Erratum: Operator ordering in the relativistic quantization: Specific heat in the Rindler frame [Phys. Rev. D 112, 044034 (2025)]

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Further analysis corrects the low-energy Airy perturbation calculation used in the Letter. We provide the resulting formal low-energy spectral expression within the same canonical-quantization prescription and independently establish that the heat capacity depends on ordering in the full square-root model specified below.

In the low-energy Rindler Hamiltonian, Eq. (13) of the Letter, the term proportional to $\eta \coloneqq \mathrm{Im}\,\gamma$ is a connection term. At the perturbative order used in Eq. (C9) of the Letter, its first-order correction vanishes, while the Thomas-Reiche-Kuhn sum rule [1] gives at second order

$$\Delta E^{(2)}_{n,\eta} = -\eta^2 \frac{\hbar^2 g^2}{2mc^4}, \tag{1}$$

which is independent of the Airy level n. This state-independent shift changes the internal energy by an additive constant but leaves the heat capacity unchanged. Therefore, the η-dependent part of Fig. 2 of the Letter is not a prediction of the low-energy Airy calculation performed there: at that order, the ordering-sensitive contribution is only a common spectral shift and drops out of C_V. The matrix-element evaluation in Eqs. (C10)–(C16) must use normalized quantum-bouncer eigenfunctions on the half-line; the whole-line integrals and discrete-delta approximations used there do not give the corresponding Airy matrix elements. The normalization of states in Eq. (C4) equals $|\mathrm{Ai}'(-a_n)|$.

The conclusion above applies only to the order retained in the original calculation. To obtain the next ordering-sensitive contribution within the same quantization prescription, we use the following low-energy expansion of the bulk Hamiltonian and perform the constant momentum shift $\hat{p} = \hat{q} - \eta\frac{\hbar g}{c^2}$, obtaining

$$\begin{aligned}\hat{H}_\eta &= mc^2 + \frac{\hat{p}^2}{2m} + mg\hat{x} + \eta\frac{\hbar g}{mc^2}\hat{p} - \frac{\hat{p}^4}{8m^3c^2} + \frac{g}{4mc^2}\{\hat{x},\hat{p}^2\} - \eta\frac{\hbar g}{2m^3c^4}\hat{p}^3 + \cdots \\ &\simeq mc^2 - \frac{\eta^2\hbar^2 g^2}{2mc^4} + \left(1 + \frac{3\eta^2\hbar^2 g^2}{2m^2c^6}\right)\frac{\hat{q}^2}{2m} + \left(1 + \frac{\eta^2\hbar^2 g^2}{2m^2c^6}\right)mg\hat{x} - \frac{\hat{q}^4}{8m^3c^2} + \frac{g}{4mc^2}\{\hat{x},\hat{q}^2\} - \frac{\eta\hbar g^2}{2mc^4}\{\hat{x},\hat{q}\}. \end{aligned} \tag{2}$$

The first four terms of Eq. (2) comprise an η-dependent Airy Hamiltonian plus a state-independent shift; the next two are ordinary relativistic corrections. The remaining term linear in η has a vanishing diagonal matrix element in every unperturbed stationary Airy state, $\langle n|\{\hat{x},\hat{q}\}|n\rangle = m\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}^2\rangle_n = 0$, while its second-order spectral contribution is of order $(\eta\frac{\hbar g}{mc^3})^2\frac{\epsilon^2_{n,0}}{mc^2}$ and is among the orders omitted below. The momentum shift is generated by a local unitary phase and preserves $\psi(0) = 0$.

We now compute corrections to the unperturbed Airy spectrum $\epsilon_{n,0}$. The state-independent terms mc^2 and $-\frac{\eta^2\hbar^2 g^2}{2mc^4}$ drop out of the specific heat. Keeping both the leading η-dependent ordering correction and the η-independent relativistic terms in Eq. (2) gives for $\epsilon_{n,0} = mgLa_n$,

$$\epsilon^{\mathrm{low}}_n(\eta) = \left[1 + \frac{5}{6}\eta^2\left(\frac{\hbar g}{mc^3}\right)^2\right]\epsilon_{n,0} + \frac{\epsilon^2_{n,0}}{30mc^2}, \tag{3}$$

where state-independent shifts have been omitted. The factor $\frac{5}{6}$ follows from the Airy rescaling of the kinetic and linear-potential coefficients in Eq. (2). A direct quadratic-form evaluation on the half-line, retaining the boundary term generated by the higher-derivative operators, gives the same coefficient $\frac{5}{6}$.

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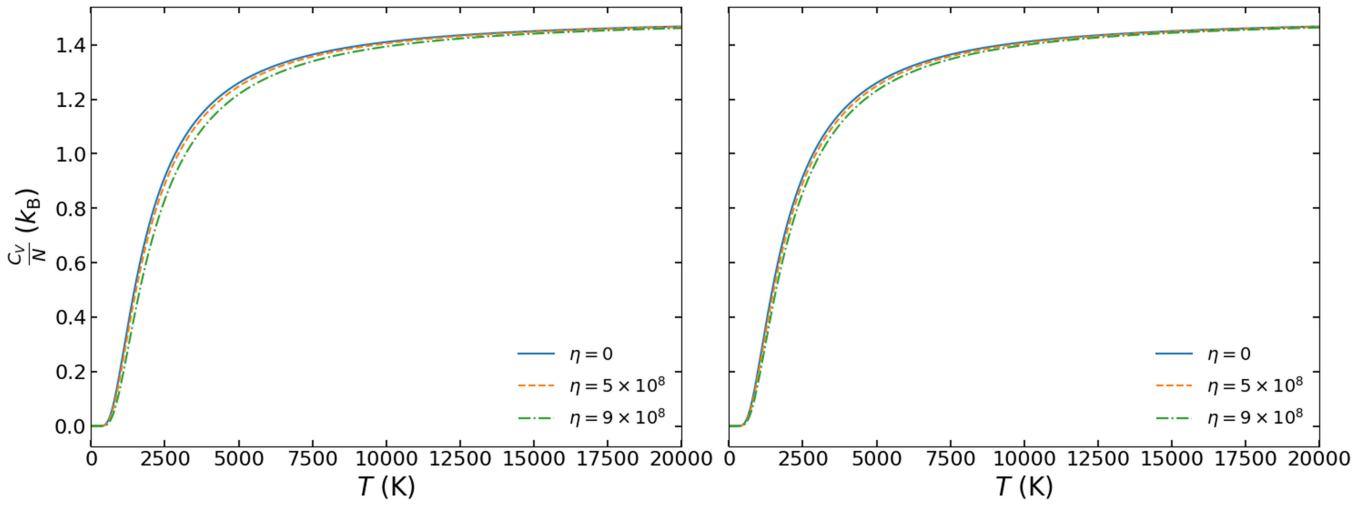


FIG. 1. Specific heat per particle for $m = m_e$, $g = 10^{20}\,\frac{\mathrm{m}}{\mathrm{s}^2}$, and $\eta \in \{0, 5 \times 10^8, 9 \times 10^8\}$ calculated from (left) the corrected low-energy spectrum in Eq. (3) and (right) the unexpanded square-root Hamiltonian with the spectral Dirichlet Laplacian.

We checked independently that this term is not an artifact of expanding the relativistic square root. We diagonalized Eq. (10) of the Letter in a finite Dirichlet cavity $0 < x < X$, using the sine eigenbasis $\phi_\ell(x) = \sqrt{\frac{2}{X}}\sin(\frac{\ell\pi x}{X})$ of the Dirichlet Laplacian $p_D^2 = -\hbar^2\partial_x^2$ with $\psi(0) = 0 = \psi(X)$, so that $\sqrt{m^2c^4 + c^2p_D^2}$ is evaluated spectrally without expansion. The half-line limit was approached by increasing the cavity size together with the number of retained modes. For $m = m_e$ and $g = 10^{20}$ ms^{-2}, the full-root first-gap shift agrees with Eq. (3) to better than 0.2% for $\eta \leq 5 \times 10^7$, while for $\eta = 5 \times 10^8$ it is about 13% smaller. Higher relativistic orders retain, but modify, the ordering effect.

Figure 1 compares the heat capacity obtained from Eq. (3) with the unexpanded square-root calculation for $\eta \in \{0, 5 \times 10^8, 9 \times 10^8\}$. Numerical convergence was checked independently with respect to the cavity size, sine basis, spectral cutoff, and thermal truncation; the largest change in the specific heat per particle under these refinements was below 3×10^{-6} over the plotted range. The maximal difference between the full-root and low-energy results are approximately $0.0039k_\mathrm{B}$ and $0.028k_\mathrm{B}$ for $\eta = 5 \times 10^8$ and 9×10^8, respectively.

Further corrections to the Letter are required. First, in Eq. (A3), the coefficient $1 - 2i$ Im α should be replaced by $1 - 2\,\mathrm{Re}\,\alpha$, while the coefficient $1 - 2i$ Im α in Eqs. (8), (9), (A4), and (A9) should read $1 + 2i$ Im α. Second, the Taylor expansions in Eqs. (7) and (A5), understood as coordinate expansions in the chosen canonical variables, should include the factor $\frac{1}{k!}$ in each term of order k. To make the temporal-ordering convention consistent with Eq. (10), γ and γ^* must also be interchanged in Eq. (5), and γ^* replaced by γ in Eq. (9). This convention change leaves the even-in-η results above unchanged.

Third, in Eq. (20) of the Letter, $\langle E\rangle = \frac{1}{Z}\sum_n E_n e^{-\beta E_n}$ and $\langle E^2\rangle = \frac{1}{Z}\sum_n E_n^2 e^{-\beta E_n}$.

Finally, in Appendix B of the Letter, the semiclassical partition function should be evaluated using the measure $\frac{\mathrm{d}x\mathrm{d}p}{2\pi\hbar}$ and the Weyl symbol of the Hamiltonian over $x \geq 0$, consistently with the quantum-bouncer boundary condition $\psi(0) = 0$, with leading nonrelativistic symbol $\frac{p^2}{2m} + mgx$, yielding $C_V = \frac{3}{2}Nk_B$ in the nonrelativistic semiclassical limit instead of the value in Eqs. (15) and (B4).

These corrections revise the quantitative predictions of the Letter, while retaining its canonical-quantization prescription. They identify the next retained ordering-dependent low-energy coefficient and establish independently that the heat-capacity dependence on ordering does not vanish identically in the specified full square-root model.

[1] M. Belloni and R. W. Robinett, J. Phys. A **42**, 075203 (2009).