

Equivalence principle violation in metric-affine gravity and finite-temperature effects

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(Received 7 January 2026; accepted 1 June 2026; published 6 July 2026)

Possible violations of the equivalence principle are investigated within the framework of metric-affine gravity, and their connections to finite-temperature effects are highlighted. Thermal corrections to particle dynamics, originally derived in a quantum-field-theory setting, can be evaluated in a purely Riemannian framework and lead to a shift in the gravitational-to-inertial mass ratio. We show that the ensuing departure from universality of free fall can be also formulated in metric-affine gravity, where the presence of the nonmetricity tensor modifies the Newtonian law in a way that closely parallels the finite-temperature scenario. Furthermore, we introduce a generalized Fermi-Walker derivative adapted to non-Riemannian contexts, which naturally reveals that no orthonormal tetrad can be propagated along an observer worldline. Although metric-affine gravity admits a pointwise realization of the Einstein equivalence principle in its gauge-theoretic, elementary-matter form, the new operator offers a direct geometric signature that this principle, in its modern formulation, is not retained, in general. Potential tests of the analyzed effects are also discussed.

DOI: [10.1103/7qp5-9kzk](https://doi.org/10.1103/7qp5-9kzk)

I. INTRODUCTION

The cornerstones of Einstein theory are the principles of relativity and general covariance, together with the equivalence principle (EP) [1]. The first asserts that *all* observers have equal status in describing physical phenomena, a condition closely connected with the second principle, which states that the fundamental laws of physics can be formulated in tensor form on a spacetime manifold $\mathcal{M}$; the EP, in its Einstein formulation, affirms that gravitational effects can be locally eliminated in a freely falling frame. Together, these tenets led Einstein to formulate general relativity (GR) as a geometric theory on a four-dimensional

(pseudo-)Riemannian manifold $\mathcal{M}$, endowed with a metric tensor $g_{\mu\nu}$ and the symmetric, metric-compatible Levi-Civita connection.

Historically, the EP appeared in progressively refined versions, reflecting the deepening of our understanding of gravitation [2]. The Newtonian EP already recognized the universality of free fall through the equivalence of inertial and gravitational mass. The weak EP sharpens this by stating that all test bodies, regardless of their composition or internal structure, fall with the same acceleration in a given gravitational field. The *modern formulation* of the Einstein EP further requires the local Lorentz invariance and the local position invariance, ensuring that nongravitational experiments yield the same outcomes in any freely falling frame. Finally, the strong EP extends these statements to include also self-gravitating bodies.

By adopting the post-Newtonian approximation, GR compliance with the strong EP has been explicitly confirmed up to the second post-Newtonian order [3] [i.e., $O(c^{-4})$ beyond the Newtonian theory]. However, it has been recently claimed that possible departures could emerge starting from the third post-Newtonian level, where inner-structure-dependent effects might potentially appear in the

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equations of motion (and the gravitational waveforms) of nonspinning compact bodies [4].

Empirical probes of the EP span laboratory and space experiments [5–10]. Violations of the weak EP are parametrized by the Eötvös ratio η , which quantifies fractional acceleration differences between test bodies of different composition. This parameter is stringently constrained by modern torsion-balance experiments [11] (e.g., Eötvös, Dicke, Braginsky, and the Eöt-Wash group), as well as by free-fall and matter-wave interferometry [12–14] and by space-based missions such as MICROSCOPE [15,16]. Additional bounds are expected from forthcoming projects, including STEP [17], Galileo Galilei [18], STE-QUEST [19], and ACES [20]. Moreover, tests of the weak and strong EP are carried out through lunar laser-ranging techniques [21–23]. Although these efforts underpin metric theories of gravity, in general, they still leave room for potential new physics beyond GR.

Deviations from the EP are predicted in several theoretical frameworks, motivated by cosmology, quantum gravity, and unification schemes [5,6,14]. Nonmetric theories introduce additional fields that couple differently to matter, leading to composition-dependent accelerations and violations of the weak EP [24,25]. Scalar-tensor theories, such as Brans-Dicke and its generalizations [26,27], predict variations in the effective gravitational constant and strong-field phenomena like the Nordtvedt effect or spontaneous scalarization in neutron stars, signaling a breakdown of the strong EP [28–30]. Vector-tensor and tensor-vector-scalar paradigms introduce preferred-frame or preferred-location effects, contravening local Lorentz invariance (i.e., the condition that experimental outcomes are independent of the velocity of particle’s freely falling frame) and local position invariance (i.e., the requirement that empirical results are unaffected by the test body location) [31–33]. Models with changing fundamental constants [34–36], including varying-speed-of-light scenarios [37], also challenge local position invariance by allowing spacetime-dependent physical constants. Furthermore, the EP can be spoiled in massive gravity and higher-order curvature theories [38–40], as well as in string-inspired effective frameworks [41].

EP breakdowns are also expected at quantum level [42]. More generally, the Einstein EP is inconsistent with scenarios in which either matter or gravity exhibit a quantum behavior, and some proposals have been advanced to extend this tenet to the quantum domain (see, e.g., Refs. [2,42–45], and references therein). Moreover, violations of the weak EP occur in effective-field-theory models of gravity, where particles with different spins experience distinct bending angles when propagating in a quantum-corrected background geometry [46–49].

A context that illustrates how quantum phenomena can lead to departures from the EP is represented by finite-temperature quantum field theory (QFT), which generalizes

standard QFT to systems in thermal equilibrium at nonzero temperature [50–53]. This framework permits the simultaneous investigation of quantum and thermal fluctuations and naturally predicts that free fall can become nonuniversal due to a temperature-induced shift between the inertial and gravitational masses of a particle [54,55]. It has been proved that this result can be recovered via an approach based on a Riemannian geometry where local Lorentz symmetry is broken by the presence of a heat bath. This analysis was first carried out in a GR setting [56] and later extended to curvature-based modified metric gravity models [57].

In this paper, we show that potential EP violations related to temperature effects can be naturally explained within the wider context of metric-affine geometry (MAG). MAG provides a gauge-theoretic formulation of the gravitational interaction analogous in spirit to Yang-Mills theories, with the non-Abelian unitary group of internal symmetries replaced by a spacetime symmetry group, i.e., the general affine group. In this framework, the metric-compatibility constraint is relaxed, and the affine connection and the metric are treated as independent variables. Consequently, the connection acquires extra components beyond the Levi-Civita part that lead to a richer geometric domain where geodesics and autoparallels no longer coincide. As we will see, these features make MAG a natural arena for exploring EP departures in the dynamics of microstructured test bodies. Moreover, we complete our analysis by introducing a generalized Fermi-Walker (FW) transport rule adapted to MAG, which provides a purely geometric demonstration of the Einstein EP breakdown in full generality.

Specifically, MAG retains the (field-theoretic) pointwise, elementary-matter version of the Einstein EP [58,59], whereby, at a given spacetime point, the non-gravitational laws for minimally coupled matter can be reduced to their special-relativistic form in a suitable local frame. By contrast, what is not, in general, valid is the modern formulation of the Einstein EP, which relies on the weak EP and the existence of local Lorentz frames extended along worldlines and, hence, involves additional assumptions that are not purely pointwise but probe the behavior of matter and frames beyond a single spacetime event. This situation stems from the fact that nonmetricity and its (nontrivial hypermomentum) couplings to matter obstruct the realization of local Lorentz frames beyond a single spacetime point, so that orthonormal tetrads cannot be preserved along worldlines and the usual local inertial-frame construction is spoiled. Interestingly, this picture aligns with recent works suggesting that possible departures from local Lorentz invariance are naturally captured in MAG by a nonvanishing nonmetricity, as opposed to the nongeometric formulations based on nondynamical Lorentz-violating tensor fields commonly employed in the literature [60,61].

The paper is organized as follows. After reviewing the key concepts of MAG, including its gauge-theoretic foundations, in Sec. II, in Sec. III, we set out gravity at finite temperature from both quantum and GR viewpoints and describe the ensuing EP departures. In Sec. IV, we demonstrate that these EP violations inherently occur in MAG and outline their potential experimental probes. Then, in Sec. V, we introduce our novel extended FW derivative, and lastly, in Sec. VI, we draw our conclusions.

Notations and conventions. We use units where $\hbar$, G , c , and the Boltzmann constant k_B are set equal to one. The metric signature is $(-+++)$. Round (respectively, square) brackets around a pair of indices stand for the usual symmetrization (respectively, antisymmetrization) procedure, i.e., $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$ [respectively, $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$].

II. METRIC-AFFINE GEOMETRY

This section has a twofold purpose. On the one hand, it provides an overview of the fundamental aspects of MAG, and, on the other, it allows us to set our conventions.

MAG involves a transition from the standard Riemannian structure of GR to a non-Riemannian scenario, where additional geometric degrees of freedom arise by allowing the connection to possess both torsion and nonmetricity. Such features are considered in Sec. II A, while the kinematical implications of this richer geometry are examined in Sec. II B. Since metric-affine gravity represents a gauge field theory *par excellence*, we conclude the section by outlining this framework in Sec. II C.

A. Torsion and nonmetricity

Riemannian geometry represents the mathematical arena of GR. In this context, the affine structure is defined by the Levi-Civita connection

$$\overset{\circ}{\Gamma}{}^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}), \quad (1)$$

which is completely determined by the metric tensor $g_{\mu\nu}$. Henceforth, Greek indices $\mu, \nu, \dots = 0, 1, 2, 3$ stand for holonomic spacetime coordinate indices, and quantities framed in Riemannian geometry are denoted with an overcircle.

In the broader MAG context, the connection $\Gamma^\alpha_{\mu\nu}$ attains the general form [62]

$$\Gamma^\alpha_{\mu\nu} = \overset{\circ}{\Gamma}{}^\alpha_{\mu\nu} + M^\alpha_{\mu\nu}, \quad (2)$$

where the non-Riemannian piece $M^\alpha_{\mu\nu}$, referred to as distortion tensor, satisfies

$$\begin{aligned} M_{\alpha[\mu\nu]} &= -\frac{1}{2}T_{\alpha\mu\nu}, \\ M_{(\alpha\mu)\nu} &= -\frac{1}{2}Q_{\nu\alpha\mu}, \end{aligned} \quad (3)$$

with

$$T^\alpha_{\mu\nu} := 2\Gamma^\alpha_{[\mu\nu]}, \quad (4)$$

$$Q_{\lambda\mu\nu} := \nabla_\lambda g_{\mu\nu} \quad (5)$$

the torsion and the nonmetricity tensors, respectively. Geometrically, $T^\alpha_{\mu\nu}$ measures the failure of the closure of the parallelogram made up of small displacement vectors and their parallel transports [63], while $Q_{\lambda\mu\nu}$ quantifies the lack of covariant conservation of the metric [64].

The definition of $M^\alpha_{\mu\nu}$ in Eq. (2) is particularly useful for two reasons. First, it allows to separate the covariant derivative into Riemannian and non-Riemannian parts:

$$\begin{aligned} \nabla_\lambda X^\alpha &= \partial_\lambda X^\alpha + \Gamma^\alpha_{\rho\lambda} X^\rho \\ &= \partial_\lambda X^\alpha + \overset{\circ}{\Gamma}{}^\alpha_{\rho\lambda} X^\rho + M^\alpha_{\rho\lambda} X^\rho \\ &=: \overset{\circ}{\nabla}_\lambda X^\alpha + M^\alpha_{\rho\lambda} X^\rho; \end{aligned} \quad (6)$$

second, the Riemann curvature tensor

$$R^\alpha_{\beta\mu\nu} := \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\beta} \quad (7)$$

can be decomposed in the following way [64]:

$$\begin{aligned} R^\alpha_{\beta\mu\nu} &= \overset{\circ}{R}{}^\alpha_{\beta\mu\nu} + \overset{\circ}{\nabla}_\mu M^\alpha_{\beta\nu} - \overset{\circ}{\nabla}_\nu M^\alpha_{\beta\mu} + M^\alpha_{\lambda\mu} M^\lambda_{\beta\nu} \\ &\quad - M^\alpha_{\lambda\nu} M^\lambda_{\beta\mu}, \end{aligned} \quad (8)$$

the only symmetry being, in general, the antisymmetry in the last two indices [64–68].

B. Kinematics

Owing to its rich structure, important implications arise at the kinematical level in MAG. Unlike in Riemannian geometries, geodesics and autoparallels no longer coincide (see Sec. II B 1), and the latter generally exhibit an acceleration term, as we will discuss in Sec. II B 2.

For further details regarding the kinematical aspects of MAG, we refer the reader to Refs. [69–71].

1. Geodesic and autoparallel curves

In GR, there is no distinction between geodesic and autoparallel structures, although they have distinct conceptual and technical origins [59]. The geodesic is an extremal curve, and its formulation, thus, arises from a variational principle. On the other hand, the autoparallel is defined as the path over which a vector is transported parallel to itself according to the connection $\Gamma^\alpha_{\mu\nu}$ of the

manifold $\mathcal{M}$, thereby generalizing the straight line notion of flat spacetime.

Based on its defining property, the geodesic equation takes the form

$$\ddot{x}^\lambda + \overset{\circ}{\Gamma}^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0, \quad (9)$$

where the dot denotes from now on differentiation with respect to the affine parameter:

$$\dot{x}_\mu \equiv \frac{dx^\mu}{ds}, \quad (10)$$

and it is clear that only the Levi-Civita connection (1) is involved. In contrast, the autoparallel equation

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \quad (11)$$

depends on the full affine connection $\Gamma^\lambda_{\mu\nu}$. It follows from identity (2) that Eq. (11) can be expressed as

$$\ddot{x}^\lambda + \overset{\circ}{\Gamma}^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -M^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \quad (12)$$

which could be seen as an initial indication of a potential EP violation within the MAG framework. We will come back to this point in Sec. IV A.

2. The anomalous acceleration

The existence of a nonzero nonmetricity tensor (5) entails significant consequences for the spacetime structure, mainly because care must be taken when evaluating the covariant derivative of covariant or contravariant vectors [cf. Eq. (6)]. In general, the main effects introduced by $Q_{\lambda\mu\nu}$ are (i) lengths and angles are typically not preserved during the parallel transport of vectors and (ii) autoparallels exhibit an acceleration term, commonly referred to as anomalous acceleration.

To illustrate property (i), consider the four-velocity vector (10). Even along an autoparallel curve, where $\dot{x}^\lambda \nabla_\lambda \dot{x}^\mu = 0$, its norm is not conserved like in GR, as one has

$$\begin{aligned} \frac{D}{ds}(g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) &= 2\dot{x}^\lambda (\nabla_\lambda \dot{x}^\mu) \dot{x}_\mu + \dot{x}^\lambda Q_{\lambda\mu\nu} \dot{x}^\mu \dot{x}^\nu \\ &= \dot{x}^\lambda Q_{\lambda\mu\nu} \dot{x}^\mu \dot{x}^\nu, \end{aligned} \quad (13)$$

where $\frac{D}{ds}(\cdot) = \dot{x}^\alpha \nabla_\alpha(\cdot)$ represents the covariant directional derivative along $\dot{x}^\alpha$. At this stage, it is thus convenient to parametrize the magnitude of $\dot{x}^\mu$ via [70]

$$\dot{x}^\mu \dot{x}_\mu = -\phi(x^\rho), \quad (14)$$

where ϕ is a positive-definite function of the spacetime point x^ρ .

The point (ii) concerns some subtleties in the expression of the four-acceleration. In GR, the commutativity between the metric and the covariant derivative implies that there is no distinction as to whether the four-acceleration definition involves the contravariant or covariant form of the four-velocity. In metric-affine gravity, however, this equivalence no longer holds, and it is possible to define two independent types of acceleration [69,70,72]: the proper acceleration

$$A^\mu := \dot{x}^\lambda \nabla_\lambda \dot{x}^\mu \quad (15)$$

and the anomalous acceleration

$$\tilde{a}_\mu := \dot{x}^\lambda \nabla_\lambda \dot{x}_\mu, \quad (16)$$

which differ owing to the nonmetricity. Specifically,

$$\begin{aligned} \tilde{a}_\mu &= \dot{x}^\lambda \nabla_\lambda (g_{\mu\nu} \dot{x}^\nu) = g_{\mu\nu} A^\nu + Q_{\lambda\mu\nu} \dot{x}^\lambda \dot{x}^\nu \\ &= A_\mu + Q_{\lambda\mu\nu} \dot{x}^\lambda \dot{x}^\nu, \end{aligned} \quad (17)$$

clearly showing that $\tilde{a}_\mu \neq g_{\mu\nu} A^\nu$. Therefore, we see that in MAG, autoparallels, i.e., curves for which $A^\mu = 0$, have, in general, a nonzero acceleration that is represented by the anomalous term (17).

An additional key difference with respect to Riemannian geometry is that both proper and anomalous accelerations are not orthogonal to the four-velocity, as it follows from the above formulas that

$$A^\mu \dot{x}_\mu = \frac{1}{2}(-\dot{\phi} - Q_{\lambda\mu\nu} \dot{x}^\lambda \dot{x}^\mu \dot{x}^\nu), \quad (18a)$$

$$\tilde{a}^\mu \dot{x}_\mu = \frac{1}{2}(-\dot{\phi} + Q_{\lambda\mu\nu} \dot{x}^\lambda \dot{x}^\mu \dot{x}^\nu), \quad (18b)$$

which, in turn, imply

$$(A^\mu + \tilde{a}^\mu) \dot{x}_\mu = -\dot{\phi}, \quad (19a)$$

$$(A^\mu - \tilde{a}^\mu) \dot{x}_\mu = -Q_{\lambda\mu\nu} \dot{x}^\lambda \dot{x}^\mu \dot{x}^\nu, \quad (19b)$$

where we have exploited Eq. (14). Bearing in mind Eq. (18a), one finds that the magnitude ϕ admits the following integral expression along an autoparallel curve C_A :

$$\phi(x^\rho) = - \int_{C_A} Q_{\lambda\mu\nu} \dot{x}^\lambda \dot{x}^\mu \dot{x}^\nu ds + k \quad (k \in \mathbb{R}), \quad (20)$$

which clearly shows that only in the Riemannian case, where $Q_{\lambda\mu\nu} = 0$, the norm of the four-velocity remains constant and can be consistently renormalized.

The concepts discussed in this section will be particularly useful for our analysis of Sec. V, where we will set

forth the generalized FW derivative and discuss the Einstein EP violation in MAG.

C. Metric-affine gauge theory of gravity

Metric-affine gravity arises as a gauge-theoretic formulation of gravitation, where spacetime geometry is described in terms of gauge potentials associated with local spacetime symmetries.

The gauge principle plays a central role in the modern description of fundamental interactions. Historically [73,74], inspired by the pioneering works of Weyl and later by the development of Yang-Mills theories for (internal) unitary symmetry groups, the gauging formalism was extended to (external) spacetime symmetries by Utiyama. The subsequent studies of Kibble and Sciama culminated in the formulation of Poincaré gauge gravity, where the spacetime connection is endowed with a nontrivial torsion but vanishing nonmetricity. The natural further step then consisted in extending the gauge pattern to accommodate a completely general affine connection [cf. Eqs. (2)–(5)]. This led to metric-affine gauge theory of gravity, which is obtained by promoting the full affine group $A(4, \mathbb{R})$ to a local gauge symmetry of spacetime [75–78]. Since $A(4, \mathbb{R})$ comprises both four-parameter translations and general linear transformations $GL(4, \mathbb{R})$, such extension of the gauge group beyond the Lorentz subgroup $SO(1, 3)$ can be interpreted physically as a possible departure from the local Lorentz invariance, which manifests in the fact that light cones cease to have the status of absolute spacetime structures [60,61]. In accordance with the gauge procedure, a nontrivial nonmetricity emerges, and the Riemann-Cartan geometry underlying Poincaré gauge gravity is extended to a MAG, where both the metric and the affine connection naturally become independent dynamical variables.

The main facets of the model are given in Sec. IIC 1, while the role of the EP and its relation with the gauge principle are discussed in Secs. IIC 2 and IIC 3.

1. The gauging procedure of the affine group

The key idea characterizing metric-affine gravity follows from the gauge-theoretic principle that the description of physics should be independent of the choice of reference frames used to express the dynamical variables of the theory.

The starting point then consists in requiring that physical matter fields be decomposed with respect to arbitrary affine frames, a construction that enables suitable $GL(4, \mathbb{R})$ representations to act on their components. An affine frame at an event $x \in \mathcal{M}$ is made up of a linear frame e_A of the local tangent space $T_x\mathcal{M}$ (capital Latin indices $A, B, \dots$ denote internal affine-frame components), together with an associated point p living in the affine tangent space $A_x\mathcal{M}$ [77]. In this framework, the *gauge principle*, thus, encodes the independence of physical results under changes of affine frames (e_A, p) , with affine transformations being

identified as gauge transformations relating equivalent frames [in more geometric terms, the affine bundle, defined as the collection of all affine tangent spaces $A_x\mathcal{M}$, is the principal bundle of the gauge procedure and its structure group is the affine group $A(4, \mathbb{R})$]. Enforcing this equivalence entails the introduction of a generalized affine connection $(\Gamma^{(L)A}_B, \Gamma^{(T)A})$, which maps infinitesimally neighboring affine tangent spaces. The $GL(4, \mathbb{R})$ -valued linear part $\Gamma^{(L)A}_B$ governs rotations and deformations of the basis vectors $e_A \in T_x\mathcal{M}$, while the translational $\mathbb{R}^4$ -valued piece $\Gamma^{(T)A}$ accounts for the shift of the reference point $p \in A_x\mathcal{M}$.

Unlike (internal) Yang-Mills theories, where unitary symmetry groups act in internal spaces, (external) gauge-gravity models require a *soldering* mechanism that relates the (internal) affine-geometric gauge structure to the (external) spacetime manifold [79]. This is achieved by identifying the affine tangent space $A_x\mathcal{M}$ with the physical tangent space $T_x\mathcal{M}$, thereby permitting the conversion of affine-frame indices $A, B, \dots$ into spacetime ones. As a result, the translational gauge invariance in the affine space is no longer realized as an independent (internal) symmetry, but a local one-to-one correspondence is established with spacetime diffeomorphisms, which can then be interpreted as locally gauged translations. Another effect of soldering is that the translational component of the generalized affine connection can be used to construct the translation-invariant anholonomic coframe ϑ^a , where $a, b, \dots = \hat{0}, \hat{1}, \hat{2}, \hat{3}$ are hereafter anholonomic spacetime tangent-frame indices. This object is related to the coordinate basis one-form dx^μ via $\vartheta^a = e_\mu^a dx^\mu$ and is commonly called soldering one-form [79], the dual frame being denoted by e_a . Moreover, the linear part of the affine connection becomes the $GL(4, \mathbb{R})$ -valued connection Γ^a_b and underlies the changes of frames under general linear transformations. Together, the pair $(\vartheta^a, \Gamma^a_b)$ constitutes the true gauge potentials of metric-affine gravity, endowed with a genuine dynamical role.

A separate procedure is necessary for the metric, since it is not determined within the affine gauging scheme but emerges naturally only after restricting the gauge symmetry to an affine-orthogonal subgroup [77].¹ This means that in MAG the existence of a dynamical and independent local metric structure can be only *postulated*. Assume that the holonomic components $g_{\mu\nu}$ with respect to some coordinate basis have been specified; these can then be expressed in terms of the coframe components e_μ^a via the following approach. At a given point $x \in \mathcal{M}$ and in a suitable gauge (referred to as orthonormal gauge [75]), which we denote by a star symbol, an orthonormal frame $\bar{e}_a$ can be selected such that

¹It has been proposed that the metric has a Goldstone nature in MAG; see Ref. [80] for further details.

$$g|_x = g_{\mu\nu} dx^\mu \otimes dx^\nu \stackrel{\star}{=} \eta_{ab} \vartheta^a \otimes \vartheta^b, \quad (21a)$$

which implies

$$g_{\mu\nu}|_x \stackrel{\star}{=} \bar{e}_\mu^a \bar{e}_\nu^b \eta_{ab}, \quad (21b)$$

with $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$. Therefore, the coframe ϑ^a fixes the metric at a single spacetime point and in a specific gauge, where it can be chosen orthonormal [75,77].

The above result stems from simple linear-algebra arguments. Indeed, rather than Eq. (21b), the most general formula providing the transition between the anholonomic and holonomic formulation is

$$g_{\mu\nu} = e_\mu^a e_\nu^b g_{ab}, \quad (22)$$

with $g_{ab} = g(e_a, e_b)$ representing, at an arbitrary point $x \in \mathcal{M}$, a symmetric and nondegenerate matrix. It is then always possible to perform a suitable local $GL(4, \mathbb{R})$ deformation to a frame where the metric is diagonal with eigenvalues normalized to ± 1 . In this new orthonormal basis, one can, thus, write $g(\bar{e}_a, \bar{e}_b) =: \eta_{ab}$, and the matrices $\bar{\Lambda}^a_b(x) \in GL(4, \mathbb{R})$ that preserve orthonormality belong to the Lorentz subgroup $SO(1, 3)$ of $GL(4, \mathbb{R})$. Hence, orthonormality is a pointwise gauge choice in MAG, and, due to the presence of nonmetricity, it is not preserved under parallel transport. This situation differs from models in which the local gauge group is reduced to $SO(1, 3)$, where ϑ^a becomes orthonormal in every allowed gauge at each spacetime point. As we will see in Sec. II C 2, relations (21) are crucial for the EP realization, as they represent the metric-affine analog of the pointwise orthonormalization underlying the construction of local inertial frames.

Once the concept of gauge potentials is enlarged to encompass, apart from ϑ^a and Γ^a_b , also the metric g_{ab} , the geometry of the spacetime is characterized accordingly by three independent field strengths: the torsion two-form $T^a = d\vartheta^a + \Gamma^a_b \wedge \vartheta^b$, the curvature two-form $R^a_b = d\Gamma^a_b - \Gamma^a_c \wedge \Gamma^c_b$, and the nonmetricity one-form $Q_{ab} = dg_{ab} - (\Gamma^c_a g_{cb} + \Gamma^b_c g_{ca})$ [their holonomic counterparts are given in Eqs. (4), (5), and (7), respectively]. The field strengths enter the $A(4, \mathbb{R})$ -gauge-invariant gravitational Lagrangian and are sourced by the canonical energy-momentum, hypermomentum, and symmetric (metric) energy-momentum tensors, which represent the matter currents of metric-affine gravity. These satisfy a set of generalized covariant Noether identities stemming from the invariance of the matter Lagrangian under local diffeomorphisms (i.e., the translation symmetry after soldering) and local $GL(4, \mathbb{R})$ transformations.

For further details, we refer the reader to Refs. [75–78,81,82], while some aspects regarding the dynamics of hypermomentum-charged test bodies will be addressed more closely in Sec. IV A.

2. The equivalence principle

In this section, we explain how the Einstein EP, in its gauge-theoretic formulation, is realized in MAG.

The starting point is that, for every event $x \in \mathcal{M}$, there exists a suitable anholonomic system where

$$g_{ab}|_x \stackrel{\star}{=} \eta_{ab}, \quad (23a)$$

$$\Gamma^a_b|_x \stackrel{\star}{=} 0. \quad (23b)$$

This is a purely pointwise gauge statement, following from the local $GL(4, \mathbb{R})$ freedom of metric-affine gravity. Indeed, under a local linear transformation, $e'_a = \Lambda^b_a e_b$ and $\vartheta'^a = (\Lambda^{-1})^a_b \vartheta^b$, so that the transformed metric at $x \in \mathcal{M}$ depends only on $\Lambda(x)$ [75]. On the other hand, the connection one-form Γ^a_b behaves inhomogeneously, and its transformation law at x also involves $\partial_\mu \Lambda(x)$. This can be seen by first writing locally $\Gamma = \Gamma_\mu dx^\mu$, where each Γ_μ is a 4×4 matrix [i.e., $(\Gamma_\mu)^a_b$]; then, under a local linear gauge transformation, one finds $\Gamma'_\mu = \Lambda^{-1} \Gamma_\mu \Lambda + \Lambda^{-1} \partial_\mu \Lambda$ [75]. Thus, to obtain Eq. (23), one first chooses $\Lambda(x)$ in such a way that $g_{ab}(x)$ is brought to Minkowski form, as we saw in Eq. (21), and then fixes $\partial_\mu \Lambda(x)$ so that the two terms in the transformed connection cancel at x , yielding $\Gamma'^a_b(x) \stackrel{\star}{=} 0$.

The simultaneous trivialization (23) of the metric and connection encodes the pointwise elementary-matter version of the Einstein EP, which asserts that locally the properties of special-relativistic matter in a noninertial frame cannot be distinguished from the properties of the same matter in a corresponding gravitational field [58,59]. Consequently, the local action of a gravitational field on elementary matter can be simulated in flat Minkowski spacetime by going over to a noninertial frame of reference, and, conversely, in curved spacetime there always exists the gauge freedom to locally eliminate the effects of a true gravitational field.

This gauge-theoretic approach to the EP reveals a close relationship with the minimal coupling principle. Indeed, the matter Lagrangian $\mathcal{L}_m$ minimally coupled to gravity involves g_{ab} and ϑ^a and depends on the connection Γ^a_b only via the gauge covariant derivative of matter fields. As a consequence, the EP entails that $\mathcal{L}_m$, along with the ensuing local form of the nongravitational field equations, matches at a given event its special-relativistic counterpart. However, the EP has a local (first-order) domain of applicability [59], and, hence, higher-order (derivative-dependent) structures, such as conservation laws involving the (canonical) energy-momentum tensor, are not constrained to take their special-relativistic form, as they probe nonlocal features of the fields. Hence, torsion and nonmetricity can still affect matter through derivative terms and couplings to intrinsic degrees of freedom, such as spin and hypermomentum (as we will discuss in Sec. IV A).

Accordingly, the EP does not imply that the field strengths vanish, and, in particular, $Q_{ab}(x) \neq dg_{ab}(x)$. Therefore, one cannot infer from conditions (23) the presence of a local (pseudo-)Euclidean structure associated with metric compatibility, consistently with the fact that the most general space with such a property is the Riemann-Cartan spacetime [58,59,83]. This is the reason why the Einstein EP, in its modern sense relying on the realizability of local Lorentz frames beyond a single point, is not valid, in general, in MAG, as we will prove in Sec. V.

3. The role of the gauge principle

From the analysis of the previous section, it is clear that in MAG the pointwise gauge version of the Einstein EP is retained and that it is implemented mathematically via the minimal coupling principle of matter to gravity [58,83]. This highlights a structural relation with the gauge principle introduced in Sec. II C 1 [73,74,76,77].

In fact, as pointed out before, the EP asserts the empirical fact that gravitational and inertial effects are locally indistinguishable. This motivates the introduction of local inertial frames where gravitation can be transformed away and, more generally, suggests a local description of spacetime geometry in terms of frames and connections.

From this standpoint, the gauge principle suggests a structurally analogous viewpoint, in which the choice of local frames, now extended to include spacetime frames, is likewise regarded as physically irrelevant. In this sense, spacetime frames can be treated on the same footing as internal frames in gauge theories. Metric-affine gravity implements this concept concretely by claiming the local indistinguishability of local affine reference frames (e_A, p) , in close analogy with Yang-Mills theory, where the corresponding indistinguishability concerns local bases in the representation space of the unitary group $SU(n)$. It then becomes natural to interpret changes of local frames as a local gauge symmetry.

Promoting such frame freedom to a local symmetry leads to the well-known gauge paradigm, namely, the requirement that invariance under a global symmetry persists under its local version. In metric-affine gravity, this tenet is applied to spacetime geometry itself and necessitates replacing ordinary derivatives by covariant ones via the introduction of gauge connections, which describe how local frames are related at neighboring points and precisely compensate for local frame changes [74]. This procedure corresponds to the minimal coupling prescription and is consistent with the Einstein EP expectation that nongravitational laws reduce locally to their special-relativistic form via a suitable gauge choice of *anholonomic* local frames, as encoded in identities (23).

III. GRAVITATION AT FINITE TEMPERATURE

In its standard formulation, GR does not account for temperature, although the interplay between gravitation and thermal phenomena plays a crucial role in a variety of contexts [84]. At finite temperature, matter no longer exists in vacuum but interacts with a heat bath, typically modeled as a background of photons or other relativistic particles. In this setup, EP violations are expected to arise, and this issue can be tackled via a formalism that permits describing quantum fields in thermal equilibrium and their coupling to gravity. The first step toward this objective is provided by finite-temperature QFT, which we briefly outline in Sec. III A. This lays the basis for the analysis of gravity in thermal environments, which we explore in Sec. III B.

A. Finite-temperature quantum field theory

Many physical phenomena of interest do not occur in vacuum but in settings characterized by a nonzero temperature T . For instance, in particle physics, this is the case in the description of the quark-gluon plasma state of matter [85]. Moreover, temperature fulfills a non-negligible role for phase transitions in models with spontaneously broken symmetries [86], in the early Universe evolution [87–89], in astrophysical compact objects such as black holes [90], and in many other fields (see Ref. [53] for further details).

The theoretical pattern to address these questions is represented by finite-temperature QFT, which extends conventional QFT concepts [91,92] to systems in thermal equilibrium at $T \neq 0$ [52,53,93]. Three principal formalisms are employed to study thermal quantum fields: the imaginary-time [94] and real-time approaches [95–98] (devised by Matsubara, and Schwinger and Keldysh, respectively) and the thermofield dynamics method [99] (introduced by Umezawa, Matsumoto, and Tachiki). For further details, we refer the reader to Refs. [50,51,100,101].

Finite-temperature QFT provides the necessary tools to compute thermal radiative corrections in particle propagators. In the next section, we discuss the gravitational implications of these effects.

B. Equivalence principle violation in finite-temperature gravity

At finite temperature, particle masses acquire additional contributions induced by interactions with the surrounding thermal bath. This phenomenon can be studied through two complementary approaches: one based on a purely quantum framework [54,55] (see Sec. III B 1) and another rooted in a classical relativistic setting [56] (see Sec. III B 2).

1. Quantum diagrammatic approach

In their seminal papers [54,55], Donoghue and collaborators exploited finite-temperature QFT tools to compute the corrections to the electron self-energy occurring in a low-temperature environment, where

$$T \ll m_0, \quad (24)$$

m_0 being the renormalized $T = 0$ electron mass. In this regime, the background temperature is mainly provided by a photon heat bath, with the effects of the electron-positron heat bath being exponentially suppressed by a factor proportional to $e^{-m_0/T}$. Under these conditions, all calculations can be consistently carried out in the low-velocity-weak-field limit.

Because of the presence of the nonzero temperature (24), the particles' Feynman propagator receives modifications involving the Bose-Einstein distribution function, which reflect the nontrivial structure of thermal vacuum. As a consequence, the inertial and gravitational masses m_i and m_g , respectively, of the electron renormalize differently, as one finds, after an elaborate Feynman-diagram analysis [54,55],

$$m_i = m_0 + \frac{\alpha\pi T^2}{3m_0}, \quad (25)$$

$$m_g = m_0 - \frac{\alpha\pi T^2}{3m_0}, \quad (26)$$

with α the fine-structure constant. This implies that the electron acceleration in a gravitational field becomes mass dependent, thereby signaling the breaking of (at least) Newtonian EP at $T \neq 0$. To leading order in the small parameter T/m_0 , the above formulas yield

$$\frac{m_g}{m_i} = 1 - \frac{2\alpha\pi T^2}{3m_0^2}, \quad (27)$$

a result that can be attributed to the fact that the heat bath singles out a preferred rest frame, leading to a lack of Lorentz invariance in the finite-temperature vacuum.

2. Classical relativistic approach

A complementary analysis was carried out by Gasperini [56], who investigated thermal effects on the gravitational acceleration of relativistic particles in the low-temperature limit (24), within a weak and static gravitational field. Unlike the previous investigation, which relies on diagrammatic techniques from finite-temperature QFT, this approach adopts a classical gravity framework where the local Lorentz symmetry is broken in the tangent space spanned by the tetrad field $\{e^\mu_a\}$ (with $a = \hat{0}, \hat{1}, \hat{2}, \hat{3}$ the anholonomic frame index), while general covariance remains intact on the spacetime manifold $\mathcal{M}$. This construction gives rise to a quasi-Riemannian scenario, which, however, does not affect directly the underlying Riemannian geometry. Specifically, the Einstein equations retain their standard form

$$\overset{\circ}{G}{}^{\mu\nu} = 8\pi\Theta^{\mu\nu}, \quad (28)$$

with $\Theta^{\mu\nu}$ the generalized energy-momentum tensor describing the effective source of gravity at finite temperature.

In this pattern, the ordinary contracted Bianchi identity $\overset{\circ}{\nabla}_\mu \overset{\circ}{G}{}^{\mu\nu} = 0$ can be still invoked, and $\Theta^{\mu\nu}$ continues to be covariantly conserved:

$$\overset{\circ}{\nabla}_\mu \Theta^{\mu\nu} = 0. \quad (29)$$

At leading order in T^2 , a detailed finite-temperature QFT calculation gives the explicit form [54,55]

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{2}{3}\alpha\pi \frac{T^2}{E^2} e^\mu_{\hat{0}} e^\nu_{\hat{0}} T^{\hat{0}\hat{0}}, \quad (30)$$

where $T^{\mu\nu} = e^\mu_a e^\nu_b T^{ab}$ is the standard particle stress-energy tensor minimally coupled to gravity, $e^\mu_{\hat{0}}$ picks out the heat-bath rest frame in the tetrad basis, and the expression for the energy E follows from the generalized dispersion relation

$$E = \sqrt{m_0^2 + |\vec{p}|^2} + \frac{2}{3}\alpha\pi T^2, \quad (31)$$

with $\vec{p}$ the momentum three-vector of the source. The tensor (30), thus, characterizes a matter distribution coupled to gravity in a generally covariant but not locally Lorentz-invariant manner: It transforms as a tensor under diffeomorphisms but not as a scalar under local Lorentz transformations that rotates the tetrad frame in the local Minkowski tangent space [102].

Applying the Papapetrou multipole-expansion method to the conservation law (29) yields the following modified worldline equation for a (spinless) test particle:

$$x^\lambda + \overset{\circ}{\Gamma}{}^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{d}{ds} \left(\frac{2}{3}\alpha\pi \frac{T^2}{mE} e^\lambda_{\hat{0}} \right) + \frac{2}{3}\alpha\pi \frac{T^2}{m^2} \overset{\circ}{\Gamma}{}^\lambda_{\mu\nu} e^\mu_{\hat{0}} e^\nu_{\hat{0}}, \quad (32)$$

where the thermal contributions induce, at lowest order in T^2/m^2 ($m := \sqrt{m_0^2 + 2\alpha\pi T^2/3}$), a mass-dependent deviation from the geodesic motion that indicates a breakdown of the universality of free fall.

A simple application illustrating this situation is the case of a test body radially evolving in the Schwarzschild background sourced by a mass M . In the weak-field approximation, Eq. (32) leads to [56,57]

$$\ddot{r} = -\frac{M}{r^2} \left(1 - \frac{2}{3}\alpha\pi \frac{T^2}{m^2} \right), \quad (33)$$

which shows that m_g/m_i matches the quantum-derived nonrelativistic formula (27). Remarkably, this result implies that, contrary to naive thermodynamic expectations, which might suggest that thermal corrections to gravitational interactions vanish in the relativistic limit, the ratio m_g/m_i exhibits a constant thermal shift independent of the particle kinetic energy.

IV. EQUIVALENCE PRINCIPLE VIOLATION FROM NONMETRICITY

In the previous section, we described how the inclusion of temperature in gravitational systems can spoil the EP. However, while in the formalism outlined in Sec. III B the underlying geometry is strictly Riemannian and departures from geodesic worldlines result from environment-induced breaking of local Lorentz symmetry in the tangent space, in this section, we turn to the metric-affine framework, whose geometric richness goes beyond Riemannian structures and the obstruction to the operational realization of local Lorentz invariance can be naturally associated with the nonmetricity.

In Sec. IV A, we show that the presence of the nonmetricity tensor provides a natural arena for possible deviations from universality of free fall by investigating the motion of a test particle endowed with a dilation charge in a static and spherically symmetric background. Then, in Sec. IV B, we discuss how the resulting EP-violation signatures could be probed experimentally through future space missions and high-precision laser ranging, and we highlight the support offered by the astrometric data-analysis software called *Planetary Ephemeris Program* (PEP).

A. Pure-dilation motion in a static and spherically symmetric background

The equations governing the trajectory of a microstructured and deformable test particle carrying spin, shear, and dilation charges in a generic non-Riemannian background² follow from the Noether conservation laws for the (canonical) energy-momentum and hypermomentum tensors, defined, respectively, as

$$\Sigma^\mu{}_\nu := \frac{\partial \mathcal{L}_m}{\partial (\nabla_\mu \psi)} \nabla_\nu \psi - \delta^\mu_\nu \mathcal{L}_m, \quad (34)$$

$$\Delta_\lambda{}^{\mu\nu} := -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta \Gamma^\lambda{}_{\mu\nu}}, \quad (35)$$

²For recent investigations on the dynamical properties of generic MAGs, including black hole solutions, accretion processes, stellar configurations, and related astrophysical processes, as well as cosmological scenarios, see, e.g., Refs. [103–110].

ψ denoting a generic matter field, $\mathcal{L}_m$ the matter Lagrangian, and $g := \det g_{\mu\nu}$. Two different but equivalent approaches can be pursued to obtain the dynamics of the test body: a covariant multipole-expansion method [111] or a direct worldline derivation using Dirac-delta distributions (supplemented by the so-called convective ansatz) [112]. Moreover, a Lagrangian formulation for a hypermomentum-charged object has been recently proposed in Ref. [113] (see also Refs. [114–116] for an analysis involving the Weyssenhoff fluid in Einstein-Cartan theory and Ref. [117] for a study of the semiclassical fermion motion in generic non-Riemannian backgrounds).

As in any gauge theory, intrinsic properties of matter act as sources for the corresponding gauge fields. In MAG, these properties are encoded in the hypermomentum tensor (35), which generalizes the notion of charge associated with internal symmetries. In particular, the spin, dilation, and shear currents represent additional fundamental characteristics of matter that couple to the affine connection, complementing the role of the energy-momentum tensor as the source of the metric field.

From a group-theoretic viewpoint, $\Delta_\lambda{}^{\mu\nu}$ admits a “fine structure” [61], since it can be decomposed into three irreducible pieces under the general linear group $GL(4, \mathbb{R})$ [75]. The antisymmetric component corresponds to the intrinsic spin of matter and couples directly to spacetime torsion. The symmetric traceless portion represents the shear degrees of freedom, which capture distortions without changing the volume and interact with the traceless part of nonmetricity. Finally, the trace term of hypermomentum describes the dilation charge (i.e., uniform changes of scale), which couples to the Weyl sector of nonmetricity:

$$Q^\lambda := g^{\mu\nu} Q^\lambda{}_{\mu\nu}. \quad (36)$$

This vector may be regarded as providing the minimal geometric mechanism to violate local Lorentz invariance [60]. As discussed before, this phenomenon can be associated with the transition from the group $SO(1,3)$ to $GL(4, \mathbb{R})$, which underlies the gauge paradigm of metric-affine gravity (see Sec. II C). This observation is relevant for the dynamics of test bodies, since the same nonmetricity components that encode this departure from local Lorentz symmetry also enter the equations of motion through the hypermomentum-nonmetricity coupling [60,61].

If one focuses on a particle with inertial mass m_i that is charged only under dilation, then the hypermomentum tensor takes the (convective) form [111,112]

$$\Delta_\lambda{}^{\mu\nu} = \delta_\lambda^\mu \Delta u^\nu, \quad (37)$$

and the worldline equation reads [111,112]

$$\frac{d^2 x^\nu}{d\tau^2} + \tilde{\Gamma}^\nu{}_{\mu\rho} u^\mu u^\rho = -\frac{\Delta}{2m_i} u_\mu \partial^{[\mu} Q^{\nu]}, \quad (38)$$

where we have employed the proper time τ to parametrize the dynamics and u^μ is the *normalized* four-velocity, defined as

$$u^\mu = \frac{dx^\mu}{d\tau} \equiv \dot{x}^\mu; \quad (39)$$

moreover, Δu^μ denotes the (conserved) dilation (vector) current, with the dilation charge density Δ being conserved along the particle path, i.e., $\dot{\Delta} = 0$ [112]. Notice that Eq. (38) bears a formal resemblance to the general autoparallel equation of MAG [cf. Eq. (12)].

Among the various hypermomentum charges appearing in MAG, the dilation coupling provides the simplest and most transparent setting for isolating nonmetricity-induced deviations from geodesic motion. We, therefore, focus on pure-dilation dynamics and investigate Eq. (38) on the equatorial $\theta = \pi/2$ plane of the static and spherically symmetric geometry:

$$ds^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2); \quad (40)$$

we also assume, consistently with the background symmetries, that the Weyl nonmetricity vector (37) attains the form

$$Q^\mu(r) = (Q^t, Q^r, 0, 0). \quad (41)$$

After a straightforward calculation, we obtain for the temporal and radial components, respectively,

$$\ddot{t} + \frac{A'}{A} \dot{t} \dot{r} = -\frac{d}{d\tau} \left(\frac{\Delta}{4m_i} Q^t \right), \quad (42a)$$

$$\ddot{r} + \frac{B}{2} [i^2 A' + i^2 (B^{-1})'] = -\frac{\Delta}{4m_i} ABi(Q^t)', \quad (42b)$$

where the prime denotes the derivative with respect to the radial variable r . Notice that the radial component Q^r gives no contribution to the test particle trajectory in the static case. Bearing in mind Eq. (42a), Eq. (42b) can be expressed as a total- r derivative:

$$\frac{B}{2} \frac{d}{dr} \left(\frac{i^2}{B} - Ai^2 \right) = 0, \quad (43)$$

whose solution is given by

$$\frac{i^2}{B} - Ai^2 = -1, \quad (44)$$

the constant factor on the right-hand side stemming from the normalization condition of the four-velocity (39). With the help of the above relation, Eq. (42b) becomes the single radial equation

$$\ddot{r} + \frac{B}{2} \left[\frac{A'}{A} + \frac{i^2}{B} \left(\frac{A'}{A} - \frac{B'}{B} \right) + \frac{\Delta}{2m_i} (Q^t)' \sqrt{A + \frac{i^2}{B} A} \right] = 0. \quad (45)$$

In order to study Eq. (45) in the weak-field-slow-motion regime, we suppose that the geometry (40) refers to a genuine gravitational field sourced by a mass M . Accordingly, we take $g_{\mu\nu}$ to be asymptotically flat, so that for $r \rightarrow +\infty$ the functions $A(r)$ and $B(r)$ behave, to leading order, as

$$\begin{aligned} A &= 1 + 2\Phi, \\ B^{-1} &= 1 - 2\Phi, \end{aligned} \quad (46)$$

with $\Phi = -M/r$ the standard Newtonian potential (in geometrized units). Under these hypotheses, and discarding higher-order contributions, Eq. (45) reduces to

$$\ddot{r} = -\Phi' - \frac{\Delta}{4m_i} (Q^t)', \quad (47)$$

showing that the additional (Lorentz-type) force modifying the particle motion is entirely controlled by the derivative of the temporal component Q^t of the Weyl vector.

The above results highlight the richness of MAG framework, which allows us to explore several dynamical scenarios, including explicit tests of the EP. Indeed, let us assume that in the weak-field limit Q^t is proportional to Φ :

$$Q^t = \chi\Phi, \quad (48)$$

where the constant χ has the dimensions of an inverse length in $G = c = 1$ units. In this way, we arrive at

$$\ddot{r} = -\frac{M}{r^2} \left(1 + \frac{\chi\Delta}{4m_i} \right), \quad (49)$$

which closely resembles Eq. (33). We, thus, obtain the EP-violation relation

$$\frac{m_g}{m_i} = 1 + \frac{\chi\Delta}{4m_i}, \quad (50)$$

which yields

$$m_g = m_i + \frac{\chi\Delta}{4}. \quad (51)$$

Equivalently, upon introducing an explicit expansion around a reference (zero-order) inertial mass m_0 (e.g., the $T = 0$ mass), we set $m_i = m_0 + O(\Delta)$ and then get, by neglecting terms $O(\Delta^2/m_0^2)$,

$$\frac{m_g}{m_i} = 1 + \frac{\chi\Delta}{4m_0}. \quad (52)$$

Thus, the metric-affine pure-dilation force reproduces, at leading order, a Newtonian law with a small rescaling governed by the dilation charge density. This outcome is closely analogous to the finite-temperature results (25)–(27), which we recall have been derived in a Riemannian setting. Our analysis, therefore, demonstrates that EP violations stemming from the lack of Lorentz invariance of the thermal vacuum admit a natural reinterpretation using MAG formalism, where, as we discussed before, departures from local Lorentz symmetry can be consistently accounted for. Moreover, this implies that non-Riemannian geometry naturally provides a classical counterpart to the quantum effects responsible for the EP departure discussed in Sec. III B 1.

Two crucial remarks are in order. First, the result (52) signals a violation of the Newtonian EP; however, if the effect described is not universal but composition or charge dependent, i.e., Δ/m_0 differs between test bodies of different composition, then also the weak EP is spoiled. Second, in the specific pure-dilation scenario examined in this section, the Weyl-type nonmetricity allows a redefinition of the affine parameter into proper time, so that a normalized four-velocity can be consistently adopted [cf. Eq. (39)]; see Ref. [112] for further details. However, in the broad metric-affine arena, nonmetricity does not preserve, in general, the four-velocity norm, as we discussed in Sec. II. For this reason, in Sec. V, we will introduce an extended FW derivative that permits addressing the EP validity in the most general MAG framework.

B. Experimental probes via laser ranging and space missions

The above results could potentially be tested using laser-ranging methods and space-based experiments.

Laser-ranging techniques currently provide one of the most sensitive empirical tests of the EP on lunar and planetary scales [21,23]. In particular, lunar laser ranging measures the round-trip time of laser pulses reflected by retroreflectors on the Moon surface, yielding subcentimeter (soon millimeter) constraints on the Earth-Moon dynamics and on key fundamental observables [118–120], including the weak and strong EP [22].

The advent of next-generation retroreflectors, most notably the *MoonLIGHT* payload integrated with a double-axis pointing actuator for precise orientation, known as MPAC (*MoonLIGHT pointing actuator*), has been designed to eliminate the pulse broadening caused by lunar librations that limit current Apollo and Lunokhod mission arrays [121]. The expected performance of these libration-free retroreflectors, enhanced by the MPAC support, is projected to enable improvements of up to 2 orders of magnitude in several tests of GR compared to the present centimeter-level

accuracy. This augmented capability directly impacts the probes of the weak and strong EP and guarantees the libration-independent operating conditions necessary to achieve millimeter-level laser-ranging performance.

The analysis of lunar laser ranging data, both historical and next-generation, is performed using the software PEP, developed at the Center for Astrophysics, Harvard and Smithsonian since the 1960s [122]. Once its modeling and implementation are further refined, PEP will allow for joint estimation of weak and strong EP violations. Furthermore, the inclusion of new normal points generated by MPAC allows PEP to extend its scope to tests of extended gravity frameworks [such as MAG, $f(R)$ models, and nonminimal coupled gravity] by incorporating additional dynamical terms directly into the equations of motion and ephemerides and by performing simultaneous fits against both standard and alternative models [123].

In addition, current and forthcoming space missions also provide an ideal laboratory for evaluating EP-violating effects. The aforementioned MICROSCOPE mission has already established the tightest bound to date on the Eötvös parameter η at the 10^{-15} level [16], a sensitivity that is expected to be improved to $\eta \sim 10^{-18}$ by STEP using superconducting, cryogenic test masses in a drag-free spacecraft [17]. Complementing these classical measurements, the European Space Agency STE-QUEST mission concept proposes to probe the universality of free fall with an accuracy of $\eta \lesssim 10^{-17}$, relying on quantum matter-wave probes instead of macroscopic test bodies [19].

In the context of metric-affine gravity, we have seen that the dilation-induced modification of the Newtonian law derived in Eq. (49) leads to the rescaling factor $\chi\Delta/(4m_0)$ in the ratio m_g/m_i , as can be read off from Eq. (52). Future space missions, which span both macroscopic and quantum regimes, together with the millimeter-level laser-ranging accuracy achievable with MPAC and the enhanced modeling provided by PEP, would make it possible to constrain this parameter to extremely small values. This, in turn, would provide an experimental window on nonmetricity signatures, the associated EP departure, and their interplay with the thermal effects discussed in Sec. III.

V. EINSTEIN EQUIVALENCE PRINCIPLE VIOLATION AND FERMI-WALKER TRANSPORT IN METRIC-AFFINE GEOMETRY

In Sec. II C, we have explained that the gauge-theoretic, elementary-matter notion of the Einstein EP can still be realized pointwise in MAG, while in Sec. IV A, we have proved how possible departures from the universality of free fall can emerge within a pure-dilation regime. In this section, we establish that the modern formulation of the Einstein EP is not valid, in general. As mentioned before, such breakdown can be traced back to the fact that normalization of the metric (as well as the trivialization

of the connection) is a point-dependent gauge choice in MAG [cf. Eqs. (21) and (23)], which is not preserved under parallel transport owing to the nonmetricity. From a gauge-theoretic perspective, this feature is intertwined with departures from local Lorentz invariance in the metric-compatible sense.

To describe this phenomenon, we introduce a generalization of the FW derivative to the MAG context. This novel construction enables a purely geometric and intrinsic analysis of potential violations of the Einstein EP within the most general metric-affine scenario. The new FW operator shows that the standard Einstein-EP local Lorentz-frame construction fails “globally,” i.e., along *generic* worldlines, thereby extending the analysis beyond the ordinary parallel transport scenario.

We first review the FW derivative in Riemannian geometry (Sec. VA), before turning to its extension to non-Riemannian settings in Secs. VB and VC. Finally, we consider possible experimental tests in Sec. VD.

A. Fermi-Walker derivative in Riemannian geometry

In GR, the FW transport rule defines a prescription for carrying a proper Lorentz frame along an arbitrary timelike worldline $\lambda(s)$ without inducing a spatial rotation of the spatial axes due to the acceleration [1,124–129].

The local Minkowski frame attached to an observer moving on $\lambda(s)$ with unit tangent four-velocity $\dot{\mathbf{x}} \equiv d\mathbf{x}/ds$ is specified via a tetrad $\{e^\mu_a\}$, which forms a local orthonormal basis in the tangent space to the spacetime manifold, with $e^\mu_{\hat{0}}$ identified with $\dot{\mathbf{x}}^\mu$. It is well known that if this frame is propagated via standard parallel transport along a nongeodesic orbit, the timelike vector $e^\mu_{\hat{0}}$ will no longer coincide with $\dot{\mathbf{x}}^\mu$, and the spatial vectors $e^\mu_{\hat{1}}$, $e^\mu_{\hat{2}}$, and $e^\mu_{\hat{3}}$ will not remain orthogonal to $\dot{\mathbf{x}}^\mu$. On the other hand, if $\{e^\mu_a\}$ is FW transported along $\lambda(s)$, the resulting frame retains its orthonormality at each point, with $e^\mu_{\hat{0}} = \dot{\mathbf{x}}^\mu$ and the spatial triad undergoing no arbitrary rotation. Physically, such a frame can be interpreted as a set of mutually orthogonal, rigid gyroscopes that keep fixed orientation relative to the observer instantaneous local rest frame while following the path $\lambda(s)$.

The FW derivative of a vector field $\mathbf{Y}$ along the timelike curve $\lambda(s)$ reads as [125]

$$\frac{D^{\text{FW}}\mathbf{Y}}{ds} = \frac{\dot{D}\mathbf{Y}}{ds} - g\left(\mathbf{Y}, \frac{\dot{D}\dot{\mathbf{x}}}{ds}\right)\dot{\mathbf{x}} + g(\mathbf{Y}, \dot{\mathbf{x}})\frac{\dot{D}\dot{\mathbf{x}}}{ds}, \quad (53)$$

where $\dot{D}(\cdot)/ds = \dot{\nabla}_{\dot{\mathbf{x}}}(\cdot)$ denotes the GR absolute derivative along $\lambda(s)$ and $g(\cdot, \cdot)$ the scalar product. It is, thus, evident that the FW derivative boils down to the usual covariant derivative if $\lambda(s)$ is a geodesic. In components, the above formula becomes

$$\dot{\mathbf{x}}^\lambda \nabla_\lambda^{\text{FW}} Y^\mu = \dot{\mathbf{x}}^\lambda \dot{\nabla}_\lambda Y^\mu + F^\mu_\nu Y^\nu, \quad (54)$$

where the skew-symmetric tensor field F^μ_ν encodes all kinematical information and takes the form

$$F^\mu_\nu = \dot{\mathbf{x}}^\mu \dot{\mathbf{x}}_\nu - \dot{\mathbf{x}}^\nu \dot{\mathbf{x}}_\mu, \quad (55)$$

with $\dot{\mathbf{x}}^\mu = \dot{\mathbf{x}}^\lambda \dot{\nabla}_\lambda \dot{\mathbf{x}}^\mu$ the Riemannian four-acceleration orthogonal to $\dot{\mathbf{x}}^\mu$. Thus, the FW covariant derivative operator (which represents formally a derivation in the tangent fiber bundles of the vector fields on the spacetime manifold $\mathcal{M}$ [1]) is given by

$$\nabla_\lambda^{\text{FW}} Y^\mu = \partial_\lambda Y^\mu + \Gamma^\mu_{\lambda\nu} Y^\nu, \quad (56)$$

where we have introduced the FW *symbols*

$$\Gamma^\mu_{\lambda\nu}^{\text{FW}} = \dot{\Gamma}^\mu_{\lambda\nu} + (\dot{\mathbf{x}}_\nu \dot{\nabla}_\lambda \dot{\mathbf{x}}^\mu - \dot{\mathbf{x}}^\mu \dot{\nabla}_\lambda \dot{\mathbf{x}}_\nu). \quad (57)$$

These give rise to a well-defined connection whose formal definition demands sophisticated mathematical tools involving the formalism of second-order tangent bundles (see Ref. [130] for further details).

The FW derivative (53) satisfies the following key properties.

- (i) The tangent vector $\dot{\mathbf{x}}$ to any timelike curve $\lambda(s)$ is FW transported:

$$\frac{D^{\text{FW}}\dot{\mathbf{x}}}{ds} = 0. \quad (58a)$$

- (ii) If $\mathbf{X}$ and $\mathbf{Y}$ are FW transported along $\lambda(s)$, then their scalar product remains constant along $\lambda(s)$:

$$\frac{dg(\mathbf{X}, \mathbf{Y})}{ds} = \frac{\dot{D}g(\mathbf{X}, \mathbf{Y})}{ds} = 0. \quad (58b)$$

- (iii) Vectors $\mathbf{Y}$ that satisfy $g(\mathbf{Y}, \dot{\mathbf{x}}) = 0$ along $\lambda(s)$ remain in the orthogonal subspace to $\dot{\mathbf{x}}$ and do not rotate within it:

$$\frac{D^{\text{FW}}\mathbf{Y}}{ds} = \dot{\mathcal{P}}_\perp \left(\frac{\dot{D}\mathbf{Y}}{ds} \right), \quad (58c)$$

$\dot{\mathcal{P}}_\perp$ denoting the projection operator in the direction orthogonal to $\dot{\mathbf{x}}$, i.e., onto the instantaneous local rest space of the observer [cf. Eq. (73b), below].

The FW derivative can be naturally extended to spacelike curves. Let $q(s)$ be a generic curve with unit tangent vector $\boldsymbol{\tau}$, not necessarily timelike. Then, the FW derivative of a vector field $\mathbf{Y}$ along $q(s)$ is

$$\frac{D^{\text{FW}}\mathbf{Y}}{ds} = \frac{\dot{D}\mathbf{Y}}{ds} + \frac{g(\mathbf{Y}, \frac{\dot{D}\boldsymbol{\tau}}{ds})}{g(\boldsymbol{\tau}, \boldsymbol{\tau})} \boldsymbol{\tau} - \frac{g(\mathbf{Y}, \boldsymbol{\tau})}{g(\boldsymbol{\tau}, \boldsymbol{\tau})} \frac{\dot{D}\boldsymbol{\tau}}{ds}, \quad (59)$$

which reduces to formula (53) when $g(\boldsymbol{\tau}, \boldsymbol{\tau}) = -1$.

When the FW derivative is to be evaluated along a congruence of null worldlines with tangent vector $\mathbf{n}$, a more careful analysis is required. In this context, a complementary null vector $\mathbf{l}$, satisfying $g(\mathbf{n}, \mathbf{l}) \neq 0$, is introduced to span the two-dimensional subspace of vectors normal to $\mathbf{n}$ [131,132]. With this auxiliary structure, it can be shown that the FW derivative of a vector field $\mathbf{Y}$ along the null congruence is

$$\begin{aligned} \frac{D^{\text{FW}}\mathbf{Y}}{ds} = & \frac{\dot{D}\mathbf{Y}}{ds} + \frac{g(\mathbf{Y}, \frac{\dot{D}\mathbf{l}}{ds})}{g(\mathbf{l}, \mathbf{n})} \mathbf{n} - \frac{g(\mathbf{Y}, \mathbf{l})}{g(\mathbf{l}, \mathbf{n})} \frac{\dot{D}\mathbf{n}}{ds} \\ & + \frac{g(\mathbf{Y}, \frac{\dot{D}\mathbf{n}}{ds})}{g(\mathbf{l}, \mathbf{n})} \mathbf{l} - \frac{g(\mathbf{Y}, \mathbf{n})}{g(\mathbf{l}, \mathbf{n})} \frac{\dot{D}\mathbf{l}}{ds}. \end{aligned} \quad (60)$$

This construction ensures a well-defined notion of transport for vectors in the transverse subspace. For further details, we refer the reader to Refs. [130,133–135].

B. Fermi-Walker derivative in metric-affine geometry

Identities (58) rely crucially on two key features of Riemannian geometry: the possibility of *always* normalizing the tangent vector $\dot{x}^\mu$ to a timelike path (i.e., $\dot{x}^\mu \dot{x}_\mu = -1$) and the ensuing orthogonality with the four-acceleration $\dot{a}_\mu$ (i.e., $\dot{x}^\mu \dot{a}_\mu = 0$). However, these two conditions generally break down in MAG, as we discussed in Sec. II B [cf. Eqs. (14) and (18)]. Therefore, in this section, we develop a FW derivative suitable for metric-affine spaces, in order to explicitly account for the nonmetricity tensor and its distinctive role.

Let there be given a curve $\gamma(s)$ with parameter s and tangent vector T^μ , which satisfies [70]

$$T^\mu T_\mu = \epsilon \phi(x^\rho), \quad (61)$$

with $\epsilon = \pm 1, 0$ whether $\gamma(s)$ is spacelike, timelike, or null, respectively. For consistency with our notation, in the case of timelike curves, we will denote the tangent vector by $\dot{x}^\mu$, which, we recall, satisfies relation (14).

In his original paper, Fermi considered only vectors orthogonal to T^μ , while Walker extended this setup to generic vectors. A straightforward way to introduce the FW derivative in GR [127,130], in line with the original ideas of Fermi and Walker, is to exploit property (58c), supplemented by the hypotheses that, along $\gamma(s)$,

- (i) the length of T^μ is conserved; and
- (ii) the scalar product $g(\mathbf{Y}, \mathbf{T})$ remains constant.

These considerations suggest that a generalized FW derivative can be properly defined starting from Eq. (58c), written in a form adapted to MAG settings.

Therefore, we introduce the projector onto the subspace orthogonal to $\mathbf{T}$:

$$\mathcal{P}_\perp^\alpha = \delta^\alpha_\beta - \frac{T^\alpha T_\beta}{\epsilon \phi}, \quad (62)$$

which allows Eq. (58c) to be reexpressed as the projected-transport condition

$$\mathcal{P}_\perp \left[\frac{D}{ds} (\mathcal{P}_\perp \mathbf{Y}) \right] = 0, \quad (63)$$

where, we recall, $D(\cdot)/ds$ involves the covariant derivative with respect to the full MAG connection $\Gamma^\alpha_{\mu\nu}$ (see Sec. II A). It is, thus, clear that $\mathcal{P}_\perp$ is well defined for paths with $\epsilon \neq 0$, to which we henceforth restrict our attention [a proper definition of the derivative along null worldlines requires a separate procedure, analogous to that employed in Eq. (60)].

In view of Eq. (61), it is clear that assumption (i) cannot be retained in MAG. On the other hand, the new FW prescription must enforce (ii) in order for a vector $\mathbf{Y}$ to remain orthogonal to $\mathbf{T}$ as it is FW transported along $\gamma(s)$. The rationale for this choice is the following. FW transport provides, in general, a way to define, point by point along a worldline, a tetrad. This scheme necessarily relies on requirement (ii), which ensures that the vectors perpendicular to the velocity remain within this orthogonal subspace throughout the evolution. Without enforcing (ii), any attempt by an observer to carry a tetrad along her or his trajectory would fail from the outset, and the EP would be immediately violated. Nevertheless, as we will show, even when (ii) is imposed, the construction of a tetrad along a curve ultimately proves impossible, thereby showing that in MAG contexts the Einstein EP is inevitably spoiled.

In view of the above considerations, drawing on Eq. (63) and exploiting hypothesis (ii) but not (i), we obtain the following generalized FW derivative of the vector Y^α along $\gamma(s)$:

$$\frac{DY^\alpha}{ds} = \frac{DY^\alpha}{ds} - \frac{1}{\epsilon \phi} \mathcal{F}^\alpha_\beta Y^\beta + \frac{(T_\beta Y^\beta)}{\epsilon^2 \phi^2} \left(T_\lambda \frac{DT^\lambda}{ds} \right) T^\alpha, \quad (64)$$

where the tensor $\mathcal{F}^\alpha_\beta$ reads as

$$\mathcal{F}^\alpha_\beta = \frac{DT^\alpha}{ds} T_\beta - T^\alpha \frac{DT_\beta}{ds} \quad (65)$$

and, contrary to Einstein gravity, is not antisymmetric [cf. Eq. (55); further details on $\mathcal{F}^\alpha_\beta$ will be given in Sec. V C]. We note that Eq. (64) reduces to Eq. (59) in the GR limit, where DT^μ/ds and DT_μ/ds become the contravariant and covariant components of the same vector [whereas in MAG they correspond, for timelike trajectories,

to the proper and anomalous accelerations, respectively; see Eqs. (15) and (16)], and the tangent T^μ is orthogonal to its covariant derivative.

The extension of the operator $\mathcal{D}/ds$ to arbitrary-rank tensors can be worked out starting from the following formulas:

$$\frac{DY_\mu}{ds} = \frac{DY_\mu}{ds} + \frac{1}{\epsilon\phi} \mathcal{F}^\alpha{}_\mu Y_\alpha - \frac{(T_\beta Y^\beta)}{\epsilon^2\phi^2} \left(T_\lambda \frac{DT^\lambda}{ds} \right) T_\mu, \quad (66a)$$

$$\begin{aligned} \frac{DA^{\mu\nu}}{ds} &= \frac{DA^{\mu\nu}}{ds} - \frac{1}{\epsilon\phi} (\mathcal{F}^\mu{}_\alpha A^{\alpha\nu} + \mathcal{F}^\nu{}_\alpha A^{\mu\alpha}) \\ &\quad + \frac{1}{\epsilon^2\phi^2} \left(T_\beta \frac{DT^\beta}{ds} \right) T_\alpha (A^{\alpha\nu} T^\mu + A^{\mu\alpha} T^\nu), \end{aligned} \quad (66b)$$

$$\begin{aligned} \frac{DA_{\mu\nu}}{ds} &= \frac{DA_{\mu\nu}}{ds} + \frac{1}{\epsilon\phi} (\mathcal{F}^\alpha{}_\mu A_{\alpha\nu} + \mathcal{F}^\alpha{}_\nu A_{\mu\alpha}) \\ &\quad - \frac{1}{\epsilon^2\phi^2} \left(T_\beta \frac{DT^\beta}{ds} \right) T^\alpha (A_{\alpha\nu} T_\mu + A_{\mu\alpha} T_\nu), \end{aligned} \quad (66c)$$

from which we derive, in particular,

$$\begin{aligned} \frac{\mathcal{D}g_{\mu\nu}}{ds} &= T^\lambda Q_{\lambda\mu\nu} - \frac{T^\lambda T^\alpha}{\epsilon\phi} (T_\mu Q_{\lambda\alpha\nu} + T_\nu Q_{\lambda\alpha\mu}) \\ &\quad - \frac{2}{\epsilon^2\phi^2} \left(T_\beta \frac{DT^\beta}{ds} \right) T_\mu T_\nu. \end{aligned} \quad (67)$$

It follows directly from the above relations that $\mathcal{D}/ds$ obeys the Leibniz rule and boils down to the ordinary derivative when acting on scalar functions.

In the case of timelike curves, for which $T^\mu \equiv \dot{x}^\mu$, with $\dot{x}^\mu \dot{x}_\mu = -\phi$ [cf. Eq. (14)], the FW derivative of the four-velocity field yields

$$\frac{\mathcal{D}\dot{x}^\alpha}{ds} = \frac{\dot{\phi}}{\phi} \dot{x}^\alpha, \quad (68a)$$

$$\frac{\mathcal{D}\dot{x}_\alpha}{ds} = 0, \quad (68b)$$

where we have exploited Eqs. (15)–(18). In other words, the tangent vector to $\gamma(s)$ is not FW conserved, unlike in Riemannian geometry.

It is worth noting that an alternative construction of the FW derivative can be provided, which can be viewed as the “dual” of the operator in Eq. (64). This second version can be derived by means of an equivalent approach, where one first evaluates the generalized FW derivative on vectors belonging to the subspace orthogonal to $\mathbf{T}$ and then extends its action to generic vectors. In this way, we obtain

$$\frac{\tilde{\mathcal{D}}Y^\mu}{ds} = \frac{DY^\mu}{ds} - \frac{1}{\epsilon\phi} \mathcal{F}^\mu{}_\alpha Y^\alpha - \frac{(T_\beta Y^\beta)}{\epsilon^2\phi^2} \left(T^\lambda \frac{DT_\lambda}{ds} \right) T^\mu, \quad (69)$$

and we find that the FW derivative $\tilde{\mathcal{D}}/ds$ is tantamount to $\mathcal{D}/ds$, in the sense that it yields the “dual” relations of those in Eq. (68):

$$\frac{\tilde{\mathcal{D}}\dot{x}^\alpha}{ds} = 0, \quad (70a)$$

$$\frac{\tilde{\mathcal{D}}\dot{x}_\alpha}{ds} = \frac{\dot{\phi}}{\phi} \dot{x}_\alpha. \quad (70b)$$

The two definitions (64) and (69) coincide on the subspace orthogonal to T^μ and differ only by the last term proportional to $1/(\epsilon^2\phi^2)$; obviously, both reduce to the standard FW transport in the metric-compatible regime. The possibility of setting up these two alternatives is fully consistent with the outcome of Ref. [136], which points out that in generic metric-affine spacetimes different, yet equally valid, constructions of a generalized FW derivative can be formulated. However, our definitions (64) and (69) have the key strength of being built from the same antisymmetric tensor (65), unlike what typically occurs in MAG [136].

Having clarified these points, we hereafter employ the derivative (64) and examine its basic properties. By construction, the analog of Eq. (58c) is satisfied, while we have already proved that the counterpart of Eq. (58a) does not hold in MAG [cf. Eq. (68)]. Likewise, if $\mathcal{D}Y^\mu/ds = \mathcal{D}X^\mu/ds = 0$, then we find that

$$\begin{aligned} \frac{d}{ds}(g_{\mu\nu} Y^\mu X^\nu) &= Q_{\lambda\mu\nu} T^\lambda Y^\mu X^\nu - \frac{Q_{\lambda\alpha\mu} T^\lambda T^\mu}{\epsilon\phi} [(T_\beta Y^\beta) X^\alpha \\ &\quad + (T_\beta X^\beta) Y^\alpha] - \frac{2}{\epsilon^2\phi^2} (X_\mu T^\mu) (Y_\nu T^\nu) T_\lambda \frac{DT^\lambda}{ds}, \end{aligned} \quad (71)$$

which is not surprising, as the corresponding GR property (58b) relies on the antisymmetry of $F^\mu{}_\nu$, a feature that is lost in MAG [see Eq. (65)]. A further crucial aspect, reflecting the basic requirements of the new FW construction, is that if $\mathcal{D}Y^\mu/ds = 0$, then

$$\frac{d}{ds}(g_{\mu\nu} Y^\mu T^\nu) = 0, \quad (72)$$

which means that, if the vector Y^μ is initially perpendicular to the tangent T^μ , FW transport along $\gamma(s)$ preserves this orthogonality during the entire evolution. However, a caveat should be mentioned at this point: In deriving the above relations, we assumed that the contravariant components of a generic vector have vanishing FW derivative. The conclusions change if one instead imposes the condition on the covariant components (e.g., $\mathcal{D}Y_\mu/ds = 0$), due to fact that the metric tensor is not FW conserved [see Eq. (67)].

As a consequence of the above formulas, the generalized FW scheme (64) fails to provide a consistent prescription for defining a tetrad³ $\{\mathbf{e}_0 = \dot{\mathbf{x}}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ that remains orthonormal everywhere on the worldline $\gamma(s)$ of an observer. This obstruction is caused by two fundamental issues: The FW derivative of $\mathbf{e}_0$ is not zero [see Eq. (68)], and the orthogonality between two spatial vectors $\mathbf{e}_i$ is not preserved throughout $\gamma(s)$, owing to Eq. (71).

The impossibility of propagating an orthonormal tetrad via the FW procedure represents a clear geometric indication of the intrinsic breakdown of the Einstein EP in generic metric-affine backgrounds. Indeed, it is known that in Einstein theory a local manifestation of the EP is provided by Riemann and Fermi normal coordinates [124,137,138] (see also Ref. [139] for a recent application), where the latter can be defined along an arbitrary worldline by resorting to the ordinary FW transport of the tetrad [124,128,140]. This standard construction, however, fails in MAG arena since, as we have shown above, the FW scheme prevents the establishment of an orthonormal frame that renders the metric locally Minkowskian on the observer trajectory. Our analysis, thus, demonstrates that the EP violation is not confined to the pure-dilation dynamics examined in Sec. IV A but is a distinctive feature of the broad MAG framework.

Some final remarks are in order. First, it is straightforward to show that the EP breaking can be equivalently described via the “dual” operator $\tilde{D}/ds$. Second, we notice that generalized notions of FW transport in non-Riemannian settings have appeared in the literature; see, e.g., Refs. [129,136,141,142] for further details. However, our projector-based construction differs in both method and emphasis: It does not assume metric compatibility, is tailored to MAG, and naturally addresses the question of EP noncompliance.

C. Projection structure of the tensor $\mathcal{F}^\alpha_\beta$

In this section, we further analyze the properties of the tensor $\mathcal{F}^\alpha_\beta$ characterizing the generalized FW derivative [see Eqs. (64) and (65)]. We first consider the Riemannian scenario in Sec. V C 1 and then turn to MAG in Sec. V C 2.

1. Riemannian spacetime

In a Riemannian spacetime, let there be given a timelike curve, and introduce the projection operators

$$\dot{\mathcal{P}}_{\parallel\mu}^\alpha = -\dot{x}^\alpha \dot{x}_\mu, \quad (73a)$$

³Here, the observer-adapted tetrad should not be confused with the frame field, dual of the coframe $\boldsymbol{\theta}^a = e_\mu^a \mathbf{d}x^\mu$ introduced in Sec. II C. Although they are objects of the same geometrical type, the former is a choice of local frame along a specific worldline, whereas the latter is not tied to a specific observer and is part of the metric-affine gauge structure.

$$\dot{\mathcal{P}}_{\perp\mu}^\alpha = \delta_\mu^\alpha + \dot{x}^\alpha \dot{x}_\mu, \quad (73b)$$

which allow any tensorial quantity to be decomposed into components parallel and orthogonal to the tangent vector $\dot{x}^\mu$ to the curve. Thus, the projection of the Riemannian acceleration along the tangent direction satisfies

$$\dot{\mathcal{P}}_{\parallel\mu}^\alpha \frac{D\dot{x}^\mu}{ds} = 0, \quad (74)$$

which, in turn, yields

$$\dot{\mathcal{P}}_{\perp\mu}^\alpha \frac{D\dot{x}^\mu}{ds} = \frac{D\dot{x}^\alpha}{ds}. \quad (75)$$

In view of Eq. (74), we now examine the tensor $\mathcal{F}^\alpha_\beta$ entering the Riemannian FW derivative (54). Bearing in mind Eq. (55) and performing a double parallel projection, we get

$$\begin{aligned} \dot{\mathcal{P}}_{\parallel\alpha}^\mu \dot{\mathcal{P}}_{\parallel\nu}^\beta \mathcal{F}^\alpha_\beta &= \dot{\mathcal{P}}_{\parallel\alpha}^\mu [(-\dot{x}^\beta \dot{x}_\nu) \dot{a}^\alpha \dot{x}_\beta \\ &\quad - (-\dot{x}^\beta \dot{x}_\nu) \dot{x}^\alpha \dot{a}_\beta] = \dot{\mathcal{P}}_{\parallel\alpha}^\mu (\dot{a}^\alpha \dot{x}_\nu) = 0, \end{aligned} \quad (76)$$

showing that $\mathcal{F}^\alpha_\beta$ possesses no component along the longitudinal direction and, thus, lies entirely in the sub-space orthogonal to $\dot{x}^\mu$.

The above results rely crucially on the metric-compatibility condition $\tilde{\nabla}_\lambda g_{\mu\nu} = 0$ and the ordinary normalization $\dot{x}^\mu \dot{x}_\mu = -1$. With these arguments in mind, we now extend the discussion to the wider context of non-Riemannian geometry.

2. Metric-affine spacetime

As we set out before, the local violation of the Lorentz invariance in metric-affine spacetime is related to the presence of the nonmetricity tensor (5) (or, equivalently, the one-form $\mathcal{Q}_{ab}$; see Sec. II C 1), which entails varying-magnitude vectors and gives rise to two different accelerations, namely, the proper and the anomalous one [cf. Eqs. (15)–(17)]. Introducing the orthogonal and longitudinal projectors along the lines of Eq. (73) [cf. Eq. (62)] and considering a timelike curve with tangent vector $\dot{x}^\mu$ satisfying the relation (14), the tangent projection of the proper acceleration yields

$$\mathcal{P}_{\parallel\mu}^\alpha \frac{D\dot{x}^\mu}{ds} = \frac{1}{2} \dot{x}^\alpha f_+, \quad (77)$$

while for the anomalous contribution we obtain

$$\mathcal{P}_{\parallel\mu}^\alpha \frac{D\dot{x}_\mu}{ds} = \frac{1}{2} \dot{x}_\alpha f_-, \quad (78)$$

where

$$f_{\pm}(x^{\rho}) := \frac{\dot{\phi}(x^{\rho})}{\phi(x^{\rho})} \pm \frac{\dot{x}^{\lambda} Q_{\lambda\mu\nu}(x^{\rho}) \dot{x}^{\mu} \dot{x}^{\nu}}{\phi(x^{\rho})}, \quad (79)$$

the orthogonal projections following immediately from $\mathcal{P}_{\perp}^{\alpha}{}_{\mu} = \delta_{\mu}^{\alpha} - \mathcal{P}_{\parallel}^{\alpha}{}_{\mu}$.

Thus, unlike the Riemannian case, where the acceleration $\hat{a}^{\mu}$ satisfies Eq. (74), in MAG both A^{μ} and $\tilde{a}_{\mu}$ carry, in general, a nonvanishing component along the tangent direction, as expected from Eq. (18). For this reason, while in the Riemannian framework one can always uniquely decompose a generic vector along the mutually orthogonal directions defined by $\hat{a}^{\mu}$ and $\dot{x}^{\mu}$, in MAG such a construction is no longer available, as neither A^{μ} nor $\tilde{a}_{\mu}$ are perpendicular to the four-velocity. This different kinematical setup inexorably leads to the new FW derivative introduced in Eq. (64), where the Riemannian tensor $F^{\mu}{}_{\nu}$ is replaced by $F^{\mu}{}_{\nu}$.

If we decompose the accelerations according to

$$\begin{aligned} A^{\alpha} &= (\mathcal{P}_{\perp}^{\alpha}{}_{\mu} + \mathcal{P}_{\parallel}^{\alpha}{}_{\mu}) A^{\mu}, \\ \tilde{a}_{\alpha} &= (\mathcal{P}_{\perp\alpha}{}^{\mu} + \mathcal{P}_{\parallel\alpha}{}^{\mu}) \tilde{a}_{\mu} \end{aligned} \quad (80)$$

and use Eq. (65), then Eqs. (77) and (78) imply that $\mathcal{F}^{\alpha}{}_{\beta}$ admits the decomposition

$$\begin{aligned} \mathcal{F}^{\alpha}{}_{\beta} &= A^{\alpha} \dot{x}_{\beta} - \dot{x}^{\alpha} \tilde{a}_{\beta} = (\mathcal{P}_{\perp}^{\alpha}{}_{\mu} A^{\mu} + \mathcal{P}_{\parallel}^{\alpha}{}_{\mu} A^{\mu}) \dot{x}_{\beta} \\ &\quad - \dot{x}^{\alpha} (\mathcal{P}_{\perp\beta}{}^{\mu} \tilde{a}_{\mu} + \mathcal{P}_{\parallel\beta}{}^{\mu} \tilde{a}_{\mu}) =: \mathcal{F}_{\perp}^{\alpha}{}_{\beta} + \mathcal{F}_{\parallel}^{\alpha}{}_{\beta}, \end{aligned} \quad (81)$$

where

$$\mathcal{F}_{\perp}^{\alpha}{}_{\beta} := \mathcal{P}_{\perp}^{\alpha}{}_{\mu} A^{\mu} \dot{x}_{\beta} - \mathcal{P}_{\perp\beta}{}^{\mu} \tilde{a}_{\mu} \dot{x}^{\alpha}, \quad (82a)$$

$$\mathcal{F}_{\parallel}^{\alpha}{}_{\beta} := \frac{1}{2} (f_{+} - f_{-}) \dot{x}^{\alpha} \dot{x}_{\beta} = \frac{\dot{x}^{\rho} Q_{\rho\sigma\tau} \dot{x}^{\sigma} \dot{x}^{\tau}}{\phi} \dot{x}^{\alpha} \dot{x}_{\beta}. \quad (82b)$$

We, thus, acknowledge a difference with the Riemannian relation (76), as $\mathcal{F}^{\alpha}{}_{\beta}$ now contains also a longitudinal component apart from the orthogonal one. In the Riemannian scenario, the former vanishes identically, while $\mathcal{F}_{\perp}^{\alpha}{}_{\beta}$ boils down to $\mathcal{F}^{\alpha}{}_{\beta}$ owing to Eq. (75).

Interestingly, in the special case where the manifold, equipped with a particular nonmetricity, admits the so-called Yano-Schrödinger connection [143–145], then the geometry satisfies

$$Q_{(\lambda\mu\nu)} = 0, \quad (83a)$$

$$\dot{\phi} = 0, \quad (83b)$$

$$\dot{x}_{\mu} \hat{a}^{\mu} = 0, \quad (83c)$$

which imply that only $\mathcal{F}_{\perp}^{\alpha}{}_{\beta}$ contributes to $\mathcal{F}^{\alpha}{}_{\beta}$, as the longitudinal part $\mathcal{F}_{\parallel}^{\alpha}{}_{\beta}$ is identically zero [see Eqs. (81) and (82)].

D. Possible experimental tests

In the previous sections, we have seen how in generic metric-affine backgrounds the standard GR notion of propagating a locally orthonormal tetrad along an observer worldline fails, and, hence, the usual construction of a local Minkowski frame (as well as the local special-relativistic interpretation of nongravitational physics) is obstructed. This setup provides an intrinsic and geometric indication that the Einstein EP is spoiled.

When the nonmetricity tensor is treated, on laboratory scales, as a weak and slowly varying *background* field, its presence effectively introduces preferred directions in the spacetime for freely falling observers, leading to orientation- and boost-dependent signatures in local measurements [146]. Despite differing from the approach presented in Refs. [60,61], this mechanism offers a direct link between nonmetricity and possible departures from local Lorentz invariance. A context where this scenario is systematically addressed is furnished by the standard-model extension, an effective field theory originally motivated by potential Lorentz-breaking phenomenological effects in string theory [147,148]. This framework is based on a Lagrangian containing all parametrized coordinate-independent Lorentz-violating terms that can be formed from known fields [147,148].

Recently, it has been argued that a background nonmetricity can induce an effective Lorentz violation in a laboratory, even when the underlying gravitational theory with nonmetricity is locally Lorentz invariant [146]. Building on this claim, a correspondence can be established between the spacetime nonmetricity and the operators responsible for Lorentz noninvariance in the standard-model extension. In this way, existing experimental bounds on effective Lorentz-symmetry breaking can be translated into laboratory constraints on nonmetricity components with high sensitivity [146]. These bounds restrict viable metric-affine models and may be interpreted as indirect constraints on Einstein-EP departures induced by nonmetricity.

Among the most relevant probes of local Lorentz invariance worth mentioning are modern Michelson-Morley-type experiments, first devised in Ref. [149], where the isotropy of the speed of light is explored by comparing the frequencies of orthogonal cryogenic optical resonators over extended periods of time, exploiting the Earth rotation. Such measurements are naturally interpreted within the standard-model-extension pattern and can be mapped onto bounds on Lorentz-violating coefficients, which, in turn, can be translated into restrictions on the nonmetricity.

More generally, precision tests of Lorentz symmetry can, therefore, serve as high-precision probes of nonmetricity. These include clock-comparison (Hughes-Drever-type

[150]) experiments [151], space-clock missions such as ACES [20], resonant-cavity experiments [152,153], spin-precession measurements [154], atom-interferometric tests [155,156], and, more recently, space-based gravitational-wave detectors [157]. Together, these experimental platforms provide a complementary and powerful avenue for assessing the validity of Einstein EP within MAG.

VI. DISCUSSION AND CONCLUSIONS

In this paper, we have investigated possible EP violations in MAG. Specifically, we have shown that universality of free fall can fail, and we have provided a proof of the breakdown of the Einstein EP in its modern formulation.

First, we have outlined the main geometrical and gauge-theoretic features of MAG in Sec. II. Then, in Sec. III, we have explained how thermal radiative corrections in finite-temperature QFT and classical effective gravity models spoil the EP by entailing a mismatch between the gravitational and inertial masses of a point particle evolving in a Riemannian background. We have then proved that these effects admit a natural geometric analog in the broader MAG framework.

Within the wide range of scenarios permitted by MAG, Sec. IV A demonstrates that the dynamics of a test body possessing a dilation charge offers a particularly clear setting for probing deviations from geodesic worldlines. The dilation current introduces an additional term in the equations of motion, resulting in a rescaling of the Newtonian law in Eq. (49), which remarkably exhibits a close similarity to the Riemannian result (33). The ensuing expressions (50)–(52) attain the same functional form as the finite-temperature formulas (25)–(27) for the ratio m_g/m_i , although their origin is different. As pointed out in Sec. IV B, the combined use of laser-ranging techniques and the software PEP could enable a test of these EP departures.

Section V highlights, *in full generality*, a direct geometric signature of the Einstein EP breakdown intrinsic to MAG by defining a novel FW derivative that extends the Riemannian FW transport prescription.

The Einstein EP is usually stated in two ways. Formulation (i) corresponds to the gauge-theoretic, elementary-matter, and purely pointwise realization of the principle relevant to minimally coupled matter and expresses the local reduction of nongravitational physics to special relativity in suitable (anholonomic) frames. In version (ii), instead, the Einstein EP is understood in modern terms as the conjunction of the weak EP, local Lorentz invariance, and local position invariance. While these notions are tantamount in standard metric theories of gravity, they need not be so in more general frameworks such as metric-affine gravity, where only (i) is retained because one can trivialize the metric and affine connection at a single event.

From a gauge-theoretic perspective, this situation stems from the extension of the gauge group from the Lorentz group $SO(1, 3)$ to the general linear group $GL(4, \mathbb{R})$, as part of the affine gauge group $A(4, \mathbb{R})$. This scenario is associated with departures from local Lorentz invariance and the emergence of nonmetricity. As a consequence, orthonormal frames can still be selected at any given spacetime event, but they cannot, in general, be extended along worldlines while preserving orthonormality, which becomes a local-at-a-point gauge choice in MAG [cf. Eq. (21)]. In contrast, in metric-compatible theories the Einstein EP is realized smoothly in local Lorentz frames at each point, as orthonormal tetrads are preserved under parallel or FW transport. In view of the analysis presented this paper, we can conclude that the local Lorentz-frame realization underlying (ii) is spoiled not only under parallel transport, but also via the new FW transport.

The FW operator introduced here is indeed tailored to MAG and permits proving that an observer cannot propagate an orthonormal tetrad basis $\{\mathbf{e}_a\}$ along his or her trajectory for two main reasons: $\mathbf{e}_0$ does not coincide with the four-velocity owing to Eq. (68); the spatial vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ fail to remain mutually orthogonal, as expressed by Eq. (71).

To the best of our knowledge, the new FW scheme (64) has not appeared in this form in the literature and, thus, can open new avenues for research on gravity in non-Riemannian geometries. The potential applications of this derivative operator, as well as the EP departures emerging in the motion of a generic hypermomentum-charged test particle in non-Riemannian backgrounds, are topics that deserve to be further examined.

ACKNOWLEDGMENTS

The authors acknowledge the support of Istituto Nazionale di Fisica Nucleare (INFN), as part of the MoonLIGHT-2 experiment in the framework of the research activities of the *Commissione Scientifica Nazionale No. 2 (CSN2)*. Authors are grateful to Marcello Miranda for fruitful discussions about nonmetricity; to Riccardo March for the critical reading of the preliminary version of the manuscript and for the useful comments provided; and to Anish Agashe for the discussion on various types of curves within the metric-affine context. S. C. thanks also the Gruppo Nazionale di Fisica Matematica (GNFM) of Istituto Nazionale di Alta Matematica (INDAM) for the support. The authors thank the anonymous referee for the useful referee report. The authors thank the support from the Agreement ASI-INFN No. 2019-15-HH.0 on “Sistemi Avanzati Laser per Applicazioni Spaziali: Attività Congiunte presso INFN-LNF” and ESA-INFN Contract No. 4000133721/21/NL/CR, INFN code TTA_20LNF_140.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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