

Domain walls from $\Sigma(36 \times 3)$, $\Delta(54)$ and $\Delta(27)$ potentials

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We consider the degenerate minima arising from scalar potentials invariant under $\Sigma(36 \times 3)$, or under its subgroups $\Delta(54)$ and $\Delta(27)$ (with or without imposed CP symmetries), for a triplet of those symmetries. In this framework, we classify the distinct domain walls between the degenerate minima and calculate the respective tensions.

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I. INTRODUCTION

Non-Abelian discrete symmetries provide a powerful framework beyond the Standard Model for addressing the flavor problem—the mystery of why there are three generations of fermions and why they possess such specific mass hierarchies and mixing patterns. One of the most significant implications is in the lepton sector, where special mixing patterns are predicted and large mixing angles are addressed with the help of nontrivial vacuum expectation values (VEVs) achieved by certain series of scalars [1–3]. In multi-Higgs models, non-Abelian discrete symmetries are also motivated in both reducing free parameters in the potential and by interesting possibilities of spontaneous CP violation [4,5].

The list of symmetries for three-Higgs doublet models (3HDMs) is listed in [6,7],¹ and some recent studies of 3HDMs are performed, e.g., in [10–18]. Solving the flavor problem in this framework was discussed recently in [19–22].

Discrete symmetries in the early Universe have interesting cosmological consequences. The spontaneous breaking of discrete symmetry generates domain walls (DWs), two-dimensional topological defects at cosmic scale [23].

They oscillate and radiate gravitational waves,² providing a cosmological probe to discrete symmetries. It was highlighted that stochastic gravitational wave backgrounds from collapsing DWs provide a good fit to data from pulsar time array measurements of NANOGrav [24]. Most studies of DW dynamics start from Z_2 DW or axion DW [25]; see [26–31] for recent updates. In more general potential in larger symmetries, DWs with more complicated properties form and their evolution dynamics can be significantly different from Z_2 DW [32]. Adjacent and nonadjacent walls are specified from the breaking of Z_N symmetries for $N > 3$ [33]. Properties of non-Abelian DWs were first studied in [34–36], with A_4 and S_4 as typical examples. Different kinds of DWs are observed due to combinations of rich vacua configurations in S_4 . Extended studies are further performed in A_4 [37]. In particular, walls separating two CP -conjugate walls are pointed out when VEVs of complex scalars in A_4 are taken into account [37]. Gravitational waves from non-Abelian DWs provide a cosmological window to the origin of the flavor problem [38,39].

In this work, we focus on properties of domain walls formed from $\Delta(27)$, $\Delta(54)$ and $\Sigma(36 \times 3)$,³ all of which are subgroups of $SU(3)$ [41], with the following isomorphism relations satisfied,

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¹See also [8,9].

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²In the radiation era, the evolution of the DW network follows the scaling behavior, with its energy density diluted much more slowly than the radiation during the Hubble expansion. To avoid its domination in the universe, the DW should collapse before the period of big bang nucleosynthesis. This condition is satisfied by including small explicit breaking in the potential.

³ $\Sigma(36 \times 3)$ is also denoted as $\Sigma(36\varphi)$ with $\varphi = 3$, and $\Sigma(36) \equiv \Sigma(36\varphi)/Z_3 \simeq (Z_3 \times Z_3) \rtimes Z_4$ [40].

$$\begin{aligned}
\Delta(27) &\simeq (Z_3 \times Z_3) \rtimes Z_3, \\
\Delta(54) &\simeq \Delta(27) \rtimes Z_2, \\
\Sigma(36 \times 3) &\simeq \Delta(54) \rtimes Z_2 \simeq \Delta(27) \rtimes Z_4. \quad (1)
\end{aligned}$$

Compared with the most well-studied groups A_4 or S_4 , these groups, due to their complex structures, provide nonrotatable special complex phases in the vacua, giving rise to geometrical CP violation [42–45].

The layout of the paper is as follows. In Sec. II we briefly review the potentials invariant under the respective symmetries and their minima. In Sec. III we classify the distinct domain walls that can form between the degenerate minima, and calculate the tensions for specific benchmark points in parameter space. We conclude in Sec. IV.

II. $\Sigma(36 \times 3)$, $\Delta(54)$ AND $\Delta(27)$ POTENTIALS

A. Generators

It will be convenient in our analysis to consider that $\Delta(27)$ (with 27 elements) is a subgroup of $\Delta(54)$ (with 54 elements), which is a subgroup of $\Sigma(36 \times 3)$ (with 108 elements).

Following the notation in [17], with $\omega = e^{i2\pi/3}$, we take the generators of $\Sigma(36 \times 3)$ for a triplet irreducible representation,

$$\begin{aligned}
a &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\
d &= \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \quad (2)
\end{aligned}$$

For $\Delta(54)$ we consider similarly the generators,

$$\begin{aligned}
a &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\
d^2 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (3)
\end{aligned}$$

It is the generator d that differentiates between the groups, with $\Delta(54)$ only having the even power d^2 , and $\Delta(27)$ having just a and b as generators.

The renormalizable scalar potential for a triplet of these groups is similar. When imposing $\Delta(27)$ invariance, the potential is accidentally invariant under generator d^2 , and thus the potential is the same as in the $\Delta(54)$ case. Throughout the paper we generally refer only to $\Delta(54)$, noting here that everything applies also to the $\Delta(27)$ case.

B. CP symmetries

The scalar potential invariant under $\Sigma(36 \times 3)$ is known to be invariant under CP , whereas the most general potential invariant under $\Delta(54)$ is not. As noted in [46], there are a few general CP transformations that do not enlarge the $\Delta(27)$ group. We list them here for convenience.

$$\begin{aligned}
S_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
S_2 &= \text{diag}(1, 1, \omega), \quad S_3 = \text{diag}(1, 1, \omega^2), \\
S_4 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad S_5 = S_4 S_1, \\
S_6 &= \frac{-i\omega}{\sqrt{3}} \begin{pmatrix} \omega^2 & 1 & 1 \\ 1 & 1 & \omega^2 \\ 1 & \omega^2 & 1 \end{pmatrix}, \quad S_7 = \frac{i\omega^2}{\sqrt{3}} \begin{pmatrix} \omega & 1 & 1 \\ 1 & 1 & \omega \\ 1 & \omega & 1 \end{pmatrix}, \\
S_8 &= \begin{pmatrix} \omega & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_9 = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
S_{10} &= S_6 S_1, \quad S_{11} = S_7 S_1. \quad (4)
\end{aligned}$$

As mentioned above, the $\Delta(27)$ and $\Delta(54)$ renormalizable potential for one triplet are the same. Here we note further that CP symmetries associated with S_0 and with S_1 act in the same way in the potential (as the generator d^2 being a symmetry of the potential). The same applies for the CP symmetry pairs S_4/S_5 , S_6/S_{10} and S_7/S_{11} .

C. Potentials

Potentials of triplets of non-Abelian symmetries have simple quadratic terms [47], which we write as,

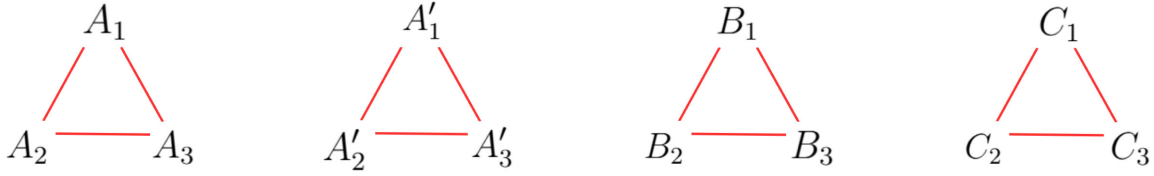
$$V = -m^2(\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3) + V_4. \quad (5)$$

The quartic terms depend on the specific symmetry.

In particular, the scalar potential of the 3HDM invariant under $\Sigma(36 \times 3)$ is,

$$\begin{aligned}
V &= -m^2[\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3] + \lambda_1[\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3]^2 \\
&\quad - \lambda_2[|\phi_1^\dagger \phi_2|^2 + |\phi_2^\dagger \phi_3|^2 + |\phi_3^\dagger \phi_1|^2 - (\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) \\
&\quad - (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) - (\phi_3^\dagger \phi_3)(\phi_1^\dagger \phi_1)] + \lambda_3[|\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_3|^2 \\
&\quad + |\phi_2^\dagger \phi_3 - \phi_3^\dagger \phi_1|^2 + |\phi_3^\dagger \phi_1 - \phi_1^\dagger \phi_2|^2]. \quad (6)
\end{aligned}$$

The most general scalar potential of the 3HDM invariant under $\Delta(54)$ is the following:

FIG. 1. Scheme of the DWs in the different alignments for the $\Delta(54)$ without CP symmetries.

$$\begin{aligned}
 V = & -m^2[\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3] + \lambda_1[\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3]^2 \\
 & - \lambda_2[|\phi_1^\dagger\phi_2|^2 + |\phi_2^\dagger\phi_3|^2 + |\phi_3^\dagger\phi_1|^2 - (\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) \\
 & - (\phi_2^\dagger\phi_2)(\phi_3^\dagger\phi_3) - (\phi_3^\dagger\phi_3)(\phi_1^\dagger\phi_1)] + \lambda_3(|\phi_1^\dagger\phi_2 - \phi_2^\dagger\phi_3|^2 \\
 & + |\phi_2^\dagger\phi_3 - \phi_3^\dagger\phi_1|^2 + |\phi_3^\dagger\phi_1 - \phi_1^\dagger\phi_2|^2) + \lambda_4((\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_3) \\
 & + (\phi_2^\dagger\phi_1)(\phi_3^\dagger\phi_1) + (\phi_3^\dagger\phi_2)(\phi_1^\dagger\phi_2)) + \text{H.c.} \quad (7)
 \end{aligned}$$

Note that both $\Sigma(36 \times 3)$ and $\Delta(54)$ include subgroup $Z_3 = \{I, \omega I, \omega^2 I\}$. Keeping in mind that the electroweak symmetry has already been satisfied, the real global symmetries to be guaranteed in Eqs. (6) and (7) are $\Sigma(36) \equiv \Sigma(36 \times 3)/Z_3 \simeq (Z_3 \times Z_3) \rtimes Z_4$ and $\Sigma(18) \equiv \Delta(54)/Z_3 \simeq (Z_3 \times Z_3) \rtimes Z_2$, respectively.

The effect of additionally imposing specific CP symmetries onto the $\Delta(54)$ potential is as follows. S_0 or S_1 force λ_4 to be real. S_2 or S_3 impose the following relation:

$$\text{Im}(\lambda_4) = \pm\sqrt{3}(\text{Re}(\lambda_4) - \lambda_3), \quad (8)$$

where signs $+$ and $-$ refer to the CP symmetries S_2 and S_3 , respectively.

The other CP symmetries, when imposed on the renormalizable potential, give rise to the $\Sigma(36 \times 3)$ potential or even to the $SU(3)$ potential.

Although the most general forms of the potentials apply for three $SU(2)$ doublets transforming as triplets of these discrete non-Abelian symmetries, in the following sections we present for simplicity results corresponding to the similar potentials for $SU(2)$ singlets, where the λ_2 is not present. The direction of the minima in the triplet direction are the same [47,48]. The VEVs and the corresponding domain wall configurations are electrically neutral.

D. Minima

The minima for the $\Delta(54)$ potential [47,48] are in the following four orbits.

$$\begin{aligned}
 \text{Alignment } A: & A_1 = (\omega, 1, 1), & A_2 = (1, \omega, 1), & A_3 = (1, 1, \omega). \\
 \text{Alignment } A': & A'_1 = (\omega^2, 1, 1), & A'_2 = (1, \omega^2, 1), & A'_3 = (1, 1, \omega^2). \\
 \text{Alignment } B: & B_1 = (1, 0, 0), & B_2 = (0, 1, 0), & B_3 = (0, 0, 1). \\
 \text{Alignment } C: & C_1 = (1, 1, 1), & C_2 = (1, \omega, \omega^2), & C_3 = (1, \omega^2, \omega). \quad (9)
 \end{aligned}$$

In order to identify what is the residual flavor symmetry subgroup for each of the minima, it is relevant to consider that the potential is invariant under a global rephasing, and to recall that we are considering the breaking of $\Sigma(18) \equiv \Delta(54)/Z_3 \simeq (Z_3 \times Z_3) \rtimes Z_2$. We list explicitly two elements preserved by one representative for each orbit, namely for A_1, A'_1, B_1, C_1 . Those two elements will generate the residual subgroup. For the other minima in the

same orbit we have elements related by conjugation, and a conjugate residual subgroup. For A_1, A'_1, B_1, C_1 , we note that d^2 (up to the global rephasing by -1) is preserved. Additionally, A_1 preserves aba , A'_1 preserves a^2ba^2 , B_1 preserves a , and C_1 preserves b , in all four cases the subgroup generated is isomorphic to S_3 , but they are different S_3 subgroups of the original group. E.g., for explicit illustration, B_1 we have the identity $e, a, a^2, d^2, ad^2, d^2a$; for C_1 we have $e, b, b^2, d^2, bd^2, d^2b$. The isomorphism between two S_3 groups is explicit from this, and the generators of the respective subgroups do not commute. Further, within each orbit, the preserved subgroup of another alignment is necessarily a conjugate subgroup and isomorphic to S_3 . Explicitly, consider the B_2 minima, which preserves bab^2 and b^2d^2b , i.e., the S_3 subgroup generated is conjugate by b with respect to the subgroup identified for the B_1 minima.

TABLE I. Table of the DW tensions and parameter values in the different alignments for the $\Delta(54)$ without CP symmetries.

Alignment	λ_3	$\text{Re}(\lambda_4)$	$\text{Im}(\lambda_4)$	σ
A	0.20	0.50	-0.50	0.54
A'	0.20	0.50	0.50	0.54
B	1.00	0.50	0.50	0.41
C	1.00	-0.50	0.50	0.75

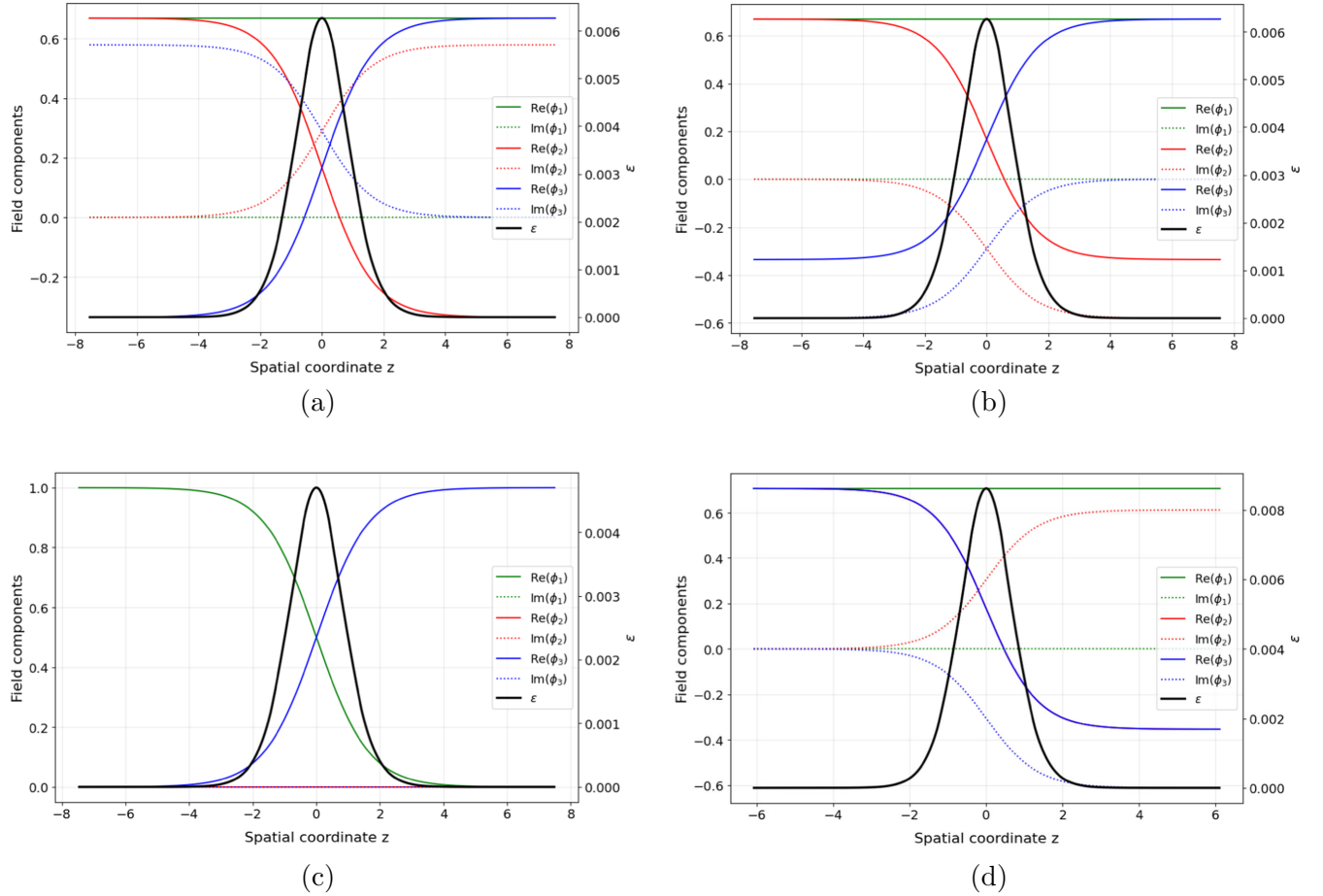


FIG. 2. Field profiles with real (solid line) and imaginary (dashed line) parts of the ϕ_i in color, and the energy density (black) as function of spatial coordinate z . Top left corresponds to orbit A, top right to orbit A', bottom left to orbit B, bottom right to orbit C.

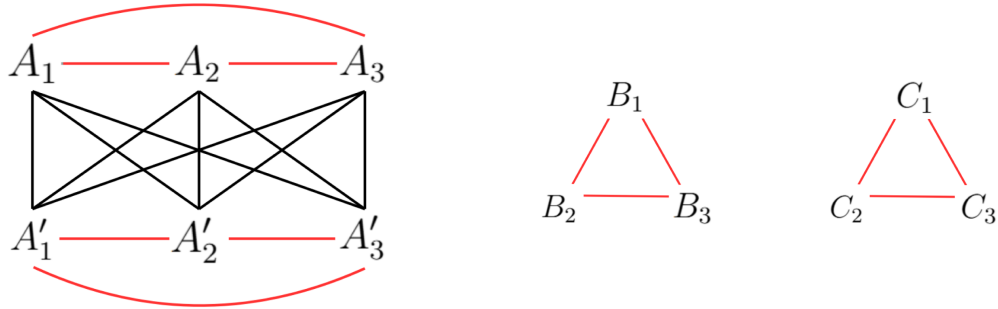


FIG. 3. Scheme of the DWs in the different alignments for the $\Delta(54)$ with trivial CP case.

Depending on the parameters of the potential, the minima are,

$$\begin{aligned}
 A: \quad & \text{Im}(\lambda_4) < 0, \quad \sqrt{3} \text{Im}(\lambda_4) < \text{Re}(\lambda_4) - 3\lambda_3, \quad \text{Im}(\lambda_4) < \sqrt{3}(\text{Re}(\lambda_4) - \lambda_3). \\
 A': \quad & \text{Im}(\lambda_4) > 0, \quad \sqrt{3} \text{Im}(\lambda_4) > -(\text{Re}(\lambda_4) - 3\lambda_3), \quad \text{Im}(\lambda_4) > -\sqrt{3}(\text{Re}(\lambda_4) - \lambda_3). \\
 B: \quad & \sqrt{3} \text{Im}(\lambda_4) > \text{Re}(\lambda_4) - 3\lambda_3, \quad \sqrt{3} \text{Im}(\lambda_4) < -(\text{Re}(\lambda_4) - 3\lambda_3), \quad \text{Re}(\lambda_4) > 0. \\
 C: \quad & \text{Im}(\lambda_4) > \sqrt{3}(\text{Re}(\lambda_4) - \lambda_3), \quad \text{Im}(\lambda_4) < -\sqrt{3}(\text{Re}(\lambda_4) - \lambda_3), \quad \text{Re}(\lambda_4) < 0.
 \end{aligned} \tag{10}$$

We obtained these constraints by computing the value of the potential at the minima for each alignment and by comparison with other alignments.

TABLE II. Table of the DW tensions and parameter values in the different alignments for the $\Delta(54)$ with trivial CP case.

Alignment	λ_3	$\text{Re}(\lambda_4)$	$\text{Im}(\lambda_4)$	σ
A or A'	-0.10	0.10	0	0.40
$A + A'$	-0.10	0.10	0	0.11
B	1.00	0.30	0	0.41
C	1.00	-0.30	0	0.57

We consider what happens to the minima when the symmetry is enlarged from $\Delta(54)$. For example, when we impose invariance under d (or, e.g., the general CP associated with S_4 , S_5), we get the $\Sigma(36 \times 3)$ potential. Through the action of d we can see that orbit B merges with orbit C . This potential is automatically also CP conserving including the trivial CP (S_0), therefore the orbit A merges with A' .

When the symmetry is enlarged just with a CP symmetry (without leading to invariance under d), we have two distinct cases in terms of orbits.

Imposing, e.g., trivial CP (S_0), A merges with A' (this happens also for S_1).

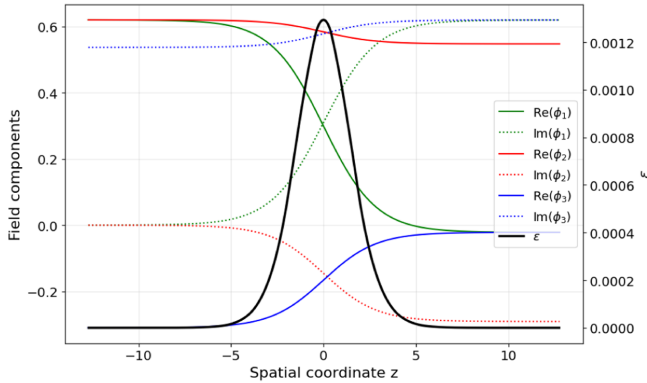


FIG. 4. Field profiles with real (solid line) and imaginary (dashed line) parts of the ϕ_i in color, and the energy density (black) as function of spatial coordinate z . The relative phase between A_3 and A'_2 is around 92° .

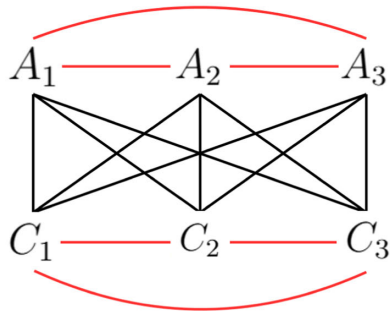


FIG. 5. Scheme of the DWs in the different alignments for the $\Delta(54)$ with S_2 CP symmetry.

Imposing either S_2 or S_3 , we have the restriction of Eq. (8). The sign $+$ makes the orbit A merge with orbit C , whereas the sign $-$ makes the alignments A' and C merge. So we find that S_2 (and S_8) merges A with C , whereas S_3 (and S_9) merges A' with C .

The other cases lead to invariance under d . The imposition of S_4 and S_5 symmetries leads to the $\Sigma(36 \times 3)$ scalar potential.

We find also that CP symmetries associated to S_6 , S_7 , S_{10} , S_{11} connect all orbits, which is understood when verifying they constrain the potential so much that we obtain accidentally the potential corresponding to the continuous $SU(3)$ symmetry.

III. DOMAIN WALLS

We perform an analysis in terms of the symmetry relations between the orbits (sets of minima that are related by symmetry). The minima in these orbits are necessarily degenerate, and during the evolution of the Universe, these potentials can give rise to DWs. Our aim here is to classify the topologically distinct cases. We do so both by considering the symmetries relating each minima, and by systematically going through each case (in the parametric region where those are the minima) and numerically estimating the tension (σ).

We created a program that numerically estimates the tension for three complex fields (six degrees of freedom). $\sigma = \int_{-\infty}^{+\infty} \epsilon dz$, where ϵ is the energy density. It numerically minimizes the potentials in Eqs. (6) and (7), and from there the DW profile can be computed numerically employing the method implemented in the CosmoTransitions [49] Python module.

Given that the potentials we are considering are invariant under a global phase, we necessarily had to take this into account, which we have done by varying the phase of one of the two minima and checking the minimum tension. It turns out that the relative phase is relevant to find the minimum tension for the particular cases where two orbits merge, as seen below.

The results now follow for each qualitatively distinct symmetry. Each $\Delta(54)$ case has a corresponding $\Delta(27)$ case as noted previously.

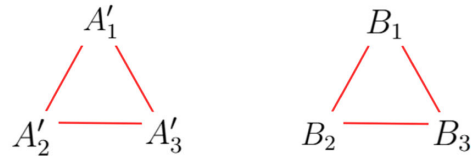


TABLE III. Table of the DW tensions and parameter values in the different alignments for the $\Delta(54)$ with S_2 CP symmetry.

Alignment	λ_3	$\text{Re}(\lambda_4)$	$\text{Im}(\lambda_4)$	σ
A or C	1.00	-0.50	-2.60	0.75
$A + C$	1.00	-0.50	-2.60	0.20
A'	0.10	0.50	0.69	0.88
B	1.00	0.30	-1.21	0.41

A. $\Delta(54)$

The results are summarized in Fig. 1, where the red lines denote that the minima in that orbit are connected by elements of $\Delta(54)$.

For each case there is only one type of DW. This is confirmed numerically. To illustrate the results, we present the DW tension (σ) in Table I, for a representative set of parameters for which the minima associated with each orbit are the true minima. For all the benchmark points, including those in the following subsections, the parameters were set to $m^2 = 1.0$ and $\lambda_1 = 1.0$.

We note that when reversing the phase of λ_4 so that either A or A' are the minima, but leaving the remaining

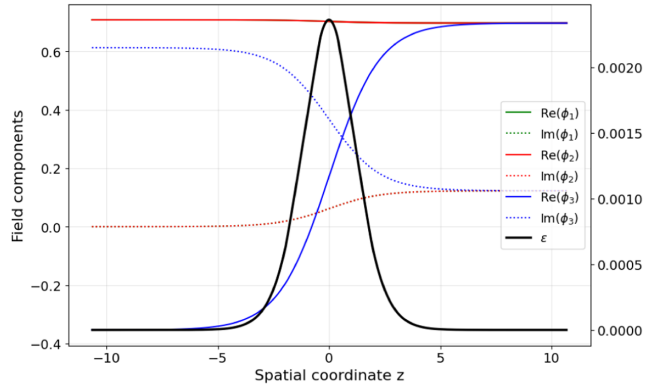


FIG. 6. Field profiles with real (solid line) and imaginary (dashed line) parts of the ϕ_i in color, and the energy density (black) as function of spatial coordinate z . The relative phase between A_3 and C_1 is around 10° .

parameters unchanged, the numerical value of the tension matches between these two cases, as seen in Table I.

We display also here, for each of the alignments, the field profiles in terms of spatial coordinate z from $-\infty$ to $+\infty$, as well as the integrand ε . Figs. 2(a)–2(d) correspond, respectively, to A , A' , B , and C . As expected from the discussion above, due to the parameters chosen for A and A' , Figs. 2(a) and 2(b) have a visible correspondence.

B. $\Delta(54)$ with trivial CP S_0

The S_0 CP symmetry, or trivial CP , imposes that λ_4 is real. Orbits A and A' merge. The results are summarized in Fig. 3.

In contrast with the general case, we find that there are two topologically distinct DWs when the minima are A and A' : DWs between two A_i minima are equivalent to DWs between two A'_i minima, shown with red lines denoting their connection through elements of $\Delta(54)$. But the DWs between an A_i and an A'_j minima are distinct, related by a CP symmetry, which we denote through black lines. When the minima are the other orbits, there is only one type of DW, as in the general case.

Table II has the numerically estimated DW tensions for representative benchmark points.

We display in Fig. 4 the field profiles and energy density for the CP DW between alignments from A and A' . For the other cases, the profiles are similar to those of Sec. III A.

C. $\Delta(54)$ with S_2 CP symmetry

The S_2 CP symmetry imposes Eq. (8) with the $+$ sign. In this case the alignments A and C merge. The results are summarized in Fig. 5.

In contrast with the general case, we find that there are two topologically distinct DWs when the minima are A and C , with the DWs between an A_i and a C_j minima, which are related by a CP symmetry (black lines), being distinct. Table III presents the numerically estimated DW tensions.

We display in Fig. 6 the field profiles and energy density for the CP DW between alignments from A and C . For the other cases, the profiles are similar to those of Sec. III A.

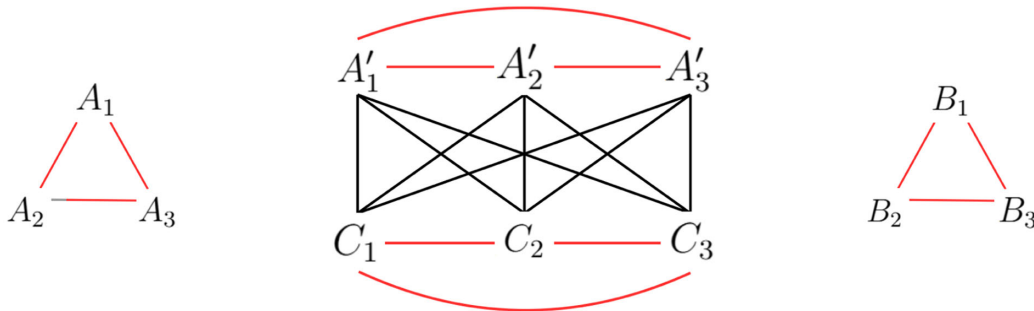


FIG. 7. Scheme of the DWs in the different alignments for the $\Delta(54)$ with S_3 CP symmetry.

TABLE IV. Table of the DW tensions and parameter values in the different alignments for the $\Delta(54)$ with S_3 CP symmetry.

Alignment	λ_3	$\text{Re}(\lambda_4)$	$\text{Im}(\lambda_4)$	σ
A	0.10	0.50	-0.69	0.88
A' or C	1.00	-0.50	2.60	0.75
$A' + C$	1.00	-0.50	2.60	0.20
B	1.00	0.30	1.21	0.41

D. $\Delta(54)$ with S_3 CP symmetry

The S_3 CP symmetry imposes Eq. (8) with the $-$ sign. In this case the alignments A' and C merge. The results are summarized in Fig. 7.

In contrast with the general case, and similarly to the S_2 case, we find that there are two topologically distinct DWs when the minima are A' and C , with the DWs between an A'_i and a C_j minima, which are related by a CP symmetry (black lines) being distinct. Table IV presents the numerically estimated DW tensions.

We had found in Table I the same tensions for A and A' , as we had selecting corresponding points in parameter space, namely by reversing only the phase of λ_4 . There is a

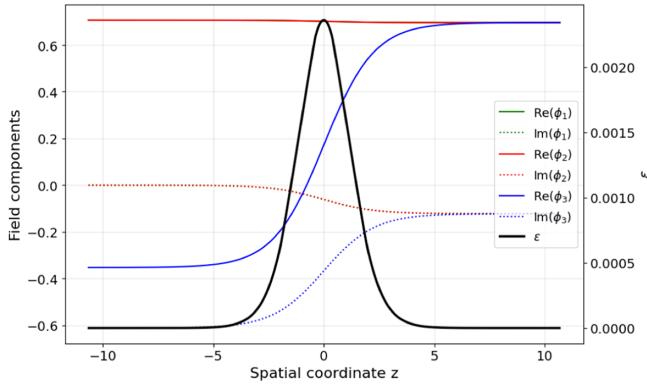


FIG. 8. Field profiles with real (solid line) and imaginary (dashed line) parts of the ϕ_i in color, and the energy density (black) as function of spatial coordinate z . The relative phase between A'_3 and C_1 is around 350° .

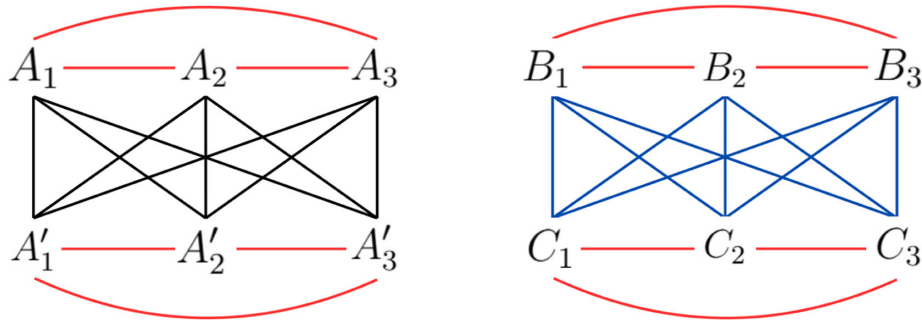


FIG. 9. Scheme of the DWs in the different alignments for the $\Sigma(36 \times 3)$ case.

similar correspondence when comparing the tensions for the CP symmetries S_2 and S_3 . For this reason, Tables III and IV have corresponding parameters. The resulting tension for the alignment B is the same in both tables, and the tension for the unmerged orbit A' of S_2 is the same as the tension for the unmerged orbit A . The tension of the merged orbits for S_2 when connected by symmetries that are CP symmetries ($A + C$) or not (A or C) are, respectively, the same as the tensions of the merged orbits for S_3 when connected by symmetries that are CP symmetries ($A' + C$) or not (A' or C).

We display in Fig. 8 the field profiles and energy density for the CP DW between alignments from A' and C . We note (as expected from the discussion above) the similarity with the results obtained for the CP DW between alignments from A and C .

E. $\Sigma(36 \times 3)$

In the $\Sigma(36 \times 3)$ case, the potential is invariant under the d generator and under CP symmetries, including the trivial CP symmetry S_0 . The orbit A merges with A' (by action of a CP symmetry) and the orbit B merges with orbit C (by action of d). The results are summarized in Fig. 9.

As expected, the scheme for alignments A and A' is the same as those in Fig. 4: the $\Sigma(36 \times 3)$ potential is a specific case of the $\Delta(54)$ potential (with $\lambda_4 = 0$).

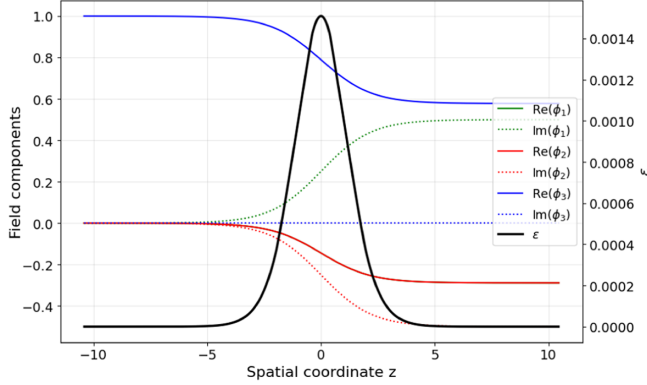
A qualitatively different result appears when the minima are B and C , where we find two distinct types of DWs. DWs between two B_i minima are equivalent to DWs between two C_i minima, related by elements of $\Delta(54)$ (red lines). DWs between a B_i and a C_j minima are distinct, and are related by d [or another element of $\Sigma(36 \times 3)$ which is not part of $\Delta(54)$], denoted by blue lines.

Table V has the numerically estimated DW tensions for representative benchmark points.

We display in Fig. 10 the field profiles and energy density for the DW between alignments from B and C . For the DW between alignments from A and A' , the profiles are similar to those found for trivial CP S_0 , as expected. For the other cases, the profiles are similar to those discussed in Secs. III A and III B.

TABLE V. Table of the DW tensions and parameter values in the different alignments for the $\Sigma(36 \times 3)$ case.

Alignment	λ_3	$\text{Re}(\lambda_4)$	$\text{Im}(\lambda_4)$	σ
A or A'	-0.10	0	0	0.38
$A + A'$	-0.10	0	0	0.11
B or C	1.00	0	0	0.41
$B + C$	1.00	0	0	0.13

FIG. 10. Field profiles with real (solid line) and imaginary (dashed line) parts of the ϕ_i in color, and the energy density (black) as function of spatial coordinate z . The relative phase between B_3 and C_2 is around 120° .

IV. CONCLUSION

We review the scalar potentials invariant under symmetries $\Sigma(36 \times 3)$ and its subgroup $\Delta(54)$. The minima of these potentials falls into orbits: sets of degenerate minima, related to each other by symmetries of the potential.

We analyze the possible domain walls that could form between degenerate minima within each orbit, considering which type of symmetry relates each pair of minima. We also calculate numerically the domain wall tensions, which we provide for benchmark points in the parameter space of the potential, where the minima belonging to each orbit are the correct minima.

The classification according to the symmetry relations between minima and the numerical calculation agrees. In the most general case we have $\Delta(54)$ without any CP symmetry. There are four orbits we label A, A', B, C , each with their respective domain wall type and tension.

Adding different CP symmetries to $\Delta(54)$ will merge two of the four general orbits. For example, the trivial CP and related CP symmetries merge A and A' into a single orbit. Other CP symmetries merge A with C ; and yet other CP symmetries merge A' with C . In these cases, we find that the tensions of the previously separate orbits become the same, but there are distinct tensions for domain walls connecting the (now degenerate) minima of what were previously separate orbits. Taking the trivial CP case: the tension between A_i and A_j becomes equal to the tension between A'_i and A'_j ; there is a different tension between A_i and A'_j .

In the most symmetric case, we have $\Sigma(36 \times 3)$, and the potential is automatically invariant under CP symmetries such as the trivial CP , which merges A and A' , leading to the same structure as above. Furthermore, the elements of $\Sigma(36 \times 3)$ that are not elements of $\Delta(54)$ merge the orbits B and C . We find that the tension between B_i and B_j becomes equal to the tension between C_i and C_j , and that there is a different tension between B_i and C_j .

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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