1\left(\begin{matrix} 1+j, D/2+j \\ D/2 \end{matrix} \middle| \frac{m_1^2}{m_2^2}\right) \right\}. \quad (C4)$$

The occurring ${}_2F_1$ functions can be transformed into truncating ${}_2F_1$ functions [see, e.g., Eq. (7.3.1.26) of Ref. [78]],

$${}_2F_1\left(\begin{matrix} 1+j, 1+\alpha+j \\ 1+\alpha \end{matrix} \middle| z\right) = (1-z)^{-1-2j} {}_2F_1\left(\begin{matrix} -j, \alpha-j \\ 1+\alpha \end{matrix} \middle| z\right), \quad (C5)$$

where $\alpha = 1 - D/2$ in the first case and $\alpha = D/2 - 1$ in the second case. The resulting finite sums can be written as

$${}_2F_1\left(\begin{matrix} -j, \alpha-j \\ 1+\alpha \end{matrix} \middle| z\right) = \sum_{l=0}^j \frac{(\alpha-j)_l}{(\alpha+1)_l} \frac{j!}{l!(j-l)!} (-z)^l, \quad (C6)$$

where the ratio of factorials is nothing but the binomial coefficient.

Using Eq. (C6), we get

$$J^{(2)}(D; 1, 1) = -i\pi^{D/2}\Gamma\left(1-\frac{D}{2}\right) \sum_{j=0}^{\infty} \frac{(k^2)^j}{(D/2)_j} \frac{1}{(m_2^2 - m_1^2)^{1+2j}} \times \left\{ (m_2^2)^{D/2-1} (2-D/2)_j \sum_{l=0}^j \frac{(1-D/2-j)_l}{(2-D/2)_l} \frac{j!}{l!(j-l)!} (-m_1^2)^l (m_2^2)^{j-l} - (m_1^2)^{D/2-1} (D/2)_j \sum_{l=0}^j \frac{(D/2-1-j)_l}{(D/2)_l} \frac{j!}{l!(j-l)!} (-m_1^2)^l (m_2^2)^{j-l} \right\}. \quad (C7)$$

Applying well-known transformations of the Pochhammer symbols, we arrive at an explicitly symmetric result,

$$J^{(2)}(D; 1, 1) = -i\pi^{D/2}\Gamma\left(1 - \frac{D}{2}\right) \sum_{j=0}^{\infty} (k^2)^j \frac{(2-D/2)_j}{(m_2^2 - m_1^2)^{1+2j}} \left\{ (m_2^2)^{D/2-1} \sum_{l=0}^j \frac{j!}{l!(j-l)!} \frac{(m_2^2)^l (m_1^2)^{j-l}}{(D/2)_l (2-D/2)_{j-l}} \right. \\ \left. - (m_1^2)^{D/2-1} \sum_{l=0}^j \frac{j!}{l!(j-l)!} \frac{(m_1^2)^l (m_2^2)^{j-l}}{(D/2)_l (2-D/2)_{j-l}} \right\}. \quad (\text{C8})$$

This result can be also presented in terms of the tadpole integrals $J^{(2)}(D; 1, 0)$ and $J^{(2)}(D; 0, 1)$,

$$J^{(2)}(D; 1, 1) = \sum_{j=0}^{\infty} (k^2)^j \frac{(2-D/2)_j}{(m_2^2 - m_1^2)^{1+2j}} \left\{ J^{(2)}(D; 0, 1) \sum_{l=0}^j \frac{j!}{l!(j-l)!} \frac{(m_2^2)^l (m_1^2)^{j-l}}{(D/2)_l (2-D/2)_{j-l}} \right. \\ \left. - J^{(2)}(D; 1, 0) \sum_{l=0}^j \frac{j!}{l!(j-l)!} \frac{(m_1^2)^l (m_2^2)^{j-l}}{(D/2)_l (2-D/2)_{j-l}} \right\}. \quad (\text{C9})$$

It is easy to check that the limit $m_1 = m_2 \equiv m$ is regular. In this case, Eq. (17) of Ref. [77] yields

$$J^{(2)}(D; 1, 1) \Big|_{k^2 \rightarrow 0, m_1 = m_2 \equiv m} = i\pi^{D/2} (m^2)^{D/2-2} \Gamma\left(2 - \frac{D}{2}\right) {}_2F_1\left(\begin{matrix} 1, 2-D/2 \\ 3/2 \end{matrix} \middle| \frac{k^2}{4m^2}\right) \\ = i\pi^{D/2} (m^2)^{D/2-2} \Gamma\left(2 - \frac{D}{2}\right) \sum_{j=0}^{\infty} \frac{(2-D/2)_j}{(3/2)_j} \left(\frac{k^2}{4m^2}\right)^j. \quad (\text{C10})$$

Using Eq. (C7), we have checked that the first 20 terms of its k^2 expansion in the limit $m_2 \rightarrow m_1$ are the same as in Eq. (C10).

Let us also consider the case $m_1 = 0, m_2 \equiv m$. In this case, using Eq. (10) of Ref. [77], we get

$$J^{(2)}(D; 1, 1) \Big|_{k^2 \rightarrow 0, m_1 = 0, m_2 \equiv m} = -i\pi^{D/2} (m^2)^{D/2-2} \Gamma\left(1 - \frac{D}{2}\right) {}_2F_1\left(\begin{matrix} 1, 2-D/2 \\ D/2 \end{matrix} \middle| \frac{k^2}{m^2}\right) \\ = -i\pi^{D/2} (m^2)^{D/2-2} \Gamma\left(1 - \frac{D}{2}\right) \sum_{j=0}^{\infty} \frac{(2-D/2)_j}{(D/2)_j} \left(\frac{k^2}{m^2}\right)^j. \quad (\text{C11})$$

Let us now consider the limit $m_1 = 0, m_2 \equiv m$ in Eq. (C7). The third line [containing $(m_1^2)^{D/2-1}$] should be omitted because it corresponds to a massless tadpole. In the sum on the second line, we only need to keep the term with $l = j$ because all the others vanish. As a result, we get

$$-i\pi^{D/2} (m^2)^{D/2-2} \Gamma\left(1 - \frac{D}{2}\right) \sum_{j=0}^{\infty} \frac{(D/2 - 1 - j)_j}{(D/2)_j} \left(-\frac{k^2}{m^2}\right)^j.$$

Transforming the Pochhammer symbol as $(D/2 - 1 - j)_j = (-1)^j (2 - D/2)_j$, we reproduce the same result as in Eq. (C11).

APPENDIX D: SPECIAL CASES OF THE TWO-POINT FUNCTION

In the special case $m_1 = 0, m_2 \equiv m$ we can use Eqs. (2.24) and (2.25) of Ref. [74]. In particular, in Eq. (2.25) an arbitrary term of the ε expansion is presented in terms of Nielsen polylogarithms $S_{a,b}(u)$, with $u \equiv k^2/m^2$. Taking into account that $S_{a,1}(u) = \text{Li}_{a+1}(u)$, we get

$$J^{(2)}(4 - 2\varepsilon; 1, 1) \Big|_{m_1 = 0, m_2 \equiv m} = i\pi^{2-\varepsilon} (m^2)^{-\varepsilon} \frac{\Gamma(1+\varepsilon)}{1-2\varepsilon} \left\{ \frac{1}{\varepsilon} - \frac{1-u}{2u\varepsilon} [(1-u)^{-2\varepsilon} - 1] \right. \\ \left. - \frac{\varepsilon(1-u)^{-2\varepsilon}}{u} \text{Li}_2(u) + \mathcal{O}(\varepsilon^2) \right\}. \quad (\text{D1})$$

The threshold corresponds to the point $u = 1$ ($k^2 = m^2$). To go beyond the threshold, we can use

$$\text{Li}_2(u) = -\text{Li}_2\left(\frac{1}{u}\right) + \frac{1}{3}\pi^2 - \frac{1}{2}\ln^2 u + i\pi \ln u.$$

Here the sign of the imaginary part is fixed according to the causal prescription ($k^2 \leftrightarrow k^2 + i0$). Using

$$\text{Li}_2(u) = -\text{Li}_2(1-u) + \frac{1}{6}\pi^2 - \ln u \ln(1-u),$$

we can get as many terms of the “on-shell” expansion in powers and logarithms of $1-u = (m^2 - k^2)/m^2$ as we like. In particular, at $k^2 = m^2$ ($u = 1$) we get

$$J^{(2)}(4-2\varepsilon; 1, 1) \Big|_{m_1=0, m_2 \equiv m, k^2=m^2} = i\pi^{2-\varepsilon}(m^2)^{-\varepsilon} \frac{\Gamma(\varepsilon)}{1-2\varepsilon}. \quad (\text{D2})$$

For $m_1 = m_2 = 0$, we can use the well-known result

$$J^{(2)}(4-2\varepsilon; 1, 1) \Big|_{m_1=m_2=0} = i\pi^{2-\varepsilon}(-k^2)^{-\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}.$$

For $k^2 = 0$ and arbitrary m_1 and m_2 , we get [see, e.g., in Ref. [77], after Eq. (21)]

$$J^{(2)}(4-2\varepsilon; 1, 1) \Big|_{k^2=0} = -i\pi^{2-\varepsilon}\Gamma(-1+\varepsilon) \frac{(m_1^2)^{1-\varepsilon} - (m_2^2)^{1-\varepsilon}}{m_1^2 - m_2^2}. \quad (\text{D3})$$

For $m_1 = m_2 \equiv m$, Eq. (D3) yields

$$J^{(2)}(4-2\varepsilon; 1, 1) \Big|_{k^2=0, m_1=m_2 \equiv m} = i\pi^{2-\varepsilon}\Gamma(\varepsilon)(m^2)^{-\varepsilon},$$

and for $m_1 = 0$, $m_2 \equiv m$ we get

$$J^{(2)}(4-2\varepsilon; 1, 1) \Big|_{k^2=0, m_1=0, m_2 \equiv m} = -i\pi^{2-\varepsilon}\Gamma(-1+\varepsilon)(m^2)^{-\varepsilon}.$$

Considering the case $m_1 \equiv m$, $m_2 = \lambda$, $k^2 = m^2$ and using the general hypergeometric representation, after some transformations we get

$$\begin{aligned} J^{(2)}(4-2\varepsilon; 1, 1) \Big|_{k^2=m^2, m_1 \equiv m, m_2 \equiv \lambda} &= i\pi^{2-\varepsilon} \left\{ \frac{\Gamma(\varepsilon)(m^2)^{-\varepsilon}}{1-2\varepsilon} \left(1 - \frac{\lambda^2}{2m^2} \right) {}_2F_1 \left(\begin{matrix} 1, \varepsilon \\ 1/2 + \varepsilon \end{matrix} \middle| \frac{\lambda^2(4m^2 - \lambda^2)}{4m^4} \right) \right. \\ &\quad + \Gamma\left(\frac{3}{2}\right) \Gamma\left(-\frac{1}{2} + \varepsilon\right) (m^2)^{-1/2} (\lambda^2)^{1/2-\varepsilon} \left(1 - \frac{\lambda^2}{4m^2} \right)^{1/2-\varepsilon} \\ &\quad \left. + \frac{\Gamma(\varepsilon)}{2m^2} (\lambda^2)^{1-\varepsilon} {}_2F_1 \left(\begin{matrix} 1, \varepsilon \\ 3/2 \end{matrix} \middle| \frac{\lambda^2}{4m^2} \right) \right\}. \end{aligned} \quad (\text{D4})$$

In the limit $\lambda \rightarrow 0$, we reproduce Eq. (D2).

Analogous results for special cases of $J^{(2)}(2-2\varepsilon; 1, 1)$ can be obtained by using Eq. (A11).

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