

Rotating Weyssenhoff fluids in linearized Einstein-Cartan theory

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The Weyssenhoff fluid expands the ordinary perfect-fluid description of general relativity by taking into account the intrinsic spin of its constituent particles. Within the linearized regime of Einstein-Cartan theory, we investigate the exterior geometry produced by isolated slowly rotating Weyssenhoff fluids, i.e., sources endowed with both macroscopic and microscopic angular momenta. Our work represents an extension of the foundational analysis of Arkuszewski, Kopczyński and Ponomarev, which was restricted to the static fluid configuration. Under the assumptions of stationarity and axial symmetry, we derive the explicit expression for the linearized stress-energy tensor of the source and obtain the corresponding metric through a multipole expansion. The ensuing solution exhibits a generalized Lense-Thirring structure, with both the gravitoelectric and gravitomagnetic sectors affected by the quantum spin and macroscopic rotation. We also discuss some phenomenological implications of the model, including the gyroscope precession and the Sagnac effect, along with potential experimental probes of the predicted corrections.

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I. INTRODUCTION

General relativity (GR) has achieved remarkable success over a broad range of domains, ranging from Solar-System observations [1] to the direct detection of gravitational waves [2], the imaging of black hole shadows [3], and the standard cosmological model [4]. Nevertheless, several theoretical considerations suggest that the underlying Riemannian geometry is sufficient to provide a correct description of gravity only in the macroscopic regime, while difficulties arise when moving to the microscopic scales. Among the various motivations for going beyond the GR paradigm, the problem of quantizing the gravitational field is perhaps the most compelling. Simple power counting arguments indicate that GR is perturbatively nonrenormalizable, since the Newtonian constant G has negative mass dimension (in natural units $\hbar = c = 1$), leading to ultraviolet divergences beyond one-loop level that cannot be canceled via a redefinition of the field parameters [5–7].

Furthermore, the well-known Hawking-Penrose theorems prove that GR is plagued by spacetime singularities, which suggest that the Universe originated from a singular state and that the final outcome of gravitational collapse may be pathological [8,9].

Abandoning Riemannian geometry provides a first attempt to address these difficulties. The simplest extension of GR consists in passing to the Riemann-Cartan spacetime, where the affine connection can acquire torsion contributions [10,11]. This framework gives rise to Einstein-Cartan (EC) gravity, which can be formulated as a gauge theory of the Poincaré group and thus represents a particular case of the broader setting of metric-affine (gauge theory of) gravity [12,13]. EC theory allows one to overcome a conceptual limitation of GR concerning microstructured matter. According to quantum field theory, particles can be classified by the irreducible unitary representations of the Poincaré group and are characterized by two fundamental quantities, namely the mass and spin. While mass(-energy) acts as the source of the spacetime curvature in GR, spin does not fulfill an independent geometrical role. This shortcoming is resolved in EC gravity, where torsion is naturally coupled (in an algebraic way) to the spin of the matter. Here, we adhere to the standard EC interpretation, and by *spin* we mean the quantum intrinsic angular momentum carried by elementary particles [10,14], as opposed to the classical orbital

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angular momentum associated with the macroscopic rotation of a body.

EC theory finds applications in a variety of physical scenarios. From a field-theoretic standpoint, torsion can provide a regularization prescription for ultraviolet-divergent integrals in loop Feynman diagrams [15], and induce non-trivial interactions with matter fields, particularly in its coupling to fermions [16–18] (see also Refs. [19,20]). From a foundational perspective, singularity theorems and focusing conditions have recently been examined in detail [21–24], partly motivated by the possibility that singularities may be avoided under suitable circumstances in gravitational collapse and black hole formation settings [25–27], as well as in cosmology [28,29]. In the latter context, torsion can generate nonsingular bouncing models [30–32], affect inflationary dynamics [33–36], account for the matter-antimatter asymmetry problem [37], and influence the late-time evolution of the Universe, leading to departures from standard dark sector phenomenology [38]. On the astrophysical side, EC gravity has been applied to the study of gravitational waves [39–43], post-Newtonian dynamics of compact binaries [44–46], and neutron star interiors [47].

A number of investigations, including many of those mentioned above, rely on the *Weyssenhoff fluid*, which generalizes the ordinary concept of perfect fluid and provides an effective macroscopic description of matter possessing intrinsic spin [10,14,48–50]. In this framework, a suitable averaging procedure over the microscopic spin degrees of freedom is adopted, and a collection of constituents with a random spin distribution is treated as a continuous medium. As a result, the average spin density tensor vanishes, while quadratic-in-spin terms survive and act as a source of torsion in the EC field equations [28].

The main deviations from GR are expected to arise at extremely high matter densities, with critical values of order 10^{50} – 10^{57} kg/m³ [10]. These severe conditions, which are typically relevant in the early Universe and final stages of gravitational collapse, make the detection of torsion effects challenging. Despite that, various empirical tests of EC theory have been proposed in the literature (see, e.g., the overview of possible measurement schemes and their limitations in Ref. [51]). These include, in particular, spin-precession and polarization experiments probing torsion-induced spin couplings [52], which can be formulated even within the weak-field approximation [53], potential modifications of electromagnetic wave propagation [38,54], as well as high-energy signatures based on effective fermion-torsion interactions; see e.g., Ref. [55] for a review.

Motivated by the importance of identifying potential scenarios where torsion might play a significant role and of devising possible experimental setups to evaluate its existence, in this paper we analyze rotating Weyssenhoff fluids, i.e., sources endowed with *both* intrinsic quantum spin *and* orbital angular momentum, within the linearized

EC approximation. This framework, to the best of our knowledge, is presented here for the first time.

The study of the exterior spacetime of a Weyssenhoff fluid sphere in linearized EC gravity was initiated by Arkuszewski, Kopczyński and Ponomarev in Ref. [56], which deals with the *static* configuration where all fluid elements are at rest with respect to an asymptotically inertial frame, i.e., the source has no macroscopic rotation. The external metric is found to be formally analogous to the Lense-Thirring solution [57], with the off diagonal components determined by the total intrinsic spin of the Weyssenhoff fluid, rather than by the classical angular momentum.

As we will show, when macroscopic rotation is switched on, the metric attains a generalized Lense-Thirring form at the lowest nontrivial multipole order. Both the gravito-electric and gravitomagnetic sectors exhibit peculiar corrections: the former acquires an additional “spin-orbit” contribution whose sign is controlled by the relative orientation of the intrinsic spin and orbital angular momentum of the fluid, while the latter depends on the linear combination of the classical and quantum spins.

The plan of the paper is thus as follows. After reviewing the main aspects of EC gravity and its linearized limit in Sec. II, we examine the external weak-field geometry of an isolated, stationary, axisymmetric, and slowly rotating Weyssenhoff fluid in Sec. III. Our approach proceeds by first computing the (metric) stress-energy tensor of the source in accordance with its symmetries, and then determining the external metric by performing a multipole expansion of the Poisson-integral solution of the linearized EC field equations, followed by an analysis of the junction conditions at the fluid-vacuum interface. In Sec. IV, we discuss the phenomenological aspects of the solution and potential tests by studying gravitomagnetic observables like the Lense-Thirring precession and Sagnac effect, as well as some spin-rotation coupling mechanisms such as the Barnett and Einstein-de Haas phenomena. Finally, in Sec. V we draw our conclusions.

Notations and conventions. We use the metric signature $(-, +, +, +)$. Greek indices take values 0, 1, 2, 3, while lowercase Latin ones take 1, 2, 3. Four-vectors are written as $a^\mu = (a^0, a^i)$. Spatial three-vectors are denoted by $\vec{a} = (a^1, a^2, a^3)$, with $\vec{a} \cdot \vec{b} := \delta_{ik} a^i b^k$, $|\vec{a}| := (\vec{a} \cdot \vec{a})^{1/2} \equiv a$, and $(\vec{a} \times \vec{b})^i := \epsilon_{ilk} a^l b^k$. An overhat symbol refers to quantities defined in terms of the Levi-Civita connection. The symmetrization and antisymmetrization procedures read as $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$ and $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$, respectively.

II. EINSTEIN-CARTAN GRAVITY AND ITS WEAK-FIELD LIMIT

The EC model represents the simplest generalization of GR that allows for a torsionful connection. Since the

underlying action is formally identical to the Einstein-Hilbert action of GR, torsion has no dynamical character and vanishes in vacuum, where the theory boils down to GR [10,11] (for further details, we refer the reader to Refs. [53,58–66]).

In this section, we briefly outline the main aspects of EC gravity relevant for our analysis. After introducing the geometric framework and setting our conventions in Sec. II A, we discuss the EC field equations in Sec. II B. Then, the linearized regime is presented in Sec. II C.

A. The geometric framework

EC gravity entails the introduction of a four-dimensional Riemann-Cartan spacetime manifold $\mathcal{M}$ endowed with a Lorentzian-signature metric $g_{\mu\nu}$ and a metric-compatible affine connection $\Gamma^\lambda_{\mu\nu}$, which admits the decomposition

$$\Gamma^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} - K_{\mu\nu}^\lambda. \quad (1)$$

Here, $\hat{\Gamma}^\lambda_{\mu\nu}$ denotes the Levi-Civita connection

$$\hat{\Gamma}^\lambda_{\mu\nu} := \frac{1}{2} g^{\lambda\alpha} (\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}), \quad (2)$$

while

$$K_{\mu\nu}^\lambda := -S_{\mu\nu}^\lambda + S_{\nu\mu}^\lambda - S_{\mu\nu}^\lambda = -K_{\mu\nu}^\lambda \quad (3)$$

represents the *contorsion tensor*, with

$$S_{\mu\nu}^\lambda := \Gamma^\lambda_{[\mu\nu]} = -S_{\nu\mu}^\lambda \quad (4)$$

the *torsion tensor*.

In our conventions, the Riemann tensor reads as [10]

$$R_{\mu\nu\lambda}^\rho = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\eta} \Gamma^\eta_{\nu\lambda} - \Gamma^\rho_{\nu\eta} \Gamma^\eta_{\mu\lambda}, \quad (5)$$

and the Einstein tensor assumes the usual form,

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad (6)$$

the Ricci tensor and Ricci scalar being given by $R_{\mu\nu} := R_{\alpha\mu}^\alpha{}_\nu$ and $R := g^{\mu\nu} R_{\mu\nu}$, respectively.

B. The field equations

Since in the EC framework the gravitational field has two geometric degrees of freedom, namely curvature and torsion, its description requires the presence of two sets of field equations. These can be derived from the total action,

$$\mathcal{S} = \frac{1}{c} \int d^4x \sqrt{-g} \left(\frac{1}{2\chi} R + \mathcal{L}_m \right), \quad (7)$$

where $\chi := 8\pi G/c^4$ and the matter Lagrangian $\mathcal{L}_m$ is assumed to be minimally coupled to gravity (see, e.g., Refs. [67–70] for extensions involving generalized gravitational actions). Varying $\mathcal{S}$ with respect to $g_{\mu\nu}$ and $K_{\mu\nu}^\lambda$ yields the EC field equations (with the one for the metric being symmetric, as expected from the symmetry of $g_{\mu\nu}$). Using the (generalized) Belinfante-Rosenfeld relation, the resulting system can be equivalently recast in the form [58]

$$G^{\alpha\beta} = \chi \mathbb{T}^{\alpha\beta}, \quad (8a)$$

$$S_{\alpha\beta}^\gamma + 2\delta^\gamma_{[\alpha} S_{\beta]\lambda}^\lambda = \chi \tau_{\alpha\beta}^\gamma. \quad (8b)$$

Here, $\mathbb{T}^{\alpha\beta}$ is the (asymmetric) *canonical energy-momentum tensor*, which is related to the (symmetric) *metric energy-momentum tensor*

$$\sqrt{-g} T^{\alpha\beta} = 2 \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g_{\alpha\beta}}, \quad (9)$$

via the aforementioned (generalized) Belinfante-Rosenfeld identity

$$\mathbb{T}^{\alpha\beta} = T^{\alpha\beta} - \nabla_\gamma^* (-\tau^{\alpha\beta\gamma} + \tau^{\beta\gamma\alpha} - \tau^{\gamma\alpha\beta}), \quad (10)$$

with $\nabla_\alpha^* := \nabla_\alpha + 2S_{\alpha\beta}^\beta$, while

$$\sqrt{-g} \tau_{\alpha\beta}^\gamma = \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta K_{\gamma}^{\beta\alpha}} \quad (11)$$

denotes the *spin angular momentum tensor*.

Starting from Eq. (8), one can derive the relations [10,58]

$$\nabla_\nu^* \mathbb{T}_\mu^\nu = 2\mathbb{T}_\lambda^\nu S_{\mu\nu}^\lambda + \tau_{\nu\rho}^\sigma R_{\mu\sigma}^{\nu\rho}, \quad (12)$$

$$\nabla_\gamma^* \tau_{\alpha\beta}^\gamma - \mathbb{T}_{[\alpha\beta]} = 0, \quad (13)$$

which represent the generalized version of the energy-momentum and total (spin plus orbital) angular momentum conservation laws, respectively.

It follows from the above formulas that the mass-energy and spin densities of matter, which act as sources of the gravitational field, are geometrically related to the space-time curvature and torsion, respectively. Another key aspect of EC gravity is that Eq. (8b) is an algebraic rather than a dynamical equation. This feature has a twofold consequence. First, the torsion can be formally eliminated in favor of the spin tensor in the field equations, which can be thus rewritten in the GR-like form

$$\hat{G}^{\mu\nu} = \chi T^{\mu\nu} + \chi^2 \tilde{\tau}^{\mu\nu}, \quad (14)$$

$\tilde{\tau}^{\mu\nu}$ being the symmetric tensor

$$\begin{aligned}\tilde{\tau}^{\mu\nu} := & -4\tau^{\mu\gamma}[\lambda\tau^{\nu\lambda}{}_{\gamma}] - 2\tau^{\mu\gamma\lambda}\tau^{\nu}{}_{\gamma\lambda} + \tau^{\gamma\lambda\mu}\tau^{\nu}{}_{\gamma\lambda} \\ & + \frac{1}{2}g^{\mu\nu}(4\tau_{\alpha}{}^{\gamma}[\lambda\tau^{\alpha\lambda}{}_{\gamma}] + \tau^{\alpha\gamma\lambda}\tau_{\alpha\gamma\lambda}).\end{aligned}\quad (15)$$

Second, torsion does not propagate since it is nonzero only inside matter, where it mediates a spin-spin contact interaction. Nevertheless, the spin of matter can still influence the spacetime geometry in vacuum, albeit only indirectly and very weakly, as can be inferred from the structure of Eq. (14), where the quadratic-in-spin corrections are suppressed by a factor χ relative to the other contributions.

C. The linearized regime

The linearized approximation of EC theory applies to the regime where both the metric and torsion can be regarded as weak (see Ref. [56] and Sec. IIIC in Ref. [40]). In such a scenario, these fields can be simultaneously expanded around the (torsionless) Minkowski background, and quantities of second order (i.e., involving quadratic contributions in the perturbations or their derivatives) can be consistently neglected.

Similarly to GR (see e.g., Refs. [57,71,72]), we introduce a *global nearly Lorentzian coordinate system* and expand the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (16)$$

where $|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric, which we use to raise and lower tensor indices. It is worth noticing that standard Cartesian coordinates $x^\mu = (ct, x, y, z)$ on the Minkowski background fall within the class of coordinates underlying Eq. (16), unlike, for example, spherical ones. To fix the ideas, for our calculations in linearized EC gravity we will use coordinates $x^\mu = (ct, x, y, z)$, unless otherwise stated.

As a consequence of Eq. (16), the inverse metric reads as $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O_g(2)$, and one thus finds

$$\hat{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\alpha}(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}) + O_g(2), \quad (17)$$

which, jointly with condition $K_{\mu\nu}{}^\lambda = O_g(1)$, leads to

$$\nabla_\mu = \partial_\mu + O_g(1). \quad (18)$$

Here, we have introduced the notation $O_g(n)$ to indicate terms of order n in the “gravitational” perturbations [e.g., $h^{\rho\sigma}h_{\mu\nu}$, $h^{\rho\sigma}S_{\mu\nu\lambda}$, $\partial_\alpha h^{\rho\sigma}S_{\mu\nu\lambda}$ represent $O_g(2)$ contributions].

Owing to the above formulas and the relation $\tilde{\tau}^{\mu\nu} = O_g(2)$ [see Eq. (15)], the EC field equations (14) take the following form in the weak-field limit:

$$\begin{aligned}-\square\bar{h}_{\mu\nu} + \partial_\sigma(\partial_\nu\bar{h}^\sigma{}_\mu + \partial_\mu\bar{h}^\sigma{}_\nu - \eta_{\mu\nu}\partial_\lambda\bar{h}^{\sigma\lambda}) + O_g(2) \\ = 2\chi T_{\mu\nu},\end{aligned}\quad (19)$$

where $\square := \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat-space d’Alembert operator,

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha{}_\alpha, \quad (20)$$

denotes the trace-reversed perturbation, and

$$T_{\mu\nu} = \mathbb{T}_{\mu\nu} + \partial_\lambda(-\tau_{\mu\nu}{}^\lambda + \tau_\nu{}^\lambda{}_\mu - \tau^\lambda{}_{\mu\nu}) + O_g(2), \quad (21)$$

which readily stems from Eq. (10) and corresponds to the linearized EC version of the Belinfante-Rosenfeld procedure.

It is thus clear that by adopting the *Lorentz gauge*

$$\partial^\nu\bar{h}_{\mu\nu} = 0, \quad (22)$$

Eq. (19) boils down, after discarding $O_g(2)$ corrections, to the flat-space wave equation

$$\square\bar{h}_{\mu\nu} = -2\chi T_{\mu\nu}. \quad (23)$$

The corresponding formal solution, appropriate for radiative problems, can be expressed via the retarded Green function as [57,72]

$$\bar{h}_{\mu\nu}(t, \vec{x}) = \frac{4G}{c^4} \int \frac{d^3x'}{|\vec{x} - \vec{x}'|} T_{\mu\nu}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}'\right), \quad (24)$$

the combination $t - c^{-1}|\vec{x} - \vec{x}'|$ representing the so-called retarded time.

A final crucial remark is in order. As discussed above, within the linearized EC approximation both the metric $h_{\mu\nu}$ and the torsion $S_{\mu\nu}{}^\lambda$ are *formally* of the same perturbative order $O_g(1)$, which justifies treating them as comparable for bookkeeping purposes. Despite that, phenomenologically the situation is different. While $h_{\mu\nu}$ can encompass both classical and quantum contributions, the latter understood as involving a factor $\hbar$, $S_{\mu\nu}{}^\lambda$ is a “fully quantum” object, being related via Eq. (8b) to the canonical spin tensor.

III. THE EXTERIOR GEOMETRY OF A SLOWLY ROTATING WEYSENHOFF-FLUID SOURCE

The key feature of the *Weyssenhoff fluid* is that it incorporates the effects of the intrinsic angular momentum of its constituents into the fluid description. As pointed out before, owing to its potentially significant implications in both high-density cosmological and astrophysical contexts, this model is extensively studied within EC gravity (see e.g., Refs. [14,49,50]).

In this section, we introduce a Weyssenhoff fluid that undergoes a macroscopic rotation and thus carries a classical angular momentum in addition to the microscopic quantum spin. To the best of our knowledge, this combined

configuration is derived here for the first time in the literature.

Specifically, we consider a *slowly rotating, stationary, axisymmetric, and isolated* Weyssenhoff source in the weak-field regime, where the linearized EC theory can be reliably invoked. After outlining the key aspects of the Weyssenhoff paradigm in Sec. III A, we determine the corresponding exterior geometry via a two-step program. First, we work out the explicit form of the metric energy-momentum tensor $T_{\mu\nu}$ in Sec. III B. Second, we formally solve the ensuing linearized EC equations by means of a multipole expansion in Sec. III C. We conclude our analysis by addressing the matching conditions across the fluid surface in Sec. III D.

As we shall see, our model generalizes the setup of Ref. [56], which deals with the gravitational field generated by a static Weyssenhoff-fluid setup.

A. A brief overview of the Weyssenhoff fluid

The fluid description of a continuous medium relies on a coarse-grained approximation of a large collection of particles, where molecular fluctuations are smoothed over and each fluid element represents a local average of the matter it contains [73,74].

The Weyssenhoff fluid furnishes a hydrodynamical model of a quantum system. Within this framework, each fluid element is characterized by an intrinsic spin density encoded by the tensor

$$s_{\mu\nu} = -\frac{1}{c^2} \tau_{\mu\nu}{}^\lambda u_\lambda = s_{[\mu\nu]}. \quad (25)$$

Here, the four velocity $u^\lambda = dx^\lambda/d\tau$, with τ the proper time measured by a comoving observer, obeys the normalization relation $u^\mu u_\mu = -c^2$ and can be arranged as

$$u^\mu = \frac{u^0}{c} (c, v^i), \quad (26)$$

$v^i = cu^i/u^0 = dx^i/dt$ being the coordinate-time three velocity and, by a slight abuse of terminology, u^0/c the Lorentz factor.¹

To correctly account for the three physical degrees of freedom associated with the spin, and to ensure the integrability of the fluid equations of motion, one typically imposes

¹Owing to the normalization condition $u^\mu u_\mu = -c^2$, the four velocity can be arranged in general as in Eq. (26), with $u^0/c = (-g_{00} - 2g_{0i}v^i/c - g_{ij}v^i v^j/c^2)^{-1/2}$. This shows that the normalization factor u^0/c depends not only on v^i , but also on the metric components, and is therefore coordinate dependent. More generally, the Lorentz factor of the fluid relative to an observer with four velocity $\mathcal{U}^\mu$ is the scalar $\gamma(\mathcal{U}) = -u_\mu \mathcal{U}^\mu/c^2$ [see e.g., Eq. (18) in Ref. [75]]. In the nearly Lorentzian coordinates adopted below, the quantity u^0/c reduces, at zeroth order in the gravitational perturbations, to the usual special-relativistic Lorentz factor [cf. Eq. (49)].

three spin supplementary constraints on $s_{\mu\nu}$ [40,41,76]. Hereafter, we will adopt the *Frenkel condition*² [49]

$$s_{\mu\nu} u^\nu = 0, \quad (27)$$

which covariantly enforces that $s_{\mu\nu}$ is spacelike in the fluid rest frame; more details will be provided in Sec. III A 1.

The Weyssenhoff fluid is characterized by the canonical stress-energy tensor [see Eq. (10)]

$$\mathbb{T}_{\mu\nu} = (\rho + p) \frac{u_\mu u_\nu}{c^2} + p g_{\mu\nu} - \frac{2}{c^2} a^\beta s_{\beta\mu} u_\nu, \quad (28)$$

where ρ and p are the total energy density and pressure, respectively, and $a^\mu = u^\lambda \nabla_\lambda u^\mu$ the four acceleration. In addition, the tensors $T_{\mu\nu}$ and $\tilde{\tau}_{\mu\nu}$ take the form [cf. Eqs. (9) and (15)]

$$\begin{aligned} T_{\mu\nu} = & (\rho + p) \frac{u_\mu u_\nu}{c^2} + p g_{\mu\nu} \\ & + 2 \left(\frac{u_\gamma u^\lambda}{c^2} - \delta^\lambda_\gamma \right) \hat{\nabla}_\lambda (s^\gamma{}_{(\mu} u_{\nu)}) \\ & - 2\chi (s^2 u_\mu u_\nu + c^2 s_{\mu\gamma} s_\nu{}^\gamma), \end{aligned} \quad (29)$$

$$\tilde{\tau}_{\mu\nu} = 2c^2 s_{\mu\gamma} s_\nu{}^\gamma + s^2 c^2 \left(\frac{u_\mu u_\nu}{c^2} - \frac{1}{2} g_{\mu\nu} \right), \quad (30)$$

where $s^2 := s^{\alpha\beta} s_{\alpha\beta}$ is the (conserved [41,49]) spin density scalar.

Combining the above formulas, one obtains from Eq. (14)

$$\begin{aligned} \hat{G}_{\mu\nu} = & \chi \left[(\rho_{\text{eff}} + p_{\text{eff}}) \frac{u_\mu u_\nu}{c^2} + p_{\text{eff}} g_{\mu\nu} \right. \\ & \left. + 2 \left(\frac{u_\gamma u^\lambda}{c^2} - \delta^\lambda_\gamma \right) \hat{\nabla}_\lambda (s^\gamma{}_{(\mu} u_{\nu)}) \right], \end{aligned} \quad (31)$$

with

$$\rho_{\text{eff}} := \rho - \rho_{\text{sc}}, \quad p_{\text{eff}} := p - p_{\text{sc}}, \quad (32a)$$

and

$$\rho_{\text{sc}} \equiv p_{\text{sc}} = \frac{1}{2} \chi c^2 s^2. \quad (32b)$$

EC field equations are thus equivalent to Einstein equations for a perfect fluid with the usual density and pressure replaced by the effective quantities (32a), along with the

²Although there exist situations in which the Frenkel condition does not hold (see e.g., Refs. [77,78]), it can be safely assumed for polarized sources, which is the regime considered in this paper.

gradient term $\widehat{\nabla}_\lambda(s^\gamma{}_{(\mu}u_{\nu)})$, which vanishes upon averaging if the spins of the individual fluid elements are randomly oriented. The contribution introduced by ρ_{sc} and p_{sc} amounts to a repulsive force that can have a crucial role in the avoidance of gravitational singularities in collapsing-star models and in the early stages of the Universe evolution [16,33,48,79]. These quadratic-in-spin corrections, which are consistent with the phenomenology expected from the Pauli exclusion principle, stem from the peculiar spin contact interactions of EC gravity.

I. More on the Frenkel condition

Let us now give further details on the physical implications of the Frenkel condition.

Similarly to the Faraday tensor [57,80], $s_{\mu\nu}$ admits the *covariant* decomposition

$$s_{\mu\nu} = \frac{1}{c}(u_\mu E_\nu - u_\nu E_\mu) + B_{\mu\nu}, \quad (33)$$

where the “electric” four-vector E_μ and the “magnetic” tensor $B_{\mu\nu} = B_{[\mu\nu]}$ are defined through

$$E_\nu = -\frac{1}{c}u^\mu s_{\mu\nu}, \quad (34)$$

$$B_{\mu\nu}u^\nu = 0. \quad (35)$$

Since both E_ν and $B_{\mu\nu}$ contain three independent components, a simple counting argument shows that one could enforce the correct number of spin degrees of freedom by imposing either $E_\nu = 0$ or $B_{\mu\nu} = 0$. The physical reason why the Frenkel condition (27) selects the former rather than the latter is that retaining a nonvanishing $B_{\mu\nu}$ permits us to properly define the Pauli-Lubanski four-vector in a covariant manner [57,81]:

$$\ell_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\frac{u^\beta}{c}s^{\mu\nu} = -\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\frac{u^\beta}{c}B^{\mu\nu}, \quad (36)$$

where $\epsilon_{\alpha\beta\mu\nu}$ is the four-dimensional totally antisymmetric Levi-Civita symbol. Thus, ℓ_α has the essential features to embody a truly covariant representation of the spin density vector: (i) it carries three independent degrees of freedom, as $\ell_\alpha u^\alpha = 0$ by construction, and (ii) it transforms as an axial vector.

This identification can be clarified as follows. If we introduce the three-vectors [40,41,50]

$$\vec{s} = (s^1, s^2, s^3) \equiv (s^{23}, s^{31}, s^{12}), \quad (37)$$

$$\vec{q} = (q^1, q^2, q^3) \equiv (s^{10}, s^{20}, s^{30}), \quad (38)$$

and exploit Eq. (26), the Frenkel condition (27) can be recast as

$$s_{i0} = -s_{ij}\frac{v^j}{c}, \quad (39a)$$

$$s_{0i}v^i = 0. \quad (39b)$$

Adopting nearly Lorentzian coordinates, we can expand identity (39a) as

$$\vec{q} = \frac{1}{c}\vec{v} \times \vec{s} + \text{O}_g(2), \quad (40)$$

where we have used the formulas

$$s^i = \frac{1}{2}\epsilon^{ilm}s_{lm} + \text{O}_g(2), \quad (41)$$

$$s_{ij} = \epsilon_{lij}s^l + \text{O}_g(2).$$

Relation (40) closely mirrors the Lorentz-transformation law connecting the electric and magnetic fields [82], which naturally suggests interpreting $\vec{q}$ and $\vec{s}$ as the electric and magnetic components of $s_{\mu\nu}$. This analogy becomes clearer in the (hypothetical) global rest frame of the fluid, where we use the symbol “ $\doteq$ ” to denote the corresponding identities. Here, E_μ , $B_{\mu\nu}$, and ℓ_μ are purely spatial, $u^\mu \doteq (c, 0, 0, 0)$, and Eqs. (37) and (38) yield

$$\vec{s} \doteq (B^{23}, B^{31}, B^{12}) \doteq \vec{\ell} + \text{O}_g(2), \quad (42)$$

$$\vec{q} \doteq (E^1, E^2, E^3). \quad (43)$$

These formulas show that $\vec{s}$ is identified with the spatial components of the Pauli-Lubanski vector, while the electric piece $\vec{q} \doteq 0$ owing to Eq. (27). However, the global frame underlying the above calculations does not exist in general, and the Frenkel condition guarantees that $\vec{q}$ vanishes only *pointwise* in the instantaneous rest frame of each fluid element, but not in an arbitrary coordinate system. This observation is crucial, as we will see in Sec. III B that the time-time component of the stress-energy tensor of a rotating Weyssenhoff fluid exhibits a peculiar (spin-orbit) correction due precisely to $\vec{q}$.

B. The stress-energy tensor

The first step in deriving the gravitational field produced by the rotating Weyssenhoff fluid is to compute the corresponding metric stress-energy tensor $T_{\mu\nu}$, which enters the right-hand side of the linearized EC field equations (23).

It follows from Eq. (29) that, in the weak-field approximation, $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = (\rho + p)\frac{u_\mu u_\nu}{c^2} + p\eta_{\mu\nu} + 2\left(\frac{u_\gamma u^\lambda}{c^2} - \delta^\lambda_\gamma\right)\partial_\lambda(s^\gamma{}_{(\mu}u_{\nu)}) + \text{O}_g(2). \quad (44)$$

At this stage, however, the above expression remains completely general. To specialize it to our configuration, it must be adapted to the assumed properties of the source. We investigate the constraints arising from stationarity and axial symmetry in Sec. III B 1, while the slow-rotation hypothesis is introduced in Sec. III B 2, where we also work out the components of $T_{\mu\nu}$ explicitly. As we will see, among the new velocity-related corrections, a novel spin-orbit coupling emerges that modifies the energy density of the source. We inspect this contribution in detail in Sec. III B 3.

1. The hypotheses of stationarity and axial symmetry

It is clear from Eq. (44) that $T_{\mu\nu}$ depends on four variables: the energy density, pressure, four velocity and spin density. Since ρ and p are scalar fields, stationarity and axial symmetry simply require them to be time independent and constant along circles centered on the symmetry axis. By contrast, the situation is slightly more involved for u^μ and $s_{\mu\nu}$, as axial symmetry imposes additional restrictions on their components.

Let us first deal with the four velocity. Differently from the static configuration [56], the presence of a macroscopic rotation entails that no (asymptotically inertial) nearly Lorentzian frame can be defined where the fluid is globally at rest [see our discussion below Eq. (41)]. Consequently, $u^\alpha(x) \neq (c, 0, 0, 0)$, except possibly on the symmetry axis. Had a coordinate system existed where $u^\alpha = (c, 0, 0, 0)$ everywhere, it would necessarily have been rotating relative to a nearly Lorentzian frame and thus would not itself qualify as nearly Lorentzian.

Given the above premises, we can obtain the explicit form of the four velocity in a straightforward way if we first adopt a set of spherical coordinates $\tilde{x}^\mu = (ct, r, \theta, \varphi)$ (the specific definition of r is not essential in the linearized regime). Supposing hereafter that the symmetry axis coincides with the z direction, a purely rotational (i.e., circular) stationary, axisymmetric source admits the general formula [cf. Eq. (11.145) in Ref. [73]]:

$$\begin{aligned} \tilde{u}^\mu(\tilde{x}) &= \frac{d\tilde{x}^\mu}{d\tau} = (\tilde{u}^0, \tilde{u}^r, \tilde{u}^\theta, \tilde{u}^\varphi) \\ &= \frac{\tilde{u}^0(r, \theta)}{c} (c, 0, 0, \omega(r, \theta)), \end{aligned} \quad (45)$$

where the time-independent function $\omega(r, \theta)$ physically represents the (counterclockwise) rotation rate of the fluid at the spacetime points specified by r and θ , as measured by an observer at infinity [cf. Eq. (11.146) in Ref. [73]]:

$$\frac{d\varphi}{dt} = \frac{d\varphi}{d\tau} \left(\frac{dt}{d\tau} \right)^{-1} = c \frac{\tilde{u}^\varphi}{\tilde{u}^0} = \omega(r, \theta). \quad (46)$$

The exact profile of ω depends both on the properties of the fluid (e.g., whether it undergoes rigid or differential

rotation) as well as on the precise representation of the variables r and θ .

Transforming expression (45) from spherical coordinates $\tilde{x}^\mu$ back to the nearly Lorentzian Cartesian system $x^\mu = (ct, x, y, z)$, and using Eq. (26), the four velocity takes the form

$$\begin{aligned} u^\mu(x) &= (u^0, u^x, u^y, u^z) \\ &= \frac{u^0}{c} (c, -\omega(r, \theta)y, \omega(r, \theta)x, 0). \end{aligned} \quad (47)$$

Therefore, the coordinate three velocity reads as

$$\vec{v} = \vec{\omega} \times \vec{x} = (-\omega y, \omega x, 0) = (v^x, v^y, v^z), \quad (48)$$

and the Lorentz factor becomes

$$\begin{aligned} \frac{u^0}{c} &= \left(1 - \frac{|\vec{v}|^2}{c^2} \right)^{-1/2} + \mathcal{O}_g(1) \\ &= \left(1 - \frac{\omega^2 \varrho^2}{c^2} \right)^{-1/2} + \mathcal{O}_g(1), \end{aligned} \quad (49)$$

with $\vec{\omega}(r, \theta) = \omega(r, \theta)\hat{z}$ and $\varrho := \sqrt{x^2 + y^2}$.

We now examine the spin density. Consistently with the axial symmetry, we assume that the Weyssenhoff fluid has spins everywhere aligned with the z axis, i.e., $s^x = s^y = 0$ [cf. Eq. (37)]. In this setup, it is natural to impose the Frenkel condition (27), which, in view of Eqs. (41) and (47), yields

$$s_{x0} + s^z x \frac{\omega}{c} + \mathcal{O}_g(2) = 0, \quad (50a)$$

$$s_{y0} + s^z y \frac{\omega}{c} + \mathcal{O}_g(2) = 0, \quad (50b)$$

$$\omega(s_{y0}x - s_{x0}y) = 0, \quad (50c)$$

$$s_{z0}u^0 + \mathcal{O}_g(2) = 0. \quad (50d)$$

Discarding second-order gravitational corrections, one immediately sees from Eq. (50d) that $s_{z0} = 0$. Hence, the only nonvanishing entries of the spin density tensor are s_{x0} , s_{y0} , and $s_{xy} \equiv s^z$, the latter being the sole independent component. This result can be easily verified. In fact, Eq. (50c) imposes no additional constraint on $s_{\mu\nu}$, as it is automatically satisfied once Eqs. (50a) and (50b) hold. Once s^z and ω are specified, Eqs. (50a) and (50b) fully determine s_{x0} and s_{y0} , which continue to vary through their dependence on the coordinates x and y even in the particular scenario where s^z and/or ω are constant. Furthermore, when $\omega \neq 0$, Eq. (50c) gives rise to the geometric relation

$$\frac{s_{y0}}{s_{x0}} = \tan \varphi, \quad \text{for } x \neq 0, \quad (51)$$

which is the expected consequence of the axial symmetry of the configuration.

A further crucial outcome of Eq. (50) is that the electric part $\vec{q} = (s^{x0}, s^{y0}, s^{z0})$ of the spin [see Eq. (38)] is in general nonvanishing whenever $\omega \neq 0$, i.e., for a rotating fluid. As pointed out before, this is perfectly compatible with the Frenkel condition.

2. The slow-rotation approximation and structure of $T_{\mu\nu}$

Having seen in the previous section how the source variables entering $T_{\mu\nu}$ are affected by the requirements of stationarity and axial symmetry, we can now introduce the additional slow-rotation hypothesis.

Accordingly, we assume that the “velocity-related” parameters $\vec{\omega}$ and $\vec{v}$ be $O_v(1)$ quantities, with $O_v(n) \equiv O(|\vec{v}|^n)$, while, like before, $h_{\mu\nu}$, $s_{\mu\nu}$, ρ , and p are regarded as $O_g(1)$ contributions. Since the linearized EC theory (see Sec. II C) does not address the full regime of strongly self-gravitating sources, the weak-field and low-velocity expansions can be treated as independent. Therefore, in this setup we retain corrections of order $O_g(1)$, $O_v(1)$, as well as their products $O_g(1)O_v(1)$, while safely discarding $O_g(2)$, $O_v(2)$, and higher. This scheme is consistent with the structure of $s_{\mu\nu}$, for which we initially take $s_{\mu\nu} = O_g(1)$, and the Frenkel condition then refines this to $s_{0i} = O_g(1)O_v(1)$ and $s_{ij} = O_g(1)$ [see Eqs. (39) and (50)].

Including mixed factors $O_g(1)O_v(1)$ is standard in linearized GR and is precisely what allows one to recover the exterior field of a weakly gravitating, rotating system (see Sec. 19.1 in Ref. [57]). Our approach extends this procedure, as keeping factors like $v^j s_{ij} \sim O_v(1)O_g(1)$ amounts to retaining $O(v)O(\hbar)$ terms that capture spin-rotation couplings (recall that $s_{\mu\nu}$ is intrinsically $\hbar$ dependent). This treatment aligns with the spirit of the static-source limit analyzed in Ref. [56].

Under the hypothesis of slow rotation, we find that the linearized stress-energy tensor (44) is (for simplicity, the remainders are omitted hereafter)³

$$T_{00} = \rho + 2c\partial_i s_{i0}, \quad (52a)$$

$$T_{0j} = -(\rho + p) \frac{v^j}{c} + c\partial_i s_{ij}, \quad (52b)$$

$$T_{ij} = p\delta_{ij} - \partial_l(s_{li}v^j) - \partial_l(s_{lj}v^i), \quad (52c)$$

which, in terms of the electric and magnetic spin three-vectors $\vec{q}$ and $\vec{s}$, becomes

$$T_{00} = \rho - 2c\vec{\nabla} \cdot \vec{q}, \quad (53a)$$

$$T_{0j} = -(\rho + p) \frac{v^j}{c} - c(\vec{\nabla} \times \vec{s})^j, \quad (53b)$$

$$T_{ij} = p\delta_{ij} + 2(\vec{v} \otimes \vec{\nabla} \times \vec{s})^{(ij)} - 2(\vec{s} \times \vec{\nabla})^{(i} v^{j)}. \quad (53c)$$

Here, the expression for T_{0j} takes into account that $s_{i0} = O_g(1)O_v(1)$ and “ $\otimes$ ” is the ordinary tensor product.

Owing to Eqs. (37) and (48), in the Cartesian coordinates $x^\mu = (ct, x, y, z)$, $T_{\mu\nu}$ explicitly reads as

$$T_{00} = \rho - 2[2s^z\omega + x\partial_x(s^z\omega) + y\partial_y(s^z\omega)], \quad (54a)$$

$$T_{0x} = (\rho + p) \frac{\omega y}{c} - c\partial_y s^z, \quad (54b)$$

$$T_{0y} = -(\rho + p) \frac{\omega x}{c} + c\partial_x s^z, \quad (54c)$$

$$T_{xx} = p - 2\partial_y(y s^z\omega), \quad (54d)$$

$$T_{xy} = y\partial_x(s^z\omega) + x\partial_y(s^z\omega), \quad (54e)$$

$$T_{yy} = p - 2\partial_x(x s^z\omega), \quad (54f)$$

$$T_{zz} = p. \quad (54g)$$

As expected, the above formulas boil down to those pertaining to nonrotating sources [56]. In contrast, when $\omega \neq 0$, $T_{\mu\nu}$ acquires additional velocity-dependent corrections. Specifically, the diagonal elements T_{xx} and T_{yy} modify the effective pressure in an anisotropic way, the mixed T_{0j} encode, as we will see, the macroscopic angular momentum of the source [cf. Eqs. (65b) and (66b) below], and the off diagonal term T_{xy} describes the xy -shear stress.

A separate discussion concerns T_{00} . It can be naturally recast as

$$T_{00} = \rho - 2\rho_{\text{so}}, \quad (55)$$

where we have introduced

$$\begin{aligned} \rho_{\text{so}} &:= c\vec{\nabla} \cdot \vec{q} = -c\partial_i s_{i0} = \partial_i(s_{ij}v^j) \\ &= 2s^z\omega + x\partial_x(s^z\omega) + y\partial_y(s^z\omega), \end{aligned} \quad (56)$$

the last equality on the first line stemming from Eq. (39a). This new contribution is nonvanishing only when both the *macroscopic* angular velocity $\vec{\omega}$ and the *microscopic* quantum spin $\vec{s}$ are nonzero, and thus behaves as an effective “spin-orbit energy density.” Being proportional to $\vec{\nabla} \cdot \vec{q}$, it is reminiscent of the charge density in electromagnetism (which involves the divergence of the electric field), and, likewise, can attain either positive or negative

³Notice that Eq. (52) is written in a form that is explicitly invariant under spatial rotations.

values. Assuming as a first approximation that $s^z\omega = \text{constant}$ (as for a uniformly polarized rigid body), the sign of ρ_{so} is determined solely by the relative orientation of $\vec{\omega}$ and $\vec{s}$, as it is positive or negative when they are aligned or antialigned, respectively. This observation points to a clear spin-orbit interpretation: the configuration having aligned $\vec{\omega}$ and $\vec{s}$ is energetically favored, since it reduces the effective energy density $\rho - 2\rho_{\text{so}}$ occurring in Eq. (55).

Interestingly, the dependence of ρ_{so} on the electric vector $\vec{q}$ allows another parallel with Maxwell theory. As explained before, $\vec{q} \neq 0$ in the nearly Lorentzian frame where the source rotates with angular velocity ω , while the Frenkel condition enforces $\vec{q} = 0$ locally in each fluid element rest frame. This behavior resembles that of the electric field generated by a rotating magnet [83]: the field is absent in the magnet rest frame but appears in a rotating one. This frame effect arises from the intertwining of the electric and magnetic fields under Lorentz transformations [82], and further supports the interpretation of Eq. (40) as providing an electromagnetic analog for $\vec{q}$ and $\vec{s}$.

3. Estimate of the spin-orbit energy density ρ_{so}

Given its central role in our analysis, in this section we take a closer look at the spin-orbit energy density ρ_{so} .

Like before, we assume $s^z\omega = \text{constant}$ for simplicity. In this setting, $\rho_{\text{so}} = 2s^z\omega$, and $T_{\mu\nu}$ becomes diagonal [cf. Eqs. (56) and (54)]:

$$T_{00} = \rho - 4s^z\omega, \quad (57a)$$

$$T_{xx} = T_{yy} = p - 2s^z\omega, \quad (57b)$$

$$T_{zz} = p, \quad (57c)$$

showing that the pressure orthogonal to the rotation axis is reduced by half the factor that affects the energy density.

Assuming a polarized source consisting of spin- $\frac{1}{2}$ particles (e.g., neutrons or electrons), the spin density can be modeled by $s^z = n\hbar/2$, while the rest energy density is $\rho = mc^2n$, where n is the particle number density and m the mass of a single constituent [10,56]. Combining these relations yields

$$\frac{\rho_{\text{so}}}{\rho} \sim \frac{\lambda_C}{c} \omega, \quad (58)$$

or, equivalently,

$$\frac{\rho_{\text{so}}}{\rho} \sim \varepsilon \frac{\lambda_C}{\mathcal{R}}, \quad (59)$$

where $\lambda_C = \hbar/(mc)$ is the reduced Compton wavelength, $\mathcal{R}$ the characteristic radius of the source, and $\varepsilon := \omega\mathcal{R}/c$ measures how close the equatorial (tangential) speed is to the relativistic limit.

Remarkably, the ratio ρ_{so}/ρ is independent of n , and hence the spin-orbit correction can, in principle, be relevant even for gravitational systems of comparatively low density. As expected, this contribution becomes more important as ω increases, and is especially significant for small, rapidly rotating objects. However, ρ_{so}/ρ cannot be arbitrarily large, for at least two reasons: (i) causality demands $\varepsilon < 1$ and (ii) for a noncollapsed source, $\mathcal{R}$ must exceed some critical value (e.g., in GR a static, spherically symmetric object must have $\mathcal{R}$ larger than the Schwarzschild radius). These physical constraints limiting ρ_{so}/ρ actually favor the self-consistency of our approach, since they help the system remain within the slow-rotation regime discussed before.

It is appropriate to make a comparison with the characteristic spin-contact density ρ_{sc} [see Eq. (32b)], which we have neglected in the linearized approach owing to its dependence on the spin density scalar s^2 . In this case, one finds

$$\frac{\rho_{\text{sc}}}{\rho} \sim \pi \ell_P^2 \lambda_C n, \quad (60)$$

with $\ell_P = \sqrt{\hbar G/c^3}$ the Planck length. Differently from Eq. (58), the spin-contact term scales linearly with n , meaning that it is significant only at extremely high densities comparable to or beyond the cutoff $n_{\text{cut}} \sim 1/(\ell_P^2 \lambda_C)$ [cf. Eq. (5.4) in Ref. [10]].

The corrections related to ρ_{so} dominate over the spin-contact ones in realistic settings, such as astrophysical scenarios comprising neutron stars. To illustrate this, let us consider the fastest-rotating neutron star observed to date, identified as PSR J1748-2446ad and with reference values $\varepsilon \sim 1/4$ and $\mathcal{R} \sim 16$ km [84]. Assuming that it is made up almost exclusively of neutrons, for which $\lambda_C \sim 2.1 \times 10^{-16}$ m, we obtain

$$\frac{\rho_{\text{so}}}{\rho} \sim 3 \times 10^{-21}, \quad (61)$$

which is roughly 20 orders of magnitude larger than the spin-contact ratio (60):

$$\frac{\rho_{\text{sc}}}{\rho} \sim 5 \times 10^{-41}, \quad (62)$$

where we have used the representative number density for neutron-star matter $n \sim 3 \times 10^{44} \text{ m}^{-3}$ (corresponding to roughly twice the nuclear saturation density [85,86]).

Remarkably, the above conclusions remain valid over a broad range of parameters. For instance, assuming again an effectively polarized neutron star and reducing ε by 1 order of magnitude, i.e., taking $\varepsilon \sim 10^{-2}$ rather than $\varepsilon \sim 1/4$, while keeping the values of all other parameters the same as before, still permits recovering the hierarchy $\rho_{\text{so}}/\rho \gg \rho_{\text{sc}}/\rho$. Indeed, Eq. (59) yields in this case $\rho_{\text{so}}/\rho \sim 1.3 \times 10^{-22}$, while ρ_{sc}/ρ remains unchanged.

A last remark is in order. Within the exact treatment of the rotating Weyssenhoff fluid, we reasonably expect that $T_{\mu\nu}$ will involve both ρ_{so} and ρ_{sc} . Accordingly, we might define an extended effective energy density $\rho'_{\text{eff}} = \rho - \rho_{\text{sc}} - 2\rho_{\text{so}} \equiv \rho_{\text{eff}} - 2\rho_{\text{so}}$ along the lines of Eq. (32a). This modified density could have implications in the study of gravitational collapse and singularity avoidance, in analogy with the nonrotating Weyssenhoff scenario; see e.g., Refs. [24,33,79].

C. Multipole expansion of the metric

With the explicit expression of the (linearized) stress-energy tensor for a slowly rotating and axisymmetric Weyssenhoff fluid at hand [see Eqs. (52)–(54)], we can now address the second step of our computational program and study the resulting exterior geometry.

As set out earlier, in the Lorentz gauge (22) the linearized EC equations take the form (23). These attain a simplified form in our setup, because stationary implies that the flat-space d'Alembert operator boils down to the flat-space Laplace operator,

$$\square \bar{h}_{\mu\nu} = \eta^{ij} \partial_i \partial_j \bar{h}_{\mu\nu} =: \Delta \bar{h}_{\mu\nu}, \quad (63)$$

since $\eta^{00} \partial_0 \partial_0 \bar{h}_{\mu\nu} = 0$.

Since the appropriate boundary conditions for an isolated source require that the metric perturbations $\bar{h}_{\mu\nu}$ vanish at spatial infinity, the formal solution of Eq. (23) in the limit (63) involves the Poisson integral

$$\bar{h}_{\mu\nu}(\vec{x}) = \frac{4G}{c^4} \int \frac{d^3 x'}{|\vec{x} - \vec{x}'|} T_{\mu\nu}(\vec{x}'), \quad (64)$$

rather than the retarded Green function (24).

In line with the fact that we are concerned with the geometry *outside* the fluid, the evaluation of the above integral can be tackled by means of a multipole expansion, where, differently from the static case [56], we allow for nonzero pressure. Following the approach outlined in Ref. [57] [see Eqs. (19.5) and (19.22) therein], we exploit the gauge freedom to perform a coordinate transformation to a new set of nearly Lorentzian coordinates $x'^{\mu} = (ct, \vec{x}')$, with $\vec{x}' = (x', y', z')$, in which the leading terms of $\bar{h}'_{00}$, $\bar{h}'_{0i}$, and $\bar{h}'_{ij}$ scale as $O(1/r')$, $O(1/r'^2)$, and $O(1/r'^3)$, respectively (with $r' := |\vec{x}'|$). Dropping the primes for simplicity, and using the definition (20), we find that the components of $h_{\mu\nu}$ in the new frame read as, to the leading nontrivial multipole order,

$$h_{00} = \frac{2G}{c^2} \frac{M - 2M_s}{r}, \quad (65a)$$

$$h_{0i} = \frac{2G}{c^3} \frac{[\vec{x} \times (\vec{J} + 2\vec{S})]_i}{r^3}, \quad (65b)$$

with

$$M := \frac{1}{c^2} \int d^3 x \rho(\vec{x}), \quad (66a)$$

$$\vec{J} := \frac{1}{c^2} \int d^3 x (\rho + p)(\vec{x} \times \vec{v}), \quad (66b)$$

$$M_s := \frac{1}{c^2} \int d^3 x \rho_{\text{so}}(\vec{x}), \quad (66c)$$

$$\vec{S} := \int d^3 x \vec{s}(\vec{x}). \quad (66d)$$

The quantities M and $\vec{J}$ are the usual mass and classical angular momentum of the source, respectively [57]. The term M_s represents an additional effective mass arising from the spin-orbit energy density (56) and hence linear in both the fluid spin and angular velocity, while $\vec{S}$ denotes the microscopic quantum spin of the matter distribution, obtained as the volume integral of the magnetic vector $\vec{s}$ [see Eq. (37)].

The solution (65) can be identified with a generalized Lense-Thirring geometry [57], where, remarkably, the Newtonian and gravitomagnetic potentials can be obtained through the replacements

$$M \rightarrow M - 2M_s, \quad \vec{J} \rightarrow \vec{J} + 2\vec{S}. \quad (67)$$

In other words, the spin-induced modifications affect both sectors of the weak-field geometry. The gravitoelectric correction is encoded in the effective mass $M - 2M_s$, whereas the gravitomagnetic field is sourced by the effective angular momentum $\vec{J} + 2\vec{S}$. We will exploit such a result in Sec. IV A.

Before proceeding further, the following points should be highlighted. First, the quantum spin $\vec{S}$ enters h_{0i} with a coefficient twice that of the classical $\vec{J}$, and an analogous structure also appears in h_{00} and h_{ij} for the mass terms. Remarkably, this feature is reminiscent of the normalization factor found in the analysis of binary dynamics in EC theory, which has been carried out, at the first post-Newtonian level, in Refs. [44–46] by applying the point-particle procedure to the Weyssenhoff fluid and within the Frenkel description. Although the two results arise in different settings, they both give precious indications of how the combined effect of the spin and macroscopic rotation affects (weak) gravitational phenomena within the Weyssenhoff-fluid scenario [and for the spin-current convention adopted in Eq. (11)].

Second, it is interesting to note that under the same assumptions as in Sec. III B 3, the neutron star PSR J1748-2446ad has a ratio S/J of the same order as the dimensionless parameter controlling the spin-orbit correction ρ_{so}/ρ [see Eq. (61)]:

$$\frac{S}{J} = \frac{5}{4} \frac{\lambda}{\epsilon \mathcal{R}} \sim 4 \times 10^{-21}. \quad (68)$$

This indicates that the spin-orbit contribution to the gravito-electric sector and the intrinsic-spin correction to the gravitomagnetic field become relevant at comparable scales.

Last, it is clear that both $\vec{J}$ and M_s vanish in the static limit. A naive application of the divergence theorem to Eqs. (56) and (66c) might suggest that M_s should be zero also in the rotating configuration. However, this need not be the case in general, since $\vec{q}$ can fail to be regular on the boundary of the fluid, as we will discuss in Sec. III D.

D. Junction conditions

In many gravitational applications, one must tackle the problem of matching two metrics smoothly across a hypersurface Σ that divides the spacetime into two regions. In such situations, the field equations are understood in a distributional sense, and singular terms are either removed or given a proper physical interpretation by imposing so-called *junction conditions*, which configure as constraints on the jumps of geometric quantities across Σ . In GR, this procedure leads to the well-known Israel formalism and the possible appearance of thin shells [87]. In EC theory, however, torsion and spin introduce additional requirements [88], which do not, in general, guarantee the continuity of the spin density tensor across Σ . When applied to our model, this implies that the divergence theorem cannot be invoked to evaluate the spin-orbit mass M_s , which is thus allowed to give a nonzero contribution to the exterior metric (65).

We briefly outline the formalism of junction conditions within EC gravity [88] in Sec. III D 1, while the application to our setup is discussed in Sec. III D 2.

1. The general setup

To fix the ideas, let Σ be a timelike hypersurface described locally by $y^1 = 0$ in a suitable coordinate system $y^\alpha = (y^0, y^1, y^2, y^3)$ defined on the Riemann-Cartan spacetime $\mathcal{M}$. We say that a function $f: \mathcal{M} \rightarrow \mathbb{R}$ is of class C_q^{-1} if it is of class C^q in $\mathcal{M}/\Sigma$, and if its derivatives up to (and including) order q admit finite left and right limits on Σ , uniformly approached from both sides. If, in addition, f is of class C^p everywhere on $\mathcal{M}$, we say that f is of class C_q^p , and we define

$$f_\pm(y^a) := \lim_{y^1 \rightarrow 0^\pm} f(y^1, y^a), \quad (69a)$$

$$[f](y^a) := f_+(y^a) - f_-(y^a), \quad (69b)$$

where $y^a = (y^0, y^2, y^3)$ are intrinsic coordinates on Σ (in this section, lowercase Latin indices from the beginning of the alphabet, $a, b, \dots = 0, 2, 3$, are used to denote quantities tangent to Σ).

A solution of the EC field equations (8) in the region $y^1 < 0$ (resp. $y^1 > 0$) is said to match a solution in the domain $y^1 > 0$ (resp. $y^1 < 0$) if, in any appropriate coordinate system, the following hold: (i) the metric $g_{\mu\nu}$ is of class C_3^0 ; (ii) the spin density tensor $\tau_{\alpha\beta}{}^\lambda$ is of class C_2^{-1} ; (iii) the canonical energy-momentum tensor $\mathbb{T}^{\mu\nu}$ is of class C_2^{-1} . These requirements are equivalent to the identities [88]

$$\left[\frac{\partial g_{ab}}{\partial y^1} \right] = -\frac{2}{g^{11}} [K_{(ab)}{}^1], \quad (70a)$$

$$[n_\lambda \tau_{\alpha\beta}{}^\lambda] = 0, \quad (70b)$$

$$\chi[n_\lambda \mathbb{T}^\lambda{}_\mu] = -[\bar{K}_{\mu\nu\lambda}] n_\rho K^{\nu\lambda\rho} - \frac{1}{2} n_\mu [\bar{K}_{\lambda\nu\rho} \bar{K}^{\rho\nu\lambda}]. \quad (70c)$$

Here, $\bar{K}_{\mu\nu}{}^\lambda$ is the projection of the contorsion tensor onto Σ , while n^μ the unit spacelike normal to Σ , which satisfies $n_\mu n^\mu = 1$ and is supposed to be of class C_3^0 . Equations (70b) and (70c) constitute the junction conditions for the matter fields $\mathbb{T}^{\mu\nu}$ and $\tau_{\alpha\beta}{}^\lambda$, and represent the only independent restrictions imposed on their jumps $[\mathbb{T}^{\mu\nu}]$ and $[\tau_{\alpha\beta}{}^\lambda]$.

In the special case where a vacuum external solution defined, say, for $y^1 > 0$ has to be matched across Σ to an inner gravitational field sourced by $\mathbb{T}^{\mu\nu}$ and $\tau_{\alpha\beta}{}^\lambda$ in $y^1 < 0$, the metric must comply with

$$g_{\mu\nu}|_+ = g_{\mu\nu}|_-, \quad (71a)$$

$$\left. \frac{\partial g_{ab}}{\partial y^1} \right|_+ = \left. \frac{\partial g_{ab}}{\partial y^1} \right|_- + \frac{2}{g^{11}} K_{(ab)}{}^1 \Big|_-, \quad (71b)$$

while the junction conditions for the matter fields reduce to

$$n_\lambda \tau_{\mu\nu}{}^\lambda|_- = 0, \quad (72a)$$

$$\left(\chi n_\lambda \mathbb{T}^\lambda{}_\mu + \frac{1}{2} n_\mu \bar{K}_{\lambda\nu\rho} \bar{K}^{\rho\nu\lambda} \right) \Big|_- = 0. \quad (72b)$$

This situation arises, for example, when the interior region is filled with a Weyssenhoff fluid, with matching conditions (72) boiling down to

$$n_\mu u^\mu|_- = 0, \quad (73a)$$

$$p|_- = \chi c^2 (n_\mu \ell^\mu)^2|_-, \quad (73b)$$

where, we recall, u^μ is the fluid four velocity, p the pressure, and ℓ_μ the Pauli-Lubanski four-vector defined in Eq. (36).

2. Application to our model

We can now apply the matching procedure to the slowly rotating Weyssenhoff fluid studied in this paper.

We introduce coordinates $y^a = (cT, R, \Theta, \Phi)$, with T timelike and (R, Θ, Φ) spacelike, and let the timelike hypersurface Σ separating the matter interior from the exterior vacuum region be defined by $R = 0$, with $y^a = (cT, \Theta, \Phi)$ the intrinsic coordinates. Owing to the symmetries of our model, Σ is invariant under translations in T and Φ . In the coordinates y^a , the normal (co)vector is given by

$$n_\mu = (0, n_R, 0, 0), \quad (74)$$

and hence the Weyssenhoff-fluid junction conditions (73) take the form

$$u^R|_- = 0, \quad (75a)$$

$$p|_- = \chi c^2 (n_R \ell^R)^2|_-. \quad (75b)$$

The first, Eq. (75a), is automatically met in our settings. In fact, recall that in spherical coordinates $\tilde{x}^\mu = (ct, r, \theta, \varphi)$ the fluid four velocity is [cf. Eq. (45)]

$$\tilde{u}^\mu = (\tilde{u}^0, 0, 0, \tilde{u}^\varphi). \quad (76)$$

The symmetries permit the identifications $T \equiv t$ and $\Phi \equiv \varphi$ (and $R \equiv r - R_\star$, $\Theta \equiv \theta$ in the spherically symmetric limit; see below). Therefore, the only nonzero Jacobian components between coordinates $\tilde{x}^\mu = (ct, r, \theta, \varphi)$ and $y^a = (ct, R, \Theta, \varphi)$ are in general

$$\frac{\partial R}{\partial r}, \quad \frac{\partial R}{\partial \theta}, \quad \frac{\partial \Theta}{\partial r}, \quad \frac{\partial \Theta}{\partial \theta}. \quad (77)$$

Therefore $\tilde{u}^r = \tilde{u}^\theta = 0$ implies $u^R = u^\Theta = 0$, and then Eq. (75a) is satisfied.

Relation (75b) constrains the *a priori* arbitrary profiles of the source pressure and spin density, and can be fulfilled only once the internal properties of the matter distribution are specified, typically via an equation of state. More importantly, it shows that the spin density tensor, as well as the pressure, can exhibit a jump discontinuity across the Weyssenhoff fluid-vacuum interface. In these cases, the divergence theorem cannot be applied to the integral (66c), and consequently M_s is, in general, nonvanishing.

To conclude, let us note that in the special scenario of the static Weyssenhoff-fluid sphere examined in Refs. [56,88], one may take $(T, R, \Theta, \Phi) \equiv (t, r - R_\star, \theta, \varphi)$ and perform the matching at the stellar surface, with Σ thereby defined by $r = R_\star$ (note that Σ is not at $r = 0$, which would be the center). Under these hypotheses, the first junction identity (73a) is readily obeyed, because the four velocity takes the simplified form $\tilde{u}^\mu = (\tilde{u}^0, 0, 0, 0)$. The second relation (73b), instead, reduces to the nontrivial Arkuszewski-Kopczyński-Ponomarev pressure condition [88].

IV. PHENOMENOLOGY AND POTENTIAL EXPERIMENTAL TESTS

As discussed before, in EC gravity torsion is intrinsically bound to matter and hence vanishes in vacuum. Consequently, outside the material source the spacetime geometry is formally Riemannian, and test bodies obey the standard GR dynamics associated with the external metric. Nevertheless, the exterior perturbation (65) retains information about the internal structure of the source, as the effective mass receives a spin-orbit correction proportional to M_s , while the gravitomagnetic sector depends on the combination $\vec{J} + 2\vec{S}$.

In this section, we explore some possible empirical consequences of these modifications and discuss potential ways to test them. Since the external geometry has the form of a generalized Lense-Thirring metric, the most natural observables are those sensitive to weak-field gravitomagnetic phenomena. We therefore analyze the precession of classical test gyroscopes (Sec. IV A) and the Sagnac effect (Sec. IV B). Last, we explore possible connections with spin-polarization mechanisms such as the Barnett and Einstein-de Haas effects (Sec. IV C).

A. Precession of classical test gyroscopes

In the context of GR, the motion of an isolated *extended* object endowed with a *macroscopic* angular momentum is described, within the pole-dipole scheme, by the *Mathisson-Papapetrou-Dixon equations*, supplemented by an appropriate spin supplementary condition [89–98]. By adopting the Chicone-Mashhoon-Punsly approximation [99,100], these equations can be consistently expanded to first order in the classical spin by assuming that the body has (i) a Møller radius much smaller than the local curvature radius and (ii) four-momentum and four-velocity vectors nearly parallel. Under these hypotheses, the world-line of the object acquires a small spin-curvature acceleration producing a slight deviation from the geodesic path, and the angular momentum undergoes Fermi-Walker transport. This is precisely the framework suitable for studying the dynamics of an ideal gyroscope, i.e., a rigid, torque-free, massive *pointlike test object with macroscopic spin* $\vec{L}$ [57,96]. In this limit, the trajectory of the gyroscope can be assigned independently of the spin dynamics, which is governed by the precession law [57,96,101,102]

$$\frac{d}{d\lambda} \vec{L} = \vec{\Omega}_p \times \vec{L}, \quad (78)$$

where λ is the proper time of an observer comoving with the body and $\vec{\Omega}_p$ the precession angular velocity evaluated with respect to distant stars.

Let us now consider the scenario in which the gyroscope is immersed in a weak gravitational field generated by a distribution of slowly moving matter. In this regime, the linearized

spacetime metric, written in Lorentz gauge, takes the standard gravitoelectromagnetic form [57,101,103–106]

$$ds^2 = -\left(1 + 2\frac{\phi}{c^2}\right)c^2 dt^2 + \frac{4}{c}(\vec{A} \cdot d\vec{x})dt + \left(1 - 2\frac{\phi}{c^2}\right)d\vec{x} \cdot d\vec{x}, \quad (79)$$

where ϕ and $\vec{A}$ are the gravitoelectric scalar potential and the gravitomagnetic vector potential, respectively, and we have employed the nearly Lorentzian coordinate system $x^\mu = (ct, \vec{x})$, with $\vec{x} = (x, y, z)$.

In this setup, it is possible to show that $\vec{\Omega}_p$ consists of three pieces, namely [57,101,102]

$$\vec{\Omega}_p = \vec{\Omega}_{\text{Th}} + \vec{\Omega}_{\text{ds}} + \vec{\Omega}_{\text{LT}}, \quad (80a)$$

known as the *Thomas, de Sitter* (or *geodetic*) and *Lense-Thirring angular velocities*, which are given by

$$\vec{\Omega}_{\text{Th}} = -\frac{1}{2}\frac{\vec{\mathcal{V}}}{c^2} \times \vec{\mathcal{A}}, \quad (80b)$$

$$\vec{\Omega}_{\text{ds}} = -\left(1 + \frac{1}{2}\right)\frac{\vec{\mathcal{V}}}{c^2} \times \vec{\nabla}\phi, \quad (80c)$$

$$\vec{\Omega}_{\text{LT}} = -\frac{1}{c}\vec{\nabla} \times \vec{A}. \quad (80d)$$

Here, $\vec{\mathcal{V}} \equiv d\vec{x}/dt$ and $\vec{\mathcal{A}} \approx d\vec{\mathcal{V}}/dt + \vec{\nabla}\phi$ are the velocity and the nongravitational part of the acceleration of the gyroscope (so that for a freely falling motion $\vec{\mathcal{A}} = \vec{0}$). For a detailed discussion on the operative meaning of $\vec{\Omega}_p$ and the various terms it comprises we refer the reader to Refs. [57,101].

Since the exterior metric (65) of the slowly rotating Weyssenhoff fluid exhibits a generalized Lense-Thirring structure, we can compare it with Eq. (79), and in this way we find the identifications (recall that $r := |\vec{x}|$)

$$\phi \equiv -\frac{G}{r}(M - 2M_s), \quad (81)$$

$$\vec{A} \equiv \frac{G}{c}\frac{\vec{x}}{r^3} \times (\vec{J} + 2\vec{S}), \quad (82)$$

where we have taken into account Eq. (67). Hence

$$\vec{\Omega}_{\text{ds}} = -\frac{3G}{2c^2}(M - 2M_s)\frac{\vec{\mathcal{V}} \times \vec{x}}{r^3}, \quad (83)$$

$$\vec{\Omega}_{\text{LT}} = -\frac{G}{c^2 r^3} \left[(\vec{J} + 2\vec{S}) - 3\frac{(\vec{J} + 2\vec{S}) \cdot \vec{x}}{r^2} \vec{x} \right], \quad (84)$$

while we recall that $\vec{\Omega}_{\text{Th}}$ can be of the same order as $\vec{\Omega}_{\text{ds}}$ for an Earth-bound observer, whereby $\vec{\mathcal{A}} \approx \vec{\nabla}\phi = -\vec{g}_\oplus$.

The de Sitter and Lense-Thirring precessions are among the main observables in weak-field gravity. The space-based mission Gravity Probe B directly measured these precessional signatures using cryogenic gyroscopes [107–109], while the LAGEOS and LARES missions probed the Lense-Thirring effect through precision laser ranging determinations of satellite orbital precessions [110–112]. Related phenomena can also be investigated through precise observations of the motion of celestial bodies [101], for instance via lunar laser ranging techniques [113]. These essentially consist in evaluating the time of flight of laser signals emitted from the Earth and retroreflected by devices placed on the Moon, and yield subcentimeter constraints on the Earth-Moon dynamics [114–116]. In this context, experiments such as MOONLIGHT, together with data-analysis frameworks like the Planetary Ephemeris Program (PEP) [117–119], can play an important role in testing torsion-induced corrections in the weak-gravity limit.

B. The Sagnac effect

While in the previous section we have dealt with the phenomenology of mechanical gyroscopes, we now turn our attention to optical gyroscopes, i.e., devices based on the *Sagnac effect* [120,121].

This phenomenon is associated with the propagation of photons in stationary, rotating spacetimes. In this setting, consider an observer $\mathcal{O}$ emitting two light rays traveling in opposite directions along a closed spatial loop $\mathcal{S}$. In general, their roundtrip (proper) times differ by an amount $\Delta\tau_S$, called *Sagnac time delay*. This quantity involves the off diagonal metric components g_{0i} of the background geometry according to the formula [101,102,122–124]

$$\begin{aligned} \Delta\tau_S &= -\frac{2}{c}\sqrt{-g_{00}(p)} \int_{\mathcal{S}} \frac{g_{0i}}{g_{00}} dl^i \\ &= -\frac{2}{c}\sqrt{-g_{00}(p)} \int_{\varphi_0}^{\varphi_0+2\pi} \frac{g_{0i}}{g_{00}} \Big|_{\mathcal{S}} \frac{dl^i}{d\varphi} d\varphi, \end{aligned} \quad (85)$$

where p denotes the position of $\mathcal{O}$ on the path $\mathcal{S}$ and $dl^i = (dl^i/d\varphi)d\varphi$ the infinitesimal coordinate displacement along $\mathcal{S}$.

If $\mathcal{S}$ is sufficiently small compared with the typical curvature scale of the spacetime, then the background metric can be locally expanded around the worldline of $\mathcal{O}$ by using proper coordinates, and curvature corrections can be neglected. In this limit, Eq. (85) reduces to

$$\Delta\tau_S = \frac{4}{c^2} \vec{\Omega} \cdot \vec{\mathcal{A}}, \quad (86)$$

with $\vec{\mathcal{A}}$ the oriented area vector enclosed by $\mathcal{S}$ and $\vec{\Omega}$ the local rotation vector of the observer spatial frame (i.e., the

rotation vector characterizing the off diagonal metric components in the proper rotating frame of $\mathcal{O}$; cf. Eq. (1.88) in Ref. [72]). The vector $\vec{\Omega}$ is related to the precession angular velocity $\vec{\Omega}_p$ introduced in Eq. (78) through the identity

$$\vec{\Omega} = \vec{\Omega}_\infty - \vec{\Omega}_p, \quad (87)$$

where $\vec{\Omega}_\infty$ is the angular velocity of $\mathcal{O}$ with respect to the fixed stars, as measured by an observer at infinity (in practice, $\vec{\Omega}_\infty$ can be determined via very-long-baseline interferometry techniques [101]).

Although formulas (85) and (86) are derived in the context of GR, they remain valid also in EC gravity, where no (minimal) coupling between the Maxwell field and the torsion is allowed [10]. Therefore, Maxwell equations retain the same form as in GR, and the geometric optics limit can be carried out in an analogous manner [57,125]. As a consequence, light rays travel on null geodesics of the spacetime metric also in EC gravity, and the computation of Eqs. (85) and (86) proceeds identically; further details can be found in Ref. [126]. Therefore, for the rotating Weyssenhoff source developed in this paper, $\Delta\tau_S$ can be computed directly by plugging the generalized Lense-Thirring metric (65), or equivalently the replacements (67), into the formulas (85)–(87).

The Sagnac effect thus configures as another means to probe torsion effects in weak gravitational fields. The devices that measure rotations through such phenomenon are known as *Sagnac gyroscopes* [120,121,127,128]. In particular, in a *ring laser gyroscope* [129–132], which is the type of device employed in the GINGER experiment [102,123,133–139], the spatial path S is realized by a set of corner mirrors, while the Sagnac time delay manifests itself in the frequency difference of counterpropagating Gaussian laser beams.

Other experiments based on different technologies, such as the fiber optic gyroscopes, and/or other interferometer configurations, like the Mach-Zehnder interferometers, evaluate the Sagnac effect either via phase shifts of continuous laser beams or via differences in the arrival times of light pulses or photons. Some applications even rely on quantum interference, where the Sagnac time delay modifies the coincidence detection probability and the interference visibility at the output of the interferometer [140–144].

C. The Barnett and Einstein-de Haas effects

One of the assumptions underlying our description of the rotating Weyssenhoff fluid is that the macroscopic and microscopic angular momenta are aligned along the same axis. From a dynamical point of view, this hypothesis can be motivated by known physical mechanisms that couple rotation and magnetization (see, for example, Refs. [145,146]). In particular, the *Barnett* [147] and the *Einstein-de Haas* effects [148–150] describe the

exchange between macroscopic rotational motion and microscopic spin degrees of freedom, and play a key role in modern experimental devices (see for example, Refs. [151–153] and references therein). Both scenarios can be understood as a consequence of the conservation of the total (macroscopic plus intrinsic) angular momentum. The Barnett effect consists in the magnetization induced by the rotation of an initially unmagnetized body, whereas the Einstein-de Haas phenomenon corresponds to the onset of mechanical rotation induced by a change in magnetization. In both cases, the magnetization vector tends to align with the rotation axis. A similar phenomenon, known as the *London moment*, manifests in superconductors, where it constitutes the physical mechanism behind the gyroscopes employed in Gravity-Probe B [154].

The abovementioned effects not only provide a physical motivation for our assumption of aligned macroscopic and microscopic spins, but may open the possibility for new laboratory setups aimed at testing gravitomagnetic effects in EC gravity (for general Earth-based laboratory applications see, e.g., Sec. 6.9 of Ref. [101]). A detailed analysis of such possibilities, however, lies beyond the scope of the present work.

V. DISCUSSION AND CONCLUSIONS

In this paper, we have studied the exterior gravitational field of a source possessing both classical and quantum spins within linearized EC gravity. Specifically, we have examined an isolated, axisymmetric Weyssenhoff fluid characterized by a macroscopic, slow rotation around the z axis, with the intrinsic spin polarized along the same direction.

The geometry has been obtained by first deriving the linearized stress-energy tensor $T_{\mu\nu}$ of the source, which depends on the four time-independent functions ρ , p , ω , and s^z , each constructed in accordance with the symmetries of the system [cf. Eqs. (52)–(54)]. Then, the corresponding metric has been computed by resorting to a multipole expansion [see Eqs. (65) and (66)]. At the leading non-trivial order, it configures as a generalized Lense-Thirring solution, where the off diagonal components involve a linear combination of the orbital and intrinsic angular momenta, while the gravitoelectric sector acquires the additional effective mass correction M_s linear in both the source spin and angular velocity. The latter contribution can be traced back to the spin-orbit energy density (56), determined by the electric piece $\vec{q}$ of the spin tensor, which is globally nonvanishing in our setup.

These spin-rotation couplings, which remarkably contribute to the gravitational field even in the weak-field regime, can be better understood if the rotating Weyssenhoff source is interpreted as the gravitational analog of a rotating magnet. In such a system, the electric field vanishes in the magnet rest frame but appears in a rotating frame. This situation mirrors the local behavior of $\vec{q}$ and is consistent with the constraints imposed by the Frenkel condition.

The phenomenology associated with the gravitational field of the rotating Weyssenhoff fluid has also been discussed. The new corrections can potentially affect standard weak-field observables, such as gyroscope precession and the Sagnac effect, and may be probed through both space-based and laboratory tests. In this regard, recent developments in condensed-matter physics may provide additional tools for exploring the connection between rotation, quantum interactions, and gravity. In particular, novel routes may emerge from spin-polarization mechanisms such as the Barnett and Einstein–de Haas phenomena. More broadly, the growing empirical control over rotating quantum systems and quantum materials, together with the observation of rotation-induced effects in interferometric and entanglement-based setups [155–159] and studies based on fast-rotating matter [160–162], may open new avenues for testing spin-rotation couplings beyond traditional astrophysical and gravitational experiments.

Although our model can furnish only a preliminary account of fast-rotating astrophysical objects, it has nevertheless shed light on the structure of spin-rotation couplings. At the leading order, the quantum spin enters the

metric with a characteristic factor of 2 relative to the orbital contribution, in agreement with the post-Newtonian analysis of compact binaries dynamics performed in Refs. [44–46]. Developing the present framework further through approaches like the post-Newtonian or the post-Minkowskian approximation schemes might reveal additional intriguing facets of the interplay between macroscopic and microscopic spins, potentially leading to new unexpected results. This is a worthwhile direction for future research.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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