

Tensor hybrid charmonia

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The mass and current coupling of the tensor hybrid charmonia H_c and $\tilde{H}_c$ with quantum numbers $J^{PC} = 2^{-+}$ and 2^{++} , as well as their full widths, are calculated in the context of the QCD sum rule method. The spectral parameters of these states are computed using the QCD two-point sum rule approach including dimension-12 terms $\sim \langle g_s^3 G^3 \rangle^2$. The full decay widths of the charmonia H_c and $\tilde{H}_c$ are evaluated by considering their kinematically allowed decay channels. In the case of the hybrid state H_c , decays to $D^{(\pm)}D^{*(\mp)}$, $D^0\bar{D}^{*0}$, and $D_s^{(\pm)}D_s^{*(\mp)}$ mesons are taken into account. The processes $\tilde{H}_c \rightarrow D^{(*)+}D^{(*)-}$, $D^{(*)0}\bar{D}^{(*)0}$, and $D_s^{(*)+}D_s^{(*)-}$ are employed to estimate the full width of the hybrid charmonium $\tilde{H}_c$. The partial widths of these decays are computed by means of the QCD three-point sum rule method, which is necessary to calculate strong couplings at the relevant hybrid-meson-meson vertices. Our predictions $m = (4.16 \pm 0.14)$ GeV and $\tilde{m} = (4.5 \pm 0.1)$ GeV for the masses and $\Gamma[H_c] = (160 \pm 30)$ MeV and $\Gamma[\tilde{H}_c] = (206 \pm 33)$ MeV for the full width of these hybrid charmonia can be useful to study and interpret various resonances in the 4–5 GeV mass range.

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I. INTRODUCTION

Investigation of structures composed of valence quarks and gluons, i.e., hybrid hadrons and their experimental discovery, is one of the important problems in the agenda of particle physics. Existence of the hybrids, which are hadrons beyond the conventional $q\bar{q}'$ and $qq'q''$ scheme, is allowed by the quantum chromodynamics and quark-gluon model. Theoretical studies of such structures have continued over the last five decades. Starting from pioneering analyses in Refs. [1,2], the physics of the hybrid mesons and baryons became a rapidly growing branch of hadronic studies [3]. Numerous publications are devoted to exploring their spectroscopic parameters, production, and decay mechanisms. To this end, researchers suggested new models and calculational schemes or adapted existing ones to embrace exotic hadrons as well (see Refs. [3–6] and references therein).

There are few experimentally observed resonances that are considered as candidates to the hybrid mesons.

The light isovector particles $\pi_1(1400)$, $\pi_1(1600)$, and $\pi_1(2015)$ with the spin-parities $J^{PC} = 1^{-+}$ are among such structures. It is remarkable that the ordinary mesons made of a quark and an antiquark cannot bear such quantum numbers. Therefore, the states $\pi_1(1400)$, $\pi_1(1600)$, and $\pi_1(2015)$ are definitely exotic particles and presumably belong to a family of hybrid mesons, though their four-quark interpretations are not excluded.

The resonance $\pi_1(1400)$ has the mass (1406 ± 20) MeV and width (180 ± 30) MeV. It was seen in the exclusive reaction $\pi^- p \rightarrow \pi^0 n \eta$ [7], and was the first candidate to the hybrid meson. The next particle from this series, $\pi_1(1600)$, was fixed by E892 Collaboration in the decay mode $\eta' \pi$ of the reaction $\pi^- p$ [8]. The structures 1^{-+} were observed and studied by this and other experimental groups in different channels as well [9–13]. The latest analysis, however, favors the existence of only one broad state, $\pi_1(1600)$ [14]. The evidence for the next exotic meson, $\pi_1(2015)$, from this family was reported in Refs. [10,11]. The isoscalar particle $\eta_1(1855)$ with $J^{PC} = 1^{-+}$ was seen quite recently by BESIII Collaboration in the radiative decay $J/\psi \rightarrow \gamma \eta_1(1855) \rightarrow \gamma \eta \eta'$ [15].

Some of the observed heavy resonances can be interpreted as candidates to hybrid mesons as well. For example, it was suggested that the resonances $\psi(4230)$ and $\psi(4360)$ may be considered as vector hybrid charmonium states $\bar{c}gc$ or as mesons with sizeable exotic hybrid

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ingredients [16,17]. A list of numerous other resonances that probably are hybrid quarkonia was presented in Ref. [18]. There are also candidates to hybrid states with baryon quantum numbers. In fact, the baryon $\Lambda(1405)$ studied by different collaborations [19–23] may be one of such exotic baryons (see, for instance, Ref. [23]).

The hybrid quarkonia $H_b = \bar{b}gb$ and $H_c = \bar{c}gc$ and mesons $H_{bc} = \bar{b}gc$ were investigated in the frameworks of diverse methods [24–49]. These publications addressed numerous problems of heavy hybrid mesons by studying their spectroscopic parameters, decay channels, and production mechanisms. These states were investigated within the QCD sum rule (SR) method, the lattice simulations, the nonrelativistic effective field theories, the Born-Oppenheimer (BO) approximation, and BO-effective field theory (BOEFT).

Results obtained for parameters of hybrid structures in the frameworks of different methods, as usual, differ from each other. Therefore, there is a necessity to perform relevant investigations with higher accuracy by including new factors in the analysis. The hybrid states H_b , H_c , and H_{bc} were also investigated in our work [6]. There, we calculated the masses and current couplings of the hybrid quarkonia H_b and H_c with spin-parities $J^{\text{PC}} = 0^{++}, 0^{+-}, 0^{-+}$, and 0^{--} and $1^{++}, 1^{+-}, 1^{-+}$, and 1^{--} . The spectral parameters of the hybrid mesons H_{bc} with $J^{\text{P}} = 0^+, 0^-, 1^+$, and 1^- were computed as well.

In the current paper, we consider the tensor hybrid charmonia H_c and $\tilde{H}_c$ with contents $\bar{c}gc$ and spin-parities $J^{\text{PC}} = 2^{-+}$ and 2^{++} , respectively. We evaluate their masses and current couplings by employing the QCD two-point SR method. In this process, we take into account nonperturbative terms $\langle g_s^3 G^3 \rangle^2$. The full decay widths of H_c and $\tilde{H}_c$ are calculated by considering their kinematically allowed decay channels. It turns out that the hybrid state H_c decays to conventional mesons through the processes $H_c \rightarrow D^{(\pm)}D^*(2010)^{(\mp)}$, $D^0\bar{D}^*(2007)^0$, and $D_s^{(\pm)}D_s^{*(\mp)}$. In the case of the hybrid charmonium $\tilde{H}_c$, modes $\tilde{H}_c \rightarrow D^{(*)+}D^{(*)-}$, $D^{(*)0}\bar{D}^{(*)0}$, and $D_s^{(*)+}D_s^{(*)-}$ are allowed decay channels. The partial widths of these processes are evaluated by means of the QCD three-point SR approach. This method is necessary to calculate strong couplings at the relevant hybrid-meson-meson vertices and, in this way, to find the width of a process under consideration.

This work is divided into five sections. In Sec. II, we calculate the masses and current couplings of the tensor mesons H_c and $\tilde{H}_c$. The decay modes of the hybrid state H_c are considered in Sec. III. The full width of $\tilde{H}_c$ is found in Sec. IV. In Sec. V, we compare our results with those obtained in other publications and make concluding notes.

II. SPECTROSCOPIC PARAMETERS OF THE TENSOR HYBRIDS H_c AND $\tilde{H}_c$

In this section, we explore the tensor hybrid charmonia H_c and $\tilde{H}_c$ by means of the QCD sum rule method [50,51]. The SR method originally was invented to investigate properties of conventional hadrons, but it can also be employed to consider multi-quark and hybrid particles as well [52–54]. It is interesting that QCD SRs were used for studying the hybrid quarkonia in early years of the method [24,25].

We derive the sum rules for the mass m and current coupling Λ of the tensor state H_c using the correlation function

$$\Pi_{\mu\nu\mu'\nu'}(p) = i \int d^4x e^{ipx} \langle 0 | \mathcal{T} \{ J_{\mu\nu}(x) J_{\mu'\nu'}^\dagger(0) \} | 0 \rangle, \quad (1)$$

where $J_{\mu\nu}(x)$ and $\mathcal{T}$ stand for the interpolating current of the hybrid state H_c and a time-ordered product of two currents, respectively.

For the tensor hybrid charmonium with the quantum numbers $J^{\text{PC}} = 2^{-+}$, the interpolating current is given by the expression

$$J_{\mu\nu}(x) = g_s \bar{c}_a(x) \sigma_{\mu\gamma 5}^{\alpha} \frac{\lambda_{ab}^n}{2} G_{a\nu}^n(x) c_b(x). \quad (2)$$

For the tensor state $J^{\text{PC}} = 2^{++}$, the interpolating current $\tilde{J}_{\mu\nu}(x)$ has the form

$$\tilde{J}_{\mu\nu}(x) = g_s \bar{c}_a(x) \sigma_{\mu\gamma 5}^{\alpha} \frac{\lambda_{ab}^n}{2} \tilde{G}_{a\nu}^n(x) c_b(x). \quad (3)$$

In Eqs. (2) and (3), $c_a(x)$ is the c quark field, and g_s is the QCD strong coupling constant. The a and b are color indices, and λ^n , $n = 1, 2, \dots, 8$ stand for the Gell-Mann matrices. By $G_{\mu\nu}^n(x)$ and $\tilde{G}_{\mu\nu}^n(x) = \epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta}(x)/2$, we denote the gluon field strength tensor and its dual field, respectively.

We start from consideration of the current $J_{\mu\nu}(x)$ and the tensor hybrid state H_c . In the SR method, one first writes $\Pi_{\mu\nu\mu'\nu'}(p)$ using the physical parameters of H_c ,

$$\begin{aligned} \Pi_{\mu\nu\mu'\nu'}^{\text{Phys}}(p) &= \frac{\langle 0 | J_{\mu\nu} | H_c(p, \epsilon) \rangle \langle H_c(p, \epsilon) | J_{\mu'\nu'}^\dagger | 0 \rangle}{m^2 - p^2} \\ &+ \dots, \end{aligned} \quad (4)$$

where m and $\epsilon = \epsilon_{\mu\nu}^{(\lambda)}(p)$ are its mass and polarization tensor, respectively. Here, the contribution of the ground-level particle H_c is shown explicitly, whereas effects due to higher resonances and continuum states are denoted by the ellipses. It is also useful to introduce the matrix element

$$\langle 0 | J_{\mu\nu} | H_c(p, \epsilon) \rangle = \Lambda \epsilon_{\mu\nu}^{(\lambda)}(p). \quad (5)$$

To find $\Pi_{\mu\nu\mu'\nu'}^{\text{Phys}}(p)$, we substitute this matrix element into Eq. (4) and perform summation over the polarization tensor using

$$\sum_{\lambda} \epsilon_{\mu\nu}^{(\lambda)}(p) \epsilon_{\mu'\nu'}^{*(\lambda)}(p) = \frac{1}{2} (\tilde{g}_{\mu\mu'} \tilde{g}_{\nu\nu'} + \tilde{g}_{\mu\nu} \tilde{g}_{\nu\mu'}) - \frac{1}{3} \tilde{g}_{\mu\nu} \tilde{g}_{\mu'\nu'}, \quad (6)$$

where

$$\tilde{g}_{\mu\nu} = -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}. \quad (7)$$

Our computations yield

$$\Pi_{\mu\nu\mu'\nu'}^{\text{Phys}}(p) = \frac{\Lambda^2}{m^2 - p^2} \left\{ \frac{1}{2} (g_{\mu\mu'} g_{\nu\nu'} + g_{\mu\nu} g_{\nu\mu'}) + \text{other structures} \right\} + \dots, \quad (8)$$

with dots standing for contributions of other structures as well as higher resonances and continuum states. Note that, after application of Eqs. (6) and (7), there appear numerous Lorentz structures in the curly brackets. The term proportional to $(g_{\mu\mu'} g_{\nu\nu'} + g_{\mu\nu} g_{\nu\mu'})$ contains contribution of only the spin-2 particle, whereas the remaining components in Eq. (8) are formed due to contributions of spin-0 and -1 states as well. Therefore, in our studies we restrict ourselves by exploring this term and the corresponding invariant amplitude $\Pi^{\text{Phys}}(p^2)$.

At the next stage of analysis, we compute the correlator $\Pi_{\mu\nu\mu'\nu'}(p)$ with some accuracy in the operator product expansion (OPE). To this end, we employ in Eq. (1) expression of the current $J_{\mu\nu}(x)$ and contract corresponding quark and gluon fields. As a result, we find

$$\Pi_{\mu\nu\mu'\nu'}^{\text{OPE}}(p) = \frac{ig_s^2}{4} \int d^4x e^{ipx} \lambda_{ab}^n \lambda_{a'b'}^m \langle 0 | G_{a\nu}^n(x) G_{a'\nu'}^m(0) | 0 \rangle \times \text{Tr} [\sigma_\mu^\alpha \gamma_5 S_c^{bb'}(x) \gamma_5 \sigma_{\mu'}^{\alpha'} S_c^{a'a}(-x)], \quad (9)$$

where $S_c^{ab}(x)$ is the c quark propagator. In our calculations, we employ the following expression for $S_c^{ab}(x)$ [$Q = c$]:

$$S_Q^{ab}(x) = i \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \left\{ \frac{\delta_{ab}(\mathbf{k} + m_Q)}{k^2 - m_Q^2} - \frac{g_s G_{ab}^{\alpha\beta} \sigma_{\alpha\beta}(\mathbf{k} + m_Q) + (\mathbf{k} + m_Q) \sigma_{\alpha\beta}}{4(k^2 - m_Q^2)^2} + \frac{g_s^2 G^2}{12} \delta_{ab} m_Q \frac{k^2 + m_Q k}{(k^2 - m_Q^2)^4} + \frac{g_s^3 G^3}{48} \delta_{ab} \frac{(\mathbf{k} + m_Q)}{(k^2 - m_Q^2)^6} \times [k(k^2 - 3m_Q^2) + 2m_Q(2k^2 - m_Q^2)](\mathbf{k} + m_Q) + \dots \right\}. \quad (10)$$

Here, we have used the short-hand notations

$$G_{ab}^{\alpha\beta} \equiv G_n^{\alpha\beta} \lambda_{ab}^n / 2, \quad G^2 = G_{\alpha\beta}^n G_n^{\alpha\beta}, \quad G^3 = f^{nml} G_{\alpha\beta}^n G^{m\beta\delta} G_\delta^{l\alpha}, \quad (11)$$

where f^{nml} are the structure constants of the color group $SU_c(3)$.

The $\Pi_{\mu\nu\mu'\nu'}^{\text{OPE}}(p)$ is a product of two factors. One of them is the term with the trace over spinor indices and consists of c quark propagators. The propagator $S_c^{ab}(x)$ has the perturbative and nonperturbative components. The latter includes terms proportional to $g_s^2 G^2$ and $g_s^3 G^3$, which between vacuum states generate the well-known gluon condensates. The term $\sim g_s G_{ab}^{\alpha\beta}$ in Eq. (10), having been multiplied with a similar component of another propagator, gives rise to the two-gluon condensate as well, which we also include in the analysis.

Our treatment of the matrix element $\langle 0 | G_{a\nu}^n(x) G_{a'\nu'}^m(0) | 0 \rangle$ also needs some comments. We replace it with the vacuum

condensate $\langle g_s^2 G^2 \rangle$ and keep the first component in the Taylor expansion at $x = 0$,

$$\langle 0 | g_s^2 G_{a\nu}^n(x) G_{a'\nu'}^m(0) | 0 \rangle = \frac{\langle g_s^2 G^2 \rangle}{96} \delta^{nm} [g_{a\alpha'} g_{\nu\nu'} - g_{a\nu'} g_{\alpha'\nu}]. \quad (12)$$

Terms obtained in this manner describe diagrams where the gluon interacts with the QCD vacuum. Alternatively, we use

$$\langle 0 | G_{a\nu}^n(x) G_{a'\nu'}^m(0) | 0 \rangle = \frac{\delta^{nm}}{2\pi^2 x^4} \left[g_{\nu\nu'} \left(g_{a\alpha'} - \frac{4x_\alpha x_{\alpha'}}{x^2} \right) + (\nu, \nu') \leftrightarrow (\alpha, \alpha') - \nu \leftrightarrow \alpha - \nu' \leftrightarrow \alpha' \right]. \quad (13)$$

Contributions obtained in this way correspond to diagrams with the full valence gluon propagator.

Having extracted the structure $(g_{\mu\mu'}g_{\nu\nu'} + g_{\mu\nu'}g_{\nu\mu'})$ from $\Pi_{\mu\nu\mu'\nu'}^{\text{OPE}}(p)$ and labeled the corresponding invariant amplitude by $\Pi^{\text{OPE}}(p^2)$, one derives SRs for the mass and current coupling of the hybrid meson H_c . To this end, we rewrite $\Pi^{\text{Phys}}(p^2)$ as the dispersion integral

$$\Pi^{\text{Phys}}(p^2) = \int_{4m_c^2}^{\infty} \frac{\rho^{\text{Phys}}(s)ds}{s - p^2} + \dots, \quad (14)$$

where m_c is the c quark mass, and the dots indicate subtraction terms required to render finite $\Pi^{\text{Phys}}(p^2)$. The spectral density $\rho^{\text{Phys}}(s)$ is equal to the imaginary part of $\Pi^{\text{Phys}}(p^2)$,

$$\rho^{\text{Phys}}(s) = \Lambda^2 \delta(s - m^2) + \rho^h(s) \theta(s - s_0), \quad (15)$$

where s_0 is the continuum subtraction parameter. The contribution of the hybrid meson H_c is represented in Eq. (15) by the pole term and separated from other effects. Contributions to $\rho^{\text{Phys}}(s)$ coming from higher resonances and continuum states are characterized by an unknown hadronic spectral density $\rho^h(s)$. It is clear that $\rho^{\text{Phys}}(s)$ leads to the expression

$$\Pi^{\text{Phys}}(p^2) = \frac{\Lambda^2}{m^2 - p^2} + \int_{s_0}^{\infty} \frac{\rho^h(s)ds}{s - p^2} + \dots \quad (16)$$

We employ, in the region $p^2 \ll 0$, the Borel transformation $\mathcal{B}$ to suppress contributions of higher resonances and continuum states. This transformation also removes subtraction terms in the dispersion integral. For $\mathcal{B}\Pi^{\text{Phys}}(p^2)$, we obtain

$$\mathcal{B}\Pi^{\text{Phys}}(p^2) = \Lambda^2 e^{-m^2/M^2} + \int_{s_0}^{\infty} ds \rho^h(s) e^{-s/M^2}, \quad (17)$$

where M^2 is the Borel parameter.

The amplitude $\Pi^{\text{OPE}}(p^2)$ can be calculated in the deep Euclidean region $p^2 \ll 0$ using the operator product expansion. The coefficient functions in OPE could be obtained using methods of perturbative QCD, whereas nonperturbative information is contained in the gluon condensates. Having calculated the imaginary part of $\Pi^{\text{OPE}}(p^2)$, we get the two-point spectral density $\rho^{\text{OPE}}(s)$.

One can also write the dispersion representation for the amplitude $\Pi^{\text{OPE}}(p^2)$ in terms of $\rho^{\text{OPE}}(s)$. Then, by equating the Borel transformations of $\Pi^{\text{Phys}}(p^2)$ and $\Pi^{\text{OPE}}(p^2)$ and using the assumption on hadron-parton duality $\rho^h(s) \simeq \rho^{\text{OPE}}(s)$ above the threshold, we can remove the second term in Eq. (17) from the right-hand side of the obtained equality and find

$$\Lambda^2 e^{-m^2/M^2} = \Pi(M^2, s_0). \quad (18)$$

Here,

$$\Pi(M^2, s_0) = \int_{4m_c^2}^{s_0} ds \rho^{\text{OPE}}(s) e^{-s/M^2}. \quad (19)$$

After simple manipulations, we get

$$m^2 = \frac{\Pi'(M^2, s_0)}{\Pi(M^2, s_0)} \quad (20)$$

and

$$\Lambda^2 = e^{m^2/M^2} \Pi(M^2, s_0), \quad (21)$$

which are the sum rules for m and Λ , respectively. In Eq. (20), we have introduced the notation $\Pi'(M^2, s_0) = d\Pi(M^2, s_0)/d(-1/M^2)$. The spectral density $\rho^{\text{OPE}}(s)$ contains the perturbative $\rho^{\text{pert}}(s)$ and nonperturbative $\rho^{\text{DimN}}(s)$ terms ($N = 4, 6, 8, 10, 12$). Explicit expressions for $\rho^{\text{OPE}}(s)$ are lengthy and not provided here.

To carry out numerical analysis, we have to fix the input parameters in Eqs. (20) and (21). For these purposes, we employ the values

$$\begin{aligned} m_c &= (1.27 \pm 0.02) \text{ GeV}, \\ \langle \alpha_s G^2/\pi \rangle &= (0.012 \pm 0.004) \text{ GeV}^4, \\ \langle g_s^3 G^3 \rangle &= (0.57 \pm 0.29) \text{ GeV}^6. \end{aligned} \quad (22)$$

Here, m_c corresponds to the running mass in the $\overline{\text{MS}}$ scheme at the scale $\mu = m_c$ [14]. The condensates $\langle \alpha_s G^2/\pi \rangle$ and $\langle g_s^3 G^3 \rangle$ were obtained from analyses of different processes [50,51,55].

Equations (20) and (21) also contain the parameters M^2 and s_0 . They have to be chosen in such a way as to guarantee the prevalence of the pole contribution (PC) in the physical quantities obtained from the SR method. The stability of these results on M^2 , as well as convergence of the OPE, are also important restrictions in numerical calculations. Numerically, these conditions can be controlled by introducing the quantities

$$\text{PC} = \frac{\Pi(M^2, s_0)}{\Pi(M^2, \infty)} \quad (23)$$

and

$$R(M^2) = \frac{\Pi^{\text{DimN}}(M^2, s_0)}{\Pi(M^2, s_0)}, \quad (24)$$

where $\Pi^{\text{DimN}}(M^2, s_0) = \sum_{N=8,10,12} \Pi^{\text{DimN}}$ is a sum of terms proportional to $\langle g_s^2 G^2 \rangle^2$, $\langle g_s^2 G^2 \rangle \langle g_s^3 G^3 \rangle$, and $\langle g_s^3 G^3 \rangle^2$, respectively.

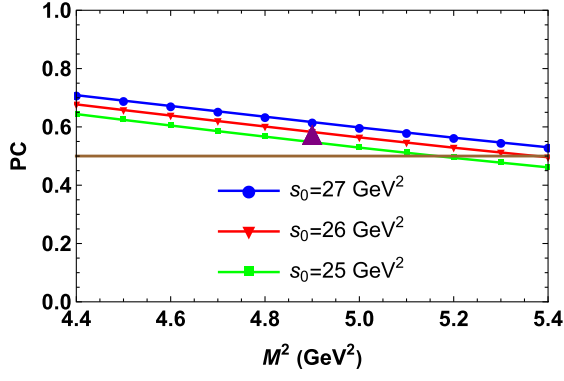


FIG. 1. PC as a function of M^2 at fixed s_0 . The red triangle marks the point $M^2 = 4.9 \text{ GeV}^2$ and $s_0 = 26 \text{ GeV}^2$.

Our numerical computations prove that in the case of H_c , the regions

$$M^2 \in [4.4, 5.4] \text{ GeV}^2, \quad s_0 \in [25, 27] \text{ GeV}^2 \quad (25)$$

satisfy restrictions of the SR analysis. Indeed, at $M^2 = 4.4 \text{ GeV}^2$ and $M^2 = 5.4 \text{ GeV}^2$, the averaged value of the pole contribution is $\text{PC} \approx 0.68$ and $\text{PC} \approx 0.50$, respectively. At $M^2 = 4.4 \text{ GeV}^2$, the parameter $|R(M^2)|$ does not exceed 0.01. The PC is shown in Fig. 1 as a function of M^2 .

The spectral parameters m and Λ are found as their average values over the regions in Eq. (25) and amount to

$$\begin{aligned} m &= (4.16 \pm 0.14) \text{ GeV}, \\ \Lambda &= (0.68 \pm 0.04) \text{ GeV}^4. \end{aligned} \quad (26)$$

The predictions in Eq. (26) correspond to the SR results at the point $M^2 = 4.9 \text{ GeV}^2$ and $s_0 = 26 \text{ GeV}^2$, where the pole contribution is $\text{PC} \approx 0.58$. This fact ensures the dominance of the pole contribution and demonstrates the ground-state nature of H_c .

Errors in Eq. (26) are generated mainly by the parameters M^2 and s_0 and the gluon condensate $\langle \alpha_s G^2 / \pi \rangle$. Uncertainties of the condensate $\langle g_s^3 G^3 \rangle$ lead to corrections, which in the case of m are very small. Throughout this work, in calculations we use the central value for $\langle g_s^3 G^3 \rangle$. The mass m is shown in Fig. 2 as a function of the parameters M^2 and s_0 .

The hybrid tensor meson $\tilde{H}_c$ with the spin-parities $J^{\text{PC}} = 2^{++}$ is explored in the same manner. In this case, $\tilde{\Pi}_{\mu\mu'\nu\nu'}^{\text{OPE}}(p)$ is obtained from Eq. (9) after substitution, $G \rightarrow \tilde{G}$. The working regions for the parameters M^2 and s_0 are

$$M^2 \in [4, 5] \text{ GeV}^2, \quad s_0 \in [26, 28] \text{ GeV}^2. \quad (27)$$

In these regions, the pole contribution changes inside of the limits $0.50 \leq \text{PC} \leq 0.68$. As a result, for the mass $\tilde{m}$ and current coupling $\tilde{\Lambda}$ of the hybrid state $\tilde{H}_c$, we get

$$\begin{aligned} \tilde{m} &= (4.5 \pm 0.1) \text{ GeV}, \\ \tilde{\Lambda} &= (0.27 \pm 0.04) \text{ GeV}^4. \end{aligned} \quad (28)$$

These results correspond to SR predictions at the point $M^2 = 4.5 \text{ GeV}^2$ and $s_0 = 27 \text{ GeV}^2$, where $\text{PC} \approx 0.59$.

III. DECAYS OF THE HYBRID MESON H_c

In this section, we consider decays of the hybrid charmonium H_c . There are two-body processes $H_c \rightarrow D^{(\pm)} D^*(2010)^{(\mp)}$, $D^0 \bar{D}^*(2007)^0$, and $D_s^{(\pm)} D_s^{*(\mp)}$, which are kinematically allowed modes of H_c . In fact, it is not difficult to see that the mass m of H_c exceeds the two-meson thresholds for these decays. The partial widths of these channels, apart from other factors, are determined by the strong couplings g_i at the corresponding H_c -meson-meson vertices. To evaluate g_i , we employ the three-point SR method, enabling us to extract information on the strong

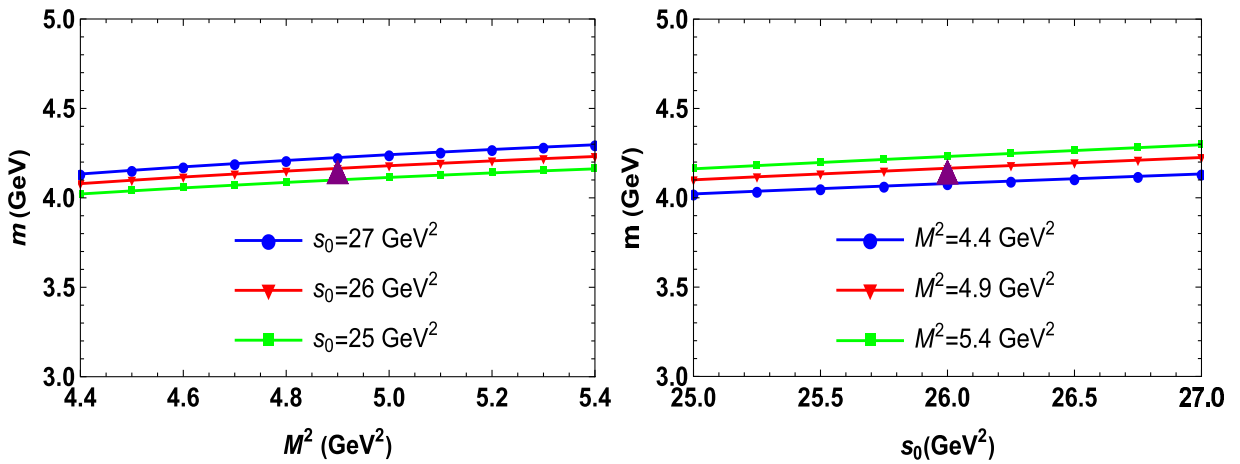


FIG. 2. Mass m vs M^2 (left) and m vs s_0 (right). The triangles on the plots show the position of H_c .

form factors $g_i(q^2)$, which at the relevant mass shells $q^2 = m_D^2$ become equal to the couplings of interest.

**A. $H_c \rightarrow D^+ D^*(2010)^-, D^- D^*(2010)^+,$
and $D^0 \bar{D}^*(2007)^0$**

Here, we are going to analyze, in expanded form, the process $H_c \rightarrow D^+ D^*(2010)^-$ and calculate its partial width $\Gamma[H_c \rightarrow D^+ D^{*-}]$. The partial width of the second decay is equal to $\Gamma[H_c \rightarrow D^+ D^{*-}]$ as well. The quark content $\bar{u}c + (\bar{c}u)^*$ of the mode $D^0 \bar{D}^*(2007)^0$ after replacement $u \rightarrow d$ becomes $\bar{d}c + (\bar{c}d)^* \rightarrow D^+ D^{*-}$. In the present work, we adopt an approximation $m_u = m_d = 0$, and use the same decay constants for the charged and neutral $D^{(*)}$ particles. The differences between the masses of the D^0 , $\bar{D}^*(2007)^0$ and D^+ , $D^*(2010)^-$ mesons are small and neglected in this work. Therefore, with high accuracy, the partial decay width of the process $H_c \rightarrow D^0 \bar{D}^*(2007)^0$ is equal to that of the first decay.

The coupling g_1 that describes the strong interaction of particles at the vertex $H_c D^+ D^{*-}$ can be evaluated using the three-point correlation function. First, we find the sum rule for the form factor $g_1(q^2)$ and consider, to this end, the correlation function

$$\Pi_{\mu\alpha\beta}(p, p') = i^2 \int d^4x d^4y e^{ip'x} e^{iqy} \times \langle 0 | T \{ J_\mu^{D^*}(x) J^\beta(y) J_{\alpha\beta}^\dagger(0) \} | 0 \rangle. \quad (29)$$

Here, $J_\mu^{D^*}(x)$ and $J^\beta(x)$ are interpolating currents of the vector $D^*(2010)^-$ and pseudoscalar D^+ mesons, respectively. They are defined as

$$J_\mu^{D^*}(x) = \bar{c}_i(x) \gamma_\mu d_i(x), \quad J^\beta(x) = \bar{d}_j(x) i \gamma_5 c_j(x), \quad (30)$$

with i and j being the color indices.

To determine the phenomenological side $\Pi_{\mu\alpha\beta}^{\text{Phys}}(p, p')$ of the sum rule, we rewrite Eq. (29) in terms of the particles' parameters. By taking into account only contributions of the ground-state particles, we recast the correlator $\Pi_{\mu\alpha\beta}(p, p')$ into the form

$$\Pi_{\mu\alpha\beta}^{\text{Phys}}(p, p') = \frac{\langle 0 | J_\mu^{D^*} | D^{*-}(p', \epsilon) \rangle \langle 0 | J^\beta | D^+(q) \rangle}{p'^2 - m_{D^*}^2} \frac{1}{q^2 - m_D^2} \times \langle D^{*-}(p', \epsilon) D^+(q) | H_c(p, \epsilon) \rangle \times \frac{\langle H_c(p, \epsilon) | J_{\alpha\beta}^\dagger | 0 \rangle}{p^2 - m^2} + \dots, \quad (31)$$

where $m_{D^*} = (2010.26 \pm 0.05)$ MeV and $m_D = (1869.5 \pm 0.4)$ MeV are the masses of the mesons D^{*-} and D^+ [14], respectively. Above, we denote by $\epsilon_\mu(p')$ the polarization vector of the meson D^{*-} .

To further simplify Eq. (31), we introduce the following matrix elements:

$$\begin{aligned} \langle 0 | J_\mu^{D^*} | D^{*-}(p', \epsilon) \rangle &= f_{D^*} m_{D^*} \epsilon_\mu(p'), \\ \langle 0 | J^\beta | D^+(q) \rangle &= \frac{f_D m_D^2}{m_c}, \end{aligned} \quad (32)$$

where $f_{D^*} = (223.5 \pm 8.4)$ MeV and $f_D = (212.0 \pm 0.7)$ MeV are decay constants of the mesons. We have to also fix the matrix element $\langle D^{*-}(p', \epsilon) D(q) | H_c(p, \epsilon) \rangle$ that is given by the expression

$$\langle D^{*-}(p', \epsilon) D(q) | H_c(p, \epsilon) \rangle = g_1(q^2) \epsilon^{\rho\sigma}(p) q_\rho \epsilon_\sigma(p'). \quad (33)$$

As a result, for $\Pi_{\mu\alpha\beta}^{\text{Phys}}(p, p')$ we get

$$\begin{aligned} \Pi_{\mu\alpha\beta}^{\text{Phys}}(p, p') &= g_1(q^2) \frac{\Lambda f_{D^*} m_{D^*} f_D m_D^2}{m_c(p^2 - m^2)(p'^2 - m_{D^*}^2)} \frac{1}{(q^2 - m_D^2)} \\ &\times \left[\frac{1}{2} g_{\mu\alpha} p'_\beta - \frac{(m^2 + m_{D^*}^2 - q^2)^2}{12 m^2 m_{D^*}^2} g_{\alpha\beta} p'_\mu \right. \\ &\left. + \text{other structures} \right] + \dots. \end{aligned} \quad (34)$$

For $\Pi_{\mu\alpha\beta}^{\text{OPE}}(p, p')$, one finds

$$\begin{aligned} \Pi_{\mu\alpha\beta}^{\text{OPE}}(p, p') &= -i \int d^4x d^4y e^{ip'x} e^{iqy} g_s \frac{\lambda_{ab}^n}{2} G_{\rho\alpha}^n(0) \\ &\times \text{Tr} \left[\gamma_\beta S_d^{ij}(x - y) \gamma_5 S_c^{ib}(y) \gamma_5 \sigma_\mu^\rho S_c^{ai}(-x) \right], \end{aligned} \quad (35)$$

where $S_d^{ij}(x - y)$ is the light quark propagator [54]. To find the SR for the form factor $g_1(q^2)$, we utilize the structures proportional to $g_{\mu\alpha} p'_\beta$ in the correlation functions and corresponding amplitudes $\Pi_1^{\text{Phys}}(p^2, p'^2)$ and $\Pi_1^{\text{OPE}}(p^2, p'^2)$. After standard operations, the sum rule for $g_1(q^2)$ reads

$$g_1(q^2) = \frac{2m_c(q^2 - m_D^2)}{\Lambda f_{D^*} m_{D^*} f_D m_D^2} e^{m^2/M_1^2} e^{m_{D^*}^2/M_2^2} \Pi_1(\mathbf{M}^2, \mathbf{s}_0). \quad (36)$$

In Eq. (36), $\Pi_1(\mathbf{M}^2, \mathbf{s}_0)$ is the Borel transformed and subtracted function $\Pi_1^{\text{OPE}}(p^2, p'^2)$. It depends on the parameters $\mathbf{M}^2 = (M_1^2, M_2^2)$ and $\mathbf{s}_0 = (s_0, s'_0)$, where the pairs (M_1^2, s_0) and (M_2^2, s'_0) correspond to the hybrid H_c and D^{*-} channels, respectively. The function $\Pi_1(\mathbf{M}^2, \mathbf{s}_0)$ amounts to

$$\Pi_1(\mathbf{M}^2, \mathbf{s}_0) = \int_{4m_c^2}^{s_0} ds \int_{m_c^2}^{s'_0} ds' \rho_1(s, s') e^{-s/M_1^2 - s'/M_2^2}. \quad (37)$$

The spectral density $\rho_1(s, s')$ is evaluated as the imaginary part of the amplitude $\Pi_1^{\text{OPE}}(p^2, p'^2)$. Calculations are

performed by taking into account nonperturbative terms up to dimension 10. The spectral density $\rho_1(s, s')$ has the components $\rho_1^{\text{pert}}(s, s')$ and $\rho_1^{\text{Dim4}}(s, s')$, which are given by the formulas

$$\rho_1^{\text{pert}}(s, s') = \frac{g_s^2 m_c^2}{32\pi^2} \int_0^1 d\alpha \int_0^{1-\alpha} \frac{d\beta \theta(N) N^2}{\alpha \beta^3 (\alpha + \beta - 1)} \quad (38)$$

and

$$\rho_1^{\text{Dim4}}(s, s') = \frac{\langle \alpha_s G^2 / \pi \rangle m_c^2}{12} \int_0^1 d\alpha \int_0^{1-\alpha} \frac{d\beta \theta(N)}{\beta^5} \times \left[g_s^2 \frac{\alpha(\alpha^4 + \beta^4)}{12\beta^2(\alpha + \beta - 1)} + \pi^2 \alpha(1 - \alpha) \right]. \quad (39)$$

In Eqs. (38) and (39), $\theta(N)$ is the unit step function with the argument

$$N = \frac{s' - (\alpha + \beta)(s' + m_c^2)}{\beta}. \quad (40)$$

As is seen, only the perturbative and dimension-4 terms contribute to the correlation function $\Pi_1(\mathbf{M}^2, \mathbf{s}_0)$. The limited number of terms in $\rho_1(s, s')$ is connected with the forms of the interpolating currents for H_c , D^+ , and D^{*-} , mathematical manipulations used to derive $\rho_1(s, s')$, and with the approximation $m_u = m_d = 0$ adopted in the present work.

In numerical computations, we choose the parameters (M_1^2, s_0) and (M_2^2, s'_0) in the following manner: in the hybrid channels, we use (M_1^2, s_0) from Eq. (25), whereas for the D^{*-} meson channel, we employ

$$M_2^2 \in [2, 4] \text{ GeV}^2, \quad s'_0 \in [5.5, 6.5] \text{ GeV}^2. \quad (41)$$

It is known that the form factor can be computed in the deep Euclidean region $q^2 \ll 0$. At the same time, the strong coupling g_1 required for our purposes is defined at the mass shell of the D^+ meson, i.e., $g_1 = g_1(m_D^2)$. To escape from these problems, it is convenient to use a variable $Q^2 = -q^2$ and denote a new function as $g_1(Q^2)$. Then, we calculate the form factor $g_1(Q^2)$ at $Q^2 = 2-20 \text{ GeV}^2$, the results of which are depicted in Fig. 3. Afterwards, we introduce the fit function $\mathcal{G}_1(Q^2, m^2)$ that, at momenta $Q^2 > 0$, leads to the same SR data, but can be extrapolated to the domain of negative Q^2 . To this end, we employ the function

$$\mathcal{G}_1(Q^2, m^2) = \mathcal{G}_1^0 \exp \left[c_1^1 \frac{Q^2}{m^2} + c_1^2 \left(\frac{Q^2}{m^2} \right)^2 \right], \quad (42)$$

where $\mathcal{G}_1^0$, c_1^1 , and c_1^2 are fitted constants. Having compared the SR output and Eq. (42), it is not difficult to find

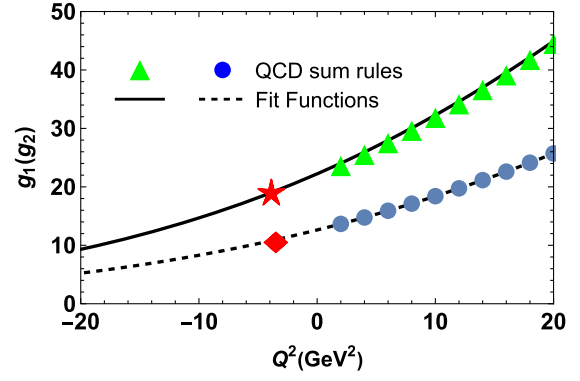


FIG. 3. QCD data and extrapolating functions $\mathcal{G}_1(Q^2)$ (dashed line) and $\mathcal{G}_2(Q^2)$ (solid line). The diamond and star fix the points $Q^2 = -m_D^2$ and $Q^2 = -m_{D^*}^2$, where the strong couplings g_1 and g_2 have been extracted.

$$\mathcal{G}_1^0 = 12.63, \quad c_1^1 = 0.69, \quad \text{and} \quad c_1^2 = -0.06. \quad (43)$$

The function $\mathcal{G}_1(Q^2, m^2)$ is also shown in Fig. 3, where one sees a reasonable agreement with the SR data. For the strong coupling g_1 , we find

$$g_1 \equiv \mathcal{G}_1(-m_D^2, m^2) = 10.96 \pm 2.32. \quad (44)$$

The width of the decay $H_c \rightarrow D^+ D^{*-}$ can be obtained by means of the formula

$$\Gamma[H_c \rightarrow D^+ D^{*-}] = g_1^2 \frac{\lambda_1}{40\pi^2 m^2} |M|^2, \quad (45)$$

where $|M|^2$ is

$$|M|^2 = \frac{1}{24m^4 m_{D^*}^2} [m^8 - 2m^2(2m_D^2 - 3m_{D^*}^2)(m_{D^*}^2 - m_D^2)^2 + (m_{D^*}^2 - m_D^2)^4 + m^6(6m_{D^*}^2 - 4m_D^2) + 2m^4(3m_D^4 - 8m_{D^*}^2 m_D^2 - 7m_{D^*}^2)]. \quad (46)$$

In Eq. (45), we also use the function $\lambda_1 = \lambda(m, m_{D^*}, m_D)$, where

$$\lambda(a, b, c) = \frac{\sqrt{a^4 + b^4 + c^4 - 2(a^2 b^2 + a^2 c^2 + b^2 c^2)}}{2a}. \quad (47)$$

Then, for the partial width of the process under consideration, we find

$$\Gamma[H_c \rightarrow D^+ D^{*-}] = (41.0 \pm 12.3 \pm 11.5) \text{ MeV}. \quad (48)$$

The theoretical errors in Eq. (48) are generated by the uncertainties in the coupling g_1 , and in the masses m , m_{D^*} , and m_D of the involved particles.

B. $H_c \rightarrow D_s^+ D_s^{*-}$ and $D_s^- D_s^{*+}$

Treatment of the decays $H_c \rightarrow D_s^+ D_s^{*-}$ and $H_c \rightarrow D_s^- D_s^{*+}$ does not differ from the analysis presented above. Here, we concentrate on the process $H_c \rightarrow D_s^+ D_s^{*-}$. The correlation function necessary to obtain the sum rule for the form factor $g_2(q^2)$ is given by the formula

$$\Pi_{\mu\alpha\beta}(p, p') = i^2 \int d^4x d^4y e^{ip'x} e^{iqy} \times \langle 0 | T \{ J_\mu^{D_s^*}(x) J_\alpha^{D_s}(y) J_\beta^\dagger(0) \} | 0 \rangle. \quad (49)$$

In Eq. (49), $J_\mu^{D_s^*}(x)$ and $J_\alpha^{D_s}(x)$ are interpolating currents for the D_s^{*-} and D_s^+ mesons, respectively. These currents have the following forms:

$$J_\mu^{D_s^*}(x) = \bar{c}_i(x) \gamma_\mu s_i(x), \quad J_\alpha^{D_s}(x) = \bar{s}_j(x) i \gamma_5 c_j(x). \quad (50)$$

The matrix elements to compute the phenomenological side of the sum rule are given as

$$0 | J_\mu^{D_s^*} | D_s^{*-}(p', \varepsilon) \rangle = f_{D_s^*} m_{D_s^*} \varepsilon_\mu(p'),$$

$$\langle 0 | J_\alpha^{D_s} | D_s^+(q) \rangle = \frac{f_{D_s} m_{D_s}^2}{m_c + m_s}. \quad (51)$$

Here, $m_{D_s^*} = (2112.2 \pm 0.4)$ MeV, $m_{D_s} = (1969.0 \pm 1.4)$ MeV, and $m_s = (93.5 \pm 0.8)$ MeV are the masses of the mesons D_s^{*-} and D_s^+ and the s quark [14], respectively. As the decay constants of these mesons, we employ $f_{D_s^*} = (268.8 \pm 6.5)$ MeV and $f_{D_s} = (249.9 \pm 0.5)$ MeV.

In this case, the correlator $\Pi_{\mu\alpha\beta}^{\text{phys}}(p, p')$ is given by Eq. (34) after evident replacements of the masses and decay constants. The function $\Pi_{\mu\alpha\beta}^{\text{OPE}}(p, p')$ has the form

$$\Pi_{\mu\alpha\beta}^{\text{OPE}}(p, p') = -i \int d^4x d^4y e^{ip'x} e^{iqy} g_s \frac{\lambda_{ab}^n}{2} G_{\rho\alpha}^n(0) \times \text{Tr}[\gamma_\beta S_s^{ij}(x-y) \gamma_5 S_c^{ib}(y) \gamma_5 \sigma_\mu^{\rho} S_c^{ai}(-x)]. \quad (52)$$

Then, the SR for the form factor $g_2(q^2)$ is

$$g_2(q^2) = \frac{2(m_c + m_s)(q^2 - m_{D_s}^2)}{\Lambda f_{D_s^*} m_{D_s^*} f_{D_s} m_{D_s}^2} e^{m^2/M_1^2} e^{m_{D_s^*}^2/M_2^2} \times \Pi_2(\mathbf{M}^2, \mathbf{s}_0). \quad (53)$$

Here,

$$\Pi_2(\mathbf{M}^2, \mathbf{s}_0) = \int_{4m_c^2}^{s_0} ds \int_{(m_c+m_s)^2}^{s_0'} ds' \rho_2(s, s') \times e^{-s/M_1^2 - s'/M_2^2}. \quad (54)$$

The spectral density $\rho_2(s, s')$ is composed of the components $\rho_2^{\text{pert}}(s, s')$, $\rho_2^{\text{Dim}^3}(s, s')$, and $\rho_2^{\text{Dim}^4}(s, s')$. The term $\rho_2^{\text{Dim}^3}(s, s')$ is proportional to the s -quark condensate $\langle \bar{s}s \rangle$. We write down below numerical values of this and other light quark condensates that appear in the calculations:

$$\langle \bar{q}q \rangle = -(0.24 \pm 0.01)^3 \text{ GeV}^3, \quad \langle \bar{s}s \rangle = (0.8 \pm 0.1) \langle \bar{q}q \rangle,$$

$$\langle \bar{q}g_s \sigma G q \rangle = m_0^2 \langle \bar{q}q \rangle, \quad m_0^2 = (0.8 \pm 0.1) \text{ GeV}^2,$$

$$\langle \bar{s}g_s \sigma G s \rangle = m_0^2 \langle \bar{s}s \rangle. \quad (55)$$

In numerical analysis, we use for the parameters M_1^2 and s_0 their values from Eq. (25), whereas in the D_s^* channel, we employ the regions

$$M_2^2 \in [2.5, 3.5] \text{ GeV}^2, \quad s_0' \in [6, 8] \text{ GeV}^2. \quad (56)$$

The SR data obtained for the form factor $g_2(Q^2)$ at $Q^2 = 2-20 \text{ GeV}^2$ can be fitted by the extrapolating function $\mathcal{G}_2(Q^2, m^2)$ with parameters $\mathcal{G}_2^0 = 22.23$, $c_2^1 = 0.68$, and $c_2^2 = -0.06$. The relevant information is presented in Fig. 3.

As a result, for the strong coupling g_2 , we get

$$g_2 \equiv \mathcal{G}_2(-m_{D_s^*}^2, m^2) = 19.0 \pm 3.2. \quad (57)$$

The partial width of this process is calculated by means of Eq. (45) with evident replacements of the particles' masses

and parameters $g_1 \rightarrow g_2$, $\lambda_1 \rightarrow \lambda_2 = \lambda(m, m_{D_s^*}, m_{D_s})$. For $\Gamma[H_c \rightarrow D_s^+ D_s^{*-}]$, we find

$$\Gamma[H_c \rightarrow D_s^+ D_s^{*-}] = (18.4 \pm 4.6) \text{ MeV}. \quad (58)$$

The errors in Eq. (58) are total uncertainties arising from the coupling g_2 and $m_{D_s^*}$, m_{D_s} .

The result of Eq. (58) is also valid for the width of the second decay $H_c \rightarrow D_s^- D_s^{*+}$. Then, the full width of the tensor hybrid charmonium H_c saturated by these five processes amounts to

$$\Gamma[H_c] = (160 \pm 30) \text{ MeV}, \quad (59)$$

which means that it is a rather broad state.

IV. FULL WIDTH OF $\tilde{H}_c$

The hybrid charmonium $\tilde{H}_c$ bears the quantum numbers $J^{PC} = 2^{++}$ and has the mass $\tilde{m} = (4.5 \pm 0.1)$ GeV. These parameters enable one to fix its numerous two-meson decay modes. It is easy to be convinced that processes $\tilde{H}_c \rightarrow D^+ D^-$, $D^0 \bar{D}^0$, $D^{*+} D^{*-}$, $D^{*0} \bar{D}^{*0}$, $D_s^+ D_s^-$, and $D_s^{*+} D_s^{*-}$ are such channels.

A. $\tilde{H}_c \rightarrow D^+ D^-$ and $\tilde{H}_c \rightarrow D^0 \bar{D}^0$

To determine the width of the decay $\tilde{H}_c \rightarrow D^+ D^-$, we start from the analysis of the correlation function

$$\tilde{\Pi}_{\alpha\beta}(p, p') = i^2 \int d^4x d^4y e^{ip'x} e^{iqy} \times \langle 0 | T \{ J^D(x) J^{D^-}(y) \tilde{J}_{\alpha\beta}^\dagger(0) \} | 0 \rangle, \quad (60)$$

with the aim to find the sum rule for the form factor $\tilde{g}_1(q^2)$ and, in this way, estimate the strong coupling $\tilde{g}_1$ of particles at the vertex $\tilde{H}_c D^+ D^-$. In Eq. (60), $J^{D^-}(x)$ is the interpolating current of the pseudoscalar D^- meson

$$J^D(x) = \bar{c}_j(x) i\gamma_5 d_j(x). \quad (61)$$

The phenomenological side of this SR is given by the formula

$$\begin{aligned} \tilde{\Pi}_{\alpha\beta}^{\text{Phys}}(p, p') &= \frac{\langle 0 | J^D | D^+(p') \rangle \langle 0 | J^{D^-} | D^-(q) \rangle}{p'^2 - m_D^2} \frac{q^2 - m_D^2}{q^2 - m_D^2} \\ &\times \langle D^+(p') D^-(q) | \tilde{H}_c(p, \epsilon) \rangle \\ &\times \frac{\langle \tilde{H}_c(p, \epsilon) | \tilde{J}_{\alpha\beta}^\dagger | 0 \rangle}{p^2 - \tilde{m}^2} + \dots \end{aligned} \quad (62)$$

The matrix elements of the $D^\pm$ mesons have the simple form $f_D m_D^2 / m_c$. The vertex $\langle D^+(p') D^-(q) | \tilde{H}_c(p, \epsilon) \rangle$ is given by the expression

$$\langle D^+(p') D^-(q) | \tilde{H}_c(p, \epsilon) \rangle = \tilde{g}_1(q^2) \epsilon_{\mu\nu}(p) p'^\mu q^\nu. \quad (63)$$

Then, the correlator becomes equal to

$$\begin{aligned} \tilde{\Pi}_{\alpha\beta}^{\text{Phys}}(p, p') &= \frac{\tilde{g}_1(q^2) \tilde{\Lambda} f_D^2 m_D^4}{m_c^2 (p^2 - \tilde{m}^2) (p'^2 - m_D^2) (q^2 - m_D^2)} \frac{1}{(q^2 - m_D^2)} \\ &\times \left\{ \frac{\tilde{\lambda}^2}{3} g_{\alpha\beta} + \left[\frac{m_D^2}{\tilde{m}^2} + \frac{2\tilde{\lambda}^2}{3\tilde{m}^2} \right] p_\alpha p_\beta + p'_\alpha p'_\beta \right. \\ &\left. - \frac{\tilde{m}^2 + m_D^2 - q^2}{2\tilde{m}^2} (p_\alpha p'_\beta + p_\beta p'_\alpha) \right\}, \end{aligned} \quad (64)$$

where $\tilde{\lambda} = \lambda(\tilde{m}, m_D, q)$.

In terms of the quark-gluon propagators, the correlation function $\tilde{\Pi}_{\alpha\beta}(p, p')$ reads

$$\begin{aligned} \tilde{\Pi}_{\alpha\beta}^{\text{OPE}}(p, p') &= \int d^4x d^4y e^{ip'x} e^{iqy} g_s \frac{\lambda_{ab}^n}{2} \tilde{G}_{\rho\alpha}^n(0) \\ &\times \text{Tr} [\gamma_5 S_d^{ij}(x-y) \gamma_5 S_c^{ib}(y) \gamma_5 \sigma_\beta^\rho S_c^{ai}(-x)]. \end{aligned} \quad (65)$$

The SR for the form factor $\tilde{g}_1(q^2)$ is derived by employing invariant amplitudes $\tilde{\Pi}_1^{\text{Phys}}(p^2, p'^2, q^2)$ and $\tilde{\Pi}_1^{\text{OPE}}(p^2, p'^2, q^2)$, which correspond to terms $g_{\alpha\beta}$ in the correlation functions

$$\tilde{g}_1(q^2) = \frac{3m_c^2(q^2 - m_D^2)}{\tilde{\Lambda} f_D^2 m_D^4 \tilde{\lambda}^2} e^{m^2/M_1^2} e^{m_D^2/M_2^2} \tilde{\Pi}_1(\mathbf{M}^2, \mathbf{s}_0, q^2). \quad (66)$$

Here, $\tilde{\Pi}_1(\mathbf{M}^2, \mathbf{s}_0, q^2)$ is the amplitude $\tilde{\Pi}_1^{\text{OPE}}(p^2, p'^2, q^2)$ after the Borel transformations and subtractions. It is determined by the spectral density $\tilde{\rho}_1(s, s', q^2)$ that contains perturbative, dimension-3, -5, and -7 terms. The dimension-5 and -7 terms are proportional to condensates $\langle \bar{q} g_s \sigma G q \rangle$ and $\langle \alpha_s G^2 / \pi \rangle \langle \bar{q} q \rangle$, respectively.

For the Borel and continuum subtraction parameters in the D^+ channel, we employ

$$M_2^2 \in [1.5, 3] \text{ GeV}^2, \quad s'_0 \in [5, 5.2] \text{ GeV}^2. \quad (67)$$

The strong coupling $\tilde{g}_1$ is fixed at the D^- meson mass shell $q^2 = m_D^2$ by means of the extrapolating function $\tilde{G}_1(Q^2, \tilde{m}^2)$ with parameters $\tilde{G}_1^0 = 10.23 \text{ GeV}^{-1}$, $\tilde{c}_1^1 = 1.91$, and $\tilde{c}_1^2 = -0.54$ (see Fig. 4). It is worth noting that the functions $\tilde{G}_i(Q^2, \tilde{m}^2)$ are given by Eq. (42) with replacement $m \rightarrow \tilde{m}$.

As a result, we get

$$\tilde{g}_1 \equiv \tilde{G}_1(-m_D^2, \tilde{m}^2) = (7.24 \pm 1.41) \text{ GeV}^{-1}. \quad (68)$$

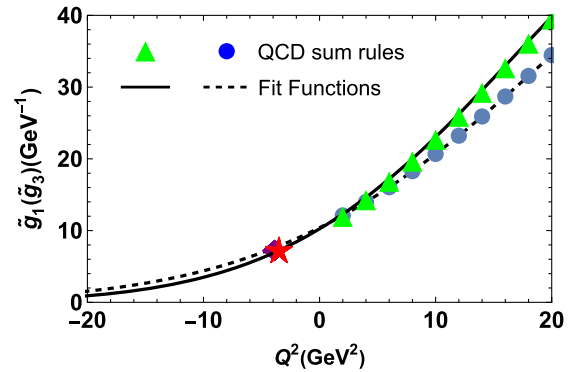


FIG. 4. SR data and functions $\tilde{G}_1(Q^2)$ (solid line) and $\tilde{G}_3(Q^2)$ (dashed line). The labels are placed at the points $Q^2 = -m_D^2$ and $Q^2 = -m_{D_s}^2$.

The partial width of the decay $\tilde{H}_c \rightarrow D^+ D^-$ is equal to

$$\Gamma[\tilde{H}_c \rightarrow D^+ D^-] = \frac{\tilde{g}_1^2 \tilde{\lambda}_1}{960\pi\tilde{m}^2} (\tilde{m}^2 - 4m_D^2)^2, \quad (69)$$

and $\tilde{\lambda}_1 = \lambda(\tilde{m}, m_D, m_D)$. Then, it is not difficult to evaluate

$$\Gamma[\tilde{H}_c \rightarrow D^+ D^-] = (42.5 \pm 16.9) \text{ MeV}, \quad (70)$$

where we present the total errors connected with $\tilde{g}_1$ and the masses $\tilde{m}$ and m_D .

The width of the process $\tilde{H}_c \rightarrow D^0 \bar{D}^0$ amounts to Eq. (69) because the quark content of the mesons $D^0 \bar{D}^0$ can be obtained from $D^+ D^-$ after replacement $d \rightarrow u$. Note that we ignore the small mass gap between $D^\pm$ and $D^0 (\bar{D}^0)$ mesons.

B. $\tilde{H}_c \rightarrow D^{*+} D^{*-}$ and $D^{*0} \bar{D}^{*0}$

Here, we consider the process $\tilde{H}_c \rightarrow D^{*+} D^{*-}$ in a detailed manner. In the case of the decay $\tilde{H}_c \rightarrow D^{*+} D^{*-}$, the SR for the strong form factor $\tilde{g}_2(q^2)$ at the vertex $\tilde{H}_c D^{*+} D^{*-}$ can be obtained from the correlation function

$$\tilde{\Pi}_{\mu\nu\alpha\beta}(p, p') = i^2 \int d^4x d^4y e^{ip'y} e^{iqx} \times \langle 0 | \mathcal{T} \{ J_\mu^{D^{*+}}(x) J_\nu^{D^{*-}}(y) \tilde{J}_{\alpha\beta}^\dagger(0) \} | 0 \rangle, \quad (71)$$

where $J_\mu^{D^{*+}}(x)$ and $J_\nu^{D^{*-}}(x)$ are the interpolating functions of the mesons $D^*(2010)^+$ and $D^*(2010)^-$, respectively. The current $J_\nu^{D^{*-}}(x)$ has been defined by Eq. (30), whereas for $J_\mu^{D^{*+}}(x)$, we have

$$J_\mu^{D^{*+}}(x) = \bar{d}_i(x) \gamma_\mu c_i(x). \quad (72)$$

The correlation function $\tilde{\Pi}_{\mu\nu\alpha\beta}(p, p')$ in terms of the physical parameters of the particles involved in this decay process is

$$\begin{aligned} \tilde{\Pi}_{\mu\nu\alpha\beta}^{\text{Phys}}(p, p') &= \frac{\langle 0 | J_\mu^{D^{*+}} | D^{*+}(p', \epsilon_1) \rangle \langle 0 | J_\nu^{D^{*-}} | D^{*-}(q, \epsilon_2) \rangle}{p'^2 - m_{D^*}^2} \frac{q^2 - m_{D^*}^2}{q^2 - m_{D^*}^2} \\ &\times \langle D^{*+}(p', \epsilon_1) D^{*-}(q, \epsilon_2) | \tilde{H}_c(p, \epsilon) \rangle \\ &\times \frac{\langle \tilde{H}_c(p, \epsilon) | \tilde{J}_{\alpha\beta}^\dagger | 0 \rangle}{p^2 - \tilde{m}^2} + \dots, \end{aligned} \quad (73)$$

where $\epsilon_{1\mu}$ and $\epsilon_{2\nu}$ are the polarization vectors of the D^{*+} and D^{*-} mesons, respectively.

The matrix elements that are used are

$$\begin{aligned} \langle 0 | J_\mu^{D^{*+}} | D^{*+}(p', \epsilon_1) \rangle &= f_{D^*} m_{D^*} \epsilon_{1\mu}(p'), \\ \langle 0 | J_\nu^{D^{*-}} | D^{*-}(q, \epsilon_2) \rangle &= f_{D^*} m_{D^*} \epsilon_{2\nu}(p'), \end{aligned} \quad (74)$$

where $m_{D^*} = (1869.5 \pm 0.4) \text{ MeV}$ and $f_{D^*} = (252.2 \pm 22.66) \text{ MeV}$ are the mass and decay constant of the mesons $D^{*\pm}$. The vertex $\langle D^{*+}(p', \epsilon_1) D^{*-}(q, \epsilon_2) | \tilde{H}_c(p, \epsilon) \rangle$ has the following form:

$$\begin{aligned} \langle D^{*+}(p', \epsilon_1) D^{*-}(q, \epsilon_2) | \tilde{H}_c(p, \epsilon) \rangle &= \tilde{g}_2(q^2) \epsilon_{\tau\rho} [\epsilon_1^* \cdot q \epsilon_2^{\tau*} p'^\rho + \epsilon_2^* \cdot p' \epsilon_1^{\tau*} q^\rho \\ &- p' \cdot q \epsilon_1^{\tau*} \epsilon_2^{\rho*} - \epsilon_1^* \cdot \epsilon_2^* p'^\tau q^\rho]. \end{aligned} \quad (75)$$

As a result, for $\tilde{\Pi}_{\mu\nu\alpha\beta}^{\text{Phys}}(p, p')$ we find the lengthy expression

$$\begin{aligned} \tilde{\Pi}_{\mu\nu\alpha\beta}^{\text{Phys}}(p, p') &= \frac{\tilde{g}_2(q^2) \tilde{\Lambda} f_{D^*}^2 m_{D^*}^2}{(p^2 - \tilde{m}^2)(p'^2 - m_{D^*}^2)(q^2 - m_{D^*}^2)} \left\{ \frac{m_{D^*}^4 - 2m_{D^*}^2(2\tilde{m}^2 + q^2) + (\tilde{m}^2 - q^2)(3\tilde{m}^2 - q^2)}{12\tilde{m}^2} \right. \\ &\times g_{\mu\nu} g_{\alpha\beta} + \frac{1}{3} g_{\alpha\beta} \left(\frac{m_{D^*}^2}{\tilde{m}^2} p_\mu p_\nu + 2p'_\mu p'_\nu \right) - \frac{1}{6\tilde{m}^2} g_{\alpha\beta} [(m_{D^*}^2 + 3\tilde{m}^2 - q^2) p_\mu p'_\nu + (m_{D^*}^2 + \tilde{m}^2 - q^2) p'_\mu p_\nu] \\ &\left. + \text{other structures} \right\}. \end{aligned} \quad (76)$$

For the QCD side of the sum rule, we obtain

$$\begin{aligned} \tilde{\Pi}_{\mu\nu\alpha\beta}^{\text{OPE}}(p, p') &= i^2 \int d^4x d^4y e^{ip'x} e^{iqy} g_s \frac{\lambda_{ab}^n}{2} \tilde{G}_{\rho\alpha}^n(0) \\ &\times \text{Tr} [\gamma_\mu S_d^{ij}(x-y) \gamma_\nu S_c^{jb}(y) \gamma_5 \sigma_\beta^\rho S_c^{ai}(-x)]. \end{aligned} \quad (77)$$

The SR for the form factor $\tilde{g}_2(q^2)$ is derived using the structures $g_{\mu\nu} g_{\alpha\beta}$ in the correlation functions. Note that

$\tilde{g}_2(q^2)$ receives contributions from the perturbative, dimension-4, -5, and -7 terms.

In numerical analysis, the parameters M_2^2 and s'_0 in the D^{*+} meson channel are chosen in the form

$$M_2^2 \in [2, 4] \text{ GeV}^2, \quad s'_0 \in [5.5, 6.5] \text{ GeV}^2. \quad (78)$$

The strong coupling $\tilde{g}_2$ amounts to

$$\tilde{g}_2 = \tilde{\mathcal{G}}_2(-m_{D^*}^2, \tilde{m}^2) = (0.84 \pm 0.16) \text{ GeV}^{-1}. \quad (79)$$

It has been estimated at the mass shell $q^2 = m_{D^*}^2$ of the D^{*-} meson by employing the interpolating function $\tilde{\mathcal{G}}_2(Q^2, \tilde{m}^2)$. The function $\tilde{\mathcal{G}}_2(Q^2, \tilde{m}^2)$ is determined by the parameters $\tilde{\mathcal{G}}_2^0 = 0.93 \text{ GeV}^{-1}$, $\tilde{c}_2^1 = 0.50$, and $\tilde{c}_2^2 = -0.27$.

The width of this decay is

$$\Gamma[\tilde{H}_c \rightarrow D^{*+} D^{*-}] = \frac{\tilde{g}_2^2 \tilde{\lambda}_2}{80\pi \tilde{m}^2} (\tilde{m}^4 - 3\tilde{m}^2 m_{D^*}^2 + 6m_{D^*}^4), \quad (80)$$

with $\tilde{\lambda}_2$ being equal to $\lambda(\tilde{m}, m_{D^*}, m_{D^*})$. Then, we get

$$\Gamma[\tilde{H}_c \rightarrow D^{*+} D^{*-}] = (36.5 \pm 11.6) \text{ MeV}. \quad (81)$$

C. Decays to charmed-strange mesons

The tensor hybrid state $\tilde{H}_c$ can also decay to charmed-strange meson pairs $\tilde{H}_c \rightarrow D_s^+ D_s^-$ and $D_s^{*+} D_s^{*-}$. Similar decays have been considered in the previous subsections. The current channels differ from those only by parameters of the mesons $D_s^\pm$ and $D_s^{*\pm}$. Therefore, it is enough to provide results of numerical computations.

The correlation function in the case of decay $\tilde{H}_c \rightarrow D_s^+ D_s^-$ contains contributions of the perturbative and dimension-3, -4, -5, and -7 terms, which are proportional to $\langle \bar{s}s \rangle$, $\langle \bar{s}g_s \sigma G s \rangle$, $\langle \alpha_s G^2/\pi \rangle$, and $\langle \alpha_s G^2/\pi \rangle \langle \bar{s}s \rangle$ condensates, respectively. The coupling $\tilde{g}_3$, responsible for strong interactions of particles at the vertex $\tilde{H}_c D_s^+ D_s^-$, is

$$\tilde{g}_3 \equiv \tilde{\mathcal{G}}_3(-m_{D_s}^2, \tilde{m}^2) = (7.60 \pm 1.28) \text{ GeV}^{-1}. \quad (82)$$

The fitting function $\tilde{\mathcal{G}}_3(Q^2)$ is determined by the coefficients $\tilde{\mathcal{G}}_3^0 = 10.41 \text{ GeV}^{-1}$, $\tilde{c}_3^1 = 1.57$, and $\tilde{c}_3^2 = -0.36$. The relevant SR predictions for the form factor $\tilde{g}_3(Q^2)$ and the function $\tilde{\mathcal{G}}_3(Q^2)$ are also plotted in Fig. 4. The width of the decay $\tilde{H}_c \rightarrow D_s^+ D_s^-$ is

$$\Gamma[\tilde{H}_c \rightarrow D_s^+ D_s^-] = (23.3 \pm 11.4) \text{ MeV}. \quad (83)$$

The process $\tilde{H}_c \rightarrow D_s^{*+} D_s^{*-}$ is determined by the following parameters:

$$\tilde{g}_4 \equiv \tilde{\mathcal{G}}_4(-m_{D_s^*}^2, \tilde{m}^2) = (0.78 \pm 0.13) \text{ GeV}^{-1} \quad (84)$$

and

$$\Gamma[\tilde{H}_c \rightarrow D_s^{*+} D_s^{*-}] = (24.1 \pm 11.5) \text{ MeV}. \quad (85)$$

Decays considered in the present section allow us to estimate the full decay width of the tensor hybrid charmonium $\tilde{H}_c$, which amounts to

$$\Gamma[\tilde{H}_c] = (206 \pm 33) \text{ MeV}. \quad (86)$$

V. DISCUSSION AND CONCLUDING NOTES

In the present article, we have evaluated the parameters of the tensor hybrid charmonia H_c and $\tilde{H}_c$. Our predictions for the masses of these states, $m = (4.16 \pm 0.14) \text{ GeV}$ and $\tilde{m} = (4.5 \pm 0.1) \text{ GeV}$, demonstrate that these structures are unstable against strong decays to conventional meson pairs.

The masses of the tensor hybrid charmonia H_c and $\tilde{H}_c$ were computed in numerous publications. Results of some of these works are collected in Table I. In the framework of the sum rule method, these particles were investigated in Refs. [32,47]. A comparison of our results for m and $\tilde{m}$ and those of Ref. [32] revealed that m was larger than the prediction made there, whereas $\tilde{m}$ was in good agreement with the results of the same paper. The masses extracted in Ref. [47] exceeded our results for m and $\tilde{m}$.

The hybrid mesons H_c and $\tilde{H}_c$ were examined in the context of the alternative methods as well. In the context of the lattice simulations, these tensor particles had approximately the masses 4.33 GeV and 4.49 GeV [31], and 4.46 GeV and 4.62 GeV [35], respectively. The lattice calculations led, as usual, to larger predictions: only the hybrid $\tilde{H}_c$ from Ref. [31] had a mass compatible with $\tilde{m}$.

The nonrelativistic effective field theory elaborated in Ref. [34] generated two sets of the masses for the charmonium hybrids with diverse spin-parities. These sets depend on the choice of the potential $V^{(0.5)}$ or $V^{(0.25)}$ in the coupled Schrodinger equation. Predictions for the masses of the hybrids H_c and $\tilde{H}_c$ are shown in Table I. It is evident that within the theoretical errors, results of the potential $V^{(0.25)}$ were consistent with our findings. Within the BOEFT method, the charmonium hybrids were examined in Ref. [44]. Our results for m and $\tilde{m}$ exceeded the mass of the tensor $\bar{c}gc$ hybrids presented there. Evidently, there is a necessity to continue studies to improve agreement between diverse predictions.

Another important measurable parameter of the hybrid mesons is the decay widths of these structures. Information on the decay pattern and full width of the hybrid states may allow one to distinguish them from other exotic mesons. For a long time, it was believed that hybrid states were

TABLE I. Predictions for the masses of the tensor $\bar{c}gc$ hybrids $J^{\text{PC}} = 2^{-+}$ and $J^{\text{PC}} = 2^{++}$. All masses are given in units of GeV.

Publications	$J^{\text{PC}} = 2^{-+}$	$J^{\text{PC}} = 2^{++}$
This work	4.16 ± 0.14	4.5 ± 0.1
Reference [32]	4.04 ± 0.23	4.45 ± 0.27
Reference [47]	4.31 ± 0.08	4.85 ± 0.06
Reference [31]	4.334 ± 0.017	4.492 ± 0.021
Reference [35]	4.456 ± 0.021	4.623 ± 0.032
Reference [34] $V^{(0.5)}$	4.05 ± 0.15	4.17 ± 0.15
Reference [34] $V^{(0.25)}$	4.15 ± 0.15	4.37 ± 0.15
Reference [44]	4.046 ± 0.030	4.232 ± 0.030

forbidden to decay into two S -wave mesons [26]. Such conclusion was made in the context of the constituent gluon and flux tube models. Recently, an analysis based on the BO approximation demonstrated that decays into two S -wave heavy-light mesons are allowed for quarkonium hybrids for a wide range of the spin-parities J^{PC} , including tensor particles with 2^{-+} and 2^{++} [45].

In the present paper, decays of the tensor charmonia have been examined within the QCD sum rule framework. Note that the QCD sum rule method relies on first principles of the field theory, and treats the heavy and light quarks as well as the gluon field on the same footing. All non-perturbative information required to calculate parameters of a decay is encoded in the vacuum expectation values of numerous quark, gluon, and mixed operators. These condensates are universal parameters, the numerical values of which were extracted from analyses of different processes.

In our article, for the first time, we calculated widths $\Gamma[\tilde{H}_c]$ and $\Gamma[H_c]$ of the tensor hybrids in the context of the SR method. In the case of the structure H_c , we studied the decays $H_c \rightarrow D^{(\pm)}D^{*(\mp)}$, $D^0\bar{D}^{*0}$, and $D_s^{(\pm)}D_s^{*(\mp)}$. The full width of H_c saturated by these decay channels amounted to $\Gamma[H_c] = (160 \pm 30)$ MeV. The width of the hybrid state $\tilde{H}_c$ was equal to $\Gamma[\tilde{H}_c] = (206 \pm 33)$ MeV, which was evaluated by taking into account the kinematically allowed processes $\tilde{H}_c \rightarrow D^+D^-$, $D^0\bar{D}^0$, $D^{*+}D^{*-}$, $D^{*0}\bar{D}^{*0}$, $D_s^+D_s^-$, and $D_s^{*+}D_s^{*-}$. Clearly, both of these hypothetical hybrid charmonia are broad structures. Predictions for these parameters can be refined by taking into consideration their other decay modes.

The diverse decay channels of the heavy hybrid mesons were previously examined in Refs. [18,42,46,48], as well. In Refs. [18,42], the authors concentrated on transitions of the heavy hybrids to standard quarkonia. The most general consideration of these decays was done in Ref. [18], in which BOEFT was applied to investigate the semi-inclusive transition rate of a heavy quarkonium hybrid H_m to $Q_n + X$, with Q_n and X being a quarkonium and any final state of light particles, respectively. It turned out that the widths of such transitions were sizeable. For instance, the

width of the decay $H_c \rightarrow \eta_c(1S) + X$ was 35_{-9-12}^{+15+16} MeV, whereas for $\tilde{H}_c \rightarrow \chi_c(1P) + X$ the authors obtained 65_{18-28}^{+41+40} MeV.

The selection rules and relative partial decay rates for decays of quarkonium hybrids with various J^{PC} into pairs of heavy-light mesons were investigated in Ref. [46]. It is worth noting that an analysis was performed by employing the Born-Oppenheimer approximation. Decays of hybrids were treated through transitions from confining BO potentials to meson-pair potentials with the same BO quantum numbers. Numerical computations were carried out in the case of the decays into pairs of $B^{(*)}$ mesons. We assumed that the same ratios were valid for decays of the hidden-charm hybrids into D meson pairs. Then, in the case of the hybrid 2^{++} , Ref. [46] predicted for decays (in $S + D$ -wave) into mesons $D\bar{D}$, $D\bar{D}^* + \bar{D}D^*$, and $\bar{D}^*D^*$ the ratios 3:9:12, i.e., decays to vector mesons are dominant channels for the $\tilde{H}_c$ state. Our results implied fulfilment of the approximate ratios 3.5:3:1:1 for decays through channels $D^+D^- + D^0\bar{D}^0$, $D^{*+}D^{*-} + D^{*0}\bar{D}^{*0}$, $D_s^+D_s^-$, and $D_s^{*+}D_s^{*-}$, respectively. As was seen, in the framework of the QCD sum rule approach, decays of the tensor hybrid charmonia $\tilde{H}_c$ into pseudoscalar and vector meson pairs contributed equally to its full width.

The investigation carried out in this paper demonstrated that there is a necessity to continue analyses of the heavy hybrid mesons in the context of different methods to refine and correct predictions for parameters of these structures. This is also required to remove observed discrepancies in outputs of various methods.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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