

Holographic one-point functions and geodesics in Schwarzschild dS₃

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Grinberg and Maldacena showed that heavy thermal one-point functions in AdS/CFT can encode complex geodesics reaching a black hole singularity. We study the de Sitter analog in three-dimensional Schwarzschild–de Sitter space, restricting to the finite cyclic quotients $dS_3/\mathbb{Z}_q$. We define a one-point function through a differentiate dictionary and show that an analogous result holds for the Bunch-Davies-prepared integral. Using the exact light-field Green’s function, the bulk integral reduces exactly to a convergent one-dimensional transform, whose large-mass limit is controlled by the near-defect region and governed by a complex saddle. Its action reproduces a complex geodesic length from future infinity to the conical defect, whose real part is the renormalized boundary-to-horizon proper time and whose imaginary part is the horizon-to-defect distance. The imaginary part appears in the one-point coefficient normalized in the Lorentzian source basis; in the Euclidean basis the same saddle contributes only the real part, the two being related by a fixed connection factor.

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I. INTRODUCTION

In the context of AdS/CFT [1–3], Grinberg and Maldacena [4] showed that the thermal one-point function of a heavy boundary operator in the dual conformal field theory encodes the complex geodesic length from the boundary insertion to the black hole singularity. Specifically, in the large-mass limit the one-point function takes the schematic form

$$\langle \mathcal{O} \rangle_{\text{AdS}} \propto \exp[-m(L_{\text{bd} \rightarrow \text{hor}} + iL_{\text{hor} \rightarrow \text{sing}})]. \quad (1)$$

Here $L_{\text{bd} \rightarrow \text{hor}}$ is the geodesic distance from the boundary to the horizon and $L_{\text{hor} \rightarrow \text{sing}}$ is the distance from the horizon to the singularity; m is the mass of the heavy probe.

This result is remarkable because the one-point function, a boundary observable, encodes geometric information about the region behind the horizon even though this region is classically inaccessible to a boundary observer. Further studies of the connection between one-point functions and bulk geodesics can be found in [5–10].

We explore an extension of this result to three-dimensional Schwarzschild–de Sitter space (SdS₃). These

geometries do not possess a curvature singularity but instead have a conical defect, which plays the role of the singularity in the Grinberg-Maldacena setup. Unlike higher-dimensional SdS geometries, SdS₃ offers two useful simplifications. First, it is locally dS₃, which makes exact analytic results available. Second, it possesses only a single cosmological horizon, further simplifying computations. We focus on a particular subclass of SdS₃—the finite cyclic quotients $dS_3/\mathbb{Z}_q$, $q \geq 2$, where $\mathbb{Z}_q$ is the cyclic group of order q . The reason for this will be explained shortly.

Unlike AdS/CFT, dS/CFT [11–16] is not an established duality. Little is known about the putative dual theory, including where it lives. We assume that it is located at the future boundary $\mathcal{I}^+$ and adopt the differentiate dictionary: the boundary one-point function is obtained by differentiating $-i \log \Psi$ (where Ψ is the scalar wave functional) with respect to the renormalized late-time source.

Following the analogous scalar construction in a BTZ background [4,17], we consider a heavy scalar φ coupled to a light scalar χ through $g\varphi\chi^2$. In three dimensions the Weyl tensor vanishes identically, so this interaction replaces the curvature source used in the higher-dimensional Grinberg-Maldacena setup.

Let P denote the future-boundary insertion. At leading order in g , the corresponding one-point function has the form

$$\langle \mathcal{O}(P) \rangle_K \propto g \int d^3x \sqrt{-g(x)} K(P; x) \langle \chi^2(x) \rangle_{\text{ren}}, \quad (2)$$

where $K(P; x)$ is the heavy field bulk-boundary kernel and $\langle \chi^2(x) \rangle_{\text{ren}}$ is the renormalized coincidence limit of the light-field two-point function on the quotient.

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The conical defect is a fixed locus of the quotient. Near this locus $\langle \chi^2(x) \rangle_{\text{ren}}$ has the expansion

$$\langle \chi^2 \rangle_{\text{ren}} = \frac{C_q}{\ell v} + O(v/\ell), \quad (3)$$

where ℓ is the de Sitter radius, v is the dimensionless transverse radius, and C_q depends only on the quotient. The $1/v$ pole is universal—it arises from the short-distance Hadamard singularity of the propagator.

This divergence compensates the small volume of the near-defect region. One then expects the large-mass behavior of (2) to be governed by the defect. This heuristic expectation is borne out by exact computation.

The integral (2) can be evaluated in closed form. At the light-field mass $m_\chi^2 \ell^2 = 1$, $\langle \chi^2 \rangle_{\text{ren}}$ on the quotient gives an elementary function of the distance from the defect and the angular integrations can then be carried out exactly. A single convergent integral remains, whose large-mass asymptotics are controlled by the defect.

The restriction to finite quotients, together with the choice of light-field mass, is what makes the above calculation exact. Because the group is finite, the coincidence limit reduces to a sum of $q - 1$ image terms. Our choice of mass then makes each term algebraic in the distance from the defect. The singularities of the integrand form a finite set of square-root loci that can be checked individually. Both these choices are used in moving the integration contour onto a section where the integrand is analytic, which allows us to complete the computation. Neither restriction affects the leading order answer, however, the exponent below is independent of q and of m_χ .

The answer depends on the choice of the quotient space kernel $K(P; x)$, or equivalently the covering space kernel (since the former is given by an image sum over the latter). The heavy field has two late-time falloffs, with the complex exponents $1 \pm i\nu$. Here $\nu = \sqrt{M^2 \ell^2 - 1}$ where M is the mass of the heavy field. Normalizing to a unit source means setting the coefficient of one of these modes to one. As the exponents are complex, the modes are branched and that coefficient becomes definite only once the branch is fixed. We use three source conventions for the covering space kernels, which we call bases. The Hartle-Hawking convention fixes the branch on the regular Euclidean cap, where the bulk-boundary invariant is positive, and continues from there. This gives the kernel K_{HH} and the source coefficient $\mathcal{J}_{E,+}$. The Lorentzian convention fixes it on the real section itself, through the $+i0$ boundary value across the light cone. This gives the kernel that we denote by K_L and the corresponding source $\mathcal{J}_+$. Its conjugate uses the $-i0$ value together with the lower preparation, giving K_L^* and $\mathcal{J}_-$. The three differ only by a branch factor. On the upper continuation the first two obey

$$K_{\text{HH}} = i e^{-\pi\nu/2} K_L. \quad (4)$$

The two conventions therefore assign different source coefficients to the same bulk field configuration: expanding a given solution in both bases gives $\mathcal{J}_+ = i e^{-\pi\nu/2} \mathcal{J}_{E,+}$. Differentiating with respect to the two coefficients then produces one-point functions differing by the same factor. For instance, the one-point functions corresponding to K_{HH} and K_L differ by a factor of $e^{-\pi\nu/2}$.

For the Lorentzian kernel K_L defined above, we find

$$\langle \mathcal{O}(P) \rangle_{K_L} \sim \mathcal{P}(\nu, P) e^{-iML_+}, \quad (5)$$

while for its conjugate K_L^* :

$$\langle \mathcal{O}(\theta_P) \rangle_{K_L^*} \sim \mathcal{P}(\nu, P) e^{-iML_-}. \quad (6)$$

$\mathcal{P}$ collects prefactors that do not affect the leading heavy-mass exponential. Here $L_\pm$ correspond to boundary-to-defect complex geodesic lengths. The geodesic calculation of Sec. IV returns both members of this pair—extremizing the length over the complexified defect worldline gives two nearest critical points related by reflection about the real section. The corresponding complex geodesic lengths are L_+ and L_- .

The real part of $L_\pm$ is the renormalized boundary-to-horizon proper time, while $|\text{Im } L_\pm| = \pi\ell/2$ is the proper distance from the cosmological horizon to the defect. Thus the boundary-to-horizon segment supplies the oscillatory phase, whereas the horizon-to-defect segment supplies the exponential weight. Their roles are reversed from the AdS case.

For K_{HH} , the one-point function behaves as $e^{-i\text{Re } L_\pm}$; only the real part of $L_\pm$ features. Thus if one were to pick the Euclidean basis, the horizon-to-defect distance would not appear in the boundary one-point function. Whereas for the Lorentzian basis or its conjugate, it would. It is possible that the dual theory would single out one basis—we report the answers for all three.

The paper is organized as follows. Section II sets up the problem: the geometry of SdS₃ and its global coordinates, the differentiate-dictionary definition of the one-point function, the three source conventions and the relations between them, and the exact light-field coincidence limit on the quotient. Section III specifies the Bunch-Davies integration contour, reduces the vertex integral to a one-dimensional radial transform, and extracts its heavy-mass asymptotics, identifying the saddle selected by the prepared state. Section IV computes the complex geodesic action independently, fixes the branch of each of the two logarithms that enter, and compares the resulting action values with the saddle exponent. Section V discusses what has been established, what depends on the source convention, and directions for extending the result. Technical material is collected in three appendixes: Appendix A constructs the light-field Green's function, Appendix B proves the

contour deformation and the exact reduction, and Appendix C rederives the geodesic pair by continuation to S^3 .

II. ONE-POINT FUNCTION FOR SdS_3 : PRELIMINARIES

A. Geometry of SdS_3

A convenient form of the Lorentzian SdS_3 metric is

$$ds^2 = -f(r)dt_s^2 + \frac{dr^2}{f(r)} + r^2 d\psi_s^2, \quad f(r) = r_c^2 - \frac{r^2}{\ell^2}, \quad r_c \in (0, 1], \quad \psi_s \sim \psi_s + 2\pi. \quad (7)$$

Its cosmological horizon and inverse temperature are

$$r_h = \ell r_c, \quad \beta = \frac{2\pi\ell}{r_c}. \quad (8)$$

For $r_c < 1$, the origin is a conical defect. The coordinate change

$$\hat{t} = r_c t_s, \quad \hat{r} = \frac{r}{r_c}, \quad \hat{\psi} = r_c \psi_s \quad (9)$$

puts the metric in the static dS_3 form

$$ds^2 = -\left(1 - \frac{\hat{r}^2}{\ell^2}\right)d\hat{t}^2 + \frac{d\hat{r}^2}{1 - \hat{r}^2/\ell^2} + \hat{r}^2 d\hat{\psi}^2, \quad (10)$$

with $\hat{\psi} \sim \hat{\psi} + 2\pi r_c$.

We specialize to

$$r_c = \frac{1}{q}, \quad q \in \mathbb{Z}_{\geq 2}, \quad (11)$$

so that the spacetime is the finite quotient $\text{dS}_3/\mathbb{Z}_q$. The quotient acts by an angular rotation at fixed Lorentzian time [18]. This construction is analogous to the quotient description of BTZ [20]; point-particle geometries in three-dimensional de Sitter gravity were studied in Ref. [21]. Note that a cone of general opening angle is not a quotient of dS_3 by a finite subgroup, and the method of images used below would not apply to it.

Near $r = 0$, setting $R_\perp := r/r_c$ gives

$$ds_{\text{spatial}}^2 = [1 + O(R_\perp^2/\ell^2)]dR_\perp^2 + \frac{R_\perp^2}{q^2} d\psi_s^2. \quad (12)$$

The deficit angle is

$$\delta_{\text{def}} = 2\pi\left(1 - \frac{1}{q}\right). \quad (13)$$

For the calculations below it is convenient to use the global covering-space coordinates (t, θ, ψ) , with $t \in \mathbb{R}$, $0 \leq \theta \leq \pi$ and $\psi \sim \psi + 2\pi$, in which

$$ds^2 = \ell^2[-dt^2 + \cosh^2 t(d\theta^2 + \sin^2 \theta d\psi^2)]. \quad (14)$$

The static and global radial coordinates are related by $\hat{r} = \ell \cosh t \sin \theta$, so the cosmological horizon $\hat{r} = \ell$ sits at $\cosh t \sin \theta = 1$. In these coordinates the quotient acts as $\psi \sim \psi + 2\pi/q$ and its fixed worldlines are the poles $\theta = 0, \pi$.

B. Differentiate-dictionary one-point function

We consider a heavy scalar φ , of mass M , and a light scalar χ on a fixed SdS_3 background, with interaction

$$S_{\text{int}} = g \int d^3x \sqrt{-g} \varphi \chi^2. \quad (15)$$

For the heavy principal-series field we write

$$\nu := \sqrt{M^2 \ell^2 - 1}, \quad \Delta_\pm := 1 \pm i\nu, \quad \nu \gg 1. \quad (16)$$

At a finite late-time surface Σ_c , let φ_c and χ_c denote the boundary values of the two fields. Their Bunch-Davies wave functional is

$$\Psi_c^{\text{BD}}[\varphi_c, \chi_c] := \langle \varphi_c, \chi_c; \Sigma_c | \text{BD} \rangle = \int_{\varphi|_{\Sigma_c} = \varphi_c}^{\text{BD}} \int_{\chi|_{\Sigma_c} = \chi_c} \mathcal{D}\varphi \mathcal{D}\chi e^{iS}. \quad (17)$$

After adding local counterterms and removing the cutoff, we denote the renormalized heavy field source by $\mathcal{J}$ and the light-field source by $\chi_{(0)}$. For a chosen heavy field bulk-boundary kernel $K(P; x)$, let $\varphi_{c,K}[\mathcal{J}]$ be the regulated heavy field datum corresponding to $\mathcal{J}$. The renormalized wave function coefficient and the associated differentiate-dictionary one-point function are

$$\mathcal{W}_K[\mathcal{J}, \chi_{(0)}] := \text{Ren}_{\Sigma_c \rightarrow \mathcal{I}^+} [-i \log \Psi_c^{\text{BD}}[\varphi_{c,K}[\mathcal{J}], \chi_c[\chi_{(0)}]]], \quad \langle \mathcal{O}(P) \rangle_K := \left. \frac{\delta \mathcal{W}_K}{\delta \mathcal{J}(P)} \right|_{\mathcal{J} = \chi_{(0)} = 0}, \quad (18)$$

where Ren denotes the cutoff removal after the addition of local counterterms, and $\chi_c[\chi_{(0)}]$ is the regulated datum corresponding to the renormalized light-field source. The subscript records the kernel used to build the regulated datum from $\mathcal{J}$; different source conventions give different $\mathcal{W}$, as we make precise in Sec. II C. Thus the “one-point function” considered here is the boundary observable defined by differentiating the Bunch-Davies wave functional; it is not a bulk in-in expectation value.

At first order in g , the perturbative expansion contains $\chi^2(x)$ at the vertex $g\varphi\chi^2$. Performing the Gaussian integral over χ replaces this pair by its two-point function at coincidence. The light-field factor entering the cubic vertex is the ordinary renormalized coincidence limit

$$\langle\chi^2(x)\rangle_{\text{ren}} := \lim_{x' \rightarrow x} [G_D^{(q)}(x, x') - G_{\text{Had}}(x, x')]. \quad (19)$$

Here, $G_D^{(q)}(x, x')$ denotes the late-time light-field Green's function on SdS_3 while G_{Had} is the local Hadamard parametrix containing the universal short-distance singularity.

$G_D^{(q)}(x, x')$ is obtained by first imposing a homogeneous Dirichlet condition at the finite final surface Σ_c and then taking $\Sigma_c \rightarrow \mathcal{I}^+$ at fixed interior points. The superscript (q) labels the quotient. The subscript D records this finite-cutoff Dirichlet prescription.

At first order in g , the corresponding contribution to the wave function coefficient is then

$$\begin{aligned} \mathcal{W}_K^{(1)}[\mathcal{J}, 0] &= g \int d^3x \sqrt{-g(x)} \varphi_K[\mathcal{J}](x) \langle\chi^2(x)\rangle_{\text{ren}}, \\ \varphi_K[\mathcal{J}](x) &= \int_{S^2/\mathbb{Z}_q} d\Omega_{P'} K(P'; x) \mathcal{J}(P'). \end{aligned} \quad (20)$$

Here P' is a boundary point and $d\Omega_{P'}$ is the unit-sphere measure. Here the integral is over the quotient space $S^2/\mathbb{Z}_q$. The kernel $K(P'; x)$ is the quotient kernel [constructed below in (50)].

Differentiating with respect to $\mathcal{J}$ gives

$$\langle\mathcal{O}(P)\rangle_K = g \int d^3x \sqrt{-g(x)} K(P; x) \langle\chi^2(x)\rangle_{\text{ren}}. \quad (21)$$

C. Choice of kernel

We now specify the three source conventions used in the paper. With bulk point at $x = (t, \theta, \psi)$ and the boundary point at $(\theta_P, \psi_P = 0)$, the de Sitter bulk-boundary invariant is

$$u(P; x) := 2[\sinh t - \cosh t(\cos\theta\cos\theta_P + \sin\theta\sin\theta_P\cos\psi)]. \quad (22)$$

Here we have used the global coordinates (14). The zero locus of $u(P; x)$ is the bulk light cone of the boundary insertion. On the single-cover hyperbolic section $t = \rho + i\pi/2$, $\rho \geq 0$, write $x_E = (\rho, \theta, \psi)$. The corresponding positive Euclidean anti-de Sitter space (EAdS) invariant is

$$U(P; x_E) := 2[\cosh\rho - \sinh\rho(\cos\theta\cos\theta_P + \sin\theta\sin\theta_P\cos\psi)]. \quad (23)$$

Thus u is adapted to Lorentzian boundary values, while U is adapted to the regular Euclidean continuation. We suppress their arguments when no ambiguity can arise. With the standard EAdS₃ unit-source normalization [22],

$$K_E^{(\Delta)}(P; x_E) = c_\Delta U(P; x_E)^{-\Delta}, \quad c_\Delta := \frac{\Gamma(\Delta)}{\pi\Gamma(\Delta-1)} = \frac{\Delta-1}{\pi}. \quad (24)$$

This formula is initially defined for $\text{Re}\Delta > 1$ and then analytically continued to the principal series. In particular,

$$c_{\Delta_+} = \frac{i\nu}{\pi}, \quad c_{\Delta_-} = -\frac{i\nu}{\pi}. \quad (25)$$

We first define the upper Euclidean source $\mathcal{J}_{E,+}$, the coefficient of the Δ_+ mode normalized as in (24), where $U > 0$ makes $U^{-\Delta_+}$ single valued. The coefficients $\mathcal{J}_+$ and $\mathcal{J}_-$ are the coefficients of the same falloffs normalized instead on the real Lorentzian section, where u changes sign across the light cone and the branch is fixed by the $+i0$ and $-i0$ boundary values respectively. The generic source $\mathcal{J}$ in (18) means the coefficient appropriate to the kernel being used.

The Hartle-Hawking kernel [23] is obtained by continuing the Euclidean unit-source kernel through the regular upper cap:

$$K_{\text{HH}}(P; x)|_{t=\rho+i\pi/2} := c_{\Delta_+} U(P; x_E)^{-\Delta_+}. \quad (26)$$

It is the kernel used when differentiating with respect to $\mathcal{J}_{E,+}$. We separately define the two Lorentzian unit-source bases by

$$\begin{aligned} K_L(P; x) &:= c_{\Delta_+} (u(P; x) + i0)^{-\Delta_+}, \\ K_L^*(P; x) &:= c_{\Delta_-} (u(P; x) - i0)^{-\Delta_-}. \end{aligned} \quad (27)$$

Here K_L is paired with $\mathcal{J}_+$ and the upper preparation. The kernel K_L^* is paired with $\mathcal{J}_-$ and the conjugate lower preparation. The displayed $i0$ specifies the distributional boundary value across the Lorentzian light cone $u = 0$.

The three definitions use the same coefficients $c_{\Delta_\pm}$, but “unit source” refers to three different asymptotic bases. Their relation is fixed by analytic continuation. Since $u(\rho + i\pi/2) = iU(\rho)$, continuation from the upper hyperbolic section gives

$$K_{\text{HH}} = e^{i\pi\Delta_+/2} K_L = ie^{-\pi\nu/2} K_L. \quad (28)$$

Define

$$f_+ := e^{i\pi\Delta_+/2} = ie^{-\pi\nu/2}. \quad (29)$$

The same bulk field configuration may then be written in either basis as

$$K_{\text{HH}}\mathcal{J}_{E,+} = K_L\mathcal{J}_+, \quad \mathcal{J}_+ = f_+\mathcal{J}_{E,+}. \quad (30)$$

At linear order in the heavy source, the chain rule gives

$$\langle \mathcal{O}(P) \rangle_{K_{\text{HH}}} = f_+ \langle \mathcal{O}(P) \rangle_{K_L}. \quad (31)$$

For the conjugate relation, let $\mathcal{J}_{E,-}$ denote the coefficient of the Euclidean-normalized Δ_- mode continued through the lower cap, $t = \rho - i\pi/2$. Since $u = -iU$ there, define

$$f_- := e^{-i\pi\Delta_-/2} = \overline{f_+} = -ie^{-\pi\nu/2}. \quad (32)$$

The lower-cap and Lorentzian source coefficients for the same bulk field configuration then obey

$$c_{\Delta_-} U^{-\Delta_-} \mathcal{J}_{E,-} = K_L^* \mathcal{J}_-, \quad \mathcal{J}_- = f_- \mathcal{J}_{E,-}. \quad (33)$$

If $\langle \mathcal{O} \rangle_{E,-}$ denotes differentiation with respect to $\mathcal{J}_{E,-}$, the corresponding chain-rule relation is

$$\langle \mathcal{O}(P) \rangle_{E,-} = f_- \langle \mathcal{O}(P) \rangle_{K_L}^{\text{lower}}. \quad (34)$$

Thus $\mathcal{J}_{E,-}$ labels a distinct lower-cap Euclidean basis. It is not notation for the numerical complex conjugate of $\mathcal{J}_{E,+}$. The two become conjugate only when the boundary data themselves obey the corresponding reality condition.

Equations (30) and (33) are changes of source basis. Identifying the Euclidean and Lorentzian source coefficients numerically while also retaining $f_{\pm}$ would compare different bulk field configurations. The factors change the angle-independent weights assigned to the one-point coefficients. They do not change the bulk saddles or their θ_P dependence.

This is also where the de Sitter normalization question differs from the usual AdS one. In AdS/CFT, the independently defined boundary CFT fixes the operator and source convention. Here the differentiate dictionary does not supply an additional canonical normalization. A dual description would be needed to make one of the bases canonical. We therefore state the source basis whenever comparing the three kernels.

D. The renormalized light-field two-point function

We now turn to the computation of $\langle \chi^2(x) \rangle_{\text{ren}}$. We start from the covering space and quotient Green's functions $G_D^{(1)}(x, x')$ and $G_D^{(q)}(x, x')$ introduced in Sec. II B. As mentioned there, the initial condition is the Bunch-Davies one; $G_D^{(q)}$ is therefore not the unrestricted Bunch-Davies Wightman function.

On the covering dS₃ geometry the global angle has period 2π , and the generator of the quotient acts as

$$\gamma: (t, \theta, \psi) \mapsto \left(t, \theta, \psi + \frac{2\pi}{q} \right). \quad (35)$$

For an ordinary quotient-invariant scalar, the method of images gives

$$G_D^{(q)}(x, x') = \sum_{n=0}^{q-1} G_D^{(1)}(x, \gamma^n x'). \quad (36)$$

The ordinary point-split renormalized coincidence limit is

$$\langle \chi^2(x) \rangle_{\text{ren}} := \lim_{x' \rightarrow x} [G_D^{(q)}(x, x') - G_{\text{Had}}(x, x')], \quad (37)$$

where G_{Had} is the local Hadamard parametrix. Hence

$$\langle \chi^2(x) \rangle_{\text{ren}} = \langle \chi^2(x) \rangle_{\text{ren}}^{(1)} + \sum_{n=1}^{q-1} G_D^{(1)}(x, \gamma^n x), \quad (38)$$

$$\langle \chi^2(x) \rangle_{\text{ren}}^{(1)} := \lim_{x' \rightarrow x} [G_D^{(1)}(x, x') - G_{\text{Had}}(x, x')]. \quad (39)$$

Thus $\langle \chi^2 \rangle_{\text{ren}}$ is the standard renormalized coincident two-point function on the quotient. The image sum is simply a convenient point-split evaluation of it. This coincidence limit is first taken away from the quotient fixed worldlines, where every nonidentity image is separated; the near-defect limit $\nu \rightarrow 0$ is taken afterward. Rotational symmetry makes the result independent of ψ .

We choose

$$m_\chi^2 \ell^2 = 1. \quad (40)$$

This is the critical value $\nu_\chi = 0$, at which the two weights $1 \pm i\nu_\chi$ coincide. Its Euclidean continuation is the Breitenlohner-Freedman value (A4) in EAdS₃. The covering-space Green's function is elementary at any mass, being the continuation of $e^{-(\Delta_\chi - 1)\sigma} / (4\pi\ell_E \sinh \sigma)$, with σ the geodesic separation. The critical value is special in that the exponential factor disappears, leaving an algebraic function of σ . As a result, the image sum (44) is a finite sum of square roots. The branch loci of the vertex integrand form a finite set that can be enumerated and shown not to obstruct the contour deformation (Appendix B). Further, the near-defect expansion (46) has no $O(v^0)$ term, so corrections to the heavy-mass limit proceed in powers of ν^{-2} .

The degenerate weight also carries a cost. Because the indicial roots coincide, a logarithmic solution appears and the Green's function is not fixed by its coincidence singularity alone. Imposing the final Dirichlet condition at finite cutoff and then removing the cutoff suppresses the logarithmic boundary solution and selects the no-logarithm Green's function. Appendix A derives this by an explicit mode construction. Here we quote the result.

For dS₃ embedding coordinates X^A , $A = 0, 1, 2, 3$, define

$$s(x, x') := \frac{\eta_{AB} X^A(x) X^B(x')}{\ell^2},$$

$$\eta_{AB} = \text{diag}(-, +, +, +), \quad s(x, x) = 1. \quad (41)$$

The covering-space Green's function is

$$G_D^{(1)}(s) = \frac{1}{4\pi\ell\sqrt{1-s^2}}. \quad (42)$$

The square root is positive for real $-1 < s < 1$. Elsewhere it is defined by continuation from that domain along the upper edge $\text{Im } t > 0$, on which the Euclidean cap lies; we refer to this throughout as the Bunch-Davies prescription, and (A22) records it explicitly for the image combination. It is the same boundary value that supplies the $+i0$ in (27). At coincidence, $G_D^{(1)} = 1/(4\pi d) + O(d)$, where d is the local proper distance. Consequently the covering-space finite part in (39) vanishes in the minimal Hadamard scheme used here.

For the n th quotient image, set

$$a_n := \sin \frac{\pi n}{q}, \quad v := \frac{\hat{r}}{\ell} = \cosh t \sin \theta,$$

$$s_n(x) := s(x, \gamma^n x) = 1 - 2v^2 a_n^2. \quad (43)$$

Equations (38) and (42) therefore give

$$\langle \chi^2(v) \rangle_{\text{ren}} = \frac{1}{8\pi\ell v} \sum_{n=1}^{q-1} \frac{1}{a_n \sqrt{1-v^2 a_n^2}}. \quad (44)$$

For sufficiently small positive v , all roots in (44) are positive. Elsewhere the expression is defined by continuation from that domain with the same Bunch-Davies prescription.

Defining

$$C_q := \frac{1}{8\pi} \sum_{n=1}^{q-1} \csc \frac{\pi n}{q}, \quad (45)$$

the near-defect series is

$$\langle \chi^2(v) \rangle_{\text{ren}} = \frac{C_q}{\ell v} + \frac{v}{16\pi\ell} \cot \frac{\pi}{2q}$$

$$+ \frac{3v^3}{256\pi\ell} \left(3 \cot \frac{\pi}{2q} - \cot \frac{3\pi}{2q} \right) + O(v^5/\ell).$$

(46)

In particular, the regular part begins at order v , rather than at order v^0 . Appendix A gives the series derivation.

Finally, on the upper hyperbolic section and its conjugate, $t = \rho \pm i\pi/2$,

$$1 - v^2 a_n^2 = 1 + a_n^2 \sinh^2 \rho \sin^2 \theta > 0 \quad (47)$$

for real ρ and θ . Hence the explicit $1/v$ pole is the only light-field singularity in the near-axis region controlling the large-mass expansion.

III. THE BUNCH-DAVIES CONTOUR AND THE HEAVY-MASS SADDLE

We now evaluate the one-point function using the exact light-field profile in (44). We first specify the spacetime integration associated with the Bunch-Davies wave functional. The resulting integral is then reduced to a single radial transform, whose heavy-mass limit identifies the saddle selected by the prepared state.

At first order in g , the interaction vertex is integrated over the same history that defines the Bunch-Davies wave functional. A regular Euclidean cap prepares the state. Lorentzian evolution carries it to the final surface at $t = t_c$. In global time this history is

$$\mathcal{C}_{\text{BD}}: t = \frac{i\pi}{2} \longrightarrow 0 \longrightarrow t_c. \quad (48)$$

We refer to $\mathcal{C}_{\text{BD}}$ as the Bunch-Davies contour. Its first segment is the regular Euclidean cap that prepares the state; its second is the Lorentzian part of the wave function evolution, along which the $+i0$ boundary values are inherited. After the local counterterms at t_c have been included, the limit $t_c \rightarrow \infty$ is taken [13,24]. A past- $i\epsilon$ representation is equivalent when it implements the same interacting Bunch-Davies state preparation.

The n th quotient image shifts the azimuth by $\delta_n = 2\pi n/q$. Its heavy-field invariant is

$$u_n := 2[\sinh t - \cosh t (\cos \theta \cos \theta_P + \sin \theta \sin \theta_P \cos(\psi + \delta_n))]. \quad (49)$$

Thus the quotient kernel is the finite image sum

$$K_L^{(q)}(P; x) = c_{\Delta_+} \sum_{n=0}^{q-1} (u_n + i0)^{-\Delta_+}. \quad (50)$$

The light-field profile is invariant under the quotient rotation. Each term in (50) therefore maps the fundamental azimuthal interval to an adjacent interval. Unfolding the image sum gives

$$\begin{aligned}
\langle \mathcal{O}(\theta_P) \rangle_{K_L} &= g\ell^3 \int_{\mathcal{C}_{\text{BD}}} dt \int_0^\pi d\theta \int_0^{2\pi/q} d\psi \cosh^2 t \sin \theta K_L^{(q)}(P; x) \langle \chi^2(t, \theta) \rangle_{\text{ren}} \\
&= g\ell^3 c_{\Delta_+} \int_{\mathcal{C}_{\text{BD}}} dt \int_0^\pi d\theta \int_0^{2\pi} d\psi \cosh^2 t \sin \theta (u + i0)^{-\Delta_+} \langle \chi^2(t, \theta) \rangle_{\text{ren}}.
\end{aligned} \tag{51}$$

Here $u := u_0$. The $+i0$ is inherited from the Lorentzian part of $\mathcal{C}_{\text{BD}}$. After unfolding, all quotient dependence is carried by the exact light-field profile.

A. Contour deformation and exact radial reduction

The integral (51) runs over the two-segment contour $\mathcal{C}_{\text{BD}}$ and over the two angles. We evaluate it in two steps. The contour is first moved onto a single hyperbolic section, on which the geometry is H^3 and the integrand is real analytic. The angular integrations are then performed in coordinates adapted to the quotient axis, which is possible because the light-field profile (44) depends only on the distance from that axis. A single radial integral survives.

For the first step, let $\mathcal{F}(t)$ denote the angularly integrated density in (51). Appendix B shows that neither a zero of the heavy invariant nor a reflected light-field branch point lies in the open strip $0 < \text{Im } t < \pi/2$, and that the connecting edge at large $\text{Re } t$ vanishes. Hence

$$\int_{\mathcal{C}_{\text{BD}}} dt \mathcal{F}(t) = \int_0^\infty d\rho \mathcal{F}\left(\rho + \frac{i\pi}{2}\right), \tag{52}$$

with orientation $\rho: 0 \rightarrow \infty$. The singularities that remain on the Lorentzian boundary of the strip—the light cone of the insertion and the reflected thresholds (A22)—are locally integrable and do not pinch the deformation. The identity is established in the regulating domain $0 < \text{Re } \Delta < 1$ and continued to $\Delta = \Delta_+$.

On the section $t = \rho + i\pi/2$ the invariants continue to $u = iU$, $v = i \sinh \rho \sin \theta$, $1 - a_n^2 v^2 = 1 + a_n^2 \sinh^2 \rho \sin^2 \theta > 0$, (53)

so that continuation from the Euclidean cap makes U positive and every light-field square root positive, and supplies an overall factor $ie^{-i\pi\Delta_+/2}$. It is convenient to work with

$$q := -iv = \sinh \rho \sin \theta, \tag{54}$$

the transverse geodesic radius measured from the quotient axis, so that $q = 0$ is the axis itself.

For the second step we use coordinates (y, q, ψ) on H^3 in which y runs along the quotient axis and q is the transverse radius, the two fixed worldlines $\theta = 0$ and $\theta = \pi$ becoming the halves $y > 0$ and $y < 0$ of one complete axis. These coordinates are adapted to both factors of the integrand: the

light-field profile is independent of y and ψ , while U factorizes so that, after a shift of y , the insertion angle appears only through an overall power of $\sin \theta_P$. Both integrations can therefore be carried out exactly, and they produce a Legendre function of q . The result, derived in Appendix B, is

$$\begin{aligned}
\langle \mathcal{O}(\theta_P) \rangle_{K_L} &= ie^{-i\pi\Delta_+/2} \frac{g\ell^2 c_{\Delta_+}}{8} \frac{\Gamma(\Delta_+/2)^2}{\Gamma(\Delta_+)} \\
&\times (\sin \theta_P)^{-\Delta_+} \sum_{n=1}^{q-1} \frac{\mathcal{R}_{\Delta_+}(a_n)}{a_n},
\end{aligned} \tag{55}$$

$$\mathcal{R}_{\Delta}(a) := \int_0^\infty dq \frac{P_{\Delta/2-1}(1+2q^2)}{\sqrt{1+a^2q^2}}, \quad a_n = \sin \frac{\pi n}{q}. \tag{56}$$

Here P_λ is the Legendre function of the first kind, and $\mathcal{R}_{\Delta}(a_n)$ is the remaining transverse integral for the n th light-field image; it retains the full radial profile, no approximation having been made. Its integrand decays as q^{-2} , so the integral converges absolutely on the principal series. Because the heavy image sum has already unfolded the quotient angular domain, and because the two fixed worldlines have become the two halves of a single axis, no residual factor of q or factor of two appears. All dependence on the insertion angle has factorized into $(\sin \theta_P)^{-\Delta_+}$, and the expression is regular at $\theta_P = \pi/2$; insertions at the fixed endpoints $\theta_P = 0, \pi$ remain excluded.

One assumption enters here. The Green's function used above is the fixed-interior limit (A3), and moving that limit through the vertex integral is not justified by pointwise convergence alone, since the finite-cutoff Dirichlet correction is nonuniform in a thin layer next to the final surface. Appendix B gives power counting that controls the fixed-mode part of this layer; the identification of (55) with the renormalized late-time coefficient assumes, as stated there, that the remaining high-angular-momentum boundary layer is local and is removed by the standard wave functional counterterms.

B. Heavy-mass limit and saddle selection

Everything now depends on the large- ν behavior of $\mathcal{R}_{1+i\nu}(a)$. Appendix B rewrites the transform as an exact cosine integral and derives, for fixed $a > 0$,

$$\mathcal{R}_{1+i\nu}(a) = \frac{1}{\nu} \left[1 + \frac{a^2}{2\nu^2} + O(\nu^{-4}) \right]. \quad (57)$$

That this is a power of ν rather than something smaller is the substance of the result. A rapidly oscillating integrand with no interior stationary point would contribute beyond all orders; the $1/\nu$ comes entirely from the endpoint $\varrho = 0$, over a region shrinking as $\varrho = O(\nu^{-1})$, as the scaling limit (B22) at fixed $\nu\varrho$ makes explicit. That endpoint is the quotient axis, and what allows it to contribute is the near-defect pole. The factor ϱ in the transverse measure would by itself suppress the endpoint; the $1/\nu$ singularity of (46) cancels it exactly, leaving the finite measure (B9). The regular part of the source begins at order ν and is therefore suppressed by two further powers of ν . The defect thus controls the leading heavy-mass behavior—not as an estimate weighed against competing regions, but as a consequence of (57), whose derivation accounts for the whole of the convergent integral (56).

Assembling (55) requires one further large- ν input,

$$\frac{\Gamma((1+i\nu)/2)^2}{\Gamma(1+i\nu)} = \sqrt{\frac{2\pi}{\nu}} e^{-i\pi/4} e^{-i\nu \log 2} [1 + O(\nu^{-1})], \quad (58)$$

whose phase $e^{-i\nu \log 2}$ is what completes $\log(2 \sin \theta_P)$ in the exponent. Collecting the numerical factors produced by (57), (58) and the kernel renormalization into

$$\tilde{C}_q := \sqrt{2\pi} e^{i\pi/4} C_q, \quad (59)$$

and using $c_{\Delta_+} = i\nu/\pi$ together with $M\ell = \nu + O(\nu^{-1})$, the exact result (55) becomes

$$\begin{aligned} \langle \mathcal{O}(\theta_P) \rangle_{K_L} &\sim \frac{\tilde{C}_q g \ell^2}{\nu^{1/2} \sin \theta_P} e^{-i\nu \log(2 \sin \theta_P)} e^{\pi\nu/2} [1 + O(\nu^{-1})] \\ &\sim \frac{\tilde{C}_q g \ell^2}{\nu^{1/2} \sin \theta_P} e^{-iML_+}. \end{aligned} \quad (60)$$

We now locate this contribution on the deformed geometry. Define

$$\rho_0 := \log \cot \frac{\theta_P}{2}. \quad (61)$$

The factorization (B10) gives, at $\varrho = 0$,

$$U = 2 \sin \theta_P \cosh(y - \rho_0). \quad (62)$$

The axial phase is stationary at $y = \rho_0$. The selected configuration therefore consists of a transverse endpoint at the quotient axis and an ordinary stationary point along that axis.

For the geodesic comparison, the nearest algebraic critical points in global coordinates are

$$t_{N,\sigma} = \rho_0 + \sigma \frac{i\pi}{2}, \quad t_{S,\sigma} = -\rho_0 + \sigma \frac{i\pi}{2}, \quad \sigma = \pm 1. \quad (63)$$

The single-cover contour (52) contains the upper representative. For $\rho_0 > 0$ it lies on the north half of the axis. For $\rho_0 < 0$ it lies on the south half. At $\rho_0 = 0$ it is the integrable center of the cap. The lower representative belongs to the conjugate preparation.

For real g , let $\langle \mathcal{O} \rangle_{K_L^+}^{\text{lower}}$ denote the coefficient of the complex-conjugate wave functional, prepared on the lower Euclidean cap. Complex conjugation then gives

$$\begin{aligned} \langle \mathcal{O}(\theta_P) \rangle_{K_L^+}^{\text{lower}} &\sim \frac{\overline{\tilde{C}_q} g \ell^2}{\nu^{1/2} \sin \theta_P} e^{+i\nu \log(2 \sin \theta_P)} e^{\pi\nu/2} \\ &\sim \frac{\overline{\tilde{C}_q} g \ell^2}{\nu^{1/2} \sin \theta_P} e^{+iML_-}. \end{aligned} \quad (64)$$

Finally, changing from the Lorentzian source $\mathcal{J}_+$ to the upper Euclidean source $\mathcal{J}_{E,+}$ according to (31) gives, with $\tilde{D}_q := i\tilde{C}_q$,

$$\langle \mathcal{O}(\theta_P) \rangle_{K_{HH}} \sim \frac{\tilde{D}_q g \ell^2}{\nu^{1/2} \sin \theta_P} e^{-i\nu \log(2 \sin \theta_P)}. \quad (65)$$

The change of source basis removes the angle-independent enhancement of the K_L coefficient without changing the selected saddle or its angular dependence.

IV. COMPLEX GEODESIC ACTION

The heavy-mass exponent in (60) contains L_+ . We now compare this quantity with the stationary values of a complex geodesic action. Two logarithms enter the comparison, and their branches are fixed in different ways. The continued H^2 distance fixes the geometric logarithm, whereas the $+i0$ Bunch-Davies boundary value fixes the logarithm in the heavy kernel.

Rotational symmetry confines the geodesic to the meridional plane through the boundary insertion. Away from the polar axis this is a surface of fixed azimuth ψ , with metric

$$ds^2 = \ell^2 (-dt^2 + \cosh^2 t d\theta^2). \quad (66)$$

One endpoint approaches $(t, \theta) = (+\infty, \theta_P)$, while the other lies on the complexified fixed locus of the quotient. Since the position of the latter endpoint is integrated over in the one-point diagram, its stationary value must be found by extremizing the geodesic length.

We first continue to the upper hyperbolic section by setting $t = \rho + i\pi/2$. Equation (66) becomes minus the metric on H^2 . Although the usual hyperbolic radius obeys $\rho \geq 0$, the continued north and south worldlines form the

two halves of one complete geodesic axis. It is convenient to parameterize that axis by a signed coordinate y :

$$(\rho, \theta) = \begin{cases} (y, 0), & y \geq 0, \\ (-y, \pi), & y \leq 0. \end{cases} \quad (67)$$

Positive and negative values cover the north and south halves, respectively, and the two meet at $y = 0$.

Place the regulated boundary endpoint at $(\rho_{\text{bdy}}, \theta_P)$. Its hyperbolic distance from the point y on the axis satisfies

$$\cosh\left(\frac{d_{H^2}}{\ell}\right) = \cosh y \cosh \rho_{\text{bdy}} - \sinh y \sinh \rho_{\text{bdy}} \cos \theta_P. \quad (68)$$

Define

$$A(y, \theta_P) := \cosh y - \sinh y \cos \theta_P. \quad (69)$$

For large ρ_{bdy} , Eq. (68) gives

$$\cosh\left(\frac{d_{H^2}}{\ell}\right) = \frac{e^{\rho_{\text{bdy}}}}{2} A(y, \theta_P) + O(e^{-\rho_{\text{bdy}}}),$$

$$d_{H^2} = \ell \rho_{\text{bdy}} + \ell \log A(y, \theta_P) + O(\ell e^{-2\rho_{\text{bdy}}}). \quad (70)$$

The subtraction is defined using the induced angular scale at the cutoff, which is $\sinh \rho_{\text{bdy}}$. The corresponding length counterterm is

$$\ell \log \sinh \rho_{\text{bdy}} = \ell(\rho_{\text{bdy}} - \log 2) + O(\ell e^{-2\rho_{\text{bdy}}}). \quad (71)$$

This convention fixes the finite additive constant. A constant rescaling of the boundary conformal metric would shift every regulated length by the same angle-independent amount. Subtracting (71) from (70), we obtain

$$L_E^{\text{reg}}(y) := \lim_{\rho_{\text{bdy}} \rightarrow \infty} [d_{H^2} - \ell \log \sinh \rho_{\text{bdy}}]$$

$$= \ell \log [2(\cosh y - \sinh y \cos \theta_P)]. \quad (72)$$

Extremizing the endpoint along the complete axis gives

$$\partial_y L_E^{\text{reg}} = 0 \Rightarrow \tanh y_* = \cos \theta_P. \quad (73)$$

The solution is $y_* = \rho_0$, where ρ_0 is defined in (61). It is unique because $\tanh y$ is monotonic, and it is the global minimum because $A(y, \theta_P)$ grows at both ends of the axis. At this point $A(y_*, \theta_P) = \sin \theta_P$, so

$$L_E^{\text{reg}}(y_*) = \ell \log(2 \sin \theta_P). \quad (74)$$

The endpoint lies on the north half for $\theta_P < \pi/2$ and on the south half for $\theta_P > \pi/2$. At the equator it reaches the common point $y = 0$.

We next return the boundary endpoint to real Lorentzian time. North-pole coordinates provide a single signed-coordinate representative of the complexified axis, for which

$$u_N(t) := 2(\sinh t - \cosh t \cos \theta_P). \quad (75)$$

When $y < 0$, the same complex point can instead be written in south-pole coordinates as $(t = -y + \sigma i\pi/2, \theta = \pi)$.

Place the boundary endpoint at (T, θ_P) , with $T \gg 1$. The de Sitter invariant between this point and $(t, 0)$ is

$$s((T, \theta_P), (t, 0)) = -\sinh T \sinh t + \cosh T \cosh t \cos \theta_P$$

$$= -\frac{e^T}{4} u_N(t) + O(e^{-T}). \quad (76)$$

The equation $\cos[D(T, t)/\ell] = s$ does not by itself fix a branch of the complex distance. We fix it before taking the Lorentzian boundary limit. On the real upper H^2 section we require $D = -id_{H^2}$, so that the continued worldline action satisfies $iD = d_{H^2} > 0$, and we then continue this branch through the same upper analytic domain. Denoting the resulting logarithm by $\log_{\text{geom}}$, its large- s behavior is

$$D(T, t) = -i\ell \log_{\text{geom}}(2s) + O(\ell s^{-2}). \quad (77)$$

With $\mathcal{L}(T, t) := iD(T, t)$, Eq. (76) then gives

$$\mathcal{L}(T, t) = \ell(T - \log 2) + \ell \log_{\text{geom}}[-u_N(t)] + O(\ell e^{-2T}). \quad (78)$$

The same boundary conformal representative is used on the Lorentzian side, where the angular scale of the late-time cutoff is $\cosh T$. Since $\log \cosh T = T - \log 2 + O(e^{-2T})$, subtracting this boundary term yields

$$L_{\text{geom}}^{\text{reg}}(t) = \ell \log_{\text{geom}}[-u_N(t)]. \quad (79)$$

The endpoint equation $u'_N(t) = 0$ has the two nearest solutions $t_{N,\sigma}$ given in (63), and at these points

$$u_N(t_{N,\sigma}) = 2\sigma i \sin \theta_P, \quad \sigma = \pm 1. \quad (80)$$

The branch fixed by the positive H^2 distance therefore gives

$$L_{\text{geom}}^{\text{reg}}(t_{N,\sigma}) = \ell \log(2 \sin \theta_P) - \sigma \frac{i\pi\ell}{2} = L_{-\sigma}. \quad (81)$$

The logarithm in the heavy kernel has a separate branch prescription. On the upper Bunch-Davies contour, $u = iU$ with $U > 0$, which fixes $\log_{\text{BD}} u = \log U + i\pi/2$. The lower preparation fixes its conjugate boundary value. At the stationary points the two branch assignments are

$$\ell \log_{\text{BD}} u_N(t_{N,\sigma}) = L_\sigma, \quad \ell \log_{\text{geom}}[-u_N(t_{N,\sigma})] = L_{-\sigma}. \quad (82)$$

The two logarithms take arguments of opposite sign, and this is not an independent choice. Equation (76) expresses the de Sitter invariant as $s = -\frac{1}{4}e^T u_N + O(e^{-T})$, so the geometric logarithm inherits $-u_N$ from the continued distance while the kernel logarithm carries u_N itself. At a fixed coordinate representative the two values therefore differ by exactly $\ell \log(-1)$, and on the branches fixed above this difference is

$$\begin{aligned} \ell \log_{\text{geom}}[-u_N(t_{N,\sigma})] - \ell \log_{\text{BD}}[u_N(t_{N,\sigma})] &= -\sigma i \pi \ell \\ &= L_{-\sigma} - L_\sigma. \end{aligned} \quad (83)$$

The shift is the separation of the two members of the pair. This is why the two prescriptions attach the labels to opposite coordinate saddles: the relative branch offset is fixed, not free. Both calculations produce the same pair $\{L_+, L_-\}$; the upper K_L coefficient selects L_+ through $\log_{\text{BD}} u$, and the conjugate preparation selects L_- . The comparison is between action values, with the branch relation (83) determined rather than assumed. Appendix C recovers the same geometric pair from the S^3 continuation.

The two parts of the complex length have a direct geometric interpretation. At the regulated boundary point the static radial coordinate is $\hat{r}_p = \ell \cosh T \sin \theta_p$, and the magnitude of the radial proper time from the horizon to this point is

$$\int_{\ell}^{\hat{r}_p} \frac{d\hat{r}}{\sqrt{\hat{r}^2/\ell^2 - 1}} = \ell \text{arccosh}(\cosh T \sin \theta_p). \quad (84)$$

After applying the same boundary subtraction, this becomes

$$\begin{aligned} \lim_{T \rightarrow \infty} [\ell \text{arccosh}(\cosh T \sin \theta_p) - \ell(T - \log 2)] \\ = \ell \log(2 \sin \theta_p). \end{aligned} \quad (85)$$

The proper distance from the defect to the cosmological horizon is

$$L_{\text{hor-def}} = \int_0^{\ell r_c} \frac{dr}{\sqrt{r_c^2 - r^2/\ell^2}} = \frac{\pi \ell}{2}. \quad (86)$$

Thus $\text{Re} L_\pm$ is the renormalized magnitude of the boundary-to-horizon radial proper time, whereas $|\text{Im} L_\pm|$ is the horizon-to-defect distance. In $e^{-iML_\pm}$, the former supplies the phase and the latter sets the exponential weight. Their roles are interchanged in the AdS black hole example.

V. DISCUSSION

Grinberg and Maldacena [4] showed that in AdS/CFT the thermal one-point function of a heavy boundary operator encodes the length of a complex geodesic running from the boundary to the black hole singularity. This paper asked whether an analogous statement can be made in de Sitter space. We found that it can.

We considered SdS_3 , where the conical defect plays the role of the singularity and took the one-point function to be defined by the differentiate dictionary at $\mathcal{I}^+$. We found that the heavy-mass limit of the one-point function is governed by a complex saddle whose action is a complex geodesic length joining future infinity to the defect. The real part is the renormalized boundary-to-horizon proper time (85) while the imaginary part is the horizon-to-defect distance (86). We showed that this agrees with the independent geodesic calculation of Sec. IV. The roles of the two segments are thus exchanged from the AdS case—here the boundary-to-horizon segment is timelike and supplies the oscillatory phase while the horizon-to-defect segment is spacelike and sets the exponential weight. That exchange is expected since cosmological and black hole horizons have causally opposite characters. The physical picture in both settings is that a source diverging at the singular locus dominates the heavy-mass limit, so that the probe is effectively localized there and its action becomes a geodesic length.

We obtain the result by evaluating the integral (2) exactly. First, the Bunch-Davies contour (48) is deformed onto a single hyperbolic section. This reduces the integral to the convergent one-dimensional transform (56) and its asymptotics follow from an endpoint expansion (as opposed to a stationary-phase estimate).

The result depends on the choice of normalization. The coefficient (60) is defined by differentiation with respect to the Lorentzian source $\mathcal{J}_+$. Reexpressing the same bulk profile in the upper Euclidean basis multiplies it by f_+ as in (31), giving (65). In that case, only $\text{Re} L_+$ survives. The imaginary part of the geodesic length is therefore visible in the Lorentzian basis and absorbed into the connection factor (29) in the Euclidean one. The final choice of basis would have to come from the dual theory.

As noted in the introduction, our computations were simplified by the restriction to finite cyclic quotients. It is interesting to note how little of the answer depends on the quotient. The exponent contains no q : $\text{Re} L_\pm$ is fixed by the insertion angle and $|\text{Im} L_\pm|$ by (86), which is independent of r_c . All q dependence resides in the prefactor $\tilde{C}_q$ and in the subleading corrections through $a_n = \sin(\pi n/q)$. Based on this, we expect the same exponent should hold for a general cone angle. The two ingredients that appear in the answer are the coefficient of the near-defect pole and the horizon-to-defect distance. The former is fixed by the local Hadamard singularity and the latter by the metric. Neither require the deficit to be of quotient type. But an exact

treatment would be unavailable, since the finite branch structure that controls the contour deformation would be replaced by a cut. We leave this for future consideration.

A further natural extension would be to the case of the general de Sitter black hole. This would be more challenging and the methods of this paper would not go through. We expect that the part about the near-defect saddle would go through. However, the global statement that this saddle exhausts the leading behavior need not carry over. We leave this for future consideration.

The connection between bulk two-point function in the heavy limit and complex geodesic distance in de Sitter has been explored in [25–27]. It will be interesting to connect these works with ours.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: CRITICAL-MASS DIRICHLET PROPAGATOR

This appendix constructs the light-field Green's function used in the main text and derives its near-defect expansion. Section A 1 solves the covering-space problem at finite cutoff and shows that removing the cutoff at fixed interior points selects the no-logarithm solution; this is the step that distinguishes the wave functional Green's function from the unrestricted Bunch-Davies Wightman function, and it is what fixes (42). Section A 2 gives the Lorentzian form and records the relation (A17) between the two. Section A 3 performs the quotient image sum, derives the near-defect series (46) whose leading $1/v$ term controls the heavy-mass limit, and establishes the large- v falloff (A23) used in the contour estimates of Appendix B.

Let $\Sigma_c := \{t = t_c\}$ be a finite late-time surface. In the wave functional path integral the final value χ_c is held fixed, so we write

$$\chi = \chi_{\text{cl}}[\chi_c] + \delta\chi, \quad \delta\chi|_{\Sigma_c} = 0. \quad (\text{A1})$$

The two-point function used at the cubic vertex is therefore the Green's function of $\delta\chi$. Denote the covering-space finite-cutoff Green's function by $G_{D,\Sigma_c}^{(1)}(x, x')$. Here the subscript D records its homogeneous final Dirichlet condition, Σ_c records the cutoff surface, and (1) labels the covering geometry. It satisfies

$$(\square_x - m_\chi^2)G_{D,\Sigma_c}^{(1)}(x, x') = \frac{i\delta^{(3)}(x, x')}{\sqrt{-g(x)}}, \quad G_{D,\Sigma_c}^{(1)}(x, x')|_{x \in \Sigma_c} = 0, \quad (\text{A2})$$

with the analogous condition in the second argument and the Bunch-Davies initial condition. A general finite-cutoff construction of such wave function Green's functions is given in Appendix B of Ref. [24].

We keep $m_\chi^2 \ell^2 = 1$ fixed and define the Green's function used in the main text by

$$G_D^{(1)}(x, x') := \lim_{\Sigma_c \rightarrow I^+} G_{D,\Sigma_c}^{(1)}(x, x'), \quad (\text{A3})$$

where the limit is taken at fixed interior points. The near-defect and heavy-mass limits are taken only afterward. Appendix B gives the boundary power counting that permits this limit to be used in the nonlocal vertex coefficient after the standard wave functional counterterms are included.

1. Finite cutoff and the no-logarithm limit

The Euclidean continuation maps the problem to EAdS₃ with positive radius ℓ_E (numerically equal to ℓ under the continuation). The continued light-field mass $m_{\chi,E}$ is at the Breitenlohner-Freedman value

$$m_{\chi,E}^2 \ell_E^2 = -1. \quad (\text{A4})$$

In global coordinates,

$$ds_E^2 = \ell_E^2(d\rho^2 + \sinh^2 \rho d\Omega_2^2), \quad 0 \leq \rho \leq R_E. \quad (\text{A5})$$

Here R_E is the Euclidean radial cutoff corresponding to the regulated final surface. We denote EAdS points by $x_E = (\rho, \Omega)$ and $x'_E = (\rho', \Omega')$, with $\Omega = (\theta, \psi)$. Expand in normalized spherical harmonics $Y_{j\mu}$, with $-\nabla_{S^2}^2 Y_{j\mu} = j(j+1)Y_{j\mu}$. Away from the source, a radial mode satisfies

$$f_j'' + 2 \coth \rho f_j' - j(j+1) \text{csch}^2 \rho f_j + f_j = 0. \quad (\text{A6})$$

Writing $f_j = \text{csch} \rho y_j$ reduces this equation to Legendre's equation. We denote the no-logarithm and logarithmic basis modes by

$$f_j^{\text{nl}}(\rho) := \text{csch} \rho P_j(\coth \rho), \quad f_j^{\text{log}}(\rho) := \text{csch} \rho Q_j(\coth \rho). \quad (\text{A7})$$

The mode $f_j^{\text{log}} = O(\rho^j)$ is regular at the center, while $f_j^{\text{nl}} = O(\rho^{-j-1})$ is singular there. At large ρ ,

$$f_j^{\text{nl}}(\rho) = 2e^{-\rho}[1 + O(e^{-2\rho})], \quad (\text{A8})$$

$$f_j^{\log}(\rho) = 2e^{-\rho}[\rho - \mathfrak{h}_j + O(\rho e^{-2\rho})], \quad \mathfrak{h}_j := \sum_{k=1}^j \frac{1}{k}, \quad \mathfrak{h}_0 := 0. \quad (\text{A9})$$

Their weighted Wronskian is

$$\sinh^2 \rho [f_j^{\text{nl}}(f_j^{\log})' - (f_j^{\text{nl}})' f_j^{\log}] = 1. \quad (\text{A10})$$

The solution that vanishes at the cutoff is

$$f_{j,R_E}^D(\rho) := f_j^{\text{nl}}(\rho) - \frac{f_j^{\text{nl}}(R_E)}{f_j^{\log}(R_E)} f_j^{\log}(\rho), \quad f_{j,R_E}^D(R_E) = 0. \quad (\text{A11})$$

Let $G_{D,R_E}^E(x_E, x'_E)$ denote the Euclidean Dirichlet Green's function on the ball $\rho \leq R_E$, normalized by $(-\nabla_E^2 + m_{\chi,E}^2)G_{D,R_E}^E = \delta_E^{(3)}(x_E, x'_E)/\sqrt{g_E}$, where $\delta_E^{(3)}$ is the coordinate delta distribution. With $\rho_< := \min(\rho, \rho')$ and $\rho_> := \max(\rho, \rho')$, it is

$$G_{D,R_E}^E(x_E, x'_E) = \frac{1}{\ell_E} \sum_{j=0}^{\infty} \sum_{\mu=-j}^j Y_{j\mu}(\Omega) Y_{j\mu}^*(\Omega') f_j^{\log}(\rho_<) \times f_{j,R_E}^D(\rho_>). \quad (\text{A12})$$

For each fixed j ,

$$\frac{f_j^{\text{nl}}(R_E)}{f_j^{\log}(R_E)} = \frac{1}{R_E - \mathfrak{h}_j} [1 + O(R_E e^{-2R_E})] \longrightarrow 0. \quad (\text{A13})$$

The fixed-interior limit consequently selects the no-logarithm inverse. We denote this Euclidean limit by

$$G_{\text{nl}}^E(\mathfrak{d}_E) := \lim_{R_E \rightarrow \infty} G_{D,R_E}^E(x_E, x'_E) = \frac{1}{4\pi \ell_E \sinh \mathfrak{d}_E}, \quad (\text{A14})$$

where the limit is taken at fixed interior points, distributionally after angular smearing, and $\mathfrak{d}_E$ is the dimensionless EAdS₃ geodesic distance between x_E and x'_E . It has the canonical $1/(4\pi d_E)$ coincidence singularity, with $d_E = \ell_E \mathfrak{d}_E$, and the pure $e^{-\mathfrak{d}_E}$ boundary falloff. The same selection follows directly from the global dS modes $\text{sech } tP_j(\tanh t)$ and $\text{sech } tQ_j(\tanh t)$. The second mode contains the late-time logarithm. In the exterior solution obtained by adding it to the no-logarithm mode so as to vanish at $t = t_c$, its coefficient is $O(t_c^{-1})$.

2. Lorentzian form and relation to the Wightman function

Continuing both the invariant and the Green's function normalization in (A14) with the Bunch-Davies prescription

gives $G_D^{(1)}$ in (42). Away from its singularities, any invariant bisolution $F(s)$ at the critical mass obeys

$$(1 - s^2)F'' - 3sF' - F = 0. \quad (\text{A15})$$

The two independent solutions are proportional to $1/\sqrt{1-s^2}$ and $\arcsin(s)/\sqrt{1-s^2}$. The latter contains the critical logarithmic falloff, so the no-logarithm condition and the canonical coincidence singularity fix $G_D^{(1)}(s)$.

For real $-1 < s < 1$, the unrestricted Bunch-Davies Wightman function is [28]

$$W_{\text{BD}}^{(1)}(s) = \frac{\pi - \arccos s}{4\pi^2 \ell \sqrt{1-s^2}}. \quad (\text{A16})$$

It follows that

$$G_D^{(1)}(s) = W_{\text{BD}}^{(1)}(s) + W_{\text{BD}}^{(1)}(-s). \quad (\text{A17})$$

The second term is homogeneous and smooth at coincidence, where it equals $1/(4\pi^2 \ell)$, but has the reflected singularity at $s = -1$. This is the critical-mass limit of the future-fast Green's function discussed in Ref. [29].

3. Finite quotient and near-defect series

For the image data in (43),

$$1 - s_n(x)^2 = 4v^2 a_n^2 (1 - v^2 a_n^2). \quad (\text{A18})$$

Substitution in (38), together with the vanishing covering-space finite part, gives (44). Within its disk of convergence,

$$\langle \chi^2(v) \rangle_{\text{ren}} = \frac{1}{8\pi \ell v} \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{4^k} v^{2k} \sum_{n=1}^{q-1} a_n^{2k-1}. \quad (\text{A19})$$

The identities

$$\sum_{n=1}^{q-1} a_n = \cot \frac{\pi}{2q}, \quad \sum_{n=1}^{q-1} a_n^3 = \frac{1}{4} \left(3 \cot \frac{\pi}{2q} - \cot \frac{3\pi}{2q} \right) \quad (\text{A20})$$

produce the series (46). The expansion contains only odd powers after its leading $1/v$ term.

For comparison, set $b_n := v a_n < 1$. The boundary-condition-dependent homogeneous term has the expansion

$$W_{\text{BD}}^{(1)}(-s_n(x)) = \frac{1}{4\pi^2 \ell} + \frac{b_n^2}{6\pi^2 \ell} + O(b_n^4/\ell). \quad (\text{A21})$$

Together with $\sum_{n=1}^{q-1} a_n^2 = q/2$, this cancels the constant and v^2 terms of the unrestricted Wightman image sum, while leaving the $1/v$ and v coefficients unchanged. Each

exact image also has a reflected square-root locus at $va_n = 1$. Equation (47) shows directly that these loci do not enter the local polar saddle neighborhoods.

On the Lorentzian edge inherited from the upper Bunch-Davies preparation, the square-root boundary value is

$$[1 - a_n^2 v^2]_{\text{BD}}^{-1/2} = [1 - a_n^2 \cosh^2 t \sin^2 \theta - i0 \sinh t]^{-1/2}. \quad (\text{A22})$$

Using $\sum_{n=1}^{q-1} \csc^2(\pi n/q) = (q^2 - 1)/3$, the exact profile has the large- v expansion

$$\langle \chi^2(v) \rangle_{\text{ren}} = \frac{i(q^2 - 1)}{24\pi\ell v^2} + O(v^{-4}/\ell). \quad (\text{A23})$$

Here the displayed sign is the future-tail sign; the conjugate preparation gives its complex conjugate. At fixed angles away from the poles and light-cone loci, this falloff makes the integrand density in (51) decay as $O(e^{-t})$.

APPENDIX B: EXACT REDUCTION OF THE BUNCH-DAVIES INTEGRAL

This appendix supplies the four technical inputs used in Sec. III. Section B 1 proves the contour deformation (52) by showing that no zero of the heavy invariant and no reflected light-field branch point enters the strip $0 < \text{Im } t < \pi/2$, and that the closing edge (B5) vanishes as $\text{Re } t \rightarrow \infty$. Section B 2 introduces coordinates adapted to the quotient axis, evaluates the axial and azimuthal integrations in (B11), and thereby reduces the vertex integral to the one-dimensional transform (56), whose absolute convergence on the principal series is established there. Section B 3 derives the large-mass expansion (57), and gives the closed form (B23) at $q = 2$ as a finite-mass check of the whole reduction. Section B 4 states the boundary power counting and the counterterm prescription under which the fixed-interior Green's function (A3) may be used inside the vertex integral.

1. Deformation to the upper hyperbolic section

Write Δ for a complex heavy weight, initially in the regulating domain $0 < \text{Re } \Delta < 1$; the physical value $\Delta = \Delta_+$ is reached by analytic continuation below. Define the angular invariant

$$\Xi := \cos \theta \cos \theta_p + \sin \theta \sin \theta_p \cos \psi. \quad (\text{B1})$$

Then $u = 2 \cosh t (\tanh t - \Xi)$. For $t = x + iy$ with $x > 0$ and $0 < y < \pi/2$,

$$\text{Im } \tanh(x + iy) = \frac{\sin(2y)}{\cosh(2x) + \cos(2y)} > 0. \quad (\text{B2})$$

Since Ξ is real, the heavy invariant has no zero in this open region.

The possible light-field branch points obey

$$\cosh t = \pm \frac{1}{a_n \sin \theta}. \quad (\text{B3})$$

The right-hand side is real and has magnitude at least one. But

$$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y. \quad (\text{B4})$$

In the open region, reality would force $x = 0$, after which the magnitude of the left-hand side is $|\cos y| < 1$. Thus no reflected light-field locus crosses the open region. On the Lorentzian boundary $y = 0$, however, reflected thresholds occur whenever $a_n \cosh x \sin \theta = 1$. These are precisely the square-root boundary values specified in (A22) and are locally integrable in the combined vertex integral. At the possible corner threshold, $\mathcal{F}(t)$ has at worst an integrable logarithm, so a small indentation gives no contribution. The heavy kernel likewise has its ordinary boundary light cone at $\Xi = \tanh x$, defined by the inherited $+i0$ prescription. In the regulating domain this singularity is locally integrable. The boundary values inherited from the upper preparation define the Lorentzian edge; since neither zero enters the open strip, these singularities do not pinch the deformation.

On the cap boundary $t = iy$, $0 < y \leq \pi/2$, the heavy invariant cannot vanish. The explicit $v = 0$ singularity on the quotient axis is locally integrable after the polar measure is included. None of these boundary singularities pinches the deformation.

At the far vertical edge $t = T + iy$, the heavy invariant is

$$u = e^{T+iy}(1 - \Xi) - e^{-T-iy}(1 + \Xi). \quad (\text{B5})$$

In this domain, for fixed $\text{Im } \Delta$ and $0 < \theta_p < \pi$, the angularly integrated edge is $O((1 + T)e^{-\text{Re } \Delta T})$. To obtain this estimate one separates the polar layers $\sin \theta = O(e^{-T})$, the boundary-light-cone layer $1 - \Xi = O(e^{-2T})$, and their complement. Their shrinking angular measure compensates the pointwise enhancement, while the square-root thresholds are integrable. The edge therefore vanishes as $T \rightarrow \infty$.

These observations prove (52) in this regulating domain and then, by analytic continuation, at $\Delta = 1 + i\nu$. The cutoff is removed at fixed ν before the large-mass expansion is taken. The endpoint $\rho = 0$ on the hyperbolic section is harmless: the volume-weighted $1/v$ singularity vanishes linearly there.

2. Cylindrical reduction

Substitution of (44) into (51) cancels the explicit factor of v and gives

$$\langle \mathcal{O}(\theta_P) \rangle_{K_L} = \frac{g\ell^2 c_\Delta}{8\pi} \sum_{n=1}^{q-1} \frac{1}{a_n} \int_{C_{BD}} dt d\theta d\psi \times \frac{\cosh t(u+i0)^{-\Delta}}{\sqrt{1-a_n^2 \cosh^2 t \sin^2 \theta}}. \quad (\text{B6})$$

The orientation of (48) becomes $\rho:0 \rightarrow \infty$. Continuing from the Euclidean cap gives $v = +i \sinh \rho \sin \theta$, so every light-field square root becomes the positive root of $1 + a_n^2 \sinh^2 \rho \sin^2 \theta$. Together with $\cosh t = i \sinh \rho$ and $(iU)^{-\Delta} = e^{-i\pi\Delta/2} U^{-\Delta}$, this fixes the overall factor $+ie^{-i\pi\Delta/2}$ below.

Introduce cylindrical coordinates (r, y, ψ) on H^3 by

$$\begin{aligned} \cosh \rho &= \cosh r \cosh y, & \sinh \rho \cos \theta &= \cosh r \sinh y, \\ \sinh \rho \sin \theta &= \sinh r. \end{aligned} \quad (\text{B7})$$

Their ranges are $r \geq 0$, $y \in \mathbb{R}$, and $0 \leq \psi < 2\pi$. The metric and volume element are

$$\begin{aligned} ds_{H^3}^2 &= \ell^2 (dr^2 + \cosh^2 r dy^2 + \sinh^2 r d\psi^2), \\ dV_{H^3} &= \ell^3 \sinh r \cosh r dr dy d\psi. \end{aligned} \quad (\text{B8})$$

In particular, the measure remaining after the explicit $1/v$ cancellation obeys

$$\sinh \rho d\rho d\theta d\psi = \cosh r dr dy d\psi. \quad (\text{B9})$$

Using $\tanh \rho_0 = \cos \theta_P$, the positive invariant factorizes as

$$U = 2 \sin \theta_P [\cosh r \cosh(y - \rho_0) - \sinh r \cos \psi]. \quad (\text{B10})$$

A shift of y removes ρ_0 from the remaining integral.

The orbit integral needed below is

$$\begin{aligned} \mathcal{A}_\Delta(r) &:= \int_{-\infty}^{\infty} dy \int_0^{2\pi} d\psi (\cosh r \cosh y - \sinh r \cos \psi)^{-\Delta} \\ &= 2^\Delta \pi \frac{\Gamma(\Delta/2)^2}{\Gamma(\Delta)} P_{\Delta/2-1}(1 + 2 \sinh^2 r). \end{aligned} \quad (\text{B11})$$

For $\text{Re} \Delta > 0$, the first line can be evaluated with a Schwinger parameter, using the modified Bessel functions I_0 and K_0 . The final line then defines its continuation to the principal series.

Combining (B6)–(B11) gives

$$\begin{aligned} \langle \mathcal{O}(\theta_P) \rangle_{K_L} &= \frac{ie^{-i\pi\Delta/2} g\ell^2 c_\Delta}{8\pi} (2 \sin \theta_P)^{-\Delta} \\ &\times \sum_{n=1}^{q-1} \frac{1}{a_n} \int_0^\infty dr \frac{\cosh r \mathcal{A}_\Delta(r)}{\sqrt{1 + a_n^2 \sinh^2 r}}. \end{aligned} \quad (\text{B12})$$

Setting $q = \sinh r$ proves (55) and (56).

For $\nu > 0$, the large-argument continuation contains the two branches $x^{-1/2+i\nu/2}$ and $x^{-1/2-i\nu/2}$ [[30], Sec. 14.8(iii)]. With $x = 1 + 2q^2$, this gives $P_{-1/2+i\nu/2}(1 + 2q^2) = O(q^{-1})$. The integrand in (55) is therefore $O(q^{-2})$. This proves absolute convergence on the principal series.

3. Large-mass expansion of the radial transform

We now prove the expansion quoted in (57). Set $\tau = \nu/2$, write $q = \sinh r$, and introduce $\eta = 2r$. The radial transform becomes

$$\begin{aligned} \mathcal{R}_{1+i\nu}(a) &= \frac{1}{2} \int_0^\infty d\eta F_a(\eta) P_{-1/2+i\tau}(\cosh \eta), \\ F_a(\eta) &:= \frac{\cosh(\eta/2)}{\sqrt{1 + a^2 \sinh^2(\eta/2)}}. \end{aligned} \quad (\text{B13})$$

Analytic continuation of the Mehler-Dirichlet formula [30] [Eq. 14.12.1] gives, for $\eta > 0$ (see [[30], Sec. 14.20(iv)]),

$$P_{-1/2+i\tau}(\cosh \eta) = \frac{\sqrt{2}}{\pi} \int_0^\eta d\omega \frac{\cos(\tau\omega)}{\sqrt{\cosh \eta - \cosh \omega}}. \quad (\text{B14})$$

The diagonal square-root singularity is integrable. After the ω integration, the absolute kernel obeys

$$F_a(\eta) \int_0^\eta \frac{d\omega}{\sqrt{\cosh \eta - \cosh \omega}} = O((1 + \eta)e^{-\eta/2}), \quad \eta \rightarrow \infty. \quad (\text{B15})$$

The double integral is therefore absolutely integrable, so its order may be reversed. This gives the exact cosine transform

$$\mathcal{R}_{1+i\nu}(a) = \frac{\sqrt{2}}{\pi} \int_0^\infty d\omega \cos(\tau\omega) H_a(\omega), \quad (\text{B16})$$

$$H_a(\omega) := \frac{1}{2} \int_\omega^\infty d\eta \frac{F_a(\eta)}{\sqrt{\cosh \eta - \cosh \omega}}. \quad (\text{B17})$$

Both H_a and its derivatives decay exponentially as $\omega \rightarrow \infty$. At the other endpoint,

$$\begin{aligned} F_a(\eta) &= 1 + \frac{1-a^2}{8} \eta^2 + O(\eta^4), \\ \cosh \eta - \cosh \omega &= \frac{\eta^2 - \omega^2}{2} \left[1 + \frac{\eta^2 + \omega^2}{12} + O(\eta^4 + \omega^4) \right]. \end{aligned} \quad (\text{B18})$$

Splitting the η integral at a fixed small positive value and integrating the displayed terms gives

$$H_a(\omega) = -\frac{1}{\sqrt{2}}\log \omega + B_a + \frac{a^2}{16\sqrt{2}}\omega^2 \log \omega + C_a\omega^2 + O(\omega^4|\log \omega|), \quad (\text{B19})$$

where B_a and C_a are independent of ω and ν .

Let $\kappa(\omega)$ be smooth, compactly supported and equal to one near $\omega = 0$. For any fixed N , analytic continuation in λ of the Mellin cosine transform gives

$$\begin{aligned} \int_0^\infty d\omega \kappa(\omega) \cos(\tau\omega) \log \omega &= -\frac{\pi}{2\tau} + O(\tau^{-N}), \\ \int_0^\infty d\omega \kappa(\omega) \cos(\tau\omega) \omega^2 \log \omega &= \frac{\pi}{\tau^3} + O(\tau^{-N}). \end{aligned} \quad (\text{B20})$$

The smooth terms may be integrated by parts repeatedly. The endpoint expansion continues in even powers of ω , with and without $\log \omega$. Its displayed remainder has cosine transform $O(\tau^{-5})$. Equations (B16) and (B19) consequently give

$$\mathcal{R}_{1+i\nu}(a) = \frac{1}{2\tau} + \frac{a^2}{16\tau^3} + O(\tau^{-5}) = \frac{1}{\nu} + \frac{a^2}{2\nu^3} + O(\nu^{-5}), \quad (\text{B21})$$

which proves (57). With $\xi := \nu\varrho$ held fixed, the Mehler-Dirichlet representation gives

$$\mathcal{P}_{-1/2+i\nu/2}(1 + 2\xi^2/\nu^2) \longrightarrow J_0(\xi), \quad (\text{B22})$$

where J_0 is the Bessel function of the first kind. This limit gives a local interpretation of the leading term. The cosine-transform proof also controls the rest of the radial integral.

For $q = 2$, one has $a_1 = 1$. Set $h_\Delta := \Delta/2$. The substitution $w = \tanh^2 r$ gives the following expression, where B is the Euler beta function:

$$\begin{aligned} \mathcal{R}_\Delta(1) &= \frac{1}{2}B\left(\frac{1}{2}, h_\Delta\right)_3 F_2\left(\begin{matrix} h_\Delta, h_\Delta, \frac{1}{2} \\ 1, h_\Delta + \frac{1}{2} \end{matrix}; 1\right) \\ &= \frac{1}{4} \frac{\Gamma(\Delta/4)\Gamma((2-\Delta)/4)}{\Gamma((4-\Delta)/4)\Gamma((2+\Delta)/4)}. \end{aligned} \quad (\text{B23})$$

The second line follows from Dixon's sum [30], Eq. 16.4.4]. For $\Delta = 1 + i\nu$, this gives

$$\mathcal{R}_{1+i\nu}(1) = \frac{1}{\nu} + \frac{1}{2\nu^3} + O(\nu^{-5}).$$

This independently confirms the radial asymptotic (57) for $a = 1$ and provides an exact finite- ν benchmark for the radial reduction. When combined with the contour factor, the normalization $c_{\Delta+}$, and the gamma-function prefactor already derived in (55), it yields the $q = 2$ specialization of (60).

4. Removal of the final cutoff

The Green's function inserted above is the fixed-interior limit in (A3). Pointwise convergence alone does not justify moving this limit through the vertex integral, because the finite-cutoff Dirichlet correction is nonuniform in a thin layer next to the final surface. The following power counting controls the fixed-mode part of this layer and isolates the remaining standard local counterterm contribution.

Let $z = e^{-\rho}$ near the hyperbolic boundary and $z_c = e^{-R_E}$ at the cutoff. Away from a quotient fixed endpoint, every nonidentity image of the critical no-logarithm Green's function obeys

$$G_D^{(1)}(x, \gamma^n x) = O(z^2). \quad (\text{B24})$$

Equations (A9) and (A13) show that, outside the layer $z = O(z_c)$, the finite-cutoff homogeneous correction is

$$O(R_E^{-1} z^2 \log^2 z). \quad (\text{B25})$$

The remaining boundary regions are integrable when the insertion is away from a quotient fixed endpoint. Let $\mathbf{y}$ denote local coordinates on the conformal boundary and let $\mathbf{y}_P$ be the coordinate of the insertion. Away from the insertion, $dV_{H^3} = O(z^{-3} dz d^2 \mathbf{y})$, the heavy kernel is $O(z)$ in magnitude, and (B24) leaves an $O(dz)$ radial density. Near the insertion set $\mathbf{y} - \mathbf{y}_P = z\mathbf{w}$; the magnitude of the kernel is then $O(z^{-\text{Re}\Delta})$ and $d^2 \mathbf{y} = z^2 d^2 \mathbf{w}$. After the intermediate angular range is included, this region is bounded by $O((1 + |\log z|)dz)$. Near a quotient-axis endpoint, the $1/\nu$ singularity cancels the polar measure and gives the same integrable logarithmic bound.

It follows that the part of the layer controlled by these estimates contributes at most $O(z_c |\log z_c|^p)$ for some fixed power p , while (B25) contributes $O(R_E^{-1})$. The fixed-mode estimates do not by themselves control angular momenta that grow with the cutoff. We use the standard wave-functional-renormalization prescription in which this remaining high-angular-momentum boundary layer is local and is removed by local counterterms [24]. Under this standard assumption, (55) is the cutoff limit of the renormalized nonlocal coefficient.

APPENDIX C: THE S^3 CONTINUATION

The geometric pair $\{L_+, L_-\}$ can also be derived through the standard Euclidean sphere continuation [26,27]. This calculation provides a second description of the complex geodesic critical points, but it uses the same geometric branch condition as Sec. IV.

On the fixed- ψ slice, the continuation $t = i\tau_E$ gives

$$ds^2 = \ell^2 (d\tau_E^2 + \cos^2 \tau_E d\theta^2). \quad (\text{C1})$$

Take the regulated boundary endpoint to be $(-iT, \theta_P)$, and represent the other endpoint on the complexified fixed axis by $(\tau_E, 0)$. If D_{S^3} denotes the analytically continued sphere distance, its invariant is

$$\mathcal{Z}(\tau_E) := \cos \frac{D_{S^3}}{\ell} = -i \sinh T \sin \tau_E + \cosh T \cos \tau_E \cos \theta_P. \quad (\text{C2})$$

For large T this becomes

$$\mathcal{Z}(\tau_E) = \frac{e^T}{2} (\cos \tau_E \cos \theta_P - i \sin \tau_E) + O(e^{-T}). \quad (\text{C3})$$

The inverse cosine in (C2) is multivalued, so its branch must be specified independently of the saddle values. On the real H^2 continuation we require $D_{S^3} = -i d_{H^2}$ and hence $i D_{S^3} = d_{H^2} > 0$. We then continue this branch through the same upper analytic domain used in Sec. IV. With the notation $\log_{\text{geom}}$ introduced there,

$$D_{S^3} = -i \ell \log_{\text{geom}}(2\mathcal{Z}) + O(\ell \mathcal{Z}^{-2}). \quad (\text{C4})$$

The continued worldline action is $\mathcal{L}_{S^3} := i D_{S^3}$. Subtracting the same boundary divergence as in the H^2 calculation gives

$$\begin{aligned} L_{S^3}^{\text{reg}}(\tau_E) &:= \lim_{T \rightarrow \infty} [\mathcal{L}_{S^3} - \ell(T - \log 2)] \\ &= \ell \log_{\text{geom}} [2(\cos \tau_E \cos \theta_P - i \sin \tau_E)]. \end{aligned} \quad (\text{C5})$$

Extremizing the endpoint along the complexified axis gives

$$\tan \tau_{E,*}^{(\sigma)} = -i \sec \theta_P, \quad \sigma = \pm 1. \quad (\text{C6})$$

At the two nearest solutions,

$$\cos \tau_{E,*}^{(\sigma)} \cos \theta_P - i \sin \tau_{E,*}^{(\sigma)} = -\sigma i \sin \theta_P, \quad (\text{C7})$$

so the geometric branch fixed above yields

$$L_{S^3,\sigma}^{\text{reg}} = \ell \log_{\text{geom}}(-2\sigma i \sin \theta_P) = \begin{cases} L_-, & \sigma = +1, \\ L_+, & \sigma = -1. \end{cases} \quad (\text{C8})$$

Using (61), their global-time representatives are

$$\tau_{E,*}^{(\sigma)} = \sigma \frac{\pi}{2} - i \rho_0, \quad i \tau_{E,*}^{(\sigma)} = t_{N,\sigma}. \quad (\text{C9})$$

These points lie on the two hyperbolic sections $t = \rho \pm i\pi/2$, on which

$$ds^2 = -\ell^2(d\rho^2 + \sinh^2 \rho d\theta^2). \quad (\text{C10})$$

The H^2 calculation is therefore a real-coordinate description of the same analytically continued S^3 critical points. On the geometric branch, the upper coordinate representative $\sigma = +1$ carries the value L_- . The Bunch-Davies kernel uses a different logarithm: its $+i0$ boundary value assigns L_+ to that representative, while the conjugate kernel assigns L_- on the lower preparation. As emphasized in (83), the comparison concerns the pair of action values and does not require the two logarithmic branches to agree at a fixed coordinate saddle.

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