

Massive tree-level splitting functions beyond kinematical limits

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We present a compact form of the massive $1 \rightarrow 3$ tree-level QCD splitting functions and discuss a decomposition of the results in terms of lower-order expressions, scalar dipole antenna functions and pure higher-order remainders. The two-gluon radiator functions introduced in this context are novel and generalize expressions obtained from the double-soft approximation. Our results are obtained without reference to soft or quasicollinear limits.

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I. INTRODUCTION

Final states with large multiplicities of jets initiated by massive partons are of particular interest to the future of the Large Hadron Collider (LHC) experimental program [1]. Both the top quark and Higgs boson preferentially decay via bottom quarks. Higgs decays to charm quark pairs have also grown in their relevance at the high-luminosity LHC (HL-LHC), following the rapid development of jet flavor tagging algorithms powered by sophisticated machine learning techniques [2,3]. Early versions of these heavy quark taggers primarily relied on identifying the decay products of heavy hadrons produced in the jets, while modern taggers exploit differences between the fragmentation patterns of light and heavy partons [3–6]. These algorithms have advanced to the point that processes originally thought impossible to study at the HL-LHC—namely, the production of Higgs boson pairs decaying via bottom quarks—will now be observed with high significance before the end of Run 5 [7]. The development of improved theory predictions that include mass effects is thus highly motivated.

At the same time, QCD jet evolution at the LHC has been measured to astonishing precision [8–19]. While the large dynamic range originating in the high center-of-mass energy of the collider makes it possible to treat the effects of heavy quark masses as a somewhat secondary problem today, the ever-increasing experimental precision will soon lead to scenarios where massless approximations can no longer be justified. Improved theory for heavy quarks should also be included consistently within the parton

shower Monte Carlo algorithms [20,21] used both to measure jet dynamics and to train the fully supervised algorithms that empower physics analyses. This will be necessary in particular to avoid potential biases and uncertainty due to the model dependence of classifiers [22–24].

In this work, we compute some important components needed for precision theory in heavy quark final states: the tree-level three-parton splitting functions and the scalar dipole antenna functions including mass effects. While the splitting functions are already known [25,26], we present them in a significantly more compact form than existing literature, which will alleviate their integration into massive infrared subtraction schemes, higher logarithmic resummation, and parton shower simulations, and provide faster evaluation and improved numerical stability in practical applications. The massive scalar dipole radiators are computed for the first time, as is the decomposition of the splitting functions into scalar radiators and pure splitting remainders.

The factorization of QCD scattering amplitudes is at the heart of most modern techniques to compute higher-order QCD corrections to scattering processes at colliders. At fixed order in perturbation theory, it allows one to construct infrared subtraction algorithms that enable the extraction of singularities from the real and virtual corrections in order to make each component finite and amenable to Monte Carlo integration over the final-state phase space [27,28]. In resummed perturbation theory, it enables the derivation of anomalous dimensions that govern the scaling of cross sections with changing resolution scale. For particles with masses, the factorization typically relies on the quasicollinear limit, which requires a simultaneous scaling of the collinear transverse momentum and the parton masses. This procedure retains the correct soft behavior of collinear scattering amplitudes at leading power, but it is somewhat unphysical, because the on-shell parton masses should

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remain, in fact, constant. As a direct consequence, one observes a noncommutativity of limits [29–31] that can only be resolved by accounting for the physical features of the scattering matrix elements lost in the limiting procedure. Here we use a different concept, which avoids the introduction of kinematical limits altogether. It is based on the observation that infrared singularities arise as a consequence of degenerate asymptotic states [32,33], such that QCD splitting functions can be computed as reduced matrix elements in a physical gauge [34–38]. Their leading singularities are determined by the semiclassical approximation [39,40], which can be derived from splitting functions in a scalar QCD, with the scalars playing the role of the semiclassical particles. This eventually allows one to derive any splitting function in terms of scalar multipole radiator functions and pure splitting remainders with a lower degree of divergence in the kinematical limits [41].

This manuscript is organized as follows: In Sec. II we provide a brief introduction to the method and compute the one-to-two massive splitting functions as simple examples. Section III presents the new results for one-to-three splitting functions with massive partons, as well as the two-emission dipole antenna radiator functions with massive partons. The decomposition of the splitting functions is discussed in Sec. V, and Sec. VI presents an outlook.

II. BACKGROUND, TECHNIQUES, AND LEADING-ORDER RESULTS

The motivation for our work stems from the simple yet important observation that the soft and collinear limits of QCD scattering amplitudes do not commute beyond leading power. To exemplify this, we consider the one-to-two parton splitting for a massive quark in the quasicollinear limit. The splitting function reads

$$\langle P_{q \rightarrow qg}(z) \rangle = C_F \left[\frac{2z}{1-z} - \frac{m_q^2}{p_q p_g} + (1-\varepsilon)(1-z) \right],$$

where p_q and p_g are the quark and gluon momenta and z is the light-cone momentum fraction of the quark with respect to the parent quark. The soft component of this splitting function is given by the $z \rightarrow 1$ limit. At leading power in the soft scaling parameter it reads

$$\langle P_{q \rightarrow q}(z) \rangle \xrightarrow{z \rightarrow 1} C_F \left[\frac{2}{1-z} - \frac{m_q^2}{p_q p_g} \right].$$

If this limit was physically meaningful, we should be able to first apply it to a complete matrix element including the quark and the gluon, and then take the quasicollinear limit on the result. The leading-power soft limit of a QCD matrix element is given in terms of the square of the soft-gluon current, $\mathbf{J}^\mu$ [42]. For a process involving a heavy quark and

a gluon in the final state, the quark-gluon quasicollinear limit on the squared current leads to

$$\mathbf{J}^\mu \mathbf{J}_\mu \xrightarrow{q \parallel g} \propto C_F \left[\frac{2z}{1-z} - \frac{m_q^2}{p_q p_g} \right].$$

The difference to the leading-power soft limit of the quasicollinear result is obvious. This simple example shows that the two kinematical limits do not commute, even at the lowest nontrivial order. The reason is that leading terms in one of the limits are subleading in the other. As a direct consequence, the leading-power expressions obscure the factorization of higher-order QCD splitting functions, and an extension to higher powers becomes mandatory to establish a systematic factorization of matrix elements at higher orders in the coupling. We will therefore refrain from using any kinematical approximation at all, and base our calculation on a different principle.

The key observation of our method is that the leading components of splitting functions in a gauge theory with spin are given in terms of the semiclassical expressions, which coincide with the results of the corresponding scalar theory [39,40]. This allows us to separate out the spin-dependent components, which do not contribute to a dipole radiation pattern. The physical picture is consistent with the well-known decomposition of the current into a charge current and a polarization current [43]. In Ref. [41], this idea was used to define a new technique to assemble splitting functions from scalar dipole radiators, lower-order splitting functions, and pure higher-order splitting remainders. Here we will extend this method to massive quarks. Because no collinear or soft limits are taken, there is no question as to how to define their extension to the quasicollinear region, where the on-shell particle masses would have to be scaled proportional to the transverse momentum. This in turn resolves common problems which arise in the definition of polarizations of the external particles and makes the splitting functions unambiguous.

It is important to note that the splitting remainders we compute are finite in all hard collinear regions for fixed parton masses due to the dead-cone effect [30,44,45]. Therefore, an infrared subtraction scheme can be devised without explicit knowledge of those quantities. However, for small parton masses, QCD scattering matrix elements containing such splittings develop logarithms of the hard scale over the parton masses, which should be resummed to correctly account for heavy jet dynamics. Treating such logarithms on the same footing with standard collinear poles also improves the behavior of (semi)numerical fixed-order calculations [46].

A. Tree-level techniques

In this section we introduce the techniques needed to perform the calculation of the tree-level splitting functions. We also derive the lowest-order results, which will be used

in Sec. III to decompose the splitting functions at higher order. Similar approaches, working directly at the level of helicity amplitudes, were introduced in [47,48]. To achieve a meaningful factorization of the results, we use physical polarization states [34–38,49–53]. The polarization sum for virtual gluons is defined in terms of a lightlike axial gauge, which is known to be ghost free [54–58],

$$\begin{aligned} d^{\mu\nu}(p, \bar{n}) &= \sum_{\lambda=\pm} \epsilon_{\lambda}^{\mu}(p, \bar{n}) \epsilon_{\lambda}^{*\nu}(p, \bar{n}) \\ &= -g^{\mu\nu} + \frac{p^{\mu} \bar{n}^{\nu} + p^{\nu} \bar{n}^{\mu}}{p \bar{n}}. \end{aligned} \quad (1)$$

Here $\bar{n}^{\mu}$ is an auxiliary lightlike vector. In the case of on-shell external partons, the polarization states are always

physical. However, the factorized expressions in Sec. V will sometimes involve external partons that are off mass shell. In such cases, we construct the corresponding on-shell momentum needed to define physical wave functions with the help of the auxiliary vector, $\bar{n}^{\mu}$. For a parton with momentum p^{μ} and on-shell mass m , this shifted momentum reads

$$\bar{p}^{\mu} = p^{\mu} - \frac{p^2 - m^2}{2p \bar{n}} \bar{n}^{\mu}. \quad (2)$$

Our calculations are based on the techniques outlined in [53]. In this method, the splitting amplitudes for quarks or gluons into l external partons with momenta $p_1 \dots p_l$ are given by

$$\begin{aligned} P_q^{ss'}(1, \dots, l) &= \delta^{ss'} \left(\frac{s_{1\dots l} - m_{1\dots l}^2}{8\pi\alpha_s \mu^{2\epsilon}} \right)^{l-1} \frac{\text{Tr}[\bar{n}\Psi(\{p_1, \dots, p_l\})\bar{\Psi}(\{p_1, \dots, p_l\})]}{\text{Tr}[\bar{n}\Psi(\{\bar{p}_{1\dots l}\})\bar{\Psi}(\{\bar{p}_{1\dots l}\})]}, \\ P_g^{\mu\nu}(1, \dots, l) &= \frac{D-2}{2} \left(\frac{s_{1\dots l}}{8\pi\alpha_s \mu^{2\epsilon}} \right)^{l-1} \frac{d^{\mu\rho}(p_{1\dots l}, \bar{n}) J_{\rho}(\{p_1, \dots, p_l\}) J_{\sigma}^{\dagger}(\{p_1, \dots, p_l\}) d^{\sigma\nu}(p_{1\dots l}, \bar{n})}{d^{\kappa\lambda}(\bar{p}_{1\dots l}, \bar{n}) J_{\lambda}(\{\bar{p}_{1\dots l}\}) J_{\tau}^{\dagger}(\{\bar{p}_{1\dots l}\}) d_{\kappa}^{\tau}(\bar{p}_{1\dots l}, \bar{n})}. \end{aligned} \quad (3)$$

Here, $s_{1\dots l} = (p_1 + \dots + p_l)^2$ is the virtuality of the l -parton final state, and $m_{1\dots l}$ is the on-shell mass of the incoming parton. The indices s and s' refer to the quark spin, while the gluon polarization indices are given by μ and ν . The tree-level currents Ψ and J can be determined using recursive techniques such as the Berends-Giele method [59–62]. We compute them in the lightlike axial gauge, Eq. (1). Only the expression for the quark current differs from the massless case discussed in [41], and the difference amounts to a trivial change in the quark propagator. We find

$$\begin{aligned} \Psi_i(p_{\alpha}) &= \sum_{\substack{\{\beta, \gamma\} \in \\ P(\alpha, 2)}} g_s T_{ij}^a \frac{i\sigma^{\mu\nu}}{p_{\alpha}^2 - m_{\alpha}^2} p_{\gamma, \nu} J_{\mu}^a(p_{\gamma}, n) \Psi_j(p_{\beta}) \\ &+ \sum_{\substack{\{\beta, \gamma\} \in \\ P(\alpha, 2)}} \left[g_s T_{ij}^a S^{\mu}(p_{\beta}, p_{\gamma}) J_{\mu}^a(p_{\gamma}, n) - \sum_{\substack{\{\delta, \epsilon\} \in \\ OP(\gamma, 2)}} \frac{g_s^2}{p_{\alpha}^2} \{T^a, T^b\}_{ij} J^{\mu, a}(p_{\delta}, n) J_{\mu}^b(p_{\epsilon}, n) \right] \Psi_j(p_{\beta}), \end{aligned} \quad (4)$$

$$\begin{aligned} J_{\mu}^a(p_{\alpha}, n) &= \sum_{\substack{\{\beta, \gamma\} \in \\ P(\alpha, 2)}} \left[g_s \frac{F_{bc}^a}{2} D_{\mu}(p_{\beta}, p_{\gamma}) J^{\rho, b}(p_{\beta}, n) J_{\rho}^c(p_{\gamma}, n) + g_s T_{ij}^a \bar{\Psi}_i(p_{\gamma}) d^{\mu\nu}(p_{\alpha}, n) \frac{\gamma_{\nu}}{p_{\alpha}^2} \Psi_j(p_{\beta}) \right] \\ &- \sum_{\substack{\{\beta, \gamma\} \in \\ P(\alpha, 2)}} \left[g_s F_{ab}^c S^{\sigma}(p_{\beta}, p_{\gamma}) J_{\sigma}^c(p_{\gamma}, n) - \sum_{\substack{\{\delta, \epsilon\} \in \\ OP(\gamma, 2)}} \frac{g_s^2}{p_{\alpha}^2} \{F^c, F^d\}_{ab} J^{\sigma, c}(p_{\delta}, n) J_{\sigma}^d(p_{\epsilon}, n) \right] d_{\mu}^{\nu}(p_{\alpha}, n) J_{\nu}^b(p_{\beta}, n). \end{aligned} \quad (5)$$

The expressions for the scalar production and decay vertices are given in complete analogy to the massless case

$$S^{\mu}(p_i, p_j) = \frac{(2p_i + p_j)^{\mu}}{p_{ij}^2 - m_{ij}^2}, \quad D^{\mu}(p_i, p_j, \bar{n}) = \frac{d_{\nu}^{\mu}(p_{ij}, \bar{n})}{p_{ij}^2} (p_i - p_j)^{\nu}. \quad (6)$$

To obtain compact expressions for the splitting functions, it will be convenient to use a Sudakov decomposition of the momenta [63]. We will restrict this decomposition to its most generic form, in particular we will not rely on an explicit definition of the transverse components. For two momenta, p_i^{μ} and p_j^{μ} , we have

$$p_i^{\mu} = z_i p_{ij}^{\mu} - \tilde{p}_{i,j}^{\mu}, \quad \text{and} \quad p_j^{\mu} = z_j p_{ij}^{\mu} - \tilde{p}_{j,i}^{\mu}, \quad (7)$$

where $p_{ij}^{\mu} = p_i^{\mu} + p_j^{\mu}$. The individual Sudakov components are

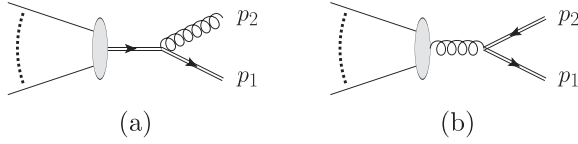


FIG. 1. Feynman diagrams contributing to the one-to-two parton splittings with massive quarks computed in Secs. II B (a) and II C (b). Double lines indicate a quark of on-shell mass m .

$$z_i = \frac{p_i \bar{n}}{p_{ij} \bar{n}}, \quad \text{and} \quad \tilde{p}_{i,j}^\mu = \frac{z_i p_j^\mu - z_j p_i^\mu}{z_i + z_j}. \quad (8)$$

This decomposition generalizes naturally to the many-parton case.

B. Quark initial state

We begin by deriving the splitting functions for one to two partons. The corresponding diagrams are shown in Fig. 1. Using Eq. (3) and the recursive definition of the quark current in Eq. (4), we find the following expression for the case of a quark-to-quark-gluon splitting,

$$\begin{aligned} P_{q \rightarrow q}(p_1, p_2) &= \frac{1}{8\pi\alpha_s \mu^{2\epsilon}} \frac{\text{Tr}[\tilde{n}\Psi(\{p_1, p_2\})\tilde{\Psi}(\{p_1, p_2\})]}{\text{Tr}[\tilde{n}\Psi(\{\bar{p}_{12}\})\tilde{\Psi}(\{\bar{p}_{12}\})]} \\ &= P_{\tilde{q} \rightarrow \tilde{q}}(p_1, p_2) + P_{q \rightarrow q}^{(f)}(p_1, p_2). \end{aligned} \quad (9)$$

Its scalar and purely fermionic components are defined as

$$\begin{aligned} P_{\tilde{q} \rightarrow \tilde{q}}(p_1, p_2) &= C_F \left[\frac{2z_1}{z_2} \left(1 - \frac{p_1^2 - m_1^2}{p_{12}^2 - m_1^2} \frac{z_{12}}{z_1} - \frac{p_2^2}{p_{12}^2 - m_1^2} \frac{z_{12}}{z_2} \right) - \frac{2m_1^2}{p_{12}^2 - m_1^2} \right], \\ \langle P_{q \rightarrow q}^{(f)}(p_1, p_2) \rangle &= C_F (1 - \epsilon) \left(\frac{z_2}{z_{12}} - \frac{z_2}{z_1} \frac{p_1^2 - m_1^2}{p_{12}^2 - m_1^2} - \frac{p_2^2}{p_{12}^2 - m_1^2} \right). \end{aligned} \quad (10)$$

These expressions are valid both for on-shell and for off-shell partons. In order to achieve a physically meaningful definition of the external wave functions in the off-shell region, we have used the shifted momentum in Eq. (2) to compute the gluon polarization tensor, $d^{\mu\nu}(p_2, \bar{n}) \rightarrow d^{\mu\nu}(\bar{p}_2, \bar{n}) = d^{\mu\nu}(p_2, \bar{n}) - p_2^\mu \bar{n}^\nu / (p\bar{n})^2$. In Sec. V we will also need the gluon-spin dependent form of Eq. (10). The complete quark-to-quark splitting tensor is given by

$$P_{q \rightarrow q}^{\mu\nu}(p_1, p_2) = P_{\tilde{q} \rightarrow \tilde{q}}^{\mu\nu}(p_1, p_2) + P_{q \rightarrow q}^{(f)\mu\nu}(p_1, p_2). \quad (11)$$

The scalar and purely fermionic components are

$$\begin{aligned} P_{\tilde{q} \rightarrow \tilde{q}}^{\mu\nu}(p_1, p_2) &= \frac{C_F}{2} (p_{12}^2 - m^2) S^\mu(p_1, p_2) S^\nu(p_1, p_2), \\ P_{q \rightarrow q}^{(f)\mu\nu}(p_1, p_2) &= -\frac{C_F}{2} \left[g^{\mu\nu} \left(\frac{z_2}{z_{12}} - \frac{z_2}{z_1} \frac{p_1^2 - m^2}{p_{12}^2 - m^2} - \frac{p_2^2}{p_{12}^2 - m^2} \right) + \frac{p_2^\mu p_2^\nu}{p_{12}^2 - m^2} + \frac{\bar{p}_{12}^\mu \bar{p}_{12}^\nu - p_{12}^\mu p_{12}^\nu}{p_{12}^2 - m^2} + \frac{\bar{p}_1^\mu \bar{p}_{12}^\nu - p_1^\mu p_{12}^\nu}{p_{12}^2 - m^2} \right], \end{aligned} \quad (12)$$

Note that the scalar result is schematically identical to the massless case [41]. It only differs in the denominator factors, which is a trivial modification arising from the difference in the retarded Green's function for massive particles. The spin-dependent terms of the splitting function exhibit minor modifications, which account for the change in the off-shellness of the heavy quark.

C. Gluon initial state

The second quark-mass-dependent one-to-two reaction we need to compute is the gluon-to-quark-antiquark splitting shown in Fig. 1(b). Using Eq. (3) and the recursive definition of the current in Eq. (5), we find the following expression:

$$P_{g \rightarrow q}^{\mu\nu}(p_1, p_2) = \frac{T_R}{2p_{12}^2} d^\mu_\rho(p_{12}, \bar{n}) \text{Tr}[(\not{\bar{p}}_1 \pm m) \gamma^\rho (\not{\bar{p}}_2 \mp m) \gamma^\sigma] d^\nu_\sigma(p_{12}, \bar{n}). \quad (13)$$

A straightforward calculation, making use of the fact that the final-state masses must be identical in QCD, gives

$$P_{g \rightarrow q}^{\mu\nu}(p_1, p_2) = T_R \left[d^{\mu\nu}(\bar{p}_{12}, \bar{n}) \left(1 - \frac{z_{12}}{p_{12}^2} \left(\frac{p_1^2 - m^2}{z_1} + \frac{p_2^2 - m^2}{z_2} \right) \right) + p_{12}^2 \frac{\bar{n}^\mu \bar{n}^\nu}{(p_{12} \bar{n})^2} - p_{12}^2 D^\mu(p_1, p_2, \bar{n}) D^\nu(p_1, p_2, \bar{n}) \right]. \quad (14)$$

In the on-shell case, and upon using a standard Sudakov decomposition of the momenta [63], Eq. (14) reduces to the familiar (massless) DGLAP splitting kernel. It is interesting to observe that the effects of parton masses in this particular case are purely kinematical. Note that Eq. (14) is still independent of the kinematics parametrization. Its spin averaged form reads

$$\langle P_{g \rightarrow q}(p_1, p_2) \rangle = T_R \left[1 - \frac{z_{12}}{p_{12}^2} \left(\frac{p_1^2 - m^2}{z_1} + \frac{p_2^2 - m^2}{z_2} \right) - \frac{2}{1 - \varepsilon} \left(\frac{z_1 z_2}{z_{12}^2} - \frac{z_1 p_1^2 + z_2 p_2^2}{z_{12} p_{12}^2} \right) \right]. \quad (15)$$

We will use the above functions in Sec. V to write the higher-order splitting functions in terms of lower-order results.

III. ONE-TO-THREE SPLITTING FUNCTIONS

In this section we provide the expressions for the one-to-three parton splittings. Our calculations are based on the

methods introduced in [41,53], in particular the use of the lightlike axial gauge, Eq. (1). We employ Eqs. (3)–(5) to determine the splitting functions in terms of Berends-Giele currents. All results computed in this subsection are known; however, we present them in a more compact form than the existing literature. We have cross-checked the expressions against Refs. [25,26] and verified that they reduce to the massless case derived in Refs. [53,64] when the parton masses are set to zero.

A. All-quark splitting functions

The flavor-changing quark-to-quark splitting function can be obtained from the product of the off-shell quark splitting function in Eq. (11) and the spin-dependent gluon-to-quark splitting function, Eq. (14). The corresponding Feynman diagram is shown in Fig. 2(a). Note that we do not make any assumptions about the on-shell quark masses, in particular m_3 can differ from m_1 . However, in QCD we always have $m_1 = m_2$. The result is

$$\begin{aligned} \langle P_{q \rightarrow q'q}(p_1, p_2, p_3) \rangle &= \frac{C_F T_R}{2} \frac{s_{123} - m_3^2}{s_{12}} \left[-\frac{t_{12,3}^2}{s_{12}(s_{123} - m_3^2)} + \frac{4z_3 + (z_1 - z_2)^2}{z_1 + z_2} - \frac{4m_3^2}{s_{123} - m_3^2} \right. \\ &\quad \left. + \left(1 - 2\varepsilon + \frac{4m_1^2}{s_{12}} \right) \left(z_1 + z_2 - \frac{s_{12}}{s_{123} - m_3^2} \right) \right]. \end{aligned} \quad (16)$$

The variable $t_{12,i}$ is defined as an extension of Eq. (22) in [53] to the massive on-shell case, using the fact that in QCD the final-state parton masses for particles 1 and 2 must be equal,¹

$$t_{12,3} = (s_{123} - m_{123}^2) S^\mu(p_3, p_{12}) s_{12} D_\mu(p_1, p_2, \vec{n}) = 2 \frac{z_1 \tilde{s}_{23} - z_2 \tilde{s}_{13}}{z_1 + z_2} + \frac{z_1 - z_2}{z_1 + z_2} s_{12}. \quad (17)$$

We have also defined the invariant masses $s_{ij} = (p_i + p_j)^2$ and the scalar products $\tilde{s}_{ij} = 2p_i p_j$. The extension of the scalar products to the three-parton case is given by $\tilde{s}_{ijk} = \tilde{s}_{ij} + \tilde{s}_{ik} + \tilde{s}_{jk}$. Equation (16) contains a component arising from the scalar radiator in Eq. (6) and the gluon splitting function in Eq. (14). It can be interpreted as a squark-to-quark splitting function and is given by

$$P_{\tilde{q} \rightarrow q'q}(p_1, p_2, p_3) = \frac{C_F T_R}{2} \frac{s_{123} - m_3^2}{s_{12}} \left[\frac{4z_3}{z_1 + z_2} - \frac{4m_3^2}{s_{123} - m_3^2} + \frac{s_{12}}{s_{123} - m_3^2} \left(1 - \frac{t_{12,3}^2}{s_{12}^2} \right) \right]. \quad (18)$$

The first one-to-three parton splitting function with a nontrivial structure is the same flavor-quark-to-three-quark splitting. There are two contributing diagrams, shown in Figs. 2(b) and 2(c). They lead to two terms of the form of Eq. (16) and an interference term,

$$\langle P_{q \rightarrow \bar{q}qq}(p_1, p_2, p_3) \rangle = (\langle P_{q \rightarrow q'q}(p_1, p_2, p_3) \rangle + \langle P_{q \rightarrow \bar{q}qq}^{(\text{id})}(p_1, p_2, p_3) \rangle) + (2 \leftrightarrow 3). \quad (19)$$

The interference contribution is given in terms of the function

¹Equation (17) differs from $t_{12,i}$ in Ref. [25] because our definition is based on the diagrammatical origin of the variable.

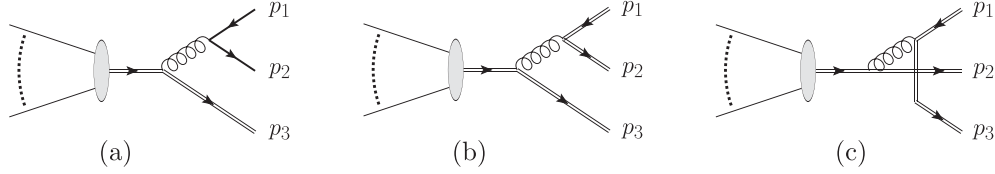


FIG. 2. Feynman diagrams contributing to the one-to-three all-quark splittings with massive quarks computed in Sec. III A. (a) Process with different on-shell masses for partons 1/2 and 3 (indicated by the double line and the thick line in the figure). (b),(c) Processes with identical on-shell masses for all quarks.

$$\begin{aligned}
 \langle P_{q \rightarrow \bar{q}qq}^{(\text{id})}(p_1, p_2, p_3) \rangle = & C_F \left(C_F - \frac{C_A}{2} \right) \left\{ -\frac{(s_{123} - m^2)^2}{s_{12}s_{13}} \frac{z_1}{2} \left[\frac{1 + z_1^2}{(1 - z_2)(1 - z_3)} - \epsilon \left(1 + 2 \frac{1 - z_2}{1 - z_3} \right) - \epsilon^2 \right] \right. \\
 & + \frac{s_{123} - m^2}{s_{12}} \left[\frac{1 + z_1^2}{1 - z_2} - \frac{2z_2}{1 - z_3} - \epsilon \left(\frac{(1 - z_3)^2}{1 - z_2} + 1 + z_1 - \frac{2z_2}{1 - z_3} \right) - \epsilon^2(1 - z_3) \right] + (1 - \epsilon) \left(\frac{2s_{23}}{s_{12}} - \epsilon \right) \\
 & + \frac{m^2}{s_{12}s_{13}} \left[s_{23} \frac{(1 + z_1)(z_1 - z_2z_3)}{(1 - z_2)(1 - z_3)} + \epsilon(z_1(s_{123} - m^2) + s_{23} + 6s_{12}) \right] \\
 & \left. - \frac{2m^2}{s_{13}} \frac{(1 + z_1)(1 + z_2z_3) - (z_2 - z_3)^2}{(1 - z_2)(1 - z_3)} + \frac{4m^4}{s_{12}s_{13}} \left(\frac{2z_2}{1 - z_3} - \frac{z_1(1 + z_2)}{1 - z_2} - \epsilon \right) \right\}. \quad (20)
 \end{aligned}$$

B. Gluon emission off quarks

In this section we present the splitting function describing gluon radiation off massive quarks. Following Ref. [53], we separate the Abelian from the non-Abelian component according to

$$\langle P_{q \rightarrow gqg}(p_1, p_2, p_3) \rangle = \langle P_{q \rightarrow gqg}^{(\text{ab})}(p_1, p_2, p_3) \rangle + \langle P_{q \rightarrow gqg}^{(\text{nab})}(p_1, p_2, p_3) \rangle. \quad (21)$$

The relevant Feynman diagrams are shown in Fig. 3. The result for the Abelian component is

$$\begin{aligned}
 \langle P_{q \rightarrow gqg}^{(\text{ab})}(p_1, p_2, p_3) \rangle = & C_F^2 \left\{ \frac{\tilde{s}_{123}^2}{2\tilde{s}_{13}\tilde{s}_{23}} z_3 \left[\frac{1 + z_3^2}{z_1z_2} - \epsilon \frac{z_1^2 + z_2^2}{z_1z_2} - \epsilon(1 + \epsilon) - \frac{4m^2}{\tilde{s}_{123}} \left(\frac{1 - z_3}{z_1z_2} + (1 - \epsilon) \frac{z_1}{z_3} \right) \right] \right. \\
 & + \frac{\tilde{s}_{123}}{\tilde{s}_{13}} \left[\frac{z_3(1 - z_1) + (1 - z_2)^3}{z_1z_2} + \epsilon^2(1 + z_3) - \epsilon(z_1^2 + z_1z_2 + z_2^2) \frac{1 - z_2}{z_1z_2} - \frac{4m^2}{\tilde{s}_{13}} \left(\frac{1 - z_2}{z_2} + (1 - \epsilon) \frac{z_2}{2} \right) \right] \\
 & \left. + \epsilon(1 - \epsilon) - \frac{\tilde{s}_{23}}{\tilde{s}_{13}} (1 - \epsilon)^2 + \frac{4m^2}{\tilde{s}_{13}} + \frac{1}{2} \left(\frac{2m^2}{\tilde{s}_{13}} + \frac{2m^2}{\tilde{s}_{23}} \right)^2 \right\} + (1 \leftrightarrow 2). \quad (22)
 \end{aligned}$$

The non-Abelian part of the splitting function is given by

$$\begin{aligned}
 \langle P_{q \rightarrow gqg}^{(\text{nab})}(p_1, p_2, p_3) \rangle = & -\frac{C_A}{2C_F} P_{q \rightarrow gqg}^{(\text{ab})}(p_1, p_2, p_3) + C_F C_A \left\{ \frac{1 - \epsilon}{4} \left(\frac{t_{12,3}^2}{s_{12}^2} + 1 \right) - (1 - \epsilon)^2 \frac{\tilde{s}_{23}}{2\tilde{s}_{13}} + \frac{\tilde{s}_{123}^2}{2s_{12}\tilde{s}_{13}} \right. \\
 & \times \left[\frac{(1 - z_3)^2(1 - \epsilon) + 2z_3}{z_2} + \frac{z_2^2(1 - \epsilon) + 2(1 - z_2)}{1 - z_3} - \frac{4m^2}{\tilde{s}_{123}} \left(1 + \frac{z_1^2}{z_2(1 - z_3)} \right) \right] \\
 & + \frac{\tilde{s}_{123}}{2s_{12}} \left[(1 - \epsilon) \frac{z_1(2 - 2z_1 + z_1^2) - z_2(6 - 6z_2 + z_2^2)}{z_2(1 - z_3)} + 2\epsilon \frac{z_3(z_1 - 2z_2) - z_2}{z_2(1 - z_3)} + \frac{4m^2}{\tilde{s}_{123}} \right] \\
 & + \frac{\tilde{s}_{123}}{2\tilde{s}_{13}} \left[(1 - \epsilon) \frac{(1 - z_2)^3 + z_3^2 - z_2}{z_2(1 - z_3)} - \epsilon \frac{2(1 - z_2)(z_2 - z_3)}{z_2(1 - z_3)} - \epsilon(1 - \epsilon)(1 - z_1) \right. \\
 & \left. - \frac{4m^2}{\tilde{s}_{13}} \left(\frac{1 - z_2}{z_2} + (1 - \epsilon) \frac{z_2}{2} \right) \right] + \frac{2m^2}{\tilde{s}_{13}} + \frac{2m^4}{\tilde{s}_{13}^2} + (1 \leftrightarrow 2) \left. \right\}. \quad (23)
 \end{aligned}$$

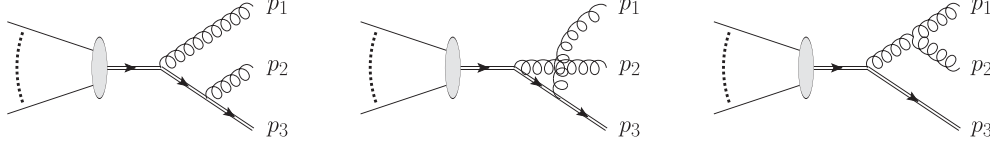


FIG. 3. Feynman diagrams contributing to the one-to-three gluon emission splittings with massive quarks, computed in Sec. III B. The double line indicates a quark of mass m .

C. Gluon emission off scalars

The results of Sec. III B are the only heavy quark splittings that overlap with the two-gluon scalar dipole radiator function. The general form of this dipole radiator will be computed in Sec. IV B. In order to prepare the discussion of its physical features, we present here the scalar splitting functions, which contribute to Eqs. (22) and (23). The Abelian component is

$$P_{\bar{q} \rightarrow g g \bar{q}}^{(\text{ab})}(p_1, p_2, p_3) = C_F^2 \left\{ \frac{\tilde{s}_{123}^2}{\tilde{s}_{13}\tilde{s}_{23}} \left[\frac{z_3^2}{z_1 z_2} - \frac{2m^2 z_3(1-z_3)}{\tilde{s}_{123} z_1 z_2} \right] + \frac{\tilde{s}_{123}}{\tilde{s}_{13}} \left[\frac{2z_3(1-z_2)}{z_1 z_2} - \frac{4m^2(1-z_2)}{\tilde{s}_{13} z_2} \right] \right. \\ \left. + (1-\epsilon) + \frac{4m^2}{\tilde{s}_{13}} + \frac{1}{2} \left(\frac{2m^2}{\tilde{s}_{13}} + \frac{2m^2}{\tilde{s}_{23}} \right)^2 \right\} + (1 \leftrightarrow 2). \quad (24)$$

In the non-Abelian case, we obtain

$$P_{\bar{q} \rightarrow g g \bar{q}}^{(\text{nab})}(p_1, p_2, p_3) = -\frac{C_A}{2C_F} P_{\bar{q} \rightarrow g g \bar{q}}^{(\text{ab})}(p_1, p_2, p_3) + C_F C_A \left\{ \frac{1-\epsilon}{4} \left(\frac{t_{12,3}^2}{s_{12}^2} + 1 \right) \right. \\ \left. + \frac{\tilde{s}_{123}^2}{s_{12}\tilde{s}_{13}} \left[\frac{z_3}{z_2} + \frac{1-z_2}{1-z_3} - \frac{2m^2}{\tilde{s}_{123}} \frac{z_1^2}{z_2(1-z_3)} \right] + \frac{\tilde{s}_{123}}{\tilde{s}_{13}} \frac{1-z_2}{z_1} \left[\frac{z_3}{z_2} - \frac{1-z_2}{1-z_3} - \frac{2m^2 z_1}{\tilde{s}_{13} z_2} \right] \right. \\ \left. + \frac{\tilde{s}_{123}}{s_{12}} \left[\frac{z_3}{z_2} - \frac{1+3z_3}{1-z_3} + \frac{2m^2}{\tilde{s}_{123}} \right] - \frac{\tilde{s}_{123}^2}{\tilde{s}_{13}\tilde{s}_{23}} \left[\frac{m^2}{s_{12}} - \frac{m^2}{\tilde{s}_{123}} \right] + \frac{2m^2}{\tilde{s}_{13}} + \frac{2m^4}{\tilde{s}_{13}^2} + (1 \leftrightarrow 2) \right\}. \quad (25)$$

D. Gluon splitting to quarks

In this section we quote the gluon splitting functions into a quark-antiquark pair and a gluon. Following the notation in [53], we separate the result into an Abelian and a non-Abelian component:

$$P_{g \rightarrow g q \bar{q}}^{\mu\nu}(p_1, p_2, p_3) = P_{g \rightarrow g q \bar{q}}^{\mu\nu(\text{ab})}(p_1, p_2, p_3) + P_{g \rightarrow g q \bar{q}}^{\mu\nu(\text{nab})}(p_1, p_2, p_3). \quad (26)$$

The Feynman diagrams for this process are shown in Fig. 4. The Abelian component is given by

$$P_{g \rightarrow g q \bar{q}}^{\mu\nu(\text{ab})}(p_1, p_2, p_3) = C_F T_R \left\{ d^{\mu\nu}(p_{123}, \vec{n}) \left[2s_{123} \left(\frac{\tilde{s}_{23}}{\tilde{s}_{12}\tilde{s}_{13}} - \frac{m^2}{\tilde{s}_{12}^2} - \frac{m^2}{\tilde{s}_{13}^2} \right) + (1-\epsilon) \left(\frac{\tilde{s}_{12}}{\tilde{s}_{13}} + \frac{\tilde{s}_{13}}{\tilde{s}_{12}} \right) - 2\epsilon \right] \right. \\ \left. + \frac{4\tilde{s}_{123}}{\tilde{s}_{12}\tilde{s}_{13}} \left(\tilde{p}_{2,13}^\mu \tilde{p}_{3,12}^\nu + \tilde{p}_{3,12}^\mu \tilde{p}_{2,13}^\nu \right) + 8m^2 \left(\frac{\tilde{p}_{2,13}^\mu \tilde{p}_{2,13}^\nu}{\tilde{s}_{13}^2} + \frac{\tilde{p}_{3,12}^\mu \tilde{p}_{3,12}^\nu}{\tilde{s}_{12}^2} \right) - (1-\epsilon) \frac{4s_{123}}{\tilde{s}_{12}\tilde{s}_{13}} \tilde{p}_{1,23}^\mu \tilde{p}_{1,23}^\nu \right\}. \quad (27)$$

The non-Abelian part is given by

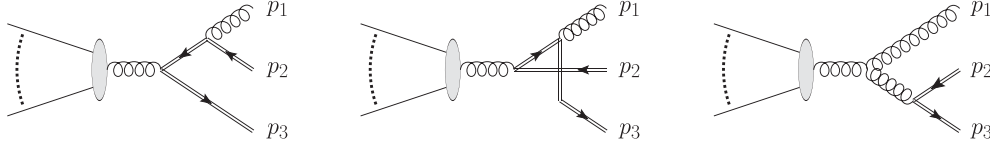


FIG. 4. Feynman diagrams contributing to the one-to-three gluon splitting into massive quarks computed in Sec. III D. The double line indicates a quark of mass m .

$$\begin{aligned}
 P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{nab})}(p_1, p_2, p_3) = & -\frac{C_A}{2C_F} P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{ab})}(p_1, p_2, p_3) + \frac{C_A T_R}{4} \left\{ \frac{\tilde{s}_{123}}{s_{23}^2} \left[-d^{\mu\nu}(p_{123}, \bar{n}) \frac{t_{23,1}^2}{\tilde{s}_{123}} - 16 \frac{1-z_1}{z_1} \tilde{p}_{2,3}^\mu \tilde{p}_{2,3}^\nu \right] \right. \\
 & + 16m^2 \left[\frac{\tilde{p}_{2,3}^\mu \tilde{p}_{2,3}^\nu}{z_1} \left(\frac{z_1}{\tilde{s}_{12}^2} - \frac{2(1-z_1)}{s_{23}^2} \right) - \frac{\tilde{p}_{1,23}^\mu \tilde{p}_{2,3}^\nu + \tilde{p}_{2,3}^\mu \tilde{p}_{1,23}^\nu}{1-z_1} \left(\frac{z_2}{\tilde{s}_{13}^2} + \frac{1-z_1}{\tilde{s}_{13}s_{23}} \right) + \frac{\tilde{p}_{1,23}^\mu \tilde{p}_{1,23}^\nu}{(1-z_1)^2} \left(\frac{z_2}{\tilde{s}_{13}} + \frac{1-z_1}{s_{23}} \right)^2 \right] \\
 & + d^{\mu\nu}(p_{123}, \bar{n}) \left[\frac{2\tilde{s}_{13}}{\tilde{s}_{12}} (1-\epsilon) + \frac{2\tilde{s}_{123}}{\tilde{s}_{12}} \left(\frac{1-z_3}{z_1(1-z_1)} - 2 \right) + \frac{2s_{123}}{s_{23}} \frac{1-z_1+2z_1^2}{z_1(1-z_1)} - 1 - \frac{4m^2 s_{123}}{\tilde{s}_{13}^2} \right. \\
 & - \left. \frac{4m^2}{\tilde{s}_{13}} \left(2 - \frac{1-z_2}{z_1(1-z_1)} \right) \right] + \frac{s_{123}}{\tilde{s}_{12}s_{23}} \left[2s_{123} d^{\mu\nu}(p_{123}, \bar{n}) \frac{z_2(1-2z_1)}{z_1(1-z_1)} - 16\tilde{p}_{3,12}^\mu \tilde{p}_{3,12}^\nu \frac{z_2^2}{z_1(1-z_1)} \right. \\
 & \left. \left. + 8(1-\epsilon) \tilde{p}_{2,13}^\mu \tilde{p}_{2,13}^\nu + 4(\tilde{p}_{2,13}^\mu \tilde{p}_{3,12}^\nu + \tilde{p}_{3,12}^\mu \tilde{p}_{2,13}^\nu) \left(\frac{2z_2(z_3-z_1)}{z_1(1-z_1)} + 1 - \epsilon \right) \right] + (2 \leftrightarrow 3) \right\}. \quad (28)
 \end{aligned}$$

Here, $\tilde{p}_{i,j}^\mu$ are the generalized transverse momenta defined in Eq. (8). They fulfill momentum conservation in the form $\tilde{p}_{i,j}^\mu + \tilde{p}_{j,i}^\mu = 0$ for any i and j , as well as $\tilde{p}_{1,23}^\mu + \tilde{p}_{2,13}^\mu + \tilde{p}_{3,12}^\mu = 0$. We have not listed explicitly $\bar{n}$ -dependent components of the splitting functions, as they vanish when multiplied by the polarization tensor of the intermediate gluon.

IV. SCALAR RADIATOR FUNCTIONS

In this section we compute the scalar radiator functions, which describe the production of one or more partons through coherent radiation off a QCD dipole antenna. In contrast to the splitting functions in Secs. II and III, they account for the precise direction of flight of the color anticharge, which must accompany the initial state charged parton in a splitting configuration such as Figs. 1(a) or 3 due to color conservation.

The simplest such function is the one-gluon radiator shown in Fig. 5. Using the formalism and notation of [42,65] generalized to the scalar QCD case [41], the dipole approximation to the complete radiator function for a single gluon can be written in terms of color insertion operators, $\mathbf{T}$, and a space-time-dependent part, $\mathcal{S}_{i,k}$, as follows:

$$S_g(\{p\}; q_1; \bar{n}) = \sum_{i,k} \hat{\mathbf{T}}_i \hat{\mathbf{T}}_k \mathcal{S}_{i,k}(q_1; \bar{n}). \quad (29)$$

For massive emitters and an on-shell gluon with momentum q_1 , the function $\mathcal{S}_{i,k}$ is given by

$$\begin{aligned}
 \mathcal{S}_{i,k}(q_1; \bar{n}) = & \frac{1}{p_{i1}^2 - m_i^2} \frac{2z_i}{z_1} \left(1 - \frac{p_k^2 - m_k^2}{p_{k1}^2 - m_k^2} \right) + \frac{1}{p_{k1}^2 - m_k^2} \frac{2z_k}{z_1} \\
 & \times \left(1 - \frac{p_i^2 - m_i^2}{p_{i1}^2 - m_i^2} \right) - \frac{4p_i p_k}{(p_{i1}^2 - m_i^2)(p_{k1}^2 - m_k^2)}. \quad (30)
 \end{aligned}$$

In order to simplify the notation in Sec. V, we also introduce a compact notation for the gluon-spin-dependent radiator function. It is given by

$$\begin{aligned}
 S_g^{\mu\nu}(\{p\}; q_1) = & \sum_{i,k} \hat{\mathbf{T}}_i \hat{\mathbf{T}}_k S_{i,k}^{\mu\nu}(q_1), \\
 \text{where } S_{i,k}^{\mu\nu}(q_1) = & S^\mu(p_i, q_1) S^\nu(p_k, q_1). \quad (31)
 \end{aligned}$$

For the decomposition of the two-gluon splitting functions in Sec. V, we will need the off-shell radiator [41] for massive particles. It is determined by using the momentum shift in Eq. (2) to define the external wave function of the

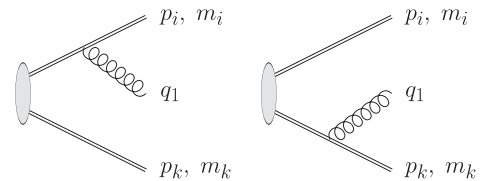


FIG. 5. Feynman diagrams contributing to the scalar radiator function for a single gluon emission. The double lines indicate quark masses of the scalar particles.

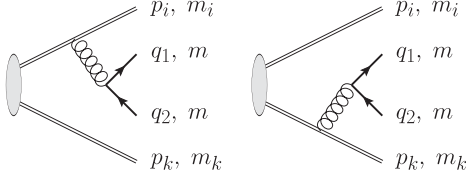


FIG. 6. Feynman diagrams contributing to the scalar radiator for a quark-antiquark pair computed in Sec. IV A. The double line indicates quark masses of the scalar particles, while the thick solid line indicates the masses of the emitted quarks.

gluon. Its space-time-dependent part reads

$$\bar{S}_{i;k}(q_1; \bar{n}) = \frac{1}{p_{i1}^2 - m_i^2} \frac{2z_i}{z_1} \left(1 - \frac{p_k^2 - m_k^2}{p_{k1}^2 - m_k^2} - \frac{q_1^2}{p_{k1}^2 - m_k^2} \frac{z_{k1}}{z_1} \right) - \frac{2p_i p_k}{(p_{i1}^2 - m_i^2)(p_{k1}^2 - m_k^2)} + (i \leftrightarrow k). \quad (32)$$

The squark splitting function in Eq. (10) is a special case of this radiator, $P_{\bar{q} \rightarrow \bar{q}}(p_1, p_2) = C_F(p_{12}^2 - m_1^2) \bar{S}_{1;1}(p_2; \bar{n})/2$.

A. Quark-antiquark radiator

The scalar dipole approximation to radiation of a quark-antiquark pair is obtained from the product of the spin-dependent one-gluon radiator in Eq. (31) and the gluon-to-quark splitting tensor in Eq. (14). The relevant Feynman diagrams are shown in Fig. 6. Due to the factorized form of the result, the color decomposition is

identical to the single gluon emission case in Eq. (31). It reads

$$\mathcal{S}_{q\bar{q}}(\{p\}; q_1, q_2; \bar{n}) = \sum_{i,k} \hat{\mathbf{T}}_i \hat{\mathbf{T}}_k T_R \mathcal{S}_{i;k}^{(q\bar{q})}(q_1, q_2; \bar{n}). \quad (33)$$

The space-time-dependent component is given by

$$\mathcal{S}_{i;k}^{(q\bar{q})}(q_1, q_2; \bar{n}) = \frac{2}{(s_{i12} - m_i^2)(s_{k12} - m_k^2)} \times \left[\frac{2}{s_{12}} \left(\frac{z_i(s_{k12} - m_k^2) + z_k(s_{i12} - m_i^2)}{z_1 + z_2} - \tilde{s}_{ik} \right) + 1 - \frac{t_{12,i} t_{12,k}}{s_{12}^2} \right]. \quad (34)$$

For $i = k$, this result is proportional to the scalar splitting function in Eq. (18), cf. Eq. (42).

B. Two-gluon radiator

The dipole approximation to the two-gluon radiator function has a more complicated structure. The relevant Feynman diagrams are shown in Fig. 7. To understand its form, it is helpful to write the result in terms of the two-gluon current [53]. It is interesting to note that this current is only minimally modified compared to its massless counterpart. For an on-shell radiator we find

$$J_{ab}^{\mu\nu}(q_1, q_2) = \sum_{i,k} \{ \hat{\mathbf{T}}_i^a, \hat{\mathbf{T}}_k^b \} \mathcal{J}_{ik}^{(ab),\mu\nu}(q_1, q_2) + \sum_i i f^{abc} \hat{\mathbf{T}}_i^c \mathcal{J}_i^{(nab),\mu\nu}(q_1, q_2), \quad (35)$$

with the color-stripped Abelian and non-Abelian components,

$$\begin{aligned} \mathcal{J}_{ik}^{(ab),\mu\nu}(q_1, q_2) &= \frac{1}{2} S^\mu(p_i, q_1) S^\nu(p_k, q_2) \\ &\quad + \frac{\delta_{ik}}{(p_i + q_{12})^2 - m_i^2} [q_1^\mu S^\mu(p_i, q_1) + q_2^\mu S^\nu(p_i, q_2) - q_1 q_2 S^\mu(p_i, q_1) S^\nu(p_i, q_2) - g^{\mu\nu}], \\ \mathcal{J}_i^{(nab),\mu\nu}(q_1, q_2) &= S_\rho(p_i, q_{12}) (d^{\rho\nu}(q_{12}) S^\mu(q_2, q_1) - d^{\rho\mu}(q_{12}) S^\nu(q_1, q_2)) \\ &\quad + \frac{1}{(p_i + q_{12})^2 - m_i^2} \left[q_1^\mu S^\mu(p_i, q_1) - q_2^\mu S^\nu(p_i, q_2) + p_i(q_2 - q_1) S^\mu(p_i, q_1) S^\nu(p_i, q_2) + \frac{t_{12,i}}{q_{12}^2} g^{\mu\nu} \right]. \end{aligned} \quad (36)$$

Changes compared to the massless result only occur due to the definition of $t_{12,i}$ [cf. Eq. (17)], and due to the difference in the propagators. The squared two-gluon current is given by [41]

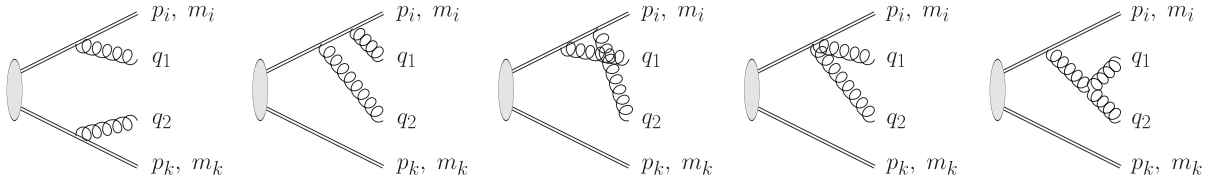


FIG. 7. Feynman diagrams contributing to the scalar radiator for two gluons, computed in Sec. IV B. The double line indicates quark masses of the scalar particles.

$$\begin{aligned}
[J_{\mu\nu}^{ab}(q_1, q_2)]^\dagger d^{\mu\rho}(q_1, \bar{n}) d^{\nu\sigma}(q_2, \bar{n}) J_{\rho\sigma}^{ab}(q_1, q_2) = & 2 \sum_{i,k} \sum_{l,m} \{ \hat{\mathbf{T}}_i^a \hat{\mathbf{T}}_l^a, \hat{\mathbf{T}}_k^b \hat{\mathbf{T}}_m^b \} \mathcal{S}_{i,k;l,m}^{(ab)}(q_1, q_2; \bar{n}) \\
& + 2 \sum_{i,k} \sum_l \{ \{ \hat{\mathbf{T}}_i^a \hat{\mathbf{T}}_l^a, \hat{\mathbf{T}}_k^b \hat{\mathbf{T}}_l^b \} + \{ \hat{\mathbf{T}}_l^a \hat{\mathbf{T}}_i^a, \hat{\mathbf{T}}_l^b \hat{\mathbf{T}}_k^b \} \} \mathcal{S}_{i,k;l}^{(ab)}(q_1, q_2; \bar{n}) \\
& + 2 \sum_{i,l} \{ \hat{\mathbf{T}}_i^a \hat{\mathbf{T}}_l^a, \hat{\mathbf{T}}_i^b \hat{\mathbf{T}}_l^b \} \mathcal{S}_{i;l}^{(ab)}(q_1, q_2) \\
& - \sum_{i,l} C_A \hat{\mathbf{T}}_i^c \hat{\mathbf{T}}_l^c [\mathcal{S}_{i;l}^{(\text{nab})}(q_1, q_2; \bar{n}) - (1 - 2\delta_{il}) \mathcal{S}_{i;l}^{(ab)}(q_1, q_2) \\
& - \mathcal{S}_{i,i;l}^{(ab)}(q_1, q_2; \bar{n}) - \mathcal{S}_{l,i;i}^{(ab)}(q_1, q_2; \bar{n}) \\
& + \mathcal{S}_{i,l;i}^{(ab)}(q_1, q_2; \bar{n}) + \mathcal{S}_{l,i;i}^{(ab)}(q_1, q_2; \bar{n}) + \mathcal{S}_{i,i;l}^{(ab)}(q_1, q_2; \bar{n}) + \mathcal{S}_{l,i;l}^{(ab)}(q_1, q_2; \bar{n})], \quad (37)
\end{aligned}$$

where we have introduced the Abelian and non-Abelian radiator functions, $\mathcal{S}(q_1, q_2; \bar{n})$. For massive on-shell particles they are defined as an extension of the results in [41],

$$\begin{aligned}
\mathcal{S}_{i,k;l,m}^{(ab)}(q_1, q_2; \bar{n}) &= \frac{1}{4} \mathcal{S}_{i;l}(q_1; \bar{n}) \mathcal{S}_{k;m}(q_2; \bar{n}), \\
\mathcal{S}_{i,k;l}^{(ab)}(q_1, q_2; \bar{n}) &= \frac{1}{\tilde{s}_{l12} \tilde{s}_{i1} \tilde{s}_{k2}} \left(\frac{\tilde{s}_{k1}}{s_{kl} \tilde{s}_{l1}} + \frac{\tilde{s}_{i2}}{s_{il} \tilde{s}_{l2}} - \frac{s_{12}}{\tilde{s}_{l1} \tilde{s}_{l2}} \right) \\
&+ \frac{1}{\tilde{s}_{l12}} \left[\frac{z_l}{z_2} \left(\frac{\tilde{s}_{il} s_{12} - \tilde{s}_{i2} \tilde{s}_{l1}}{\tilde{s}_{i1} \tilde{s}_{l1} \tilde{s}_{l2}} - \frac{z_l s_{12}}{2 z_1 \tilde{s}_{l1} \tilde{s}_{l2}} \right) + \frac{z_i \tilde{s}_{l1} + z_l \tilde{s}_{i1} - z_1 \tilde{s}_{il}}{z_2 \tilde{s}_{i1} \tilde{s}_{l1}} \right. \\
&\left. - \frac{\tilde{s}_{ik}}{2 \tilde{s}_{i1} \tilde{s}_{k2}} \left(1 + \frac{m_k^2 + m_l^2}{\tilde{s}_{ik}} \frac{2 \tilde{s}_{i2}}{\tilde{s}_{l2}} \right) + \frac{s_{12}(s_{il} + \tilde{s}_{il})(m_k^2 + m_l^2)}{2 \tilde{s}_{i1} \tilde{s}_{k2} \tilde{s}_{l1} \tilde{s}_{l2}} + \left(\begin{array}{c} (i \leftrightarrow k) \\ (1 \leftrightarrow 2) \end{array} \right) \right], \\
\mathcal{S}_{i;k}^{(ab)}(q_1, q_2) &= \frac{(\tilde{s}_{i1} \tilde{s}_{k2} - \tilde{s}_{i2} \tilde{s}_{k1})^2 - 2 s_{12} \tilde{s}_{ik} (\tilde{s}_{i1} \tilde{s}_{k2} + \tilde{s}_{i2} \tilde{s}_{k1}) + s_{12}^2 \tilde{s}_{ik}^2}{\tilde{s}_{i12} \tilde{s}_{k12} \tilde{s}_{i1} \tilde{s}_{k1} \tilde{s}_{i2} \tilde{s}_{k2}} + \frac{2 s_{12}}{\tilde{s}_{i12} \tilde{s}_{k12}} \left(\frac{m_i^2}{\tilde{s}_{i1} \tilde{s}_{i2}} + \frac{m_k^2}{\tilde{s}_{k1} \tilde{s}_{k2}} \right) + \frac{2(1 - \varepsilon)}{\tilde{s}_{i12} \tilde{s}_{k12}}, \quad (38)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{S}_{i;k}^{(\text{nab})}(q_1, q_2; \bar{n}) &= \mathcal{S}_{i;k}^{(\text{nab},p)}(q_1, q_2; \bar{n}) - 2(1 - \varepsilon) \mathcal{S}_{i;k}^{\mu\nu}(q_{12}; \bar{n}) D_\mu(q_1, q_2, \bar{n}) D_\nu(q_1, q_2, \bar{n}) \\
&+ \frac{1}{4} (\tilde{\mathcal{S}}_{i;i}(q_{12}; \bar{n}) + \tilde{\mathcal{S}}_{k;k}(q_{12}; \bar{n}) - 2 \tilde{\mathcal{S}}_{i;k}(q_{12}; \bar{n})) \left[\mathcal{S}_{i;i}(q_2; \bar{n}) + \mathcal{S}_{1;1}(q_2; \bar{n}) - 2 \mathcal{S}_{i;1}(q_2; \bar{n}) \right. \\
&\left. - \frac{1}{2} (\mathcal{S}_{i;i}(q_2; \bar{n}) + \mathcal{S}_{k;k}(q_2; \bar{n}) - 2 \mathcal{S}_{i;k}(q_2; \bar{n})) + (i \leftrightarrow k) + (1 \leftrightarrow 2) + \left(\begin{array}{c} (i \leftrightarrow k) \\ (1 \leftrightarrow 2) \end{array} \right) \right], \quad (39)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}_{ik}^{(\text{nab},p)}(q_1, q_2; \bar{n}) = & \left[\frac{3\tilde{s}_{ik}}{4\tilde{s}_{i1}\tilde{s}_{k1}} + \frac{2s_{12}}{z_{12}^2} \left(\frac{z_i}{\tilde{s}_{i12}} - \frac{z_k}{\tilde{s}_{k12}} \right) \left(\frac{z_{i12}}{\tilde{s}_{i12}} - \frac{z_{k12}}{\tilde{s}_{k12}} \right) \right] \left(\frac{2\tilde{s}_{i1}}{s_{12}\tilde{s}_{i2}} - \frac{\tilde{s}_{ik}}{\tilde{s}_{i2}\tilde{s}_{k2}} \right) \\
& + \frac{1}{\tilde{s}_{i12}\tilde{s}_{k12}} \left\{ \frac{3}{2} + \frac{3}{2} \frac{\tilde{s}_{ik}^2}{\tilde{s}_{i1}\tilde{s}_{k1}} + \frac{\tilde{s}_{ik}}{2s_{12}} - \frac{z_2}{z_1} + \frac{\tilde{s}_{k1}\tilde{s}_{i2} - \tilde{s}_{ik}s_{12}}{2\tilde{s}_{i1}\tilde{s}_{k2}} \left(1 - \frac{\tilde{s}_{ik}}{s_{12}} \right) + \frac{\tilde{s}_{ik}}{\tilde{s}_{i1}} \left(1 - \frac{3\tilde{s}_{i12}}{s_{12}} \right) \right. \\
& - \frac{\tilde{s}_{i2}}{\tilde{s}_{i1}} \frac{z_i}{z_2} \left(2 + \frac{z_1}{z_i} + \frac{\tilde{s}_{i12}}{\tilde{s}_{i2}} + \frac{\tilde{s}_{k12}}{\tilde{s}_{i2}} \right) + \frac{2\tilde{s}_{k1}}{s_{12}} \left(\frac{z_1 - z_2}{z_1} - \frac{z_2 - z_i}{z_2} - \frac{z_i\tilde{s}_{i12}}{z_2\tilde{s}_{i1}} + \frac{\tilde{s}_{i2}}{\tilde{s}_{i1}} - \frac{\tilde{s}_{i1}}{\tilde{s}_{i2}} \frac{2z_i + z_2}{z_1} \right) \\
& - \frac{z_{12}^2}{2z_1z_2} \frac{t_{12,i}t_{12,k}}{s_{12}^2} - \frac{t_{12,k}}{2s_{12}} \left[\frac{\tilde{s}_{i1}}{\tilde{s}_{i2}} \frac{z_{i2}}{z_1} \left(1 - \frac{z_i s_{12}}{z_2\tilde{s}_{i1}} \right) + \frac{2z_{12}}{z_1} \left(1 + \frac{\tilde{s}_{i1}}{s_{12}} \right) + \frac{3z_i}{z_2} - (1 \leftrightarrow 2) \right] \\
& - \frac{m_i^2}{\tilde{s}_{i1}} \left[\frac{2t_{12,k}}{s_{12}} - \frac{2s_{ik}s_{12}}{\tilde{s}_{i2}^2} \left(1 + \frac{s_{12}}{\tilde{s}_{k1}} + \frac{\tilde{s}_{i2}\tilde{s}_{k1}}{\tilde{s}_{i1}\tilde{s}_{k2}} \right) \right] \left. \right\} + \frac{z_i^2}{2\tilde{s}_{i1}\tilde{s}_{i2}z_1z_2} \left(1 + \frac{\tilde{s}_{i1} - \tilde{s}_{i2}\tilde{s}_{k1} - \tilde{s}_{k2}}{\tilde{s}_{i12}} \right) \\
& + \frac{4m_i^2}{\tilde{s}_{i12}^2} \frac{\tilde{s}_{i1}\tilde{s}_{k2} + \tilde{s}_{i2}\tilde{s}_{k1} - \tilde{s}_{ik}s_{12}}{s_{12}\tilde{s}_{i1}\tilde{s}_{k1}} + \frac{m_i^2}{\tilde{s}_{i1}\tilde{s}_{i12}} \left[\frac{4}{\tilde{s}_{k2}} \left(\frac{s_{ik}}{\tilde{s}_{i2}} - \frac{\tilde{s}_{k1}}{s_{12}} + \frac{\tilde{s}_{k2}z_1}{s_{12}z_2} - \frac{\tilde{s}_{k1}}{2\tilde{s}_{i1}} \right) - \frac{1}{\tilde{s}_{i2}} \frac{2z_i}{z_1} + \frac{1}{\tilde{s}_{i1}} \frac{2z_{i1}}{z_2} \right. \\
& - \frac{2\tilde{s}_{i1} + s_{12}}{\tilde{s}_{i1}} \left(\frac{m_i^2}{\tilde{s}_{i2}^2} + \frac{2(m_i^2 + m_k^2)}{\tilde{s}_{i2}\tilde{s}_{k2}} \right) \left. \right] + (i \leftrightarrow k) + (1 \leftrightarrow 2) + \left(\begin{matrix} (i \leftrightarrow k) \\ (1 \leftrightarrow 2) \end{matrix} \right). \quad (40)
\end{aligned}$$

As in the massless parton case, the radiators define the splitting functions in Eqs. (24) and (25). We find

$$\begin{aligned}
P_{\bar{q} \rightarrow g\bar{q}\bar{q}}^{(\text{ab})}(p_1, p_2, p_i) &= (s_{i12} - m_i^2)^2 C_F^2 (\mathcal{S}_{i,i;i}^{(\text{ab})}(p_1, p_2; \bar{n}) + 2\mathcal{S}_{i,i;i}^{(\text{ab})}(p_1, p_2; \bar{n}) + \mathcal{S}_{i,i}^{(\text{ab})}(p_1, p_2)), \\
P_{\bar{q} \rightarrow g\bar{q}\bar{q}}^{(\text{nab})}(p_1, p_2, p_i) &= -(s_{i12} - m_i^2)^2 \frac{C_F C_A}{4} \mathcal{S}_{i,i}^{(\text{nab})}(p_1, p_2; \bar{n}) - \frac{C_A}{4C_F} P_{\bar{q} \rightarrow g\bar{q}\bar{q}}^{(\text{ab})}(p_1, p_2, p_i). \quad (41)
\end{aligned}$$

In the double-soft gluon limit, Eq. (39) reduces to the massive double-soft eikonal computed in Appendix B of Ref. [66] (see also [67]).

V. COMPOSITION OF ONE-TO-THREE SPLITTING FUNCTIONS

In this section we address the main question of this work, namely how the one-to-three parton splitting functions can be decomposed in terms of the scalar dipole radiators derived in Sec. IV and the lower-order splitting functions in Sec. II. The results we will derive are structurally identical to the massless case, which was determined based on the Feynman diagrams that enter the calculations. This provides a strong cross-check of our technique, as the remainder functions we will obtain are required to reduce to their massless equivalent in the case of massless external partons.² We will make this structure apparent in all our results.

A. All-quark splitting functions

We begin with the simplest case, which is the all-quark splitting function with distinct flavor quarks. Only the diagram in Fig. 2(a) contributes to this result; therefore, the splitting function admits a trivial decomposition, which is given by

$$\langle P_{q \rightarrow \bar{q}' q' q}(p_1, p_2, p_3) \rangle = \frac{s_{123} - m_3^2}{s_{12}} \left[\frac{C_F}{2} (s_{123} - m_3^2) \mathcal{S}_{3;3}^{\mu\nu}(p_{12}) + \langle P_{q \rightarrow q}^{(f)\mu\nu}(p_3, p_{12}) \rangle \right] P_{g \rightarrow q, \mu\nu}(p_1, p_2). \quad (42)$$

We note that the scalar contribution can be written in terms of the two-quark scalar radiator in Eq. (34), or in terms of the scalar splitting function, Eq. (18):

$$P_{\bar{q} \rightarrow \bar{q}' q' \bar{q}}(p_1, p_2, p_3) = \frac{C_F (s_{123} - m_3^2)^2}{2 s_{12}} \mathcal{S}_{3;3}^{\mu\nu}(p_{12}) P_{g \rightarrow q, \mu\nu}(p_1, p_2) = \frac{C_F T_R}{4} (s_{123} - m_3^2)^2 \mathcal{S}_{3;3}^{(q\bar{q})}(p_1, p_2; \bar{n}). \quad (43)$$

For the all-quark splitting function with same flavor quarks, we obtain

²No limiting procedure is involved when determining the massless results from the massive ones, the masses can simply be set to zero.

$$\begin{aligned} \langle P_{q \rightarrow \bar{q}qq}(p_1, p_2, p_3) \rangle = & \left\{ \frac{s_{123} - m^2}{s_{12}} \left[\frac{C_F}{2} (s_{123} - m^2) \mathcal{S}_{3;3}^{\mu\nu}(p_{12}) + \langle P_{q \rightarrow q}^{(f)\mu\nu}(p_3, p_{12}) \rangle \right] P_{g \rightarrow q, \mu\nu}(p_1, p_2) \right. \\ & \left. + (2 \leftrightarrow 3) \right\} + \langle P_{q \rightarrow \bar{q}qq}^{(p)}(p_1, p_2, p_3) \rangle. \end{aligned} \quad (44)$$

The pure splitting contribution is a consequence of interference between the diagrams in Figs. 2(b) and 2(c), and is given in terms of Eq. (20) as

$$\langle P_{q \rightarrow \bar{q}qq}^{(p)}(p_1, p_2, p_3) \rangle = \langle P_{q \rightarrow \bar{q}qq}^{(\text{id})}(p_1, p_2, p_3) \rangle + (2 \leftrightarrow 3). \quad (45)$$

B. Gluon emission off quarks

The quark-to-quark-gluon splitting function overlaps with the scalar dipole radiator for two-gluon emission in Sec. IV B. It allows the following decomposition [41]:

$$\begin{aligned} \langle P_{q \rightarrow gqg}^{(\text{ab})}(p_1, p_2, p_3) \rangle = & \left[\frac{C_F^2}{4} \tilde{s}_{123}^2 \mathcal{S}_{13;13}(p_2; \bar{n}) \mathcal{S}_{3;3}(p_1; \bar{n}) + \frac{C_F}{2} \frac{\tilde{s}_{123}^2}{\tilde{s}_{13}} \mathcal{S}_{13;13}(p_2; \bar{n}) \langle P_{q \rightarrow q}^{(f)}(p_3, p_{12}) \rangle \right. \\ & + \frac{C_F}{2} \tilde{s}_{123} \langle P_{q \rightarrow q}^{(f)}(p_{13}, p_2) \rangle \mathcal{S}_{3;3}(p_1; \bar{n}) + \frac{\tilde{s}_{123}}{\tilde{s}_{13}} \langle P_{q \rightarrow q}^{(f)}(p_{13}, p_2) \rangle \langle P_{q \rightarrow q}^{(f)}(p_3, p_{12}) \rangle + (1 \leftrightarrow 2) \Big] \\ & + P_{q \rightarrow gqg}^{(\text{ab}, p)}(p_1, p_2, p_3) + \langle P_{q \rightarrow gqg}^{(\text{ab}, p, f)}(p_1, p_2, p_3) \rangle, \end{aligned} \quad (46)$$

where the pure scalar and fermionic components are given by

$$\begin{aligned} P_{q \rightarrow gqg}^{(\text{ab}, p)}(p_1, p_2, p_3) = & P_{q \rightarrow gqg}^{(\text{ab})}(p_1, p_2, p_3) - \frac{C_F^2}{4} \tilde{s}_{123}^2 [\mathcal{S}_{13;13}(p_2; \bar{n}) \mathcal{S}_{3;3}(p_1; \bar{n}) + \mathcal{S}_{23;23}(p_1; \bar{n}) \mathcal{S}_{3;3}(p_2; \bar{n})] \\ = & C_F^2 \left\{ \frac{\tilde{s}_{123}^2}{\tilde{s}_{13} \tilde{s}_{23}} \left[\frac{z_3^2}{z_1 z_2} - \frac{2m^2}{\tilde{s}_{123}} \frac{z_3(1 - z_3)}{z_1 z_2} \right] - \frac{\tilde{s}_{123}}{\tilde{s}_{13}} \frac{2z_3(1 - z_2)}{z_1 z_2} + \frac{4z_3}{z_1 z_2} \right. \\ & \left. + (1 - \varepsilon) + \frac{4m^2}{\tilde{s}_{13}} \frac{z_3 - (1 - z_2)^2}{z_1 z_2} + \frac{4m^4}{\tilde{s}_{13} \tilde{s}_{23}} \right\} + (1 \leftrightarrow 2) \end{aligned} \quad (47)$$

and

$$\begin{aligned} \langle P_{q \rightarrow gqg}^{(\text{ab}, p, f)}(p_1, p_2, p_3) \rangle = & C_F^2 (1 - \varepsilon) \left\{ \frac{\tilde{s}_{123}^2}{2\tilde{s}_{13} \tilde{s}_{23}} z_3 \left(\frac{(z_1 + z_2)^2}{z_1 z_2} + \varepsilon \right) + \left(\frac{s_{12}}{\tilde{s}_{13}} + \frac{z_1 z_2}{(1 - z_2)^2} \right) (1 - \varepsilon) \right. \\ & - \frac{\tilde{s}_{123}}{\tilde{s}_{13}} \left[(1 - z_2) \left(\frac{(z_1 + z_2)^2}{z_1 z_2} + 1 \right) + (1 - \varepsilon) \left(\frac{z_1 z_2}{1 - z_2} - z_3 \right) \right] + \frac{2z_1}{z_2} \left(\frac{1}{1 - z_2} + \frac{z_3}{1 - z_1} \right) \\ & \left. - \frac{2m^2}{\tilde{s}_{13}} \frac{z_2(1 + z_3)}{1 - z_2} - \frac{m^2 s_{12}}{\tilde{s}_{13} \tilde{s}_{23}} (1 - z_3) \right\} + (1 \leftrightarrow 2). \end{aligned} \quad (48)$$

For the non-Abelian quark-to-gluon splitting, we find the following decomposition [41]:

$$\begin{aligned} \langle P_{q \rightarrow gqg}^{(\text{nab})}(p_1, p_2, p_3) \rangle = & \frac{C_F C_A}{4} \tilde{s}_{123}^2 [\bar{\mathcal{S}}_{3;3}(p_{12}; \bar{n}) (\mathcal{S}_{1;1}(p_2; \bar{n}) + \mathcal{S}_{2;2}(p_1; \bar{n}) - \mathcal{S}_{1;3}(p_2; \bar{n}) - \mathcal{S}_{2;3}(p_1; \bar{n})) \\ & + 2(1 - \varepsilon) \mathcal{S}_{3;3}^{\mu\nu}(p_{12}; \bar{n}) D_\mu(p_1, p_2, \bar{n}) D_\nu(p_1, p_2, \bar{n})] \\ & + \frac{C_A}{2} \tilde{s}_{123} [\langle P_{q \rightarrow q}^{(f)}(p_3, p_{12}) \rangle (\mathcal{S}_{1;1}(p_2; \bar{n}) + \mathcal{S}_{2;2}(p_1; \bar{n}) - \mathcal{S}_{1;3}(p_2; \bar{n}) - \mathcal{S}_{2;3}(p_1; \bar{n})) \\ & + 2(1 - \varepsilon) \langle P_{q \rightarrow q}^{(f)\mu\nu}(p_3, p_{12}) \rangle D_\mu(p_1, p_2, \bar{n}) D_\nu(p_1, p_2, \bar{n})] \\ & - \frac{C_A}{2C_F} [P_{q \rightarrow gqg}^{(\text{ab}, p)}(p_1, p_2, p_3) + \langle P_{q \rightarrow gqg}^{(\text{ab}, p, f)}(p_1, p_2, p_3) \rangle] \\ & + P_{q \rightarrow gqg}^{(\text{pnab}, p)}(p_1, p_2, p_3) + \langle P_{q \rightarrow gqg}^{(\text{pnab}, p, f)}(p_1, p_2, p_3) \rangle. \end{aligned} \quad (49)$$

where the pure scalar and fermionic components are given by

$$\begin{aligned}
P_{\bar{q} \rightarrow gg\bar{q}}^{(\text{pnab}, p)}(p_1, p_2, p_3) &= -\frac{C_F C_A}{4} \tilde{s}_{123}^2 [\mathcal{S}_{3;3}^{(\text{nab})}(p_1, p_2; \bar{n}) + \mathcal{S}_{3;3}^{(\text{ab})}(p_1, p_2) + 2\mathcal{S}_{3;3;3}^{(\text{ab})}(p_1, p_2; \bar{n}) \\
&\quad + \mathcal{S}_{3;3;3}^{(\text{ab})}(p_1, p_2; \bar{n}) + \tilde{\mathcal{S}}_{3;3}(p_{12}; \bar{n})(\mathcal{S}_{2;2}(p_1; \bar{n}) + \mathcal{S}_{1;1}(p_2; \bar{n}) - \mathcal{S}_{2;3}(p_1; \bar{n}) - \mathcal{S}_{1;3}(p_2; \bar{n})) \\
&\quad + 2(1 - \epsilon) \mathcal{S}_{3;3}^{\mu\nu}(p_{12}; \bar{n}) D_\mu(p_1, p_2, \bar{n}) D_\nu(p_1, p_2, \bar{n})] + \frac{C_A}{2C_F} P_{\bar{q} \rightarrow gg\bar{q}}^{(\text{ab}, p)}(p_1, p_2, p_3) \\
&= C_F C_A \left\{ \frac{1 - \epsilon}{4} + \left(\frac{\tilde{s}_{123}}{\tilde{s}_{13}} \frac{z_1}{1 - z_3} - 1 \right) \left(\frac{\tilde{s}_{123}}{s_{12}} \frac{1 - z_1}{z_2} - \frac{2m^2}{s_{12}} \frac{z_1}{z_2} - 1 \right) - 1 \right. \\
&\quad \left. + \left[\frac{\tilde{s}_{13} - \tilde{s}_{23}}{\tilde{s}_{13}} \frac{z_1 - z_2}{z_2} - \frac{s_{12}}{\tilde{s}_{13}} \left(\frac{1 - z_3}{z_2} + \frac{2z_3}{z_1} \right) \right] \frac{z_3}{(1 - z_3)^2} + \frac{2z_3}{z_1 z_2} - \frac{2m^2}{\tilde{s}_{13}} \frac{1}{z_2} \right\} + (1 \leftrightarrow 2)
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
\langle P_{q \rightarrow ggq}^{(\text{pnab}, p, f)}(p_1, p_2, p_3) \rangle &= C_F C_A (1 - \epsilon) \left\{ \frac{z_2}{z_1} \left(\frac{1}{1 - z_1} + \frac{z_3}{1 - z_2} \right) + \left(\frac{2z_2}{z_1} + 1 \right) \left(1 - \frac{3}{4} \frac{\tilde{s}_{123}}{s_{12}} (1 - z_3) \right) \right. \\
&\quad + \frac{1 - \epsilon}{2} \left[1 + \frac{s_{12}}{\tilde{s}_{13}} + \frac{z_1}{1 - z_2} \left(\frac{z_2}{1 - z_2} - \frac{\tilde{s}_{123}}{\tilde{s}_{13}} \right) \right] + \frac{\tilde{s}_{123}}{2\tilde{s}_{13}} \left(\frac{\tilde{s}_{123}}{s_{12}} - \frac{1 - z_2}{z_1} \right) \left(\frac{(1 - z_3)^2}{z_2} + \frac{z_2^2}{1 - z_3} \right) \\
&\quad \left. + \left(\frac{\tilde{s}_{123}}{s_{12}} \frac{1 - z_3}{z_1} + \frac{\tilde{s}_{123}}{\tilde{s}_{13}} \frac{1 - z_2}{z_1} - \frac{\tilde{s}_{123}^2}{s_{12} \tilde{s}_{13}} \right) \left(1 - z_3 - \frac{s_{12}}{\tilde{s}_{123}} \right) + \frac{m^2}{\tilde{s}_{13}} \frac{z_1 - z_2}{1 - z_2} - \frac{3}{4} \right\} + (1 \leftrightarrow 2).
\end{aligned} \tag{51}$$

C. Gluon splitting to quarks

The Abelian gluon-to-quark splitting tensor can be assembled from lower-order components as follows:

$$\begin{aligned}
P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{ab})}(p_1, p_2, p_3) &= \left[\frac{C_F}{2} s_{123} P_{g \rightarrow q}^{\mu\nu}(p_{12}, p_3) (\mathcal{S}_{2;2}(p_1; \bar{n}) - \mathcal{S}_{2;3}(p_1; \bar{n})) \right. \\
&\quad \left. + \frac{s_{123}}{\tilde{s}_{12}} P_{g \rightarrow q}^{\mu\nu}(p_{12}, p_3) \langle P_{q \rightarrow q}^{(f)}(p_2, p_1) \rangle + (2 \leftrightarrow 3) \right] + P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{ab}, p)}(p_1, p_2, p_3) + \dots
\end{aligned} \tag{52}$$

In the massive case, the pure splitting tensor is given by

$$\begin{aligned}
P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{ab}, p)}(p_1, p_2, p_3) &= C_F T_R \left\{ -d^{\mu\nu}(p_{123}, \bar{n}) \left[\frac{(\tilde{s}_{123} - \tilde{s}_{23})^2}{2\tilde{s}_{12}\tilde{s}_{13}} (1 + \epsilon) - \frac{z_1}{1 - z_3} \left(\frac{1}{1 - z_3} - \frac{2m^2}{\tilde{s}_{12}} \right) (1 - \epsilon) \right] \right. \\
&\quad - d^{\mu\nu}(p_{123}, \bar{n}) \left[\frac{\tilde{s}_{13}}{\tilde{s}_{12}} \frac{z_2}{z_1(1 - z_2)} + \frac{\tilde{s}_{23}}{\tilde{s}_{12}} \frac{1 - 2z_2}{1 - z_2} - \frac{1}{1 - z_3} \left(\frac{z_2}{z_1} - \frac{2m^2}{\tilde{s}_{12}} \right) + 1 \right] \\
&\quad + \left(d^{\mu\nu}(p_{123}, \bar{n}) - \frac{4\tilde{p}_{3,12}^\mu \tilde{p}_{3,12}^\nu}{\tilde{s}_{123}} \right) \left[\frac{\tilde{s}_{123}}{\tilde{s}_{12}} \left(\frac{z_1 - z_2}{z_1} - (1 - \epsilon) \frac{z_1}{1 - z_3} \right) + \frac{\tilde{s}_{123}}{\tilde{s}_{13}} \frac{1 - z_2}{z_1} \right] \\
&\quad \left. + 2\tilde{p}_{1,23}^\mu \tilde{p}_{1,23}^\nu \frac{\tilde{s}_{123}}{\tilde{s}_{12}\tilde{s}_{13}} \left(\epsilon - \frac{2m^2}{\tilde{s}_{123}} (1 - \epsilon) \right) \right\} + (2 \leftrightarrow 3).
\end{aligned} \tag{53}$$

The non-Abelian splitting tensor admits the following decomposition:

$$\begin{aligned}
P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{nab})}(p_1, p_2, p_3) &= \frac{C_A s_{123}^2}{2 s_{23}} [\mathcal{S}_{1;1}^{\alpha\beta}(p_{23}) P_{g \rightarrow q, \alpha\beta}(p_2, p_3) d^{\mu\nu}(p_{123}, \bar{n}) + \mathcal{S}_{23;23}(p_1; \bar{n}) P_{g \rightarrow q}^{\mu\nu}(p_2, p_3)] \\
&\quad + \frac{C_A s_{123}^2}{2 s_{23}} 2(1 - \epsilon) D^\mu(p_1, p_{23}, \bar{n}) D^\nu(p_1, p_{23}, \bar{n}) \langle P_{g \rightarrow q}(p_2, p_3) \rangle \\
&\quad + \frac{C_A}{4} s_{123} [P_{g \rightarrow q}^{\mu\nu}(p_{12}, p_3) + P_{g \rightarrow q}^{\mu\nu}(p_2, p_{13})] \mathcal{S}_{2;3}(p_1; \bar{n}) - \frac{C_A}{2C_F} P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{ab}, p)}(p_1, p_2, p_3) \\
&\quad + P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{pnab}, p)}(p_1, p_2, p_3) + \dots,
\end{aligned} \tag{54}$$

where the pure splitting tensor is given by

$$\begin{aligned}
P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{pnab},p)}(p_1, p_2, p_3) = & \frac{C_A T_R}{2} \left\{ -d^{\mu\nu}(p_{123}, \bar{n})(1-\varepsilon) \left(\frac{s_{123}}{\tilde{s}_{12}} \frac{z_1}{1-z_3} - \frac{\tilde{s}_{13}}{\tilde{s}_{12}} - \frac{z_1}{(1-z_3)^2} \right) \right. \\
& - d^{\mu\nu}(p_{123}, \bar{n}) \left[\frac{1-2z_1}{z_1} \left(\frac{s_{123}}{\tilde{s}_{12}} \frac{1-z_3}{1-z_1} + \frac{s_{123}}{s_{23}} - \frac{s_{123}^2}{\tilde{s}_{12}s_{23}} \frac{z_2}{1-z_1} \right) - \frac{2}{z_1} + 1 - \frac{1}{1-z_3} \left(\frac{2z_2}{z_1} - \frac{2m^2}{\tilde{s}_{12}} \right) \right] \\
& + \frac{4\tilde{p}_{1,23}^\mu \tilde{p}_{1,23}^\nu}{s_{23}} \left[\frac{2z_2 z_3}{(1-z_1)^2} - (1-\varepsilon) + \frac{4m^2}{\tilde{s}_{13}} \frac{z_2}{1-z_1} \right] + \frac{2(\tilde{p}_{2,13}^\mu \tilde{p}_{3,12}^\nu + \tilde{p}_{3,12}^\mu \tilde{p}_{2,13}^\nu) s_{123}}{s_{23} \tilde{s}_{12}} \left[\frac{2z_2 z_3 - z_1}{z_1} + 1 - \varepsilon \right] \\
& - \frac{4\tilde{p}_{3,12}^\mu \tilde{p}_{3,12}^\nu}{s_{23}} \left[\frac{2z_2}{z_1} \left(\frac{s_{123}}{\tilde{s}_{12}} \frac{z_2}{1-z_1} - \frac{s_{23}}{\tilde{s}_{12}} \right) - (1-\varepsilon) \left(\frac{s_{123}}{\tilde{s}_{13}} + \frac{s_{23}}{\tilde{s}_{12}} \frac{z_1}{1-z_3} \right) \right] \\
& \left. - \frac{8}{z_1} \frac{\tilde{p}_{2,3}^\mu \tilde{p}_{2,3}^\nu}{s_{23}} + \frac{8m^2}{\tilde{s}_{12}} \frac{\tilde{p}_{1,23}^\mu \tilde{p}_{2,3}^\nu + \tilde{p}_{1,23}^\nu \tilde{p}_{2,3}^\mu}{s_{23}} \right\} + (2 \leftrightarrow 3). \tag{55}
\end{aligned}$$

The ellipses in Eqs. (52) and (54) stand for terms proportional to $\bar{n}^\mu$ or $\bar{n}^\nu$, which vanish when contracted with a gluon polarization tensor.

D. Kinematical limits of pure splitting remainders

In Ref. [41], it was found that the massless pure splitting remainders corresponding to $\langle P_{q \rightarrow q\bar{q}q}^{(p)} \rangle$, $\langle P_{q \rightarrow gq\bar{q}}^{(\text{ab},p,f)} \rangle$, $\langle P_{q \rightarrow gq\bar{q}}^{(\text{pnab},p,f)} \rangle$, $P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{ab},p)}$, and $P_{g \rightarrow gq\bar{q}}^{\mu\nu(\text{pnab},p)}$ in Eqs. (45), (48), (51), (53), and (55) have at most subleading singularities in the soft-gluon limits (if applicable) and in the collinear limits. This statement still holds for the massive pure splitting remainders, if the collinear limit is replaced by the quasicollinear one. In particular, Table I of Ref. [41] remains unchanged if the collinear limit is replaced by the quasicollinear limit, and the massive extensions of the pure splitting functions are used. In complete analogy, the functions $P_{\bar{q} \rightarrow gq\bar{q}}^{(\text{ab},p)}$ and $P_{\bar{q} \rightarrow gq\bar{q}}^{(\text{pnab},p)}$ in Eqs. (47) and (50) have leading soft singularities, which arise from the interference contributions of the leading diagrams in the axial gauge. The scalar splitting functions should therefore be expressed in terms of the scalar radiator functions by making use of Eqs. (41). In summary, we find that the methodology for the computation of splitting functions first introduced in Ref. [41] applies to both massive and massless QCD, because it is based on a physical, rather than a purely mathematical understanding of the origin of infrared singularities.

VI. CONCLUSIONS

Using a recently introduced method for the decomposition of splitting functions into scalar dipole radiators, lower-order expressions, and pure splitting remainders, we derived a representation of the massive one-to-three-parton splittings that makes their factorization manifest. The massive two-gluon radiators used in this context were computed in a process-independent form for the first time, to our knowledge. The pure splitting remainders exhibit at most subleading singularities in the (double-)soft gluon limits and in the quasicollinear limits. Kinematical limits are, however, not

used in any of our calculations, which implies that our results remain valid across the complete physical phase space.

As a byproduct of our analysis, we derived the massive QCD one-to-three parton splitting functions in a significantly more compact form than existing literature. This will reduce evaluation time and increase numerical stability in practical applications. We have also obtained the first (to our knowledge) independent cross-check of the massive, triple-collinear splitting functions that do not exist in the SCET-based approach of [26], in particular Eqs. (5.9) and (5.10) of [25].

We expect the novel techniques introduced in this manuscript to provide a path toward higher-order computations which do not require sectorization or other forms of overlap removal for different singular phase-space configurations. At the same time, the splitting functions derived here will be helpful for higher logarithmic resummation of soft gluon effects on QCD observables. These methods will help drive improvements in the modeling of heavy quark jet formation, leading to better precision in analyses of top quark and electroweak boson production at the LHC.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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