

Consistent chiral limit in the on-shell renormalized quark-meson model with ChPT scaling

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The infrared regularized and the large N_c standard $U(3)$ chiral perturbation theory (ChPT) predicted scaling of the pion, kaon decay constants f_π , f_K , and $M_\eta^2 = m_\eta^2 + m_{\eta'}^2$, has been used in the renormalized $2 + 1$ flavor quark-meson (RQM) model to find a consistent path to the chiral limit as the π and K meson masses approach zero. Using the inputs from the above two prescriptions of the ChPT, the Columbia plots are drawn and compared in the $m_\pi - m_K$ and the light strange quark mass $m_{ud} - m_s$ planes, free from any ambiguity or heuristic adjustment in the model parameter fixing away from the physical point. The on-shell renormalization of the parameters and consistent treatment of the divergent quark one-loop vacuum fluctuations in the RQM model make the axial $U_A(1)$ anomaly significantly stronger, and the light (strange) explicit chiral symmetry breaking strength becomes weaker by a small (relatively large) amount. The first-order chiral transition region in the $m_\pi - m_K$ and $\mu - m_K$ planes of the Columbia plots generated using the inputs from both versions of the ChPT increases due to the above novel features of the RQM model. Comparing our results (in the presence of the axial $U_A(1)$ anomaly) regarding the softening effect of the quark one-loop vacuum fluctuations on the chiral transition, it looks overestimated in a recent functional renormalization group study. The $U_A(1)$ symmetry and the $SU_A(3)$ chiral symmetry restoration in the chiral limit takes place at the same transition temperature.

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I. INTRODUCTION

The chiral phase transition is the focus of intensive research since the universality [1] predictions that it is first order in the chiral limit of the u , d , s quark masses $m_u = m_d = 0 = m_s$ and second order of the $O(4)$ universality class if $m_{u,d} = 0$, $m_s = \infty$ with the strong $U_A(1)$ anomaly [2] (η' mass $m_{\eta'}(T_c) \gg T_c$) at the chiral symmetry restoring critical temperature T_c . The second-order transition turns first order at a triple point for some finite $m_s = m_s^{\text{TCP}}$ ($m_K = m_K^{\text{TCP}}$) in the light chiral limit $m_{u,d} = 0$ ($m_\pi = 0$). The mass dependence of the order of the phase transition, the structures of the critical lines and the surfaces at zero, and the real/imaginary chemical potentials μ is summarized in the Columbia plot [3], which is well understood in the heavy quark mass limit both from the continuum and lattice quantum chromodynamics (LQCD) studies [4–8]. The LQCD has settled that the chiral symmetry restoration at

the physical point for $\mu = 0$ is a crossover transition with pseudo-critical temperature $T_\chi \equiv 155 \pm 2$ MeV [9–13].

The precise mapping of the critical line separating the region of the first-order transition from that of the crossover for small masses in the $m_{u,d} - m_s$ ($m_\pi - m_K$) plane is quite challenging in the LQCD [14–21] as the results show large variations due to the strong cutoff and discretization effects. The crossover at $\mu = 0$ turns first order for the critical quark (pion) mass $m_q(m_\pi) < m_q^c(m_\pi^c)$ for $N_f = 3$ degenerate flavor of quarks. The LQCD studies conducted between 2001 and 2017 confirm the first-order region close to the chiral limit and find the m_π^c in the range of 290–50 MeV [15–23]. A recently improved LQCD $N_f = 3$ study [24] gets no direct evidence of the first-order transition for $80 \text{ MeV} \leq m_\pi \leq 140 \text{ MeV}$. Using Möbius domain wall fermions, Ref. [25] finds small $m_q^c \leq 4$ MeV, while another LQCD chiral limit study suggests that the chiral transition could be of the second order. Although the improved LQCD studies suggest a small or no first-order region in some cases, the picture is not very clear as the results still show large discrepancies because of the notoriously difficult problem of implementing the chiral fermions on the lattice. Apart from the LQCD, an $N_f = 3$ study in the Dyson-Schwinger approach [26] and another study in Ref. [27] find the second-order transition. Very recently, a functional

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renormalization group (FRG) study in local potential approximation (LPA) [28,29] showed that in contrast to the ϵ expansion prediction [1], the transition can be of the second order for $N_f \geq 2$ if the $U_A(1)$ symmetry gets restored at the T_χ . But they have added a note of caution for drawing final conclusions as the LPA completely neglects the wave function renormalization. The thermal fate of the axial $U_A(1)$ anomaly is still unsettled [30] as some studies find that it is relevant at the critical point T_χ [31–36] while others [24,37–40] claim that it vanishes. Novel signals, if anomaly strength becomes very weak at the T_χ , has been conjectured in very recent works such as Ref. [41]. In another very recent work, Ref. [42] has explored the relation between the terms which break the anomalous $U_A(1)$ symmetry and the order of the chiral transition.

The QCD phase transition studies in the framework of the effective models, like the linear sigma model (LSM) [43–53], the quark-meson (QM) model [54–68], or the Nambu-Jona-Lasinio [69–71] (NJL) model and the non-perturbative FRG technique models [72–92], have a long history of complementing LQCD studies with new insights. Refining the QM model into a valuable tool of chiral limit studies, one faces the challenge of finding the quark (pion and kaon) mass dependence of the model parameters. The often used method [43,45,61,71,72,78], called the fixed-ultraviolet (UV) scheme [23], relies on changing the light (strange) explicit chiral symmetry breaking strengths $h_x(h_y)$ while all other parameters are kept the same as the ones at the physical point. In this scheme, the chiral limit can be explored only for very large scalar σ meson masses $m_\sigma \geq 800$ MeV because the spontaneous chiral symmetry breaking (SCSB) gets lost in the chiral limit as the mass parameter m^2 turns positive for the $m_\sigma < 800$ MeV [23,61]. Here, it is relevant to mention that the connection of the scalar σ meson state occurring in the QM model like Lagrangian to the physical $\sigma/f_0(500)$ state is not clear and remains unsettled as yet. The physical $\sigma/f_0(500)$ state quoted by the particle data group (PDG) is a broad state showing up as a resonance in the $\pi\pi$ scattering, and its properties as well as the debates regarding its status can be found in the excellent review by J. R. Pelaez in Ref. [93]. In order to do phenomenological investigations and calibrate the results, one usually takes different values of mass m_σ for the scalar σ meson and see how it influences the chiral transition. In a recent FRG QM model study under the LPA of the Columbia plot, Resch *et al.* [23] proposed the ChPT motivated fixed- f_π scheme where they heuristically adjust the initial effective action to the larger scales ($\Lambda' > \Lambda$) for every smaller mass point in the path to the chiral limit such that the f_π always retains its physical value and hence the SCSB is not lost for the scalar σ mass range $m_\sigma = 400$ –600 MeV. The changing scale ($\Lambda' > \Lambda$) accounts for the change in parameters (which are

kept the same as at the physical point) when the explicit chiral symmetry breaking, non-strange direction strength h_x , and the strange direction strength h_y , decrease.

The present study aims to generate the left corner of the Columbia plot and find a consistent path to the chiral limit using the exact chiral effective potential of the RQM model [94,95] whose parameters are renormalized on-shell after the consistent treatment of the quark one-loop vacuum fluctuation. Using the $\mathcal{O}(\frac{1}{f_\pi})$ accurate results of the two ChPT prescriptions determining the (m_π, m_K) dependence of the f_π, f_K , and M_η^2 , namely, the large N_c standard U(3) ChPT [96–101] and the infrared regularized U(3) ChPT [102–105], whose first use for finding the (m_π, m_K) dependence of the tree-level parameters of the sigma model was done in Ref. [48], the RQM model parameter fixing away from the physical point has been kept free from any ambiguity and heuristic adjustment. After renormalization, the axial $U_A(1)$ anomaly gets significantly stronger in the RQM model while the explicit chiral symmetry breaking strength $h_x(h_y)$ at the physical point gets reduced by a small (relatively large) amount. In contrast, the c and $h_x(h_y)$ do not change from their tree-level values at the physical point in the FRG study under LPA where the wave function renormalization is completely ignored and the initial conditions (action of the low-energy model at scale Λ) are not uniquely determined by solving the RG evolution of the QCD flow starting from a microscopic QCD action at a large perturbative energy scale $k \gg 1$ GeV. The tree-level anomaly strength c also increases toward the chiral limit due to scaling of the f_π, f_K , and M_η in both the infrared regularized and large N_c standard ChPT prescriptions.

We will explore how the above mentioned new features of the RQM model impact the nature and extent of the first- or second-order chiral transition in the Columbia plot computed for the $m_\sigma = 400$ and 530 MeV using the (m_π, m_K) dependence of the f_π, f_K , and M_η from the input set M_η -I of the infrared regularized ChPT in the presence as well as the absence of the $U_A(1)$ anomaly. The input set M_η -II of the large N_c standard U(3) ChPT scaling of the f_π, f_K and M_η for the (m_π, m_K) dependence will also be used to compute the Columbia plot in the presence of the $U_A(1)$ anomaly when the $m_\sigma = 530$ MeV. The $m_\pi - m_K$ plane Columbia plot for the input set M_η -II will be compared with that obtained from the input set M_η -I, and results of the plots from both the input sets will be contrasted with the results of the e-MFA FRG study of Ref. [23] done for the $m_\sigma = 530$ MeV. The Columbia plots will be drawn in the light-strange quark mass $m_{ud} - m_s$ plane for both the input sets M_η -I as well as M_η -II and the results will be compared. Relevant LQCD study results with references will also be presented in a tabular form for comparison. In order to know the degree of the $U_A(1)$ symmetry restoration and the strength of the $U_A(1)$ anomaly in the path to the

chiral limit, the temperature variation of η' and η meson curvature masses will also be studied for both the input sets M_{η} -I and -II.

The paper is arranged in the following order: Sec. II gives a very brief presentation of the $SU_L(3) \times SU_R(3)$ RQM model. The brief derivation of the curvature masses of the pseudoscalar mesons is given in Sec. II A. The ChPT scaling of the f_π , f_K , and M_η are presented in the Sec. II B, where the infrared regularized U(3) ChPT formulas are described in Sec. II B I, while the large N_c standard U(3) ChPT formulas are given Sec. II B II. Section III A presents results for the consistent path to the chiral limit and results for the input sets M_{η} -I and M_{η} -II when $m_\sigma = 530$ MeV are compared in Sec. III A I. Section III B discusses the results for the Columbia plots obtained in the $m_\pi - m_K$ plane using the input set M_{η} -I for different values of the scalar σ meson mass m_σ in the presence as well as the absence of the $U_A(1)$ anomaly. The $m_\pi - m_K$ plane Columbia plot for the input set M_{η} -II in the presence of the $U_A(1)$ anomaly when $m_\sigma = 530$ MeV is presented in Sec. III B I, where it is compared with the corresponding Columbia plot for the input set M_{η} -II. The Columbia plots in the quark mass $m_{ud} - m_s$ plane for both the input sets M_{η} -I and M_{η} -II are compared in Sec. III B II. In order to study the $U_A(1)$ symmetry restoration trend when one goes to the chiral limit, the temperature variations of the curvature masses of the η' and η mesons have been presented in Sec. III C for both the input sets M_{η} -I as well as M_{η} -II and the corresponding results will be compared. Finally, the summary of the whole work is presented in Sec. IV.

II. THE RQM MODEL

The RQM model Lagrangian [44,61,65] is

$$\mathcal{L}_{\mathcal{QM}} = \bar{\psi}[i\gamma^\mu D_\mu - gT_a(\sigma_a + i\gamma_5\pi_a)]\psi + \mathcal{L}(\mathcal{M}). \quad (1)$$

$$\begin{aligned} \mathcal{L}(\mathcal{M}) = & \text{Tr}(\partial_\mu \mathcal{M}^\dagger \partial^\mu \mathcal{M} - m^2(\mathcal{M}^\dagger \mathcal{M})) \\ & - \lambda_1 [\text{Tr}(\mathcal{M}^\dagger \mathcal{M})]^2 - \lambda_2 \text{Tr}(\mathcal{M}^\dagger \mathcal{M})^2 \\ & + c[\det \mathcal{M} + \det \mathcal{M}^\dagger] + \text{Tr}[H(\mathcal{M} + \mathcal{M}^\dagger)]. \end{aligned} \quad (2)$$

The Yukawa coupling g couples the flavor triplet quark fields ψ (color N_c -plet Dirac spinor) to the nine scalar (pseudo-scalar) meson ξ fields $\sigma_a(\pi_a)$ of 3×3 complex matrix $\mathcal{M} = T_a \xi_a = T_a(\sigma_a + i\pi_a)$. $T_a = \frac{\lambda_a}{2}$, where λ_a ($a = 0, 1, \dots, 8$) are Gell-Mann matrices with $\lambda_0 = \sqrt{\frac{2}{3}}\mathbb{I}_{3 \times 3}$. The field ξ picks up the nonzero vacuum expectation value $\bar{\xi}$ in the 0 and 8 directions. The condensates $\bar{\sigma}_0$ and $\bar{\sigma}_8$ break the $SU_L(3) \times SU_R(3)$ chiral symmetry spontaneously while the external fields $H = T_a h_a$ with $h_0, h_8 \neq 0$ break it explicitly. The change from the singlet octet (0,8) to the nonstrange strange basis (x, y) gives $x(h_x) = \sqrt{\frac{2}{3}}\bar{\sigma}_0(h_0) + \frac{1}{\sqrt{3}}\bar{\sigma}_8(h_8)$ and $y(h_y) = \frac{1}{\sqrt{3}}\bar{\sigma}_0(h_0) -$

$\sqrt{\frac{2}{3}}\bar{\sigma}_8(h_8)$. Considering mesons at the mean-field level with the thermal and quantum fluctuations of the quarks/anti-quarks, the grand potential [61,65] is the sum of the vacuum effective potential $U(x, y)$ and the quark/anti-quark contribution $\Omega_{q\bar{q}}$ at finite temperature T and quark chemical potential μ_f ($f = u, d, s$),

$$\Omega_{\text{MF}}(T, \mu) = U(x, y) + \Omega_{q\bar{q}}(T, \mu; x, y). \quad (3)$$

$$\Omega_{q\bar{q}}(T, \mu; x, y) = \Omega_{q\bar{q}}^{\text{vac}} + \Omega_{q\bar{q}}^{T, \mu}. \quad (4)$$

$$\begin{aligned} U(x, y) = & \frac{m^2}{2}(x^2 + y^2) - h_x x - h_y y - \frac{c}{2\sqrt{2}}x^2 y + \frac{\lambda_1}{2}x^2 y^2 \\ & + \frac{1}{8}(2\lambda_1 + \lambda_2)x^4 + \frac{1}{8}(2\lambda_1 + 2\lambda_2)y^4. \end{aligned} \quad (5)$$

$$\Omega_{q\bar{q}}^{\text{vac}} = -2N_c \sum_f \int \frac{d^3 p}{(2\pi)^3} E_q \theta(\Lambda_c^2 - \vec{p}^2). \quad (6)$$

$$\Omega_{q\bar{q}}^{T, \mu} = -2N_c \sum_{f=u,d,s} \int \frac{d^3 p}{(2\pi)^3} T [\ln g_f^+ + \ln g_f^-]. \quad (7)$$

$$\Omega_{\text{QM}}(T, \mu, x, y) = U(x, y) + \Omega_{q\bar{q}}^{T, \mu}. \quad (8)$$

The $g_f^\pm = [1 + e^{-E_f^\pm/T}]$, where $E_f^\pm = E_f \mp \mu_f$ and $E_f = \sqrt{p^2 + m_f^2}$, is the quark/anti-quark energy. The light (strange) quark mass is $m_{u/d} = \frac{qx}{2}$ ($m_s = \frac{qy}{\sqrt{2}}$) and $\mu_u = \mu_d = \mu_s = \mu$. The quark one-loop vacuum term with ultraviolet cut-off Λ_c in Eq. (6) is dropped for the standard mean field approximation (s-MFA) of the quark meson (QM) model grand effective potential in Eq. (8). Several studies have used the minimal subtraction scheme to regularize the quark one-loop vacuum divergences after including the vacuum fluctuations in the extended mean field approximation (e-MFA) [106–113], but the effective potential $\Omega_{\text{vac}}(x, y) = U(x, y) + \Omega_{q\bar{q}}^{\text{vac}}$ in their treatment turns inconsistent as they fix the model parameters using the curvature masses of the mesons, which are obtained by taking the double derivatives of the effective potential with respect to the different fields at its minimum. The above-mentioned inconsistency becomes apparent when one notes that the calculation of the curvature masses involves the evaluation of the meson self-energies at zero momentum because the effective potential is the generator of the n-point functions of the theory at vanishing external momenta [114–122]. It has to be noted that the pole definition of the meson mass is the physical and gauge invariant [123,124] one.

Here, we will employ the consistent e-MFA RQM model effective potential calculated in our very recent works [94,95] after relating the counter-terms in the $\overline{\text{MS}}$ scheme

to those in the on-shell (OS) scheme [115,118–122]. The relations between the renormalized parameters of both the schemes are determined when the physical quantities (the on-shell pole masses of the $m_\pi, m_K, m_\eta, m_{\eta'}$, and m_σ and the pion and kaon decay constants f_π and f_K) are put into the relation of the $\overline{\text{MS}}$ running couplings and mass parameter. These relations are used as input when the effective potential is calculated using the modified minimal subtraction procedure. After the cancellation of the $1/\epsilon$ divergences, the vacuum effective potential $\Omega_{\text{vac}} = U(x_{\overline{\text{MS}}}, y_{\overline{\text{MS}}}) + \Omega_{\overline{\text{MS}}}^{q,\text{vac}} + \delta U(x_{\overline{\text{MS}}}, y_{\overline{\text{MS}}})$ in the $\overline{\text{MS}}$ scheme has been rewritten in Ref. [94] in terms of the scale Λ independent constituent quark mass parameters $\Delta_x = \frac{g_{\overline{\text{MS}}} x_{\overline{\text{MS}}}}{2}$ and $\Delta_y = \frac{g_{\overline{\text{MS}}} y_{\overline{\text{MS}}}}{\sqrt{2}}$ as the following:

$$\begin{aligned} \Omega_{\text{vac}}(\Delta_x, \Delta_y) = & \frac{m_0^2}{g_0^2} (2\Delta_x^2 + \Delta_y^2) - 2\frac{h_{x0}}{g_0} \Delta_x - \sqrt{2}\frac{h_{y0}}{g_0} \Delta_y \\ & - 2\frac{c_0}{g_0^2} \Delta_x^2 \Delta_y + 4\frac{\lambda_{10}}{g_0^4} \Delta_x^2 \Delta_y^2 + 2\frac{(2\lambda_{10} + \lambda_{20})}{g_0^4} \Delta_x^4 \\ & + \frac{(\lambda_{10} + \lambda_{20})}{g_0^4} \Delta_y^4 + \frac{2N_c \Delta_x^4}{(4\pi)^2} \\ & \times \left[\frac{3}{2} + \ln\left(\frac{\Lambda^2}{m_u^2}\right) + \ln\left(\frac{m_u^2}{\Delta_x^2}\right) \right] \\ & + \frac{N_c \Delta_y^4}{(4\pi)^2} \left[\frac{3}{2} + \ln\left(\frac{\Lambda^2}{m_u^2}\right) + \ln\left(\frac{m_u^2}{\Delta_y^2}\right) \right]. \end{aligned} \quad (9)$$

$$\begin{aligned} \Omega_{\text{vac}}^{\text{RQM}}(x, y) = & \frac{(m^2 + m_{\text{FIN}}^2)}{2} (x^2 + y^2) - (h_x + h_{x\text{FIN}})x - (h_y + h_{y\text{FIN}})y - \frac{(c + c_{\text{FINTOT}})}{2\sqrt{2}} x^2 y + \frac{(\lambda_1 + \lambda_{1\text{FIN}})}{2} x^2 y^2 \\ & + \frac{\{2(\lambda_1 + \lambda_{1\text{FIN}}) + (\lambda_2 + \lambda_{2\text{FIN}})\} x^4}{8} + (\lambda_1 + \lambda_{1\text{FIN}} + \lambda_2 \\ & + \lambda_{2\text{FIN}}) \frac{y^4}{4} + \frac{N_c g^4 (x^4 + 2y^4)}{8(4\pi)^2} \left[\frac{3}{2} - C(m_\pi^2) - m_\pi^2 C'(m_\pi^2) \right] \\ & - \frac{N_c g^4}{8(4\pi)^2} \left[x^4 \ln\left(\frac{x^2}{f_\pi^2}\right) + 2y^4 \ln\left(\frac{2y^2}{f_\pi^2}\right) \right]. \end{aligned} \quad (11)$$

$$\Omega_{\text{RQM}}(T, \mu, x, y) = \Omega_{\text{vac}}^{\text{RQM}}(x, y) + \Omega_{q\bar{q}}^{T,\mu}. \quad (12)$$

The search of the grand potential minima $\frac{\partial \Omega_{\text{RQM}}}{\partial x} = \frac{\partial \Omega_{\text{RQM}}}{\partial y} = 0$ for the Eq. (12) gives the T and μ dependence of the x and y .

A. The curvature masses of the pseudoscalar mesons

The curvature of the grand potential in Eq. (12) at its global minimum gives the temperature and chemical potential dependence of the scalar and pseudoscalar masses of the mesons,

The condition that the minimum of the effective potential of the RQM model does not shift from that of the QM model fixes the scale Λ_0 [94,95] as

$$\ln\left(\frac{\Lambda_0^2}{m_u^2}\right) + C(m_\pi^2) + m_\pi^2 C'(m_\pi^2) = 0. \quad (10)$$

The terms $C(m_\pi^2)$ and $C'(m_\pi^2)$ and the derivations for the renormalized parameters $m_0^2 = (m^2 + m_{\text{FIN}}^2)$, $h_{x0} = (h_x + h_{x\text{FIN}})$, $h_{y0} = (h_y + h_{y\text{FIN}})$, $\lambda_{10} = (\lambda_1 + \lambda_{1\text{FIN}})$, $\lambda_{20} = (\lambda_2 + \lambda_{2\text{FIN}})$, and $c_0 = (c + c_{\text{FINTOT}})$ are given in detail in Refs. [94,95]. The m_{FIN}^2 , $h_{x\text{FIN}}$, $h_{y\text{FIN}}$, $\lambda_{1\text{FIN}}$, $\lambda_{2\text{FIN}}$, and c_{FINTOT} are the finite on-shell corrections in the parameters at the scale Λ_0 . The experimental values of the pseudoscalar meson masses $m_\pi, m_K, m_\eta, m_{\eta'}$ ($m_\eta^2 + m_{\eta'}^2$), the scalar σ mass m_σ , and the f_π, f_K as input determine the tree-level QM model quartic couplings λ_1, λ_2 , mass parameter m^2 , h_x, h_y , and the coefficient c of the 't Hooft determinant term for the $U_A(1)$ axial anomaly [44,61].

Although the f_π, f_K , and g get renormalized due to the dressing of the meson propagator in the on-shell scheme, they do not change as the $g_{\overline{\text{MS}}} = g_{\text{ren}} = g_0 = g$, $x_{\overline{\text{MS}}} = x$, and $y_{\overline{\text{MS}}} = y$ at Λ_0 . Further, $x_{\overline{\text{MS}}} = f_{\pi,\text{ren}} = f_\pi$ and $y_{\overline{\text{MS}}} = \frac{2f_{K,\text{ren}} - f_{\pi,\text{ren}}}{\sqrt{2}} = \frac{2f_K - f_\pi}{\sqrt{2}}$ at the minimum. Using $\Delta_x = \frac{gx}{2}$ and $\Delta_y = \frac{gy}{\sqrt{2}}$, the vacuum effective potential in Eq. (9) can be written in terms of x and y as

$$\begin{aligned} m_{\alpha,ab}^2|_{T,\text{QM/RQM}} = & \frac{\partial^2 \Omega_{\text{QM/RQM}}(T, \mu; x, y)}{\partial \xi_{\alpha,a}^2 \partial \xi_{\alpha,b}^2} \Big| \\ = & m_{\alpha,ab}^2|_{\text{QM/RQM}} + (\delta m_{\alpha,ab}^T)^2. \end{aligned} \quad (13)$$

$\alpha = s$ and p , respectively, denote the scalar and pseudo-scalar meson. The vacuum meson masses $m_{\alpha,ab}^2$ of the QM model are evaluated in Refs. [44,61] from the second derivatives of the pure meson potential $U(x, y)$. Their expressions can be found in the tables of Refs. [65,94,112]. Mass modifications $(\delta m_{\alpha,ab}^T)^2$ due to the quark/anti-quark contributions at the finite temperature (chemical potential)

are calculated in Ref. [61] as the following:

$$(\delta m_{\alpha,ab}^T)^2|_{\text{QM}} = \frac{\partial^2 \Omega_{q\bar{q}}^T(T, \mu, x, y)}{\partial \xi_{\alpha,a} \partial \xi_{\alpha,b}} \Big|_{\min} \quad (14)$$

$$= 3 \sum_{f=x,y} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_f} \times \left[(n_f^+ + n_f^-) \left(m_{f,ab}^2 - \frac{m_{f,a}^2 m_{f,b}^2}{2E_f^2} \right) + (b_f^+ + b_f^-) \left(\frac{m_{f,a}^2 m_{f,b}^2}{2E_f T} \right) \right],$$

$$\text{where } n_f^\pm = \frac{1}{1 + e^{E_f^\pm/T}} \quad \text{and} \quad b_f^\pm = n_f^\pm (1 - n_f^\pm). \quad (15)$$

$m_{f,a}^2 \equiv \partial m_f^2 / \partial \xi_{\alpha,a}$ is the first partial derivative, and $m_{f,ab}^2 \equiv \partial^2 m_f^2 / \partial \xi_{\alpha,a} \partial \xi_{\alpha,b}$ is the second partial derivative of the square of quark mass in the different directions of the meson fields $\xi_{\alpha,b}$. These derivatives have been calculated and presented in Table III of Ref. [61].

The vacuum ($T, \mu = 0$) curvature masses of the mesons in the RQM model are different from their pole masses due to the on-shell parameter fixing. In the RQM model, the curvature meson mass matrix is calculated by taking the double derivative of Eq. (9) with respect to the meson fields ξ ,

$$m_{\alpha,ab}^2|_{\text{RQM}} = m_{\alpha,abc}^2 = \frac{\partial^2 \Omega_{\text{vac}}(\Delta_x, \Delta_y)}{\partial \xi_{\alpha,a} \partial \xi_{\alpha,b}} \Big|_{\min} = (m_{\alpha,ab}^0)^2 + (\delta m_{\alpha,ab}^{v,\text{RQM}})^2. \quad (16)$$

Here, c denotes the curvature masses for the RQM model. The $(m_{\alpha,ab}^0)^2$ mass expressions are the same as for the QM model vacuum masses $(m_{\alpha,ab})^2$, but the superscript 0 stands for the evaluation of the parameters at the scale Λ_0 in the $\overline{\text{MS}}$ scheme. The RQM model quark one-loop vacuum corrections for the meson masses are

$$(\delta m_{\alpha,ab}^{v,\text{RQM}})^2 = \frac{N_c}{8\pi^2} \sum_{f=u,d,s} \left[\left(\frac{\partial \Delta_f^2}{\partial \xi_{\alpha,a}} \right) \left(\frac{\partial \Delta_f^2}{\partial \xi_{\alpha,b}} \right) \ln \left(\frac{\Lambda_f^2}{\Delta_f^2} \right) + \Delta_f^2 \left(1 + \ln \left(\frac{\Lambda_f^2}{\Delta_f^2} \right) \right) \frac{\partial^2 \Delta_f^2}{\partial \xi_{\alpha,a} \partial \xi_{\alpha,b}} \right]. \quad (17)$$

The $f = (u, d) = x$ and $f = s = y$. The first $\partial \Delta_f^2 / \partial \xi_{\alpha,a} \equiv \partial m_f^2 / \partial \xi_{\alpha,a}$ and the second $\partial^2 \Delta_f^2 / (\partial \xi_{\alpha,a} \partial \xi_{\alpha,b}) \equiv \partial^2 m_f^2 / (\partial \xi_{\alpha,a} \partial \xi_{\alpha,b})$ partial derivatives of the scale independent squared quark masses with respect to the different meson fields $\xi_{\alpha,b}$ can be taken from Table III of Ref. [61]. Using the condition in Eq. (10), the above corrections have been evaluated in Ref. [94] at the scale $\Lambda = \Lambda_0$. The RQM model expressions of the vacuum curvature masses of the pseudo-scalar mesons as taken from [94] are given below,

$$(m_{p,11c})^2 = m_0^2 + \lambda_{10}(x^2 + y^2) + \frac{\lambda_{20}x^2}{2} - \frac{c_0y}{\sqrt{2}} + \frac{N_c g^4 x^2}{32\pi^2} \left[1 + 2 \ln f_\pi - C(m_\pi^2) - m_\pi^2 C'(m_\pi^2) - 2 \ln x \right], \quad (18)$$

$$(m_{p,44c})^2 = (m_{K,c})^2 = m_0^2 + \lambda_{10}(x^2 + y^2) + \frac{\lambda_{20}}{2} (x^2 - \sqrt{2}xy + 2y^2) - \frac{c_0}{2}x + \frac{N_c g^4}{32\pi^2} \left[(x^2 - \sqrt{2}xy + 2y^2) \left(1 + 2 \ln f_\pi - C(m_\pi^2) - m_\pi^2 C'(m_\pi^2) \right) - \frac{2}{(x + \sqrt{2}y)} \left(x^3 \ln x + (\sqrt{2}y)^3 \ln \sqrt{2}y \right) \right], \quad (19)$$

$$(m_{p,00c})^2 = m_0^2 + \lambda_{10}(x^2 + y^2) + \frac{\lambda_{20}}{3} (x^2 + y^2) + \frac{c_0}{3} (2x + \sqrt{2}y) + \frac{N_c g^4}{48\pi^2} \left[(x^2 + y^2) \left(1 + 2 \ln f_\pi - C(m_\pi^2) - m_\pi^2 C'(m_\pi^2) \right) - 2 \left(x^2 \ln x + y^2 \ln \sqrt{2}y \right) \right], \quad (20)$$

$$(m_{p,88c})^2 = m_0^2 + \lambda_{10}(x^2 + y^2) + \frac{\lambda_{20}}{6} (x^2 + y^2) - \frac{c_0}{6} (4x - \sqrt{2}y) + \frac{N_c g^4}{96\pi^2} \left[(x^2 + 4y^2) \left(1 + 2 \ln f_\pi - C(m_\pi^2) - m_\pi^2 C'(m_\pi^2) \right) - 2 \left(x^2 \ln x + 4y^2 \ln \sqrt{2}y \right) \right], \quad (21)$$

$$(m_{p,08c})^2 = \frac{\sqrt{2}\lambda_{20}}{6} (x^2 + y^2) - \frac{c_0}{6} (\sqrt{2}x - 2y) + \frac{\sqrt{2}N_c g^4}{96\pi^2} \left[(x^2 - 2y^2) \left(1 + 2 \ln f_\pi - C(m_\pi^2) - m_\pi^2 C'(m_\pi^2) \right) - 2 \left(x^2 \ln x - 2y^2 \ln \sqrt{2}y \right) \right]. \quad (22)$$

The curvature masses of the η and η' mesons are written below, where the $-$ sign gives the m_η and the $+$ sign gives $m_{\eta'}$,

$$m_{\eta(\eta'),c}^2 = \frac{1}{2} \left[(m_{p,00c}^2 + m_{p,88c}^2) \mp \sqrt{(m_{p,00c}^2 - m_{p,88c}^2)^2 + 4m_{p,08c}^4} \right]. \quad (23)$$

The temperature (chemical potential) dependence of the x and y generates the temperature (chemical potential) dependence in the meson masses. Including the medium contribution $(\delta m_{a,ab}^T)^2$, which remains the same in the QM and RQM model, meson curvature masses at finite temperature (chemical potential) in the RQM model are obtained as

$$m_{a,ab}^2|_{T,RQM} = m_{a,ab}^2 + (\delta m_{a,ab}^T)^2|_{QM}. \quad (24)$$

Including the finite temperature contributions for the 00,88 and 08 components in Eq. (23), one can find the temperature dependence of the η and η' mesons.

Note that when one uses Eq. (11) to evaluate the equations of motion $\frac{\partial \Omega_{vac}^{RQM}}{\partial x} = 0 = \frac{\partial \Omega_{vac}^{RQM}}{\partial y}$, one gets the renormalized explicit chiral symmetry breaking strengths h_{x0} and h_{y0} as the following:

$$h_{x0} = m_{\pi,c}^2 f_\pi, \quad (25)$$

$$h_{y0} = \left(\sqrt{2} f_K m_{K,c}^2 - \frac{f_\pi}{\sqrt{2}} m_{\pi,c}^2 \right). \quad (26)$$

The pion and kaon curvature masses $m_{\pi,c}$ and $m_{K,c}$ as derived in Ref. [94] have the following expressions:

$$m_{\pi,c}^2 = m_\pi^2 \left\{ 1 - \frac{N_c g^2}{4\pi^2} m_\pi^2 C'(m_\pi^2, m_u) \right\}, \quad (27)$$

$$m_{K,c}^2 = m_K^2 \left[1 - \frac{N_c g^2}{4\pi^2} \left\{ \mathcal{C}(m_\pi^2, m_u) + m_\pi^2 C'(m_\pi^2, m_u) - \left(1 - \frac{(m_s - m_u)^2}{m_K^2} \right) \mathcal{C}(m_K^2, m_u, m_s) + \right. \right. \\ \left. \left. \times \left(1 - \frac{f_\pi}{f_K} \right) \left(\frac{m_u^2 - m_s^2 + 2m_s^2 \ln(\frac{m_s}{m_u})}{m_K^2} \right) \right\} \right]. \quad (28)$$

It is a nice consistency check to have verified that when $x = f_\pi$ and $y = \frac{(2f_K - f_\pi)}{\sqrt{2}}$ in the vacuum, the expression of $m_{p,11c}^2$ in Eq. (18) gives the same value as $m_{\pi,c}^2$ in Eq. (27), while the expression of $m_{p,44c}^2$ in Eq. (19) gives the value that is equal to the kaon curvature mass square $m_{K,c}^2$ of Eq. (28).

B. The ChPT scaling of f_π, f_K , and $M_\eta^2 = m_\eta^2 + m_{\eta'}^2$

Since the low-energy degrees of freedom in the $2+1$ flavor QM model are scalar and pseudo-scalar mesons rather than the u , d , and s quarks of the perturbative QCD regime, the lower left corner of the Columbia plot and current quark mass ($m_{u/d} - m_s$ plane) sensitivity of the chiral transition in the chiral limit can be explored in the equivalent description of the $m_\pi - m_K$ plane. The experimental information in tune with the underlying theory of QCD is available only at the physical point where the nonstrange and strange direction explicit chiral symmetry breaking source strengths $h_x = f_\pi m_\pi^2$, $h_y = \sqrt{2} f_K m_K^2 - f_\pi m_\pi^2 / \sqrt{2}$, and the model parameters can be determined. Different strategies are employed to fix the model parameters as it is *a priori* not clear how they change as the system is tuned away from the physical point. The most commonly used strategy [43,45,61,71,72,78] in the literature, called the fixed-ultraviolet (UV) scheme [23], is based on the assumption that the change in current quark masses of the QCD can be directly mapped onto a change of symmetry breaking source strengths. Hence, one varies the $h_x(h_y)$ while keeping all other parameters the same as the ones at the physical point such that the initial effective action does not change in the UV [23]. The fixed-UV scheme does not work for the range of the scalar σ masses $m_\sigma = 400-800$ MeV because the spontaneous chiral symmetry breaking (SCSB) is lost in the chiral limit as the mass parameter m^2 becomes positive for the moderate m_σ [23,61]. This problem is circumvented in the chiral limit studies of Ref. [61] by choosing quite a large $m_\sigma \geq 800$ MeV.

In a nonperturbative FRG study of the Columbia plot, the first of its kind, Resch *et al.* [23] proposed a fixed- f_π scheme motivated by the findings of ChPT where the initial effective action is heuristically adjusted to the larger scales ($\Lambda' > \Lambda$) as one takes smaller masses toward the chiral limit in the Columbia plot such that the infra-red (IR) pion decay constant f_π always remains fixed to its physical value. The fixed- f_π scheme resolves the problem of the fixed-UV scheme by construction as the SCSB is not lost and the condensates being intimately related to the fixed f_π do not drop to zero. The change of scale ($\Lambda' > \Lambda$) in the fixed- f_π scheme is equivalent to the change of parameters (which retain the same value at Λ in the fixed UV-scheme) when $h_x(h_y)$ is decreased away from the physical point. Even though the fixed- f_π scheme is physically reasonable, it suffers from the problem of a heuristic determination of the initial effective action [23]. Below, we will be employing more accurate methods of fixing the parameters unambiguously by using the ChPT predicted scaling of the decay constants f_π, f_K , and $M_\eta^2 = m_\eta^2 + m_{\eta'}^2$ when the m_π and m_K and therefore the explicit symmetry breaking strengths h_x and h_y go to zero in the RQM model as one

moves away from the physical point toward the chiral limit in the Columbia plot.

The quark loop responsible for the $U_A(1)$ anomaly is suppressed in the large N_c limit, and the chiral symmetry of the QCD Lagrangian gets enhanced to $U_L(3) \times U_R(3)$. The effective Lagrangian can be constructed using the nonet of the Goldstone bosons where the η' meson becomes the ninth addition to the $SU_A(3)$ octet of (π, K, η) . In Ref. [96], a systematic expansion of the Green functions of the $U(3)$ ChPT in powers momenta, quark masses, and the $\mathcal{O}(p^2)$ parameter $\frac{1}{N_c}$ was introduced and got firmly established in Refs. [97–101]. It is argued in Refs. [102–104] that the large N_c arguments are not necessary for constructing the effective Lagrangian with the singlet field as the additional $\frac{1}{N_c}$ counting scheme is imposed only to ensure that loops with an η' are suppressed by powers of $\frac{1}{N_c}$. They point out that the mass of the η' particle that introduces an additional low-energy scale of about 1 GeV is proportional to $\frac{1}{N_c}$. Hence, treating it perturbatively, the η' can be systematically included in the ChPT without using the $\frac{1}{N_c}$ counting rules. The alternate method of treating massive fields was put on a firm basis in Ref. [105], where the loop integrals are evaluated using a modified regularization scheme, the so-called infrared regularization, in which Lorentz and chiral invariance are kept at all stages [102,103]. Dealing with the inclusion of loops and renormalization issues, the authors in Ref. [103] discussed that the η' loops do not contribute (at the fourth-order Lagrangian) to infrared regularization, and the η' can be treated as a background field while the contribution of the Goldstone boson (π, K, η) loops get evaluated. The above approach is different from the large N_c standard $U(3)$ ChPT works in Refs. [99–101], where the authors have treated the η' mesons on the same footing as the original Goldstone Bosons and included the η' in loops. The quark mass and condensate dependent expressions of the decay constants f_π, f_K and the masses m_π, m_K , and M_η^2 for performing the chiral limit study in the RQM model are described below in subsection I and subsection II, respectively, for the infrared regularized $U(3)$ ChPT [102–104] and the large N_c standard $U(3)$ ChPT [99–101]. The chiral limit and Columbia plot in the $m_\pi - m_K$ and $m_{ud} - m_s$ planes will be investigated in both of the above prescriptions and results will be compared.

1. M_η^2 from infrared regularized $U(3)$ ChPT inputs

Taking the expressions of M_η^2 from the infrared regularized $U(3)$ ChPT, this ChPT-based parameter fixing procedure was originally proposed in Ref. [48] for the linear sigma model studied in the optimized perturbation theory framework. In this approach, first one needs the functional forms $f_\pi(m_\pi, m_K)$ and $f_K(m_\pi, m_K)$, for which it is sufficient to use the $SU_L(3) \times SU_R(3)$ ChPT framework. The one-loop ChPT calculation [96] gives the following

expressions of $m_\pi^2, m_K^2, f_\pi, f_K$ with $\mathcal{O}(\frac{1}{f^2})$ accuracy in terms of the eight parameters $f, A, q, M_0, L_4, L_5, L_6, L_8$,

$$m_\pi^2 = 2A \left[1 + \frac{1}{f^2} \left\{ \mu_\pi - \frac{\mu_\eta}{3} + 16A(2L_8 - L_5) + 16A(2+q)(2L_6 - L_4) \right\} \right], \quad (29)$$

$$m_K^2 = A(1+q) \left[1 + \frac{1}{f^2} \left\{ \frac{2}{3}\mu_\eta + 8A(1+q)(2L_8 - L_5) + 16A(2+q)(2L_6 - L_4) \right\} \right], \quad (30)$$

$$f_\pi = f + \frac{1}{f} [-2\mu_\pi - \mu_K + 8AL_5 + 8A(2+q)L_4], \quad (31)$$

$$f_K = f \left[1 + \frac{1}{f^2} \left\{ -\frac{3}{4}(\mu_\pi + \mu_\eta + 2\mu_K) + 8A(2+q)L_4 + 4A(1+q)L_5 \right\} \right]. \quad (32)$$

The $\mu_{\text{PS}} = \frac{m_{\text{PS}}^2}{32\pi^2} \ln(\frac{m_{\text{PS}}^2}{M_0^2})$ in the above equations are the chiral logarithms at the scale M_0 , where for m_{PS}^2 , the leading order squared mass of the corresponding pseudo-scalar octet meson is substituted. It is to be noted that the chiral constants L_i do not vary with the pseudo-scalar masses. The quark mass and quark condensate dependences of the above one-loop ChPT expressions of m_π^2, m_K^2, f_π and f_K are governed by the parameters $q = 2m_s/(m_u + m_d) = m_s/m_{ud}$ and $A = B(m_u + m_d)/2 = Bm_{ud}$, where the chiral limit value of the quark condensate $\langle \bar{u}u \rangle$ determines B . Inverting Eqs. (29) and (30) to order $\mathcal{O}(\frac{1}{f^2})$ accuracy, the q and A can be expressed below in terms of the masses m_π, m_K, m_η , and chiral constants L_i ,

$$A = \frac{m_\pi^2}{2} \left[1 - \frac{1}{f^2} \left\{ \mu_\pi - \frac{\mu_\eta}{3} + 8m_\pi^2(2L_8 - L_5) + 8(2m_K^2 + m_\pi^2)(2L_6 - L_4) \right\} \right], \quad (33)$$

$$q + 1 = \frac{2m_K^2}{m_\pi^2} \left[1 - \frac{1}{f^2} \left\{ \mu_\eta - \mu_\pi + 8(m_K^2 - m_\pi^2)(2L_8 - L_5) \right\} \right]. \quad (34)$$

The leading order relations of the above two equations are sufficient for finding the following $\mathcal{O}(\frac{1}{f^2})$ accurate m_π, m_K -dependence of f_π, f_K from Eqs. (31) and (32),

$$f_\pi = f - \frac{1}{f} [2\mu_\pi + \mu_K - 4m_\pi^2(L_4 + L_5) - 8m_K^2 L_4], \quad (35)$$

$$f_K = f \left[1 - \frac{1}{f^2} \left\{ \frac{3}{4} (\mu_\pi + \mu_\eta + 2\mu_K) - 4m_\pi^2 L_4 - 4m_K^2 (2L_4 + L_5) \right\} \right]. \quad (36)$$

With the input $f_\pi = 93$ MeV, $f_K = 113$ MeV, $m_\pi = 138$ MeV, $m_K = 495.6$ MeV, $m_\eta = 547.8$ MeV, and $M_0 = 4\pi f_\pi \equiv 1168$ MeV and $f = 88$ MeV, one gets the constants L_4 and L_5 at the physical point as

$$L_4 = -0.7033 \times 10^{-3}, \quad L_5 = 0.3708 \times 10^{-3}. \quad (37)$$

The values of A and q that one takes at the physical point, control the constants L_6 and L_8 . The strange to average nonstrange quark mass ratio has the value $q = 24.9$, similar to Ref. [48] as it is close to the lattice determination and compatible with the range $20 \leq q \leq 34$ indicated by the PDG listing [125]. Choosing the leading order ChPT value of A in the physical point $A = A^{(0)}$ and then using the phenomenological values of m_π^2, m_K^2 with the Gell-Mann-Okubo formula for m_η^2 in the $\mathcal{O}(\frac{1}{f^2})$ accurate expressions of A and q , one gets

$$L_6 = -0.3915 \times 10^{-3}, \quad L_8 = 0.511 \times 10^{-3}. \quad (38)$$

The values of M_0 , f , and L_i can be used further for the continuation of A and q from the physical point to an arbitrary point in the $m_\pi - m_K$ plane.

Note that the physical states of η and η' are the mixtures of the states η_0 and η_8 . Apart from the values m_π^2 and m_K^2 , the numerical input of the $m_{\eta,00}^2 + m_{\eta,88}^2$ is needed to fix the quartic coupling λ_2 in the sigma/QM model [45,61,64,65]. Once λ_2 is known, the 't Hooft cubic coupling c gets fixed easily. We know the experimental masses m_η and $m_{\eta'}$ only at the physical point. The numerical value of the total sum $m_\eta^2 + m_{\eta'}^2 = m_{\eta,00}^2 + m_{\eta,88}^2 = M_\eta^2$ is relevant for fixing λ_2 . Even if we do not know the $m_\eta^2(m_\pi, m_K)$ and $m_{\eta'}^2(m_\pi, m_K)$ individually, the knowledge of the sum $M_\eta^2(m_\pi, m_K)$ as input is necessary and sufficient for fixing the λ_2 as one moves away from the physical point.

Finding the functional form $M_\eta^2(m_\pi, m_K)$, requires the application of the infrared regularized $U_L(3) \times U_R(3)$ ChPT, which relies on similar steps as in the previous description, but the mass mixing in the η_0, η_8 sector makes it a little involved. The formulas, spread over several papers [99,102,103], are collected in Ref. [48] as per their relevance. The experimental values of masses $m_\eta, m_{\eta'}$ with the choice of mixing angle $\theta_\eta = -20^\circ$ give the values of $m_{\eta,00}^2, m_{\eta,88}^2, m_{\eta,08}^2$ at the physical point. When the ChPT expressions of $m_{\eta,00}^2, m_{\eta,88}^2, m_{\eta,08}^2$ are made equal to their

respective numerical values at the physical point, the four chiral constants $L_7, v_0^{(2)}, v_2^{(2)}, v_3^{(1)}$ appearing in those expressions get restricted by the three relations. Note that the constant $v_0^{(2)}$ accounts for the $U_A(1)$ anomaly contribution to the η' mass, which is determined largely by the topological characteristics of the gluonic configurations. Hence, it would be insensitive against the quark mass variations. Choosing the large N_c relation for $v_0^{(2)} = -29.3f^2$, the remaining three chiral constants as reported in Ref. [48] are

$$L_7 = -0.2272 \times 10^{-3}, \quad v_3^{(1)} = 0.095, \quad v_2^{(2)} = -0.1382. \quad (39)$$

The sum of Eqs. (B10) and (B11) in Ref. [48] gives the following (m_π, m_K) dependence of the M_η^2 :

$$\begin{aligned} M_\eta^2 = & 2m_K^2 - 3v_0^{(2)} + 2(2m_K^2 + m_\pi^2)\{3v_2^{(2)} - v_3^{(1)}\} \\ & + \frac{1}{f^2} \left[8v_0^{(2)}(2m_K^2 + m_\pi^2)(3L_4 + L_5) + m_\pi^2(\mu_\eta - 3\mu_\pi) \right. \\ & - 4m_K^2\mu_\eta + \frac{16}{3} \{(6L_8 - 3L_5 + 8L_7)(m_\pi^2 - m_K^2)^2 \\ & \left. + 2L_6(m_\pi^4 - 2m_K^4 + m_\pi^2 m_K^2) + L_7(m_\pi^2 + 2m_K^2)^2 \} \right]. \end{aligned} \quad (40)$$

The η mass occurs at the leading order in the chiral logarithm, i.e., $(m_\eta^{(0)})^2 = (4m_K^2 - m_\pi^2)/3$. Hence, the functions $f_\pi(m_\pi, m_K)$, $f_K(m_\pi, m_K)$, and $M_\eta^2(m_\pi, m_K)$ are applicable only when $4m_K^2 > m_\pi^2$. Eq. (40), together with Eqs. (35) and (36), enable the calculation of the RQM/QM model couplings λ_2, c, m^2 , and λ_1 in the $m_\pi - m_K$ plane.

The chiral symmetry breaking strengths are reduced by reducing the m_π and m_K away from the physical point for the chiral limit studies. The results in the $m_\pi - m_K$ plane can be translated to the light-strange quark mass $m_{us} - m_s$ plane using Eqs. (33) and (34), whose multiplication gives the $\mathcal{O}(1/f^2)$ expression of $A(q+1) = B(m_s + m_{ud})$ in terms of the π , the K masses, and the chiral constants. The $m_\pi = 138$ MeV determines the constant B from Eq. (33) as $A = Bm_{ud}$ by taking the $m_{ud} = 4$ MeV at the physical point. Knowing the constant B , quark masses away from the physical point get easily evaluated to generate the Columbia plot in the $m_{ud} - m_s$ plane.

2. f_π, f_K, m_π, m_K , and M_η^2 from large N_c $U(3)$ ChPT

The following expressions of the pion, the kaon decay constants, and the masses f_π, f_K and m_π^2, m_K^2 are taken from Refs. [100,101]:

$$f_\pi = f \left(1 + 4 \frac{L_5}{f^2} m_\pi^2 \right), \quad (41)$$

$$f_K = f \left(1 + 4 \frac{L_5}{f^2} m_K^2 \right), \quad (42)$$

$$m_\pi^2 = 2m_{ud}B \left[1 + \frac{16m_{ud}B}{f^2} (2L_8 - L_5) \right], \quad (43)$$

$$m_K^2 = m_{ud}B(1+q) \left[1 + \frac{8m_{ud}B}{f^2} (1+q)(2L_8 - L_5) \right]. \quad (44)$$

Only the dominant low-energy constants appear in the expressions as the L_5 and L_8 are $\mathcal{O}(N_c)$ while the rest, $L_{4,6,7}$, are subleading and suppressed in the large N_c standard U(3) ChPT. As defined earlier, the parameter $q = m_s/m_{ud}$ and B is determined by the quark condensate $\langle \bar{u}u \rangle$ in the chiral limit. Taking $f_\pi = 93$ MeV with $f_K = 1.223f_\pi$ and using Eqs. (41) and (42) for the physical $m_\pi = 138$ MeV and $m_K = 496$ MeV, one finds $f = 91.2599$ MeV and $L_5 = 2.084659 \times 10^{-3}$. The octet η_8 and singlet η_0 masses [100,101] are written as below

$$m_{88}^2 = \frac{(4m_K^2 - m_\pi^2)}{3} + \frac{4}{3}(m_K^2 - m_\pi^2)\Delta_M, \quad (45)$$

$$m_{00}^2 = \frac{(2m_K^2 + m_\pi^2)}{3}(1 - 2\Delta_N) + \frac{2}{3}(m_K^2 - m_\pi^2)\Delta_M - 3v_{02}, \quad (46)$$

$$\Delta_M = \frac{8}{f^2}(m_K^2 - m_\pi^2)(2L_8 - L_5), \quad (47)$$

$$\Delta_N = 3v_{31} - \frac{12}{f^2}v_{02}L_5, \quad (48)$$

$$M_\eta^2 = m_{00}^2 + m_{88}^2 = 2m_K^2 - \frac{2}{3}(m_K^2 - m_\pi^2)y + 2(m_K^2 - m_\pi^2)\Delta_M - \frac{2}{3}(2m_K^2 + m_\pi^2)\Delta_N. \quad (49)$$

The $\frac{2(m_K^2 - m_\pi^2)}{3}y = 3v_{02}$ in Eq. (49) defines y . The η' mass $m_{\eta'}^2 = (M_\eta^2 - m_\pi^2)$ where the physical η has the following mass:

$$m_\eta^2 = m_K^2 - \frac{(m_K^2 - m_\pi^2)}{3} \left[y + (\sqrt{9 + 2y + y^2}) \right] + \Delta_M + (m_K^2 - m_\pi^2) \left(1 - \frac{9 + y}{3\sqrt{9 + 2y + y^2}} \right) - \frac{\Delta_N}{3} \left\{ 2m_K^2 + m_\pi^2 - \frac{3(2m_K^2 - 3m_\pi^2) - y(2m_K^2 + m_\pi^2)}{\sqrt{9 + 2y + y^2}} \right\}. \quad (50)$$

The five parameters v_{02} , v_{31} , L_8 together with the quark mass parameters $m_{ud}B$ and q can be expressed in terms of

five independent observables $m_\pi, m_K, m_\eta, m_{\eta'}$, and the $\eta - \eta'$ mixing angle θ . Thus, large N_c ChPT is not predictive at this level. In order to get predictions from the large N_c ChPT, Ref. [100] imposed that $\mathcal{O}(\delta)$ corrections are not too large so that the large N_c expansion makes sense [101]. Keeping the mixing angle in the range $20^\circ < \theta < 24^\circ$, Ref. [100] gives the following range of the parameters fit for the physical observables:

$$0.980 \leq \frac{2m_{ud}B}{m_\pi^2} \leq 0.988,$$

$$18.3 \leq (q - 1) \leq 20.9,$$

$$26 \leq \frac{-v_{02}}{f^2} \leq 29,$$

$$-0.164 \leq v_{31} \leq -0.161,$$

$$1.35 \times 10^{-3} \leq L_8 \leq 1.57 \times 10^{-3}. \quad (51)$$

The experimental masses $m_\eta = 547.5$ MeV and $m_{\eta'} = 957.78$ MeV give $M_\eta = (\sqrt{m_\eta^2 + m_{\eta'}^2}) = 1103.22$ MeV. With this same numerical value of the M_η , when the pole masses $m_\eta = 528.48$ MeV and $m_{\eta'} = 968.41$ MeV are used as input in the RQM model (with f_π, f_K as above) for calculating the η, η' self-energy corrections and the renormalized parameters, the same pole masses for the η and η' are reproduced in the output after adding the corresponding self-energy corrections to their respective masses calculated from the new set of renormalized parameters. For getting the $M_\eta = 1103.22$ MeV at the physical point from the ChPT expression given in Eq. (49), one finds the following numerical values of the U(3) ChPT parameters:

$$L_8 = 1.44 \times 10^{-3}, \quad v_{31} = -0.1637, \quad \text{and} \quad v_{02} = -29.1f^2. \quad (52)$$

It is nice to point out that with the above set of parameter values at the physical point, the ChPT expressions in Eqs. (50) and (49) give the η and η' masses as $m_\eta = 527.82$ MeV and $m_{\eta'} = 968.77$, which are almost the same as the corresponding masses obtained in the RQM model. Note that the QM model parameters give $m_\eta = 538.96$ MeV and $m_{\eta'} = 962.62$ MeV in the output when $M_\eta = 1103.22$ MeV at the physical point.

One can find the light and strange quark masses corresponding to the reduced masses of pions and kaons from the Eqs. (43) and (44). Solving the quadratic Eq. (43) and taking the positive root, one gets $m_{ud}B$ whose value for the $m_\pi = 138$ MeV will give the constant B if one takes $m_{ud} = 4$ MeV at the physical point. The positive root of the quadratic Eq. (44) gives numerical values of $Bm_{ud}(q+1) = B(m_{ud} + m_s)$ for different kaon masses. Thus, knowing B , one can know the m_{ud} and m_s for

different values of the m_π and m_K as one moves away from the physical point in the Columbia plot.

III. RESULTS AND DISCUSSION

A. The Consistent path to the chiral limit

It is worth emphasizing that the consistent treatment of the quark one-loop vacuum fluctuation and the proper renormalization of the parameters in the RQM model gives rise to a stronger axial $U_A(1)$ anomaly as the 't Hooft coupling c increases significantly while the explicit chiral symmetry breaking strength in the nonstrange direction h_x decreases by a small amount (same as in the two-flavor RQM model [121]) and the strength h_y in the strange direction gets reduced by a relatively large amount because after the renormalization, the pion and kaon curvature masses of $m_{\pi,c} = 135.95$ MeV and $m_{K,c} = 467.99$ MeV are, respectively, smaller by 2.05 MeV and 28.01 MeV from their corresponding pole masses of $m_\pi = 138.0$ MeV and $m_K = 496.0$ MeV [94,95]. Increased c_0 and decreased $h_{x0}(h_{y0})$ both have the propensity to make the chiral transition stronger while the renormalized quartic couplings λ_{10} , λ_{20} and mass parameter m_0^2 significantly soften the chiral transition. The overall softening influence of the quark one-loop vacuum fluctuation on the chiral transition looks moderate in the RQM model [95] when it is compared to the studies [106–113] where the QM model parameters in the presence of the vacuum fluctuations are fixed using the curvature masses of the mesons. The above-mentioned novel features of the RQM model will have important consequences for the chiral limit study, the location of the tricritical point (TCP) at $\mu = 0$ in the light chiral limit, and the value of the pion (light quark) critical mass m_π^c (m_{ud}^c) beyond which the chiral transition is a

smooth crossover on the three flavor chiral limit line ($m_\pi = m_K$, i.e., $m_{ud} = m_s$) in the Columbia plot.

The light chiral limit in the RQM model is given by $h_{x0} = 0$ [Eq. (25)], while putting $h_{y0} = 0$ in Eq. (26) defines the line of the strange chiral limit. The reduced chiral symmetry breaking strengths h_{x0}^* and h_{y0}^* (h_x^* and h_y^*) and the corresponding RQM (QM) model parameters λ_{20} , c_0 , λ_{10} , $m_0^2(\lambda_2, c, \lambda_1, m^2)$ for the chiral limit study are calculated using the two prescriptions of the ChPT. The use of infrared regularized U(3) ChPT prescription described in Sec. II B I gives us the input set M_η – I when the changes in pion and kaon decay constants f_π and f_K are calculated by putting the values of the reduced pion and the kaon masses m_π^* and m_K^* , respectively, into Eqs. (35) and (36), while M_η is calculated using Eq. (40). The large N_c standard U(3) ChPT framework given in the Sec. II B II provides us the input set M_η – II when the changing f_π and f_K are computed by substituting the values of the reduced masses m_π^* and m_K^* , respectively, in Eqs. (41) and (42), whereas M_η is evaluated using Eq. (49).

The starred π and K masses are reduced in a fixed ratio, i.e., $\frac{m_\pi^*}{m_\pi} = \frac{m_K^*}{m_K} = \beta$. Apart from providing us a direct path that connects the physical point to the chiral limit when one moves away from the physical point just by changing the only one quantity as the fraction β instead of changing both the m_π and m_K randomly, this choice ensures that $\frac{m_\pi^*}{m_K^*}$ remains equal to the ratio $\frac{m_\pi}{m_K}$ [61] at the physical point, where the ChPT parameter $q = \frac{2m_s}{m_u+m_d} = \frac{m_s}{m_{ud}}$ as the ratio of the strange quark mass m_s to the average of light quark mass m_{ud} , also gets fixed. Further, β , as defined above, also facilitates the comparison of our results with those of Refs. [23,61] for the fractions defined analogously. Table I presents the

TABLE I. Chiral limit path is made by reducing the π , K meson starred masses as $\frac{m_\pi^*}{m_\pi} = \frac{m_K^*}{m_K} = \beta \leq 1$ such that $\frac{m_\pi^*}{m_K^*} = \frac{m_\pi}{m_K}$, where $m_\pi = 138$ and $m_K = 496$ MeV at the physical point [61]. Equations (35) and (36) with the m_π^* , m_K^* for any β give the corresponding f_π , f_K for the infrared regularized U(3) ChPT input set M_η –I, where the corresponding M_η gets calculated using Eq. (40). The experimental values $(m_\eta, m_{\eta'}) = (547.5, 957.8)$ MeV give the M_η –Expt = 1103.22 MeV, while the infrared regularized U(3) ChPT expression in Eq. (40) gives M_η –I = 1035.55 MeV for the physical m_π , m_K . The RQM{QM} model parameters presented here are λ_{20} , c_0 , h_{x0} , h_{y0} $\{\lambda_2, c, h_x, h_y\}$. $\delta_1 = 7.283 \times 10^{-6}$; $\delta_2 = 16.574 \times 10^{-6}$.

β	f_π (MeV)	f_K (MeV)	M_η –I: (MeV)	$\lambda_{20}\{\lambda_2\}$	$c_0\{c\}$ (MeV)	$h_{x0}^*\{h_x^*\}$ (MeV ³)	$h_{y0}^*\{h_y^*\}$ (MeV ³)
1	92.9737	113.2635	1035.55	35.47 {54.46}	7390.50 {3913.39}	(119.79) ³ {(120.98) ³ }	(324.39) ³ {(336.65) ³ }
1	92.9737	113.2635	650.43	90.81 {83.76}	0 {0}	”	”
0.5	92.7862	102.1041	865.99	–12.95 {16.59}	7705.88 {4241.24}	(75.78) ³ {(76.16) ³ }	(197.66) ³ {(204.63) ³ }
0.3742	91.510	97.723	845.54	–25.01 {5.86}	7833.91 {4505.99}	(62.4) ³ {(62.49) ³ }	(162.07) ³ {(166.16) ³ }
0.3	90.6629	95.1438	837.31	–32.64 {–1.17}	7901.53 {4675.25}	(53.71) ³ {(53.76) ³ }	(139.46) ³ {(142.1) ³ }
0.2	89.5151	91.9039	829.99	–43.65 {–11.62}	7970.71 {4895.85}	(40.84) ³ {(40.85) ³ }	(106.03) ³ {(107.17) ³ }
0.15	88.9846	90.4869	827.71	–49.53 {–17.32}	7992.99 {4992.98}	(33.65) ³ {(33.66) ³ }	(87.39) ³ {(87.996) ³ }
0.05	88.1655	88.3998	825.32	–62.97 {–30.57}	8011.99 {5132.15}	(16.13) ³ {(16.13) ³ }	(41.92) ³ {(41.97) ³ }
0.007246	88.0055	88.0129	825.049	–72.80 {–40.35}	8012.25 {5156.06}	(4.44) ³ {(4.44) ³ }	(11.561) ³ {(11.562) ³ }
0.000202	88 + δ_1	88 + δ_2	825.04	–79.67 {–47.2}	8012.18 {5156.80}	(0.409) ³ {(0.409) ³ }	(1.064) ³ {(1.064) ³ }
1	92.9737	113.2635	M_η : Expt = 1103.22	31.19 {47.93}	7962.25 {4785.76}	(119.79) ³ {(120.98) ³ }	(324.39) ³ {(336.65) ³ }

TABLE II. Other RQM{QM} model parameters λ_{10} and $m_0^2 \{\lambda_1 \text{ and } m^2\}$, which depend on the value of the scalar σ meson mass m_σ , have been presented in this table for $m_\sigma = 400$ MeV and $m_\sigma = 530$ MeV.

β	f_π (MeV)	f_K (MeV)	M_η -I: (MeV)	$m_\sigma = 400$ (MeV)		$m_\sigma = 530$ (MeV)	
				$\lambda_{10}\{\lambda_1\}$	$m_0^2\{m^2\}$ (MeV ²)	$\lambda_{10}\{\lambda_1\}$	$m_0^2\{m^2\}$ (MeV ²)
1	92.9737	113.2635	1035.55	1.27 $\{-9.87\}$	(447.71) ² {(467.34) ² }	4.499 $\{-5.58\}$	(378.99) ² {(378.06) ² }
1	92.9737	113.2635	650.43	-27.68 $\{-25.06\}$	-(154.53) ² {(311.53) ² }	-24.18 $\{-20.65\}$	-(292.08) ² {(140.06) ² }
0.5	92.7862	102.1041	865.99	20.19 $\{6.04\}$	(217.28) ² {(283.01) ² }	23.96 $\{10.499\}$	-(93.16) ² {(118.64) ² }
0.3742	91.510	97.723	845.54	25.35 $\{11.05\}$	(157.17) ² {(245.25) ² }	29.57 $\{15.67\}$	-(182.89) ² {-(60.06) ² }
0.3	90.6629	95.1438	837.31	28.6 $\{14.3\}$	(118.4) ² {(225.6) ² }	33.15 $\{19.05\}$	-(214.51) ² {-(108.35) ² }
0.2	89.5151	91.9039	829.99	33.17 $\{19.00\}$	(52.76) ² {(203.43) ² }	38.22 $\{23.94\}$	-(245.43) ² {-(142.06) ² }
0.15	88.9846	90.4869	827.71	35.52 $\{21.46\}$	-(36.92) ² {(194.83) ² }	40.75 $\{26.52\}$	-(254.90) ² {-(153.35) ² }
0.05	88.1655	88.3998	825.32	40.55 $\{26.69\}$	-(79.90) ² {(184.28) ² }	46.4 $\{31.86\}$	-(273.52) ² {-(162.75) ² }
0.007246	88.0055	88.0129	825.049	43.92 $\{30.1\}$	-(84.02) ² {(182.92) ² }	49.87 $\{35.30\}$	-(275.99) ² {(164.29) ² }
.000202	$88 + \delta_1$	$88 + \delta_2$	825.04	46.21 $\{32.39\}$	-(84.12) ² {(182.89) ² }	52.16 $\{37.60\}$	-(276.07) ² {-(164.32) ² }
1	92.9737	113.2635	M_η : Expt	1.09 $\{-6.48\}$	(444.12) ² {(495.28) ² }	4.32 $\{-2.21\}$	(375.01) ² {(412.65) ² }

RQM{QM} model parameters $\lambda_{20}, c_0, h_{x0}, h_{y0} \{\lambda_2, c, h_x, h_y\}$ for f_π, f_K, M_η of the input set M_η -I when $\beta = 1, 0.5, 0.3742, 0.3, 0.2, 0.15, 0.05, 0.007246$, and 2.02×10^{-4} , while Table II shows the corresponding RQM {QM} model parameters $\lambda_{10}, m_0^2 \{\lambda_1, m^2\}$ for the σ masses $m_\sigma = 400$ MeV and $m_\sigma = 530$ MeV. $\beta = 1$ signifies the physical point, where $m_\pi = 138.0$ MeV, $m_K = 496.0$ MeV. The second row of Tables I and II shows the RQM {QM} model parameters at the physical point when the axial $U_A(1)$ anomaly is absent, i.e., $c = 0$, while the last row presents the parameters for the M_η : Expt in the presence of the axial $U_A(1)$ anomaly, and f_π, f_K are the same as in the input set M_η -I. With the input of $M_\eta = \sqrt{(m_\eta^2 + m_{\eta'}^2)} = 1103.22$ MeV (termed as M_η : Expt) for the experimental $(m_\eta, m_{\eta'}) = (547.5, 957.8)$ MeV at the physical point, the parameters of the QM model give $(m_\eta, m_{\eta'}) = (538.54, 962.85)$ MeV in the output, while the renormalized parameters of the RQM model give $(m_\eta, m_{\eta'}) = (527.82, 968.76)$ MeV in the output. The results are consistent because the calculated M_η in the output of the QM

(RQM) model is the same as in the input. The ChPT expression in Eq. (40) gives a slightly smaller value of the $M_\eta = 1035.55$ MeV for the input set M_η -I at the physical point, and with this M_η in the input, the QM model parameters give $(m_\eta, m_{\eta'}) = (528.76, 890.79)$ MeV in the output, while the renormalized parameters of the RQM model give $(m_\eta, m_{\eta'}) = (520.62, 895.17)$ MeV in the output. Here, also the output M_η is same as in the input. Table I shows the systematic reduction of the nonstrange [strange] direction explicit chiral symmetry breaking strengths $h_{x0}^* \{h_x^*\} [h_{y0}^* \{h_y^*\}]$ in the RQM {QM} model for the input set M_η -I as the pion and kaon masses get decreased by the fraction β . The RQM model parameters $\lambda_{20}, c_0, \lambda_{10}, m_0^2, h_{x0}, h_{y0}$ for f_π, f_K, M_η obtained from the input set M_η -II with the $\beta = 1, 0.5, 0.36887, 0.3, 0.15, 0.05, 0.00725$ are presented in Table III. The experimental M_η :Expt = 1103.22 MeV at the physical point is reproduced for the input set M_η -II by taking the values of the large N_c standard U(3) ChPT parameters and chiral constants in the range given in Refs. [100,101]. The M_η :Expt = 1103.22 MeV for

TABLE III. Equations (41) and (42) with the m_π^* and m_K^* for any β give the corresponding f_π, f_K in the large N_c standard U(3) ChPT and the M_η computed from Eq. (49) in this context has been termed as M_η -II. The parameters and chiral constants of the standard U(3) ChPT at the physical point, which reproduce the experimental M_η :Expt = 1103.22 MeV, give the $(m_\eta, m_{\eta'}) = (527.82, 968.77)$ MeV from Eqs. (49) and (50). The RQM model parameters presented here are $\lambda_{20}, c_0, \lambda_{10}, m_0^2, h_{x0}$ and h_{y0} .

β	f_π (MeV)	f_K (MeV)	M_η -II: (MeV)	λ_{20}	c_0	λ_{10}	m_0^2	h_{x0}^* (MeV ³)	h_{y0}^* (MeV ³)
1	93.0	113.7389	1103.22	33.522	7330.557	5.105	(374.128) ²	(119.804) ³	(323.531) ³
0.5	91.695	96.8797	913.684	32.176	7916.86	10.317	-(83.669) ²	(75.676) ³	(199.838) ³
0.36887	91.497	94.3185	885.715	30.978	8077.842	12.039	-(181.369) ²	(61.811) ³	(162.140) ³
0.3	91.417	93.283	874.485	30.377	8149.262	12.859	-(209.694) ²	(53.864) ³	(140.878) ³
0.15	91.299	91.766	858.115	29.36	8260.449	14.214	-(247.063) ²	(33.939) ³	(88.361) ³
0.05	91.264	91.316	853.285	29.026	8294.868	14.655	-(257.538) ²	(16.318) ³	(42.422) ³
0.00725	91.260	91.261	852.695	29.057	8299.125	14.685	-(258.807) ²	(4.504) ³	(11.707) ³

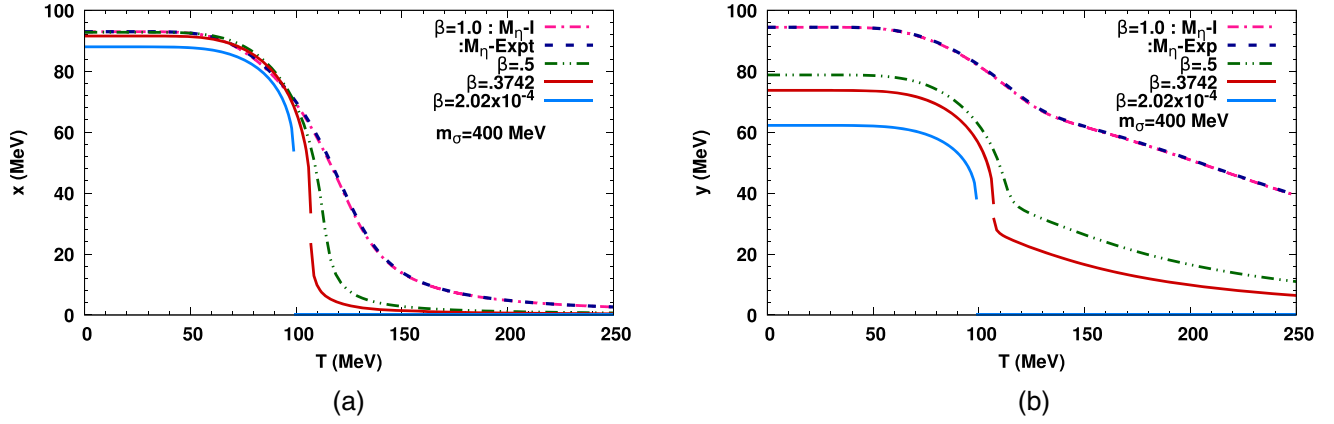


FIG. 1. Temperature variations of x and y [(a) and (b)] at $\mu = 0$ for the ratio $\frac{m_\pi^*}{m_K^*} = \beta = 1, 0.5, 0.3742$, and 2.02×10^{-4} in the infrared regularized U(3) ChPT. (a) Non strange chiral condensate. (b) Strange chiral condensate.

the experimental $(m_\eta, m_{\eta'}) = (547.5, 957.8)$ MeV and f_π, f_K for $\beta = 1$ from the input set M_η -II give the $(m_\eta, m_{\eta'}) = (527.82, 968.77)$ MeV from the standard U(3) ChPT expressions in Eqs. (49) and (50), whereas the corresponding RQM model calculation gives almost the same $(m_\eta, m_{\eta'}) = (528.48, 968.41)$ MeV.

Figures 1(a) and 1(b) show the plots of temperature variations of the light and strange condensate x and y in respective order for the $m_\sigma = 400$ MeV when the infrared regularized U(3) ChPT input set M_η : I has been used for calculation. The x (y) plot at the physical point $\beta = 1$ for the input set M_η : I in Fig. 1(a) [Fig. 1(b)], is overlapping with the corresponding temperature plot obtained after taking M_η : Expt, whereas the f_π, f_K are the same as in the input set M_η : I. The physical point crossover transition becomes sharper as β is decreased to 0.5, and it turns second order (the Z_2 critical point) when the $(m_{\pi, \text{cep}}^*, m_{K, \text{cep}}^*) = (52.38, 188.25)$ MeV at $\beta_{\text{cep}}|_{(m_\sigma=400 \text{ MeV}-M_\eta: \text{I})} = 0.3795$. The corresponding plot of $x(y)$ is not shown in Fig. 1(a)

[Fig. 1(b)]. When $\beta = 0.3742$, the respective temperature variations of the x and y in Figs. 1(a) and 1(b), show a small first-order transition gap, which becomes huge, signaling a very strong first-order transition in the chiral limit, where a very small difference of $f_\pi = (88 + 7.283 \times 10^{-6})$ MeV from the $f_K = 88 + 16.574 \times 10^{-6}$ MeV (for $\beta = 2.02 \times 10^{-4}$ with the $\{m_\pi^*, m_K^*\} = \{0.1, .028\}$ MeV) enables the calculation of the parameters, and the symmetry breaking strengths h_{x0} and h_{y0} are put to zero.

1. Comparing chiral limit studies for the input set M_η - I and M_η - II when $m_\sigma = 530$ MeV

The temperature variation plots of the light and strange condensates x and y obtained from the input set M_η : I have been compared, respectively, in Figs. 2(a) and 2(b) with the corresponding temperature variation plots of the x and y , computed using the large N_c standard U(3) ChPT input set M_η : II when the $m_\sigma = 530$ MeV and the fraction $\beta = 1, 0.5, 0.3$ and 0.00725 . The $x(y)$ plot from the input set M_η : I

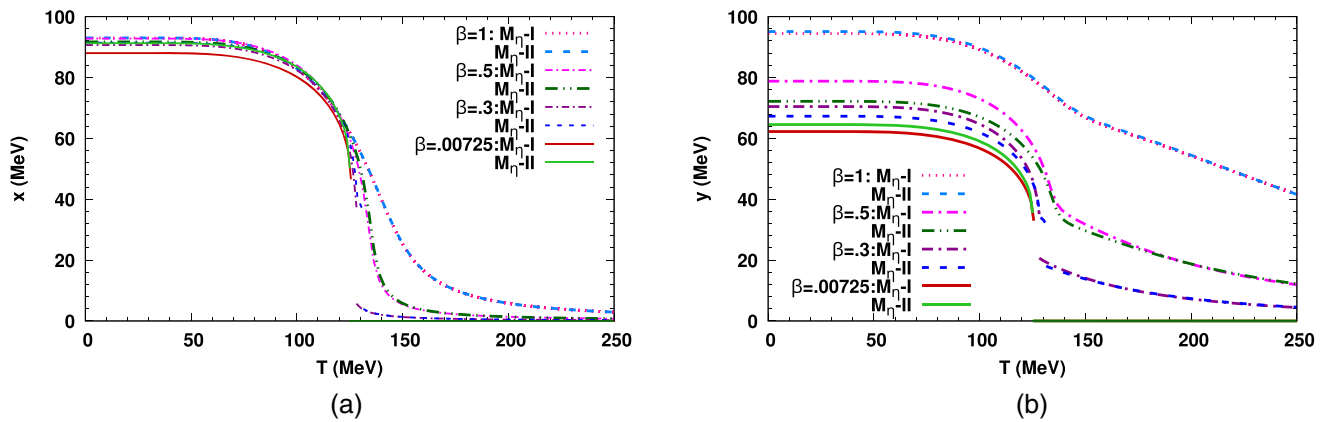


FIG. 2. x and y temperature variations [(a) and (b)] at $\mu = 0$ for the $m_\sigma = 530$ MeV with ratios $\frac{m_\pi^*}{m_K^*} = \beta = 1, 0.5, 0.3$, and 0.00725 using the infrared regularized U(3) ChPT input M_η -I and the large N_c standard U(3) ChPT input M_η -II. (a) Non-strange chiral condensate. (b) Strange chiral condensate.

and M_η : II completely overlap with each other at the physical point $\beta = 1$ in Figs. 2(a) and 2(b). When the m_π , m_K are reduced to half, i.e., $\beta = 0.5$, the temperature variation of the x for the input set M_η : I starts from the higher value 92.79 MeV in Fig. 2(a) and remains above the x of the input set M_η : II (which begins from the 91.69 MeV) before the chiral crossover has taken place. The x temperature variation of the input set M_η : I for $\beta = 0.3$ ($\beta = .00725$ for $(m_\pi^*, m_K^*) = (1, 3.6)$ MeV very close to the chiral limit) starts from a lower value 90.66 {88.005} MeV in Fig. 2(a), and it remains below {distinctly below} the x of the input set M_η : II [which begins from the 91.42 {91.26} MeV] before the first-order (very strong first order) chiral transition. The x temperature plots of both the input sets for every β merge with each other after the chiral transition. The temperature variation for the strange condensate y from the input set M_η : I for $\beta = 0.5$ ($\beta = 0.3$) begins from a higher value 78.79 {70.45} MeV in Fig. 2(b), and it remains noticeably above {above} the y of the input set M_η : II (which begins from the 72.17 {67.28} MeV) before the crossover (first-order) chiral transition. The preceding pattern gets reversed very close to the chiral limit when $\beta = 0.00725$, for which the y temperature variation of the input set M_η : I starts from a lower value 62.24 MeV in Fig. 2(b), and it remains below the y of the input set M_η : II (which begins from the 64.53 MeV) before the very strong first-order chiral transition takes place. The y temperature variations for both the input sets for every β become overlapping after the chiral transition.

For $m_\sigma = 530$ MeV, the crossover transition ends at the second order Z_2 critical point for $(m_{\pi,\text{cep}}^*, m_{K,\text{cep}}^*) = (50.26, 180.64)$ MeV at $\beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{I})} = 0.3642$ if the infrared regularized U(3) ChPT input set M_η : I is used, whereas one finds the Z_2 critical point for $(m_{\pi,\text{cep}}^*, m_{K,\text{cep}}^*) = (50.9, 182.96)$ MeV at $\beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{II})} = 0.36887$ if the large N_c standard U(3) ChPT input set M_η : II is used. The chiral transition becomes first order for the $\beta < \beta_{\text{cep}}$. Here, it is worth emphasizing that the input set M_η : II yields a marginally stronger first-order strength for the chiral transition as $\beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{I})} < \beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{II})}$, and one gets the first-order transition by marginally smaller reduction of π , K masses when the input set M_η : II is used. The lower value of $m_\sigma = 400$ MeV gives rise to a stronger first-order chiral transition whose strength becomes weaker for the higher $m_\sigma = 530$ MeV as it is evident from the result obtained using the input set M_η : I that $\beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{I})} = 0.3642 < \beta_{\text{cep}}|_{(m_\sigma=400 \text{ MeV}-M_\eta: \text{I})} = 0.3795$.

Here, it is important to recall that in the e-MFA QM model FRG work of Ref. [23], they changed the scale $\Lambda = (1143, 1100, 1000, 700)$ MeV for adjusting the initial

action to keep f_π fixed when the nonstrange and strange direction chiral symmetry breaking source strengths, defined as j_x and j_y in their work, are reduced by tuning a single parameter $\alpha = (0, 0.04, 0.17, 1)$. The physical point crossover transition for $\alpha = 1$ turns second order for the critical $\alpha_c = .04$ to become the first-order transition for $\alpha_c < .04$ when $m_\sigma = 530$ MeV in their work. The α in their study is related to the parameter β of our work as the $\beta = \sqrt{\alpha}$. Comparing our results with those of Ref. [23], one finds that the second-order Z_2 critical point in their e-MFA QM model study occurs for a significantly smaller $\beta_{\text{cep}} = \sqrt{\alpha_c} = 0.2$, corresponding to quite small pion and kaon masses $(m_{\pi,\text{cep}}^*, m_{K,\text{cep}}^*) = (27.6, 99.2)$ MeV. Thus, the effect of the quark one-loop vacuum fluctuation on the chiral transition is quite moderate in our studies with both the input sets M_η : I and M_η : II as we are getting a noticeably larger extent of the first-order region, while the extent of the first-order transition in the e-MFA QM model work of Ref. [23] is small due to the large softening effect of the quark one-loop vacuum correction on the strength of the chiral transition. Note that we are getting a slightly larger extent of the first-order region in our study when we are using the large N_c standard ChPT input set M_η : II. In order to see the effect of the fermionic vacuum fluctuation in proper perspective, it is pointed out that in the QM model result of Fig. (9) in Ref. [61], where Dirac's sea contribution is neglected under the s-MFA, one finds that the crossover transition ends at the second-order transition for the significantly larger fraction $\beta_{\text{cep}} = .488$, giving noticeably larger π and K masses $(m_{\pi,\text{cep}}^*, m_{K,\text{cep}}^*) = (67.34, 242.05)$ MeV where they have taken $m_\sigma = 800$ MeV in their calculation so that the SCSB is not lost. The comparison of the RQM model results for the present chiral limit studies using the input sets M_η : I and M_η : II with the e-MFA QM model FRG study and the s-MFA QM model work has been presented in the Table (IV).

For the exact chiral limit, the $f_\pi = f_K$, the $m_\eta = 0$, and the heavy $m_{\eta'} \equiv 825$ MeV and $m_{\eta''} \equiv 853$ MeV, respectively, for the input set M_η : I and M_η : II signifies the $U_A(1)$ anomaly. The problem of the loss of the SCSB when the $m_\pi, m_K \rightarrow 0$ in the chiral limit for the moderate $m_\sigma = 400$ –800 MeV gets cured after using the infrared regularized U(3) ChPT scaling [48] and the large N_c standard U(3) ChPT scaling [100,101] of the m_π, m_K, M_η^2 . This stands in contrast to the s-MFA studies [44,61], where the SCSB occurs for very large σ meson masses $m_\sigma \geq 800$ MeV and also to the case of the fixed- f_π scheme FRG study performed using the LPA in the [23] where the SCSB is retained for the $m_\sigma = 400$ –600 MeV at each step of the computation by adjusting the UV scale Λ of the action heuristically for each and every lower mass point in the Columbia plot.

B. The Columbia plot for different m_σ

The calculations of different regions in the Columbia plot require implementing conditions which are different from the earlier method of reducing the m_π and m_K simply by changing the one parameter, i.e., the fraction β . To compute the tricritical line that demarcates the second order from the first order transition region in the vertical $\mu - m_K$ plane, one has to work in the light chiral limit ($m_\pi = 0 \Rightarrow h_{x0} = h_x = 0$). The chiral critical line (shown by the solid black line in the figures below) that separates the first order from the crossover transition region in the horizontal $m_\pi - m_K$ plane at $\mu = 0$ is the locus of the Z_2 critical end points (cep), which are obtained by fixing the $m_{\pi, \text{cep}}$ and finding the $m_{K, \text{cep}}$ at which the crossover transition for $m_K > m_{K, \text{cep}}$ turns second order to become first order for the $m_K < m_{K, \text{cep}}$. The blue dashed dotted lines in the figures have been drawn by implementing the condition of the strange chiral limit $h_{y0} = 0 = h_y$. The shaded area of the figures signify the region in which the chiral symmetry breaking strength of the strange direction h_y becomes negative. The green dashed line in the figures discussed below are the $SU(3)$ symmetric lines which go to the chiral limit and are given by the condition $m_\pi = m_K$.

The chiral transition for the light chiral limit ($m_\pi = 0$) in Figs. 3(a) and 4(a) obtained for the input set M_η -I, respectively, for $m_\sigma = 400$ and 530 MeV at $\mu = 0$ is of the second order for the $m_K \geq 496$ MeV as in [23] and also predicted in Ref. [1], while it is first order in the s-MFA QM

model independent of the m_K and the $U_A(1)$ anomaly strength [61]. For the $m_K < 496$ MeV, the second-order transition line of the $O(4)$ universality ends at the blue dot of the tricritical point $m_K^{\text{TCP}} = 242.6$ MeV for $m_\sigma = 400$ MeV and $m_K^{\text{TCP}} = 235.7$ for $m_\sigma = 530$ MeV, respectively, in Figs. 3(a) and 4(a), where the transition turns first order and becomes stronger until the chiral limit is reached. The chiral critical solid black line (of the Z_2 universality) that separates the crossover from the first-order region in the $m_\pi - m_K$ plane at $\mu = 0$ terminates on the strange chiral limit line at the terminal pion mass $m_\pi^t \equiv 168.39$ MeV in Fig. 3(a) for $m_\sigma = 400$ MeV. The m_π^t is a mass beyond which the transition is a smooth crossover everywhere. For the higher mass of the scalar σ meson $m_\sigma = 530$ MeV, we get smaller $m_\pi^t \equiv 160.75$ MeV in Fig. 4(a). The intersection of the $SU(3)$ symmetric $m_\pi = m_K$ chiral limit path drawn with the green dashed line with the solid black line in Figs. 3(a) and 4(a) gives the critical pion mass m_π^c , where the boundary of the first-order region ends in the $m_\pi - m_K$ plane at $\mu = 0$. We find that $m_\pi^c = 134.16$ MeV for $m_\sigma = 400$ MeV in Fig. 3(a), and $m_\pi^c = 128.38$ MeV for $m_\sigma = 530$ MeV in Fig. 4(a). The s-MFA study of Ref. [61] for $m_\sigma = 800$ MeV finds $m_\pi^c \equiv 150$ MeV. The chiral matrix model study using the mean-field analysis gives $m_\pi^c \equiv 110$ MeV, as mentioned in Ref. [41]. The first-order region in Fig. 3(a) is significantly large in our work, with the $U_A(1)$ anomaly for the smaller σ meson $m_\sigma = 400$ MeV mass as we are finding that $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) \equiv (242.6, 168.39, 134.16)$ MeV. For

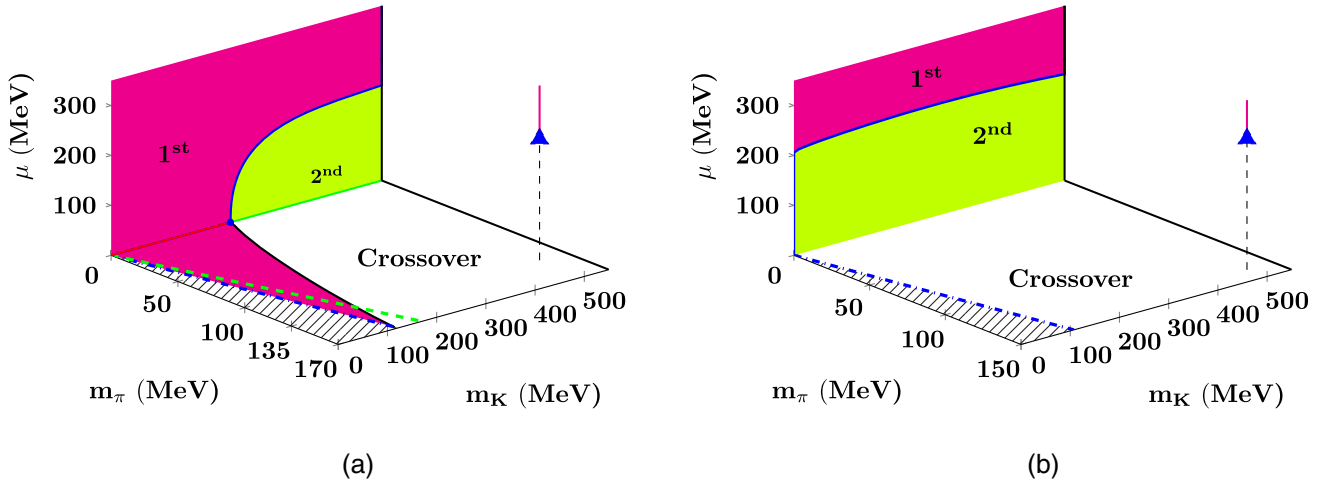


FIG. 3. $\mu - m_K$ and $m_\pi - m_K$ plane depict the respective chiral transition for the $m_\pi = 0$ and $\mu = 0$. Results with the $U_A(1)$ anomaly, i.e., $c_0 \neq 0$, are presented in (a), while results without the $U_A(1)$ anomaly, i.e., $c_0 = 0$, are shown in (b). The solid black second-order $Z(2)$ critical line in (a) separating the crossover from the first-order region, intersects the $m_\pi = m_K$ green dashed line at the critical pion mass $m_\pi^c = 134.16$ MeV, and it terminates at the terminal pion mass $m_\pi^t = 168.39$ MeV. The solid blue line of the tricritical points [which starts at the blue dot $m_K^{\text{TCP}} = 242.6$ MeV in (a)], separates the second- and first-order regions in both Figs.(a) and (b). The h_{y0} is negative in the dashed area below the dashed dotted blue line for the strange chiral limit $h_{y0} = 0$. The black dashed vertical line at the physical point in Figs. (a) and (b) shows the crossover transition, which ends at the critical end point in the blue triangle, and the solid red line shows the first-order transition. Parameters of RQM model away from the physical point are obtained using the infrared regularized $U(3)$ ChPT when the scalar σ mass $m_\sigma = 400$ MeV. (a) Columbia plot: With $U_A(1)$ anomaly. (b) Columbia plot: without $U_A(1)$ anomaly.

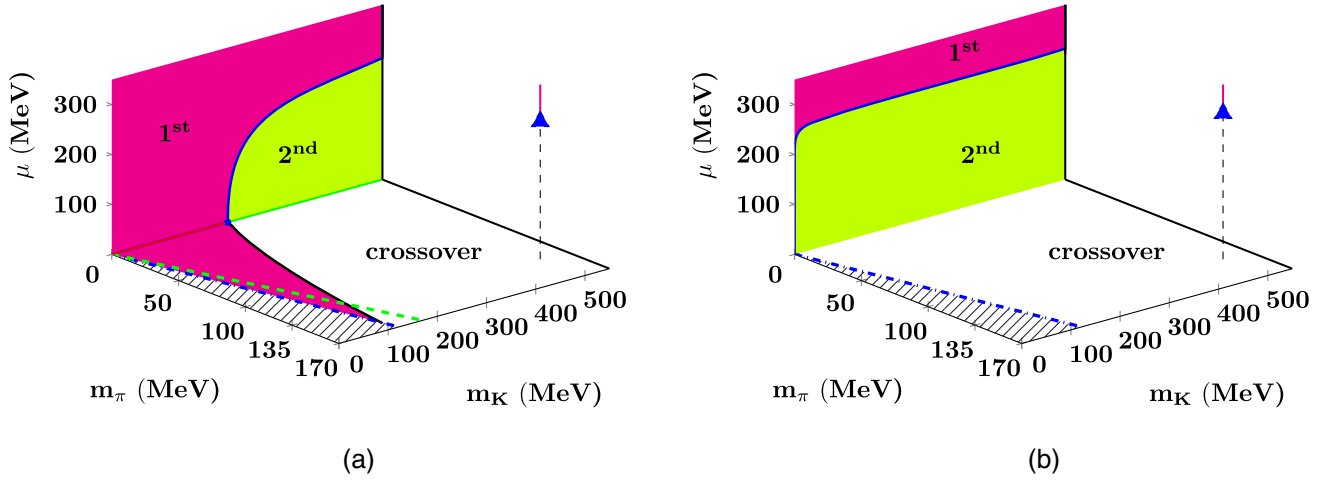


FIG. 4. $\mu - m_K$ and $m_\pi - m_K$ plane presents the respective chiral transition for $m_\pi = 0$ and $\mu = 0$ when $m_\sigma = 530$ MeV and the RQM model parameters away from the physical point are calculated using the infrared regularized U(3) ChPT. The solid black second-order Z_2 critical line in (a) demarcating the crossover from the first-order region, intersects the $m_\pi = m_K$ green dashed line at the critical pion mass $m_\pi^c = 128.38$ MeV, and it terminates at the terminal pion mass $m_\pi^t = 160.75$ MeV. The solid blue line of the tricritical points [which starts at the blue dot $m_K^{\text{TCP}} = 235.7$ MeV in (a)], separates the second- and first-order regions in both (a) and (b). Other lines, regions, and features are defined similar to Fig. 3. (a) Columbia plot: With $U_A(1)$ anomaly. (b) Columbia plot: without $U_A(1)$ anomaly.

the larger σ meson $m_\sigma = 530$ MeV mass, the first-order region in Fig. 4(a) of our work gets a small reduction as we are getting $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) \equiv (235.7, 160.75, 128.38)$ MeV.

The robust first-order region in the $\mu - m_K$ plane (for the $m_\pi = 0$) of both Figs. 3(a) and 4(a) near the chiral limit increases on increasing the chemical potential μ . Since the chiral critical surface has a positive curvature [126], its intersection with the dashed black vertical line of the crossover transition at the physical mass point marks the existence of a critical end point in a solid blue arrow in the $\mu - T$ plane of Figs. 3(a) and 4(a). The positive slope of the chiral tricritical line (above the second-order and below the first-order area) starting at $m_K^{\text{TCP}} = 242.6, \mu = 0$ MeV for the case of $m_\sigma = 400$ MeV in Fig. 3(a), keeps on decreasing for the larger μ and m_K , and it shows near saturation. $\mu = 184.7$ MeV at $m_K = 450$ MeV on the tricritical line becomes $\mu = 188.25$ MeV at $m_K = 500$ MeV with a slope of .071, and this slope becomes smaller as it is .049 when the $\mu = 188.25$ MeV at $m_K = 500$ MeV becomes $\mu = 190.7$ MeV at $m_K = 550$ MeV. This tricritical line is expected to be connected to the tricritical point of the two-flavor chiral limit [127] at some higher value of μ and m_K . The tricritical line starts at the $m_K^{\text{TCP}} = 235.7, \mu = 0$ MeV when $m_\sigma = 530$ MeV in Fig. 4(a), and it also has a positive slope, but it does not show the saturation trend like the one that we get for the $m_\sigma = 400$ MeV case. The slope 0.1852 between the points $(\mu = 219.14, m_K = 400.0)$ MeV and $(\mu = 228.4, m_K = 450.0)$ MeV on the tricritical line in Fig. 4(a) decreases to 0.1478 when one goes from the point $(\mu = 228.4, m_K = 450.0)$ MeV to the point $(\mu = 235.79, m_K = 500.0)$ MeV,

but it becomes 0.1566 between the point $(\mu = 235.79, m_K = 500.0)$ MeV and $(\mu = 243.62, m_K = 550.0)$ MeV. The solid blue arrow at the physical point marks the critical end point position in Fig. 3(a) at the $\mu_{\text{cep}} = 243.29$ MeV for the $m_\sigma = 400$ MeV case. The critical end point in Fig. 4(a) for the $m_\sigma = 530$ MeV case is marked by the solid blue arrow at $\mu_{\text{cep}} = 275.96$ MeV.

Figure 3(b) for $m_\sigma = 400$ MeV and Fig. 4(b) for $m_\sigma = 530$ MeV show that in the absence of the $U_A(1)$ anomaly, the first-order region in the $m_\pi - m_K$ plane at $\mu = 0$ disappears, unlike the predictions of Ref. [1] and the s-MFA result in Ref. [61]. This is the effect of the quark one-loop vacuum fluctuation, which is similar to the results of the e-MFA QM model FRG study in Ref. [23]. We get a second-order chiral transition everywhere in the $\mu - m_K$ plane, independent of the value of the m_K in the light chiral limit $m_\pi = 0$ for the chemical potentials $\mu < \mu_c$, and the first-order transition appears again for the very high $\mu_c = 204.6$ MeV for $m_\sigma = 400$ MeV and $\mu_c \sim 210$ MeV for $m_\sigma = 530$ MeV in the $SU(3)$ chiral limit ($m_\pi = 0 = m_K$). For $m_\sigma = 400$ MeV, the line of the tricritical points, separating the second-order chiral transition region below it from the first-order transition region above has a small positive slope until $\mu_c = 223.2, m_K = 240$ MeV, which becomes constant for the range $m_K = 240 - 350$ MeV and then decreases very slowly when the μ_c becomes 213.2 MeV at $m_K = 550$ MeV. The tricritical line for the $m_\sigma = 530$ MeV case in the Fig. 4(b) has a nearly constant slope after $m_K \geq 150$ MeV as the $\mu_c = 259.33$ MeV at the $m_K = 150$ MeV changes by only about 1.2 MeV to become $\mu_c = 260.5$ MeV at $m_K = 500$ MeV. The solid blue arrow marks the critical end point position

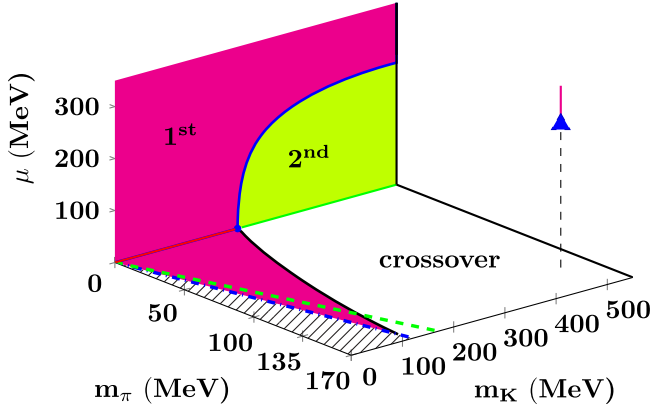
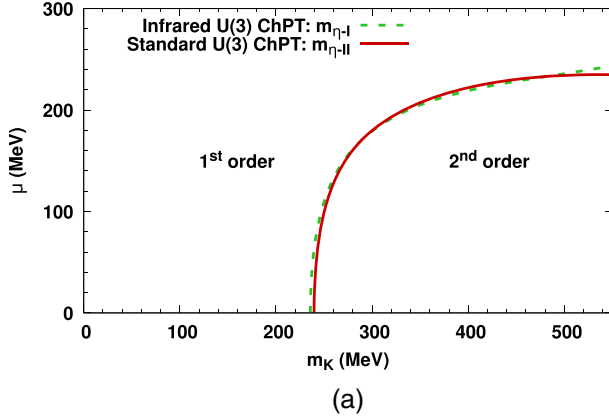


FIG. 5. $\mu - m_K$ and $m_\pi - m_K$ plane presents the respective chiral transition for $m_\pi = 0$ and $\mu = 0$ when $m_\sigma = 530$ MeV and the RQM model parameters away from the physical point are calculated using the standard large N_c U(3) ChPT. The solid black second-order Z_2 critical line that demarcates the crossover region from the first-order area intersects the $m_\pi = m_K$ green dashed line at the critical pion mass $m_\pi^c = 128.04$ MeV, and it terminates at the terminal pion mass $m_\pi^t = 161.35$ MeV. The solid blue line of the tricritical points starts at the blue dot $m_K^{\text{TCP}} = 239.58$ MeV. Other lines, regions, and features are defined similar to Fig. 4(a).

corresponding to the physical point at $\mu_{\text{cep}} = 262.8$ MeV in Fig. 3(b) for $m_\sigma = 400$ MeV and $\mu_{\text{cep}} = 292.2$ MeV in Fig. 4(b) for $m_\sigma = 530$ MeV.

1. Comparing plots for the input set $M_\eta\text{-I}$ and $M_\eta\text{-II}$

Figure (5) for $m_\sigma = 530$ MeV depicts the $\mu - m_K$ plane and $m_\pi - m_K$ plane of the Columbia plot, respectively, for



the light chiral limit $m_\pi = 0$ and for the zero chemical potential $\mu = 0$ obtained using the large N_c standard U(3) ChPT input set $M_\eta\text{-II}$ for calculating the RQM model parameters away from the physical point. The solid blue line of the tricritical points in the vertical $\mu - m_K$ plane starts from the blue dot $m_K^{\text{TCP}} = 239.58$ MeV at $\mu = 0$. The second-order Z_2 critical solid black line demarcating the first-order area from the crossover region in the horizontal $m_\pi - m_K$ plane, intersects the $SU(3)$ symmetric chiral limit path $m_\pi = m_K$ in green dashed line at the critical pion mass $m_\pi^c = 128.04$ MeV, and it terminates on the strange chiral limit $h_y = 0$ at the terminal pion mass $m_\pi^t = 161.35$ MeV beyond which the transition is crossover everywhere in the $m_\pi - m_K$ plane.

In order to compare the Columbia plots in the three dimensional Figs. 4(a) and (5) obtained, respectively, from the infrared regularized ChPT input set $M_\eta\text{-I}$ and the standard ChPT input set $M_\eta\text{-II}$, the vertical $\mu - m_K$ plane and horizontal $m_\pi - m_K$ plane of both Figs. 4(a) and (5) have been plotted, respectively, in Figs. 6(a) and 6(b). The blue solid line of the tricritical points that separates the first- and second-order regions in the vertical $\mu - m_K$ planes of Figs. 4(a) and (5) has been replotted, respectively, as the dashed green line (beginning from the $m_K^{\text{TCP}} = 235.7$ for the input set $M_\eta\text{-I}$) and the solid red line (starting from $m_K^{\text{TCP}} = 239.58$ for the input set $M_\eta\text{-II}$) in Fig. 6(a). The dashed green and solid red line are almost overlapping for $m_K < 450$ MeV, and the real difference emerges for $m_K > 450$ as the red line shows saturation for the range $m_K = 450\text{--}550$ MeV, while the green dashed line does

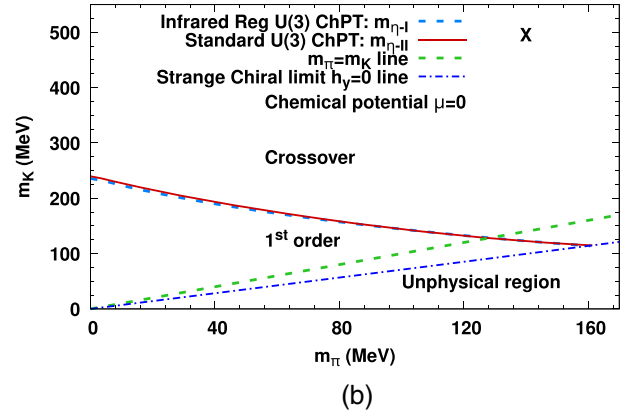


FIG. 6. Solid red line, being the locus of the tricritical points for the standard large N_c U(3) ChPT inputs, demarcates the second-order region from the first-order region. It begins from $m_K^{\text{TCP}} = 239.58$ MeV on the m_K axis in (a), while the dashed line in green for the corresponding calculation with the infrared regularized U(3) ChPT inputs, starts from $m_K^{\text{TCP}} = 235.7$ MeV. In (b), the solid red line that separates the crossover from the first-order area is the locus of the second-order Z_2 critical points for the standard large N_c U(3) ChPT inputs. The dashed line in sky blue shows the corresponding line for the infrared regularized U(3) ChPT inputs. The red solid and the sky blue dashed lines intersect the $m_\pi = m_K$ axis in the green dashed line, respectively, at the critical pion mass of $m_\pi^c = 128.04$ MeV and $m_\pi^c = 128.33$ MeV. These lines terminate on the strange chiral limit $h_y = 0$ dashed dotted line in blue, respectively, at the terminal pion mass of $m_\pi^t = 161.35$ MeV and $m_\pi^t = 160.75$ MeV. (a) Shows the comparison of plots in the vertical $\mu - m_K$ planes of the Columbia plots in the Figs. 4(a) and (5). (b) Shows the comparison of plots in the horizontal $m_\pi - m_K$ planes of the Columbia plots in the Figs. 4(a) and (5).

not. The red line has the slope of only 0.078 in going from point $(m_K = 450.0, \mu = 230.22)$ MeV to $(m_K = 500.0, \mu = 234.12)$ MeV, while the slope of the dashed green line is almost double of the above slope as it is 0.1478 in going from point $(m_K = 450.0, \mu = 228.4)$ MeV to $(m_K = 500.0, \mu = 235.79)$ MeV. The slope of the green dashed line slightly increases to 0.1566 when it goes from point $(m_K = 500.0, \mu = 235.79)$ MeV to $(m_K = 550.0, \mu = 243.62)$ MeV, while the average slope of the solid red line decreases and becomes only 0.0136 when it goes from point $(m_K = 500.0, \mu = 234.12)$ MeV to $(m_K = 550.0, \mu = 234.8)$ MeV. The slope of the red line becomes zero in the range $m_K = 520.0$ – 550.0 MeV as the chemical potential becomes almost constant at $\mu = 234.8$ MeV in this range. Thus, the tricritical line in the $\mu - m_K$ plane shows saturation for the large N_c standard ChPT input set M_η -II and does not saturate when the infrared regularized ChPT input set M_η -I is used for the calculation.

The solid black line of the $Z(2)$ critical points that separates the first-order region from the crossover transition area in the horizontal $m_\pi - m_K$ planes of Figs. 4(a) and (5) has been replotted, respectively, as the dashed line in sky blue for the input set (M_η -I) and the solid red line for the input set (M_η -II) in Fig. 6(b). Lying marginally above the dashed sky blue line for the $m_\pi < 80$ MeV, the solid red line completely overlaps with it for the range $m_\pi = 80$ – 160 MeV. The red solid and the sky blue dashed lines intersect the $SU(3)$ symmetric $m_\pi = m_K$ axis with the green dashed line, respectively, at the critical pion mass of $m_\pi^c = 128.04$ MeV and $m_\pi^c = 128.33$ MeV, and these lines terminate on the strange chiral limit $h_y = 0$ dash dot blue line respectively at the terminal pion mass of $m_\pi^t = 161.35$ MeV and $m_\pi^t = 160.75$ MeV. Thus, it is evident from Fig. 6(b) that the results for the value of the m_π^c, m_π^t and the extent of the first-order or crossover region in the $m_\pi - m_K$ plane are almost the same whether

one uses the infrared regularized $U(3)$ ChPT input set M_η -I or the large N_c standard $U(3)$ ChPT input set M_η -II for performing the chiral limit study.

Whether the input set M_η -I or the input set M_η -II is used for finding the RQM model parameters away from the physical point, the first-order region in our work for the $m_\sigma = 530$ MeV is significantly larger than the first-order region seen in Fig. (3a) of the QM model e-MFA study done after switching off the quantum and thermal fluctuations of the mesons in the advanced FRG framework of Ref. [23]. Table IV shows the comparison of our results with those of the FRG work where one finds $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) \equiv (169, 110, 86)$ MeV for $m_\sigma = 530$ MeV, while we find $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) \equiv (235.7, 160.75, 128.38)$ MeV for the infrared regularized ChPT input set M_η -I and $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) \equiv (239.58, 161.35, 128.04)$ MeV for the standard ChPT input set M_η -II.

Since the exact on-shell renormalization of the parameters in the RQM model gives a significantly stronger $U_A(1)$ anomaly strength c together with the weaker light and strange chiral symmetry breaking strengths h_x and h_y , the softening effect of the quark one-loop vacuum fluctuation on the chiral transition in the RQM model becomes moderate. Note that the strengths c , h_x , and h_y do not change either in the e-MFA QM model FRG study under the LPA or in the renormalized QM model potential of Ref. [108], which gives similar effects, as argued in Ref. [23]. Also, the changing anomaly strength c becomes stronger as shown in Table I for smaller masses in the Columbia plot due to the consistent ChPT scaling of m_π, m_K, M_η in the QM/RQM model parameter fixing. Thus, the softening effect of the quark one-loop vacuum fluctuation on the chiral transition in the e-MFA QM model FRG work using the LPA looks overestimated, similar to what has been shown in our recent RQM model studies [94,95]

TABLE IV. This table presents the quantities $m_K^{\text{TCP}}, m_s^{\text{TCP}}, m_\pi^t, m_{ud}^t, m_\pi^c, m_{ud}^c$ and the $\beta_{\text{cep}}: m_{\pi,\text{cep}}^*, m_{K,\text{cep}}^*$ for the input set M_η -I and M_η -II in the RQM model. Some of these quantities, as obtained in the e-MFA QM model FRG study for $m_\sigma = 530$ MeV and the QM model s-MFA study for $m_\sigma = 800$ MeV, are also presented for comparison. The second-order chiral transition and the TCP is absent on the $m_\pi = 0$ axis in the QM model s-MFA study. The last two rows show the critical pion (light quark) mass m_π^c (m_{ud}^c) for three degenerate flavors at $\mu = 0$ from different LQCD studies whose results are labeled with the corresponding fermion implementation and the number of time slices N_t .

Model	m_σ	m_K^{TCP}	m_s^{TCP}	m_π^t	m_{ud}^t	m_π^c	m_{ud}^c	β_{cep}	$m_{\pi,\text{cep}}^*$	$m_{K,\text{cep}}^*$
RQM: M_η : I	400	242.6	25.53	168.39	6.13	134.16	3.84	0.3795	52.38	188.25
RQM: M_η : I	530	235.7	24.08	160.75	5.58	128.38	3.51	0.3642	50.26	180.64
RQM: M_η : II	530	239.58	23.47	161.35	5.44	128.04	3.45	0.36778	50.9	182.96
e-MFA:QM-FRG [23]	530	169	...	110	...	86	...	0.20	27.6	99.2
s-MFA:QM [61]	800	No TCP	...	177	...	150	...	0.488	67.34	242.05
LQCD Study	$N_t = 4$, Standard	$N_t = 4$, p4	$N_t = 6$, Standard	$N_t = 6$, Stout	Wilson Clover	$N_t = 6$, HISQ	Wilson Clover	m_{ud}^c : Möbius		
	Staggered m_π^c	Staggered m_π^c	Staggered m_π^c	Staggered m_π^c	$N_t = 6$, m_π^c	Staggered m_π^c	$N_t = 8$, m_π^c	Domain Wall		
Ref.	~290 [15]	~67 [16]	~150 [17]	~0 [18]	~300 [19]	≤ 50 [20]	≤ 170 [21]	≤ 4 [25]		

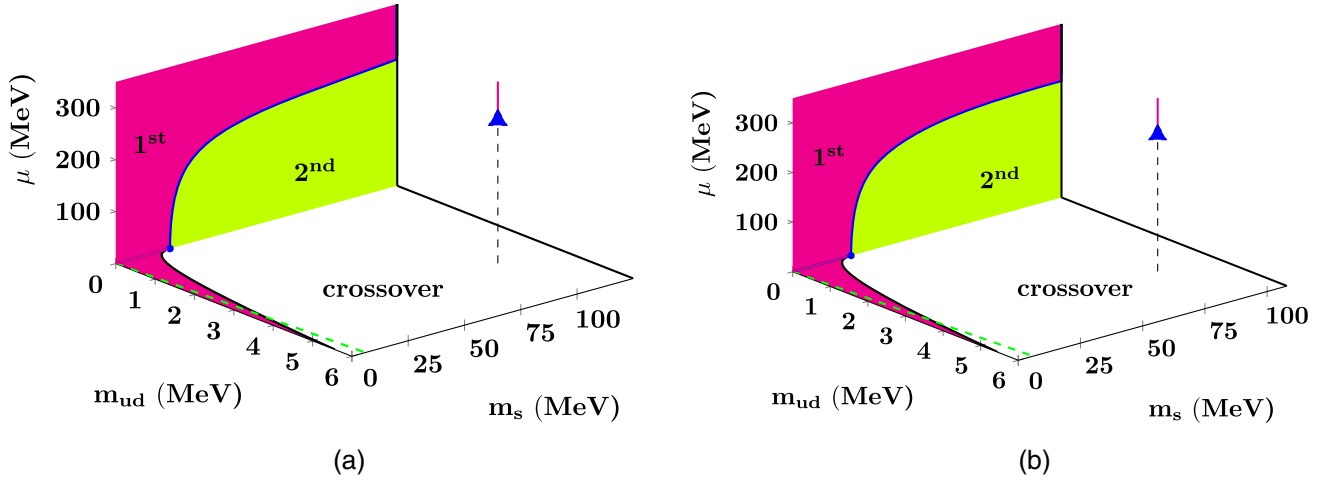


FIG. 7. $\mu - m_s$ and $m_{ud} - m_s$ plane presents the respective chiral transition for $m_{ud} = 0$ and $\mu = 0$ for the scalar σ meson mass $m_\sigma = 530$ MeV. The solid blue line of the tricritical points demarcating the second-order region from the first-order region in the $\mu - m_s$ plane starts from $m_s^{\text{TCP}} = 24.08$ MeV and $m_s^{\text{TCP}} = 23.47$ MeV, respectively in (a) and (b). The solid black second-order Z_2 critical line demarcating the crossover from the first-order region intersects the $m_{ud} = m_s$ green dashed line in (a) and (b), respectively, at the critical light quark mass $m_{ud}^c = 3.51$ MeV and $m_{ud}^c = 3.45$ MeV, while it terminates, respectively, on the m_{ud} axis (for $m_s = 0$) at the terminal light quark mass $m_{ud}^t = 5.58$ MeV and $m_{ud}^t = 5.44$ MeV. The black dashed vertical line at the physical point in the (a) and (b) shows the crossover transition, which ends at the critical end point in the solid blue triangle, and the solid red line shows the first-order transition. (a) Columbia plot using the infrared regularized U(3) ChPT inputs. (b) Columbia plot using the standard large N_c U(3) ChPT inputs.

for the e-MFA QM model works in the Refs. [108,109,111,112], which use the curvature masses of the mesons to fix the parameters of the model. The full FRG study where meson thermal and vacuum quantum fluctuations are also included finds very low $m_\pi^c \equiv 17$ MeV, for which Pisarski *et al.* [41] point out that it is partly because the approximation used in Ref. [23] is known to overestimate the mesonic fluctuations that tend to soften the chiral transition [128]. Note also that Ref. [29] has cautioned that the LPA in the FRG studies completely neglects the wave function renormalization, which might affect the final result.

2. Columbia plots in the quark mass $m_{ud} - m_s$ and $\mu - m_s$ plane for the input set M_η -I and M_η -II

The Columbia plot in the $m_\pi - m_K$ plane can be mapped into the light-strange quark mass $m_{us} - m_s$ plane by the respective use of Eqs. (33) and (34) for the infrared regularized U(3) ChPT input set M_η -I and Eqs. (43) and (44) for the standard U(3) ChPT input set M_η -II as described, respectively, at the end of the Sec. II B-I and Sec. II B-II. Mapping of the results from the $m_\pi - m_K$ to $m_{us} - m_s$ plane has been done by fixing the physical point $m_{ud} = 4$ MeV for both the input sets. Figures 7(a) and 7(b) show the Columbia plots in the quark mass $m_{ud} - m_s$ and $\mu - m_s$ plane, respectively, for the input set M_η -I and M_η -II when $m_\sigma = 530$ MeV. The $\mu - m_s$ plane depicts the chiral transition for the light chiral limit $m_{ud} = 0$ while the chiral transition at zero chemical potential $\mu = 0$ has been

presented in the $m_{ud} - m_s$ plane. The solid blue line of the tricritical points in the $\mu - m_s$ plane that demarcates the second-order region from the first-order region in the $\mu - m_s$ plane starts from tricritical point $m_s^{\text{TCP}} = 24.08$ MeV and $m_s^{\text{TCP}} = 23.47$ MeV at the $m_{ud} = 0$ axis, respectively, in Figs. 7(a) and 7(b). In the $m_{ud} - m_s$ plane for $\mu = 0$, the solid black second-order Z_2 critical line differentiates the crossover from the first-order region. This line intersects the $SU(3)$ symmetric $m_{ud} = m_s$ green dashed line in Figs. 7(a) and 7(b), respectively, at the critical light quark mass $m_{ud}^c = 3.51$ MeV and $m_{ud}^c = 3.45$ MeV and it terminates on the m_{ud} axis (for $m_s = 0$), respectively, at the terminal light quark mass $m_{ud}^t = 5.58$ MeV and $m_{ud}^t = 5.44$ MeV. The crossover transition at the physical point in Figs. 7(a) and 7(b) is depicted by the black dashed vertical line, which ends in the solid blue triangle depicting the critical end point, and the solid red line shows the first order transition.

Thus, when the infrared regularized ChPT input set M_η -I is used to find the RQM model parameters for the chiral limit study, one finds $(m_s^{\text{TCP}}, m_{ud}^t, m_{ud}^c) = (24.08, 5.58, 3.51)$ MeV corresponding to the $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) = (235.7, 160.75, 128.38)$ MeV, while the use of standard ChPT input set M_η -II yields $(m_s^{\text{TCP}}, m_{ud}^t, m_{ud}^c) = (23.47, 5.44, 3.45)$ MeV corresponding to the $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) = (239.58, 161.35, 128.04)$ MeV. Note that the lower $m_K^{\text{TCP}} = 235.7$ MeV gives higher $m_s^{\text{TCP}} = 24.08$ MeV for the input set M_η -I, while the higher $m_K^{\text{TCP}} = 239.58$ MeV gives a lower $m_s^{\text{TCP}} = 23.47$ MeV for the input set M_η -II. This

apparent contradictory looking result gets explained if one notes that when the light quark mass is fixed to $m_{ud} = 4$ MeV at the physical point, the solution of Eqs. (33) and (34) of the input set M_η -I gives a larger strange quark mass $m_s = 99.7$ MeV, while the solution of Eqs. (43) and (44) of the input set M_η -II gives a moderate strange quark mass $m_s = 86.2$ MeV at the physical point. If one matches the value of the strange quark mass at the physical point for both the input sets by taking $m_s = 86.2$ MeV at the physical point for the input set M_η -I also, one needs to take a lower value of light quark mass $m_{ud} = 3.46$ MeV at the physical point such that the m_s becomes 86.2 MeV. Performing the input set M_η -I calculation taking $m_{ud} = 3.46$ MeV, one finds that the $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) = (235.7, 160.75, 128.38)$ MeV gives $(m_s^{\text{TCP}}, m_{ud}^t, m_{ud}^c) = (20.85, 4.83, 3.04)$ MeV, which is lower than the input set M_η -II result $(m_s^{\text{TCP}}, m_{ud}^t, m_{ud}^c) = (23.47, 5.44, 3.45)$ MeV. Note that the chiral constants and parameters of the input set M_η -II gives $q = \frac{m_s}{m_{ud}} = 21.56$ at the physical point, and this value lies in its appropriate range given in Ref. [100,101]. Here, it is worthwhile to point out that the critical light quark mass value in either case of our study falls in the range $m_{ud}^c = 3.04\text{--}3.51$ MeV, while the recent LQCD study in Ref. [25], which uses Möbius domain wall fermions, finds that $m_{ud}^c \leq 4$ MeV. Results from different LQCD studies for the critical pion mass m_π^c and the critical light quark mass m_{ud}^c are compiled in the Table IV for ready reference.

C. $m_\eta, m_{\eta'}$, and $U_A(1)$ restoration in chiral limit

Several of the theoretical studies [129–132] in the ambit of the effective theory framework have investigated the chiral limit behavior of the $U_A(1)$ restoring observables, such as the topological susceptibility or the η' meson mass

at finite temperature. Comparing them to the quark condensate or to the other chiral restoring functions, such as the scalar susceptibility [133], one can determine the strength of the $U_A(1)$ breaking at the chiral transition. In order to study the extent of the $U_A(1)$ symmetry restoration in the chiral limit path, the T dependences of the η' and η curvature masses have been explored below.

The significantly larger 't Hooft coupling c , due to the proper renormalization of the quark one-loop vacuum fluctuation in the RQM model gives rise to a larger curvature mass for the η' meson in the vacuum, which is quite different from its pole mass as reported in Ref. [94]. The f_π, f_K in Table I and Eq. (40) for the infrared regularized ChPT framework give the M_η -I = 1035.55 MeV at the physical point $\beta = 1$, which is slightly different from the experimental M_η : Expt = 1103.22 MeV. The parameters of the RQM model with the input M_η -I = 1035.55 MeV give the η and η' vacuum curvature masses as $m_{\eta,c} = 526.03$ MeV and $m_{\eta',c} = 1153.01$ MeV [shown in Table V]. The large N_c standard ChPT f_π, f_K [in Table III and Eq. (49)] reproduce the M_η : Expt as the M_η -II = 1103.22 MeV at the physical point. The use of this input in the RQM model gives the η and η' vacuum curvature masses as $m_{\eta,c} = 523.22$ MeV and $m_{\eta',c} = 1149.95$ MeV (Table V), which are marginally lower than the values obtained for the M_η -I input set. Note that in contrast to the η' curvature mass, the η vacuum curvature mass is only marginally different from its pole mass. Since the $U_A(1)$ anomaly strength gets significantly diminished due to a large melting of the light and strange condensates, the η' meson curvature masses $m_{\eta',c}$ -I and $m_{\eta',c}$ -II get noticeably reduced at finite temperature where the chiral symmetry restoring transition takes place.

TABLE V. η, η' meson curvature masses together with their mass difference in the vacuum $T = 0$ and at the temperature T are denoted in the respective order as $m_{\eta,c}|_{T=0}$, $m_{\eta',c}|_{T=0}$, $\Delta m_{\eta,c}|_{T=0}$ and $m_{\eta,c}|_T$, and $m_{\eta',c}|_T$ and $\Delta m_{\eta,c}|_T$. T is the temperature at which the η' meson curvature mass becomes minimum. Results are presented for the input set M_η -I and M_η -II. $m_\sigma = 530$ MeV.

$\frac{m_\pi^*}{m_\pi} = \frac{m_K^*}{m_K} = \beta$	$m_{\eta',c} _{T=0}$	$m_{\eta,c} _{T=0}$	$\Delta m_{\eta,c} _{T=0}$	T_c^x	T	$m_{\eta',c} _T$	$m_{\eta,c} _T$	$\Delta m_{\eta,c} _T$
$\beta = 1$: M_η - I	1153.01	526.03	626.98	139.42	162.55	895.02	723.82	171.2
$\beta = 1$: M_η - II	1149.95	523.22	626.73	139.47	162.57	893.75	718.99	174.76
$\beta = 0.50$: M_η - I	1089.29	269.26	820.03	133.97	148.64	656.71	487.03	169.6
$\beta = 0.50$: M_η - II	1082.31	279.36	802.95	135.05	147.59	642.90	507.76	135.14
$\beta_{\text{cep}} = 0.3642$: M_η - I	1070.25	199.59	870.66	129.336	139.1	562.33	416.97	145.36
$\beta_{\text{cep}} = 0.36887$: M_η - II	1074.38	207.61	866.77	130.267	138.26	555.47	434.95	120.52
$\beta = 0.30$: M_η - I	1061.08	165.75	895.33	128.21	133.2	510.26	379.36	130.9
$\beta = 0.30$: M_η - II	1071.39	169.31	902.08	131.47	131.84	502.54	391.64	110.9
$\beta = 0.15$: M_η - I	1040.61	84.3	956.31	127.5	127.5	379.24	326.83	52.41
$\beta = 0.15$: M_η - II	1067.21	84.97	982.24	128.1	128.1	385.06	338.04	47.02
$\beta = 0.05$: M_η - I	1030.48	28.32	1002.16	125.6	125.6	310.5	302.98	7.52
$\beta = 0.05$: M_η - II	1066.02	28.35	1037.67	125.94	125.94	325.06	318.16	6.9
$\beta = 0.00725$: M_η - I	1028.47	4.11	1024.36	125.59	125.59	300.86	300.73	0.13
$\beta = 0.00725$: M_η - II	1065.87	4.11	1061.76	125.07	125.07	313.59	313.45	0.14

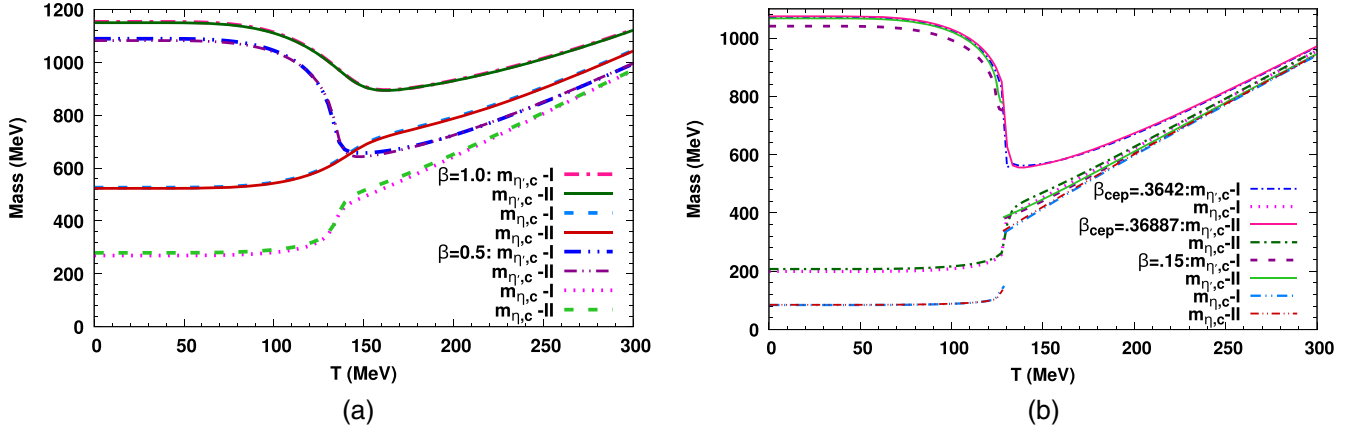


FIG. 8. Temperature plots of the η , η' curvature masses $m_{\eta,c}$ -I,II and $m_{\eta',c}$ -I,II. Tables I and II contain the RQM model parameters for different β with input set M_η -I (infrared regularized ChPT), while Table III gives parameters for input set M_η -II (standard ChPT). The $m_{\eta',c}$ -I ($m_{\eta,c}$ -I) plot overlaps with the $m_{\eta',c}$ -II ($m_{\eta,c}$ -II) plot for $\beta = 1$ ($m_\pi = 138, m_K = 496$) MeV in (a), and for $\beta = 0.5$ ($m_\pi = 69, m_K = 248$) MeV, the $m_{\eta',c}$ -I ($m_{\eta,c}$ -I) plot, while being nearly overlapping, is slightly above (below) the $m_{\eta',c}$ -II ($m_{\eta,c}$ -II) plot. The $m_{\eta',c}$ -II ($m_{\eta,c}$ -II) plot for $\beta_{\text{cep}} = .36887$ ($m_\pi = 50.9, m_K = 182.96$) MeV in (b) nearly overlaps with the $m_{\eta',c}$ -I ($m_{\eta,c}$ -I) plot for $\beta_{\text{cep}} = .3642$ ($m_\pi = 50.26, m_K = 180.24$) MeV while being slightly above it. The crossover turns first order at β_{cep} . The $m_{\eta',c}$ -I plot lies below the $m_{\eta',c}$ -II plot, while the $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots are coincident before the first-order transition (large mass gap) in (b) for $\beta = 0.15$ ($m_\pi = 20.7, m_K = 74.4$) MeV. After the chiral transition, the overlapped $m_{\eta',c}$ -I and $m_{\eta',c}$ -II plots have small mass differences with the degenerate $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots. (a) The $m_{\eta,c}$ -I,II and $m_{\eta',c}$ -I,II plots for $\beta = 1, 0.5$. The I(II) stands for using the infrared regularized (standard) U(3) ChPT input. (b) The ($m_{\eta,c}, m_{\eta',c}$)-I plots for $\beta_{\text{cep}} = .3642$ and ($m_{\eta,c}, m_{\eta',c}$)-II plots for $\beta_{\text{cep}} = .36887$. The $m_{\eta,c}$ -I,II and $m_{\eta,c}$ -I,II plots for $\beta = .15$.

Being a measure of the $U_A(1)$ anomaly strength, the η' vacuum curvature mass of $m_{\eta',c} - I(m_{\eta',c} - II) = 1153.01(1149.95)$ MeV for the physical point with the M_η -I (M_η -II) input set gets reduced by 257.99 (256.2) MeV to become minimum 895.02 (893.75) MeV at $T = 162.55(162.57)$ MeV near the pseudo-critical temperature $T_c^x = 139.42(139.47)$ MeV of the chiral transition for $m_\sigma = 530$ MeV. Table V shows that the corresponding η meson curvature mass $m_{\eta,c}$ -I ($m_{\eta,c}$ -II), increases with temperature to become 723.82 (718.99) MeV at the above-mentioned T , where the difference $\Delta m_{\eta,c}|_T$ between the η' and η meson curvature masses for the M_η -I (M_η -II) input set becomes only 171.2 (174.76) MeV from its vacuum value of the $\Delta m_{\eta,c}|_{T=0} = 626.98(626.73)$ MeV. The mass difference $\Delta m_{\eta,c}|_T$ is an indicator of the extent and degree of the $U_A(1)$ symmetry restoration at temperature T . When the π and K meson masses are reduced to half of their physical point values, i.e., $\beta = 0.5$, the $m_{\eta',c} - I(m_{\eta',c} - II) = 1089.29(1082.31)$ MeV gets reduced by 432.58 (439.41) MeV to become minimum 656.71 (642.90) MeV at $T = 148.64(147.59)$ MeV near the pseudo-critical temperature $T_c^x = 133.97(135.05)$ MeV. The corresponding η mass $m_{\eta,c}$ -I ($m_{\eta,c}$ -II) in Table V, after increasing with the temperature, becomes 487.03 (507.76) MeV at the above-mentioned T , where the difference $\Delta m_{\eta,c}|_T$ between the η' and η meson curvature masses for the M_η -I (M_η -II) input set, becomes only 169.6 (135.14) MeV from its vacuum value of the

$\Delta m_{\eta,c}|_{T=0} = 820.03(802.95)$ MeV. The temperature variations of the η' and η meson curvature masses have been plotted in Fig. 8(a) for the physical point $\beta = 1.0$ and $\beta = 0.5$. Note that in the $\beta = 1$ case the temperature variation of the $m_{\eta',c}$ -I ($m_{\eta,c}$ -I) for the input set M_η -I, overlaps with the $m_{\eta',c}$ -II ($m_{\eta,c}$ -II) plot of the input set M_η -II, and the approach of the $m_{\eta,c}$ -I ($m_{\eta,c}$ -II) temperature variation toward the $m_{\eta',c}$ -I ($m_{\eta',c}$ -II) plot is quite slow and gradual for temperatures higher than T . For $\beta = 0.5$, the $m_{\eta',c}$ -I ($m_{\eta,c}$ -I) temperature variation, while being nearly overlapping, is slightly above (below) the $m_{\eta',c}$ -II ($m_{\eta,c}$ -II) temperature plot in Fig. 8(a), and it is evident that for higher temperatures ($> T$) the approach of $m_{\eta,c}$ -I ($m_{\eta,c}$ -II) temperature variation toward the $m_{\eta',c}$ -I ($m_{\eta',c}$ -II) plot becomes faster compared to the case of $\beta = 1$.

When $\beta = \beta_{\text{cep}} = 0.3642\{0.36887\}$ with the $(m_\pi, m_K) = (50.26, 180.64)\{(50.9, 182.96)\}$ MeV, the crossover chiral transition becomes second order for the input set M_η -I [M_η -II] [corresponding temperature variations of the condensates x and y are shown in Figs. 2(a) and 2(b)]. For the $\beta = \beta_{\text{cep}}$, $m_{\eta',c} - I(m_{\eta',c} - II) = 1070.25(1074.38)$ MeV gets reduced by 507.92 (518.91) MeV to become minimum 562.33 (555.47) MeV at $T = 139.1(138.26)$ MeV near $T_c^x = 129.34(130.27)$ MeV. The corresponding η mass $m_{\eta,c}$ -I ($m_{\eta,c}$ -II) in Table V, after increasing with temperature, becomes 416.97 (434.95) MeV at the above mentioned T , where the difference $\Delta m_{\eta,c}|_T$ between the η' and η meson curvature masses for the M_η -I (M_η -II) input set, gets

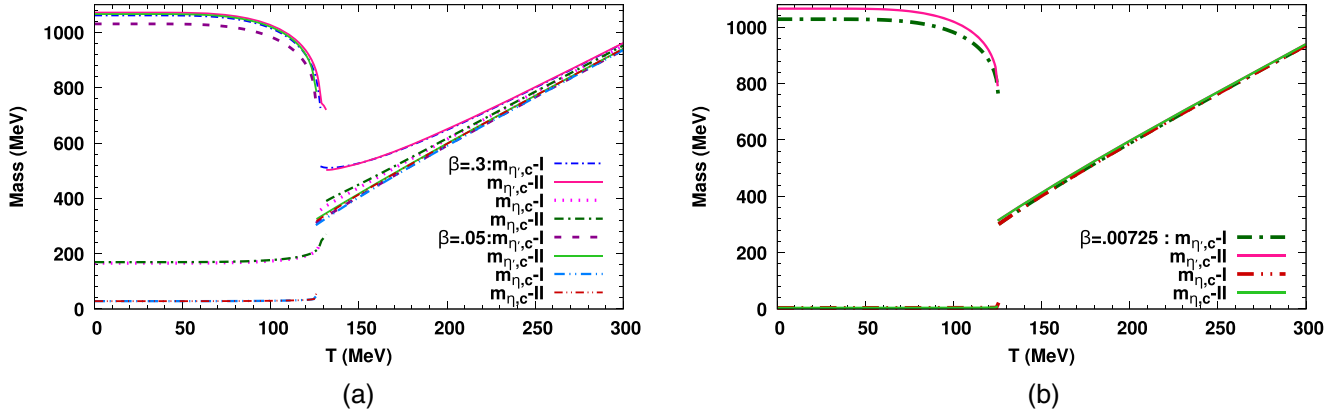


FIG. 9. $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots are coincident for $\beta = 0.3$ ($m_\pi = 41.4$, $m_K = 148.8$) MeV and $\beta = 0.05$ ($m_\pi = 6.9$, $m_K = 24.8$) MeV, while the $m_{\eta',c}$ -I plot lies slightly (noticeably) below the $m_{\eta',c}$ -II plot for $\beta = 0.3$ (0.05) in the (a) before the first-order transition. The $m_{\eta',c}$ -I and $m_{\eta',c}$ -II plots for $\beta = 0.05$ become almost overlapping after the transition and have very small differences with the degenerate $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots in (a). The $m_{\eta',c}$ -I plot lies noticeably below the $m_{\eta',c}$ -II plot for $\beta = 0.00725$ ($m_\pi = 6.9$, $m_K = 24.8$) MeV, while the $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots are coincident before the first-order mass gap appears in (b). All the masses become degenerate after the chiral transition, signaling restoration of the $U_A(1)$ axial symmetry. The scalar σ meson mass $m_\sigma = 530$ MeV. (a) The $m_{\eta,c}$ -I,II and $m_{\eta',c}$ -I,II plots for $\beta = 0.3, 0.05$. (b) The $m_{\eta,c}$ -I,II and $m_{\eta',c}$ -I,II plots for $\beta = 0.00725$.

further reduced to 145.36 (120.52) MeV from its value $\Delta m_{\eta,c}|_{T=0} = 870.66(866.77)$ MeV in the vacuum. The above discussed trend of mass decrease in the $m_{\eta',c} - I(m_{\eta',c} - II)$, and corresponding pattern of mass increase in the $m_{\eta,c} - I(m_{\eta,c} - II)$ temperature variations for temperatures ($T < T_c^X$) is presented in Table V for $\beta = 0.3, 0.15, 0.05, 0.00725 < \beta_{\text{cep}}$. Before the chiral transition, the $m_{\eta',c}$ -II ($m_{\eta,c}$ -II) plot for $\beta_{\text{cep}} = .36887$ in Fig. 8(b) nearly overlaps with the $m_{\eta',c}$ -I ($m_{\eta,c}$ -I) plot for $\beta_c = .3642$ while remaining slightly above it. After the chiral transition, the nearly overlapping plots of the $m_{\eta,c}$ -I and $m_{\eta,c}$ -II converge rapidly toward the almost degenerate $m_{\eta',c}$ -I and $m_{\eta',c}$ -II temperature plots in Fig. 8(b). In order to show the contrasting difference with the $\beta = \beta_{\text{cep}}$ plots, the plots for $\beta = 0.15$ has also been shown in Fig. 8(b). The $m_{\eta',c}$ -I plot lies below the $m_{\eta',c}$ -II plot, while the $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots are coincident before the first-order transition (large mass gap) in Fig. 8(b) for $\beta = 0.15$ ($m_\pi = 20.7$, $m_K = 74.4$) MeV. After the chiral transition, the overlapped $m_{\eta',c}$ -I and $m_{\eta',c}$ -II plots have quite a small mass difference with the degenerate $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots.

The vacuum value of the mass difference $\Delta m_{\eta,c}|_{T=0} = 895.33(902.08)$, $956.31(982.24)$, $1002.16(1037.67)$, and $1024.36(1061.76)$ MeV for the input set $M_\eta - I(M_\eta - II)$, drops down successively to the $\Delta m_{\eta,c}|_T = 130.9(110.9)$, $52.41(47.02)$, $7.52(6.9)$ and $0.13(.14)$ MeV, as shown in the Table V, when the m_π and m_K are reduced respectively by the fraction $\beta = 0.3, 0.15, 0.05$ and 0.00725 to the 30, 15, 5 and .725 percent of their physical point value. The contrasts of the plots for the $\beta = 0.3$ and $\beta = 0.05$ are shown in Fig. 9(a). The $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots are coincident for $\beta = 0.3$ and $\beta = 0.05$, while the $m_{\eta',c}$ -I plot

lies slightly (noticeably) below the $m_{\eta',c}$ -II plot for $\beta = 0.3$ (0.05) in Fig. 9(a) before the first-order transition. Recall that the chiral transition turns first order for $\beta = 0.3$ as evident from the gaps in the $m_{\eta',c}$ -I ($m_{\eta',c}$ -II) and $m_{\eta,c}$ -I ($m_{\eta,c}$ -II) temperature variations in Fig. 9(a), and the minimum in the $m_{\eta',c}$ -I ($m_{\eta',c}$ -II) occurs at the $T = 133.2(131.84)$ MeV, which is quite close to the chiral transition temperature $T_c^X = 128.21(131.47)$ MeV. For the cases when $\beta \leq 0.15$, the $m_{\eta',c}$ -I ($m_{\eta',c}$ -II) drops down with larger mass gaps of stronger first-order transitions in Figs. 8(b), 9(a), and 9(b), and the minimum in the $m_{\eta',c}$ -I ($m_{\eta',c}$ -II) temperature variation occurs exactly at the chiral transition temperature, i.e., $T_c^X = T$ as shown in the Table V for the $\beta = 0.15, .05$, and 0.00725 . The $m_{\eta',c}$ -I and $m_{\eta',c}$ -II plots for the $\beta = 0.05$ become almost overlapping after the transition and have very small differences with the degenerate $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots in Fig. 9(a).

It is clear from Figs. 8(a) and 8(b), 9(a), and 9(b), and Table V that the temperature variations of the $m_{\eta,c}$ for both the input sets $M_\eta - I$ and $M_\eta - II$, start approaching the corresponding $m_{\eta',c}$ temperature plots at lower temperatures, and the $U_A(1)$ symmetry restoration trend sets up early and shifts to the smaller temperatures for the $\beta \leq 0.3$. The η and η' curvature mass difference $\Delta m_{\eta,c}$ at $T = 250$ MeV for the input set $M_\eta - I(M_\eta - II)$ is 16.45 (16.54) MeV, 4.19 (4.35) MeV, and 0.47 (0.5) MeV, respectively, for $\beta = 0.30, .15$, and 0.05 . The $\Delta m_{\eta,c}$ is only 6 (6.17) MeV at a lower temperature $T = 225$ MeV for the $\beta = 0.15$, and it becomes only 3.2 (3.2) MeV at $T = 150$ MeV (which is just 25 MeV higher than the corresponding the T_c^X) for $\beta = 0.05$. Here, it is relevant to emphasize that the use of the infrared regularized $U(3)$

ChPT input (M_η -I) and the large N_c standard U(3) ChPT input (M_η -II) give almost the same and overlapping temperature variations of the η and η' masses after the chiral transition has taken place when different β fractions are taken for the chiral limit study.

Very close to the chiral limit when $\beta = 0.00725$ with the $m_\pi = 1$ and $m_K = 3.6$ MeV, the $m_{\eta',c}$ -I plot lies noticeably below the $m_{\eta',c}$ -II plot, while the $m_{\eta,c}$ -I and $m_{\eta,c}$ -II plots are coincident before a very large first-order mass gap appears in Fig. 9(b). The $m_{\eta,c}$ -I ($m_{\eta,c}$ -II) temperature variation in Fig. 9(b) merges completely with the temperature plot of $m_{\eta'}$ -I ($m_{\eta'}$ -II) just after the chiral transition. Thus, it is interesting to observe and emphasize that the $U_A(1)$ symmetry restoration and the chiral $SU_A(3)$ symmetry restoration take place exactly at the same temperature, i.e., the chiral transition temperature $T_c^\chi = 125.59$ MeV (see Table V) when one reaches the chiral limit in the RQM model.

IV. SUMMARY

The lower left corner of the Columbia plot is generated to consistently explore the chiral limit using the exact chiral effective potential of the RQM model [94,95] whose parameters are renormalized on-shell after the proper treatment of the quark one-loop vacuum fluctuation. The $\mathcal{O}(\frac{1}{f_\pi^2})$ accurate results of the large N_c standard U(3) ChPT [96–101] and the infrared regularized U(3) ChPT [48,102–105] has been used, respectively, to constitute the input set M_η -II and input set M_η -I for finding the (m_π, m_K) dependence of f_π , f_K , and M_η^2 for the computation of the Columbia plot. When the expressions of f_π , f_K , and M_η^2 from either of the input set M_η -I or the input set M_η -II is used in the RQM model, its parameter fixing is rendered free from any ambiguity and heuristic adjustment as the π , K meson masses are reduced to zero as one moves away from the physical point for performing the chiral limit study.

A direct path from the physical point to the chiral limit in the Columbia plot is adopted, where symmetry breaking strengths h_x and h_y are reduced by defining the reduced pion and kaon masses m_π^* and m_K^* as $\frac{m_\pi^*}{m_\pi} = \frac{m_K^*}{m_K} = \beta$. This choice ensures that the $\frac{m_\pi^*}{m_K^*}$ is equal to the ratio $\frac{m_\pi}{m_K}$ at the physical point where the ChPT parameter $q = \frac{2m_s}{m_u+m_d}$, as the ratio of the strange quark mass to the average of the light quark mass also gets fixed. The temperature variations of the light (strange) condensate $x(y)$ has been computed, respectively, for $m_\sigma = 400$ with $\beta = 1, 0.5, 0.3742$, and 2.02×10^{-4} in the input set M_η -I and $m_\sigma = 530$ with the $\beta = 1, 0.5, 0.3$, and 0.00725 for both the input set M_η -I and M_η -II. The chiral crossover transition of the physical point when $\beta = 1$ becomes sharper for the decreased $\beta = 0.5$ and it turns second order at the Z_2 critical point given, respectively, by

$\beta_{\text{cep}}|_{(m_\sigma=400 \text{ MeV}-M_\eta: \text{I})} = 0.3795$ with $(m_{\pi,\text{cep}}^*, m_{K,\text{cep}}^*) = (52.38, 188.25)$ MeV for $m_\sigma = 400$ MeV and $\beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{I})} = 0.3642$ with $(m_{\pi,\text{cep}}^*, m_{K,\text{cep}}^*) = (50.26, 180.64)$ MeV for $m_\sigma = 530$ MeV when the input set M_η -I is used. One finds the Z_2 critical point for the $(m_{\pi,\text{cep}}^*, m_{K,\text{cep}}^*) = (50.9, 182.96)$ MeV at $\beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{II})} = 0.36887$ if the standard ChPT input set M_η : II is used. Note that the lower $m_\sigma = 400$ MeV generates a stronger first order chiral transition whose strength is weaker for the higher $m_\sigma = 530$ MeV, as evident from the input set M_η : I result that $\beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{I})} = 0.3642 < \beta_{\text{cep}}|_{(m_\sigma=400 \text{ MeV}-M_\eta: \text{I})} = 0.3795$. It is also noteworthy that the input set M_η : II yields a marginally stronger first-order transition, as $\beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{I})} = 0.3642 < \beta_{\text{cep}}|_{(m_\sigma=530 \text{ MeV}-M_\eta: \text{II})} = 0.36887$. On comparison, one finds that the second-order Z_2 critical point in the e-MFA QM model study of Ref. [23] occurs for a significantly smaller $\beta_{\text{cep}} = \sqrt{\alpha_c} = 0.2$, corresponding to quite small pion and kaon masses $(m_{\pi,\text{cep}}^*, m_{K,\text{cep}}^*) = (27.6, 99.2)$ MeV. It is important to emphasize that in comparison to the e-MFA QM model FRG work, the first-order region in the RQM model work is considerably larger, whether one uses the input set M_η : I or M_η : II.

The kaon mass (strange quark mass m_s) on the light chiral limit line $m_\pi = 0$ ($m_{ud} = 0$) of the Columbia plot where the second-order transition turns first order defines the tricritical point as $m_K = m_K^{\text{TCP}}(m_s = m_s^{\text{TCP}})$. The chiral critical line of the $Z(2)$ universality from the first-order region in the $m_\pi - m_K$ ($m_{ud} - m_s$) plane at the $\mu = 0$, terminates on the strange chiral limit line $h_{y0} = 0$ ($m_s = 0$ line) at the terminal π mass m_π^t (light quark mass m_{ud}^t), beyond which the transition is a smooth crossover everywhere. This Z_2 chiral critical line intersects the $SU(3)$ symmetric $m_\pi = m_K$ ($m_{ud} = m_s$) chiral limit line also at the critical pion mass m_π^c (light quark mass m_{ud}^c) where the boundary of first-order region ends.

The Columbia plot in our work with the $U_A(1)$ anomaly has a significantly large first-order region for the scalar σ meson mass $m_\sigma = 400$ MeV as we are finding the $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) \equiv (242.6, 168.39, 134.16)$ MeV for the infrared regularized ChPT input set M_η -I. For the larger σ mass $m_\sigma = 530$ MeV, the first-order region gets a little reduced as we are finding the $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) \equiv (235.7, 160.75, 128.38)$ MeV for the input set M_η -I. If we use the standard ChPT input set M_η -II for $m_\sigma = 530$ MeV case in our work, we find a slightly larger first order region as we are getting the $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) \equiv (239.58, 161.35, 128.04)$ MeV. Thus, whether the input set M_η -I or the input set M_η -II is used for finding the RQM model parameters away from the physical point, the first-order region in our work for the $m_\sigma = 530$ MeV is significantly larger than the first-order region seen in

Fig. 3(a) of the QM model e-MFA study done after switching off the quantum and thermal fluctuations of the mesons in the advanced FRG framework of Ref. [23], where they are finding the $(m_K^{\text{TCP}}, m_\pi^t, m_\pi^c) \equiv (169, 110, 86)$ MeV. The above result gets explained if one recalls again that the softening effect of the quark one-loop vacuum fluctuation on the chiral transition is moderate in the RQM model as the on-shell renormalization of its parameters generate a significantly stronger axial $U_A(1)$ anomaly, whereas the strange and nonstrange direction explicit chiral symmetry breaking strengths h_x and h_y also get reduced. The softening influence of the quark one-loop vacuum fluctuation in the e-MFA FRG work using the LPA looks overestimated similar to what has been shown in our recent RQM model studies [94,95] for the e-MFA QM model works in Refs. [108,109,111,112], where the $U_A(1)$ anomaly strength c and $h_x(h_y)$ do not change as they use curvature meson masses to fix the model parameters.

The first-order regions in the $m_\pi - m_K$ plane at $\mu = 0$ for $m_\sigma = 400$ and 530 MeV in our work for the input set M_η -I, disappear when the $U_A(1)$ anomaly is absent. This is the effect of the quark one-loop vacuum fluctuation, which is similar to the results of the e-MFA QM model FRG study in Ref. [23]. We get a second-order chiral transition everywhere in the $\mu - m_K$ plane for $\mu < \mu_c$, irrespective of the m_K value in the light chiral limit $m_\pi = 0$. The first-order transition appears again for the very high chemical potentials.

The positive slope of the tricritical line keeps on decreasing to become small for the larger μ and m_K , and it shows the near saturation for the smaller σ mass $m_\sigma = 400$ MeV when the input set M_η -I is used and the $U_A(1)$ anomaly is present in the RQM model. For the larger $m_\sigma = 530$ MeV, the $\mu - m_K$ plane tricritical lines for the input set M_η -I and M_η -II are almost overlapping for the $m_K < 450$ MeV, and the real difference emerges for the $m_K > 450$ MeV when the tricritical line obtained from the standard ChPT input set M_η -II shows saturation for the range $m_K = 450\text{--}550$ MeV, and its slope becomes zero in the range $m_K = 520.0\text{--}550.0$ MeV, in which the chemical potential stays almost constant at $\mu = 234.8$ MeV, while the tricritical line for the infrared regularized input set M_η -I does not show saturation as its average slope increases to 0.1566 when it goes from point $(m_K = 500.0, \mu = 235.79)$ MeV to the $(m_K = 550.0, \mu = 243.62)$ MeV.

One finds $(m_s^{\text{TCP}}, m_{ud}^t, m_{ud}^c) = (24.08, 5.58, 3.51)$ MeV in the light-strange quark mass $m_{ud} - m_s$ plane of the

Columbia plot for the case of $m_\sigma = 530$ MeV when the input set M_η -I is used with $(m_{ud}, m_s) = (4, 99.7)$ MeV at the physical point, while the input set M_η -II with the $(m_{ud}, m_s) = (4, 86.2)$ MeV at the physical point gives $(m_s^{\text{TCP}}, m_{ud}^t, m_{ud}^c) = (23.47, 5.44, 3.45)$ MeV. If the physical point value is taken as $(m_{ud}, m_s) = (3.46, 86.2)$ MeV for the input set M_η -I such that the s quark mass becomes equal to $m_s = 86.2$ MeV for both the input sets, one finds $(m_s^{\text{TCP}}, m_{ud}^t, m_{ud}^c) = (20.85, 4.83, 3.04)$ MeV, which is lower than the input set M_η -II result of $(m_s^{\text{TCP}}, m_{ud}^t, m_{ud}^c) = (23.47, 5.44, 3.45)$ MeV. Note that the critical light quark mass value in either case of our study falls in the range $m_{ud}^c = 3.04\text{--}3.51$ MeV, while the recent LQCD study in Ref. [25] using Möbius domain wall fermions finds that $m_{ud}^c \leq 4$ MeV.

The temperature variations of the η curvature mass $m_{\eta,c}$ for both the input sets M_η : I and M_η : II start approaching the corresponding η' curvature mass $m_{\eta',c}$ temperature plots at lower temperatures, and the $U_A(1)$ symmetry restoration trend sets up early and shifts to the smaller temperatures when the fraction $\beta = \frac{m_\pi^*}{m_\pi} = \frac{m_K^*}{m_K}$ is reduced for the chiral limit study. The η and η' curvature mass difference $\Delta m_{\eta,c}$ for the input set M_η : I (M_η : II) is only 6 (6.17) MeV at $T = 225$ MeV for $\beta = 0.15$, and it becomes only 3.2 (3.2) MeV at $T = 150$ MeV for $\beta = 0.05$. Note that the input set M_η -I and the input M_η -II give almost the same and overlapping temperature variations of the η and η' masses after the chiral transition has taken place for different β fractions. Very close to the chiral limit when $\beta = 0.00725$ for $m_\pi = 1$ and $m_K = 3.6$, the $m_{\eta,c}$ -I ($m_{\eta,c}$ -II) temperature variation merges completely with the temperature plot of $m_{\eta'}$ -I ($m_{\eta'}$ -II) just after the chiral transition. Thus, it is interesting to observe and emphasize that the $U_A(1)$ symmetry restoration and the chiral $SU_A(3)$ symmetry restoration take place exactly at the same temperature, i.e., the chiral transition temperature $T_c^X = 125.59$ MeV when one reaches the chiral limit in the RQM model.

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