

Casimir effect in Yang-Mills theories in three and four dimensions

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We calculate the Casimir energy for the configuration of two parallel plates coupled to non-Abelian gauge fields with a Yang-Mills action. We consider both $2 + 1$ and $3 + 1$ dimensions in the manifestly gauge-invariant formalism we have pursued over the last several years which allows us to factor out the gauge degrees of freedom. A boundary action in the functional integral, equivalent to the insertion of operators representing the plates, is used to enforce the required boundary conditions for the gauge fields. The result is for a kinematic regime corresponding to the exchange of gluons with a dynamically generated mass. We find good agreement in $2 + 1$ dimensions and reasonable agreement in $3 + 1$ dimensions with lattice-based numerical evaluations.

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I. INTRODUCTION

The Casimir effect has recently emerged as an interesting probe of certain nonperturbative aspects of non-Abelian gauge theories in both three and four dimensions. There have been several lattice-based numerical simulations of geometrical setups including the classic parallel plate arrangement (or parallel wires in three dimensions) [1–3]. The results on the Casimir energy, say, between the two parallel plates, have been quite revealing. In the perturbative regime where the propagating particles can be taken to be massless gluons, we should expect the familiar result for massless fields with the power law falloff of the Casimir energy as a function of the separation of the plates. On the other hand, in the confining regime, the only correlation between the plates must involve the exchange of massive colorless glueballs and hence we should expect exponential falloff with the exponent related to the lowest glueball mass. The actual simulation yields a result which agrees with neither of these cases. There is exponential falloff, unlike the massless case, but the exponent is much smaller than what is expected from the glueball exchange. In fact, simulations seem to capture an intermediate regime where the gluon should be treated as a massive particle, with the nonperturbative dynamical

generation of mass, but can still be considered as a propagating degree of freedom.

The natural question then is whether we can understand the dynamical generation of mass, starting from first principles and then use that result to calculate the Casimir energy. Here we will argue that this is indeed possible. In fact, in the case of $(2 + 1)$ -dimensional Yang-Mills theory, a nonperturbative Hamiltonian analysis led to a prediction for the propagator mass of the gluon [4–6]. The resulting expression for the Casimir energy was shown to agree remarkably well with the lattice-based numerical results [7]. While our Hamiltonian analysis applies to all gauge groups, at the time of Ref. [7], the lattice calculation of the Casimir energy was available only for the case of $SU(2)$ Yang-Mills theory [1], and so this was the case for which the comparison was made. Recently, there have been estimates of the Casimir energy for the case of $SU(3)$ [3]. In the following, we will therefore first present the comparison of these results with our Hamiltonian analysis. We note that the authors of [1,3] used a fitting formula for their results, which is different from ours. The mass defined by the exponential falloff was interpreted as due to a state bound to the plates. The interpretation we give here in terms of exchange of gluons with dynamically generated mass is more natural and also supported by the Hamiltonian analysis. The agreement of our formula with the data is very good as well.

A further point is that the mass parameter is related to the magnetic mass of the $(3 + 1)$ -dimensional theory in the high-temperature limit. The argument is that at high temperatures the $(3 + 1)$ -dimensional theory will reduce to a 3d Euclidean theory, which via the usual Wick rotation can be related to the Hamiltonian analysis we have done.

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The propagator mass then controls the exponential falloff in the correlators of the 3d Euclidean theory, which in turn implies that it is the magnetic screening mass of the $(3 + 1)$ -dimensional theory [8,9]. It is important that the value of the mass obtained in the Hamiltonian analysis [6] simultaneously gives good agreement for the Casimir energy [7] and also for estimates of the magnetic screening mass by other methods [9].

Turning to $3 + 1$ dimensions, we note that an analysis similar to what was done for $2 + 1$ dimensions is possible. The idea is to consider the complex Euclidean version of the spacetime manifold and view it as a complex manifold. A simple choice is then the manifold $\mathbb{CP}^2$, where we are able to obtain a parametrization of gauge potentials similar to what was used in the Hamiltonian analysis in $2 + 1$ dimensions [10]. The complex structure is crucial for this and the parametrization allows for factoring out the gauge degrees of freedom without having to use gauge fixing. The gauge-invariant volume element was calculated and shown to require the specification of a dynamical mass for the gluons. The four-dimensional theory, as is well known, requires dimensional transmutation: one needs to specify an input parameter with the dimensions of mass to define the theory. Conventionally, this is taken to be Λ_{QCD} , defined via one-loop running of the coupling constant. However, in our formulation, the dynamical mass of the gluons serves as the input parameter. In principle, Λ_{QCD} is then calculable in terms of this dynamical mass, if one performs perturbative calculations. We have not done this yet. But the key point is that our analysis justifies calculating the Casimir energy in terms of a massive field with two polarizations. The resulting formula, we will show, is in reasonable agreement with the available lattice data from [3] for $SU(2)$ and from [2,3] for $SU(3)$.

There are several approaches to the computation of the Casimir energy. The most common method is to calculate the vacuum energy with the field modes restricted to obey appropriate boundary conditions at the locations of the plates [11]. The difference with the true vacuum, which does not have any such boundary conditions for the modes, is the Casimir energy. An alternative is to view the plates as physical entities and calculate their interaction energy, via van der Waals type forces. Yet another approach is to impose the boundary conditions by constraints and obtain the partition function and free energy. This last method has the advantage of bypassing normalization issues in the functional integral, dealing directly with correlations between the plates. This is what we shall use in what follows.

In Sec. II, we give the general method of calculating the Casimir energy in terms of the constraints as explained above, for both scalar fields and for gauge fields. The specific calculation for $2 + 1$ dimensions is given in Sec. III and for $3 + 1$ dimensions in Sec. IV. We also discuss the dependence of the Casimir energy on different boundary

conditions. The comparison of these results with the lattice-based numerical evaluation is done in Sec. V, with a number of relevant comments. We give a resumé of our conclusions in Sec. VI. The paper includes a short Appendix on how the results of our analysis on $\mathbb{CP}^2$ can be expressed in a form which has 4d covariance, suitable for application to the calculation of the Casimir energy.

II. CONSTRAINTS AND BOUNDARY ACTIONS

In this section we will frame the calculation of the Casimir energy by imposing boundary conditions as constraints in the functional integral. The essence of the calculation can then be reduced to that of a two-point function for the gauge potentials. To see how this works out in general, we will start with a massless scalar field described by the Euclidean action

$$\mathcal{S} = \int d^4x \frac{1}{2} (\partial\phi)^2. \quad (1)$$

We will consider the parallel plate geometry where we impose the boundary condition that ϕ vanishes on the plates. (As is well known, the self-adjointness of the Laplacian allows for an infinity of boundary conditions. Boundary actions suitable for the general case in the context of Casimir energy have been discussed in [12]; see also [13]. But here we will focus on fields vanishing on the plates.) In turn, this requires expanding in terms of Dirichlet modes and then calculating the determinant from the functional integral. However, we can also impose the boundary condition by a functional δ function on the plates. Thus we consider the functional integral

$$Z = \int [D\phi][D\sigma] e^{-\mathcal{S}} e^{i \int \sigma \phi}. \quad (2)$$

We take the plates to be at $x_3 = 0, R$ and the field σ is a three-dimensional field on the plates. Actually, we have two independent fields $\sigma_1 = \sigma(\vec{x}, x_3 = 0)$ and $\sigma_2 = \sigma(\vec{x}, x_3 = R)$. Thus

$$i \int \sigma \phi = i \int d^3x \sigma_1(\vec{x}) \phi(\vec{x}, 0) + i \int d^3x \sigma_2(\vec{x}) \phi(\vec{x}, R). \quad (3)$$

Integration over σ gives the δ -function setting $\phi = 0$ on the plates. As for the integration over the ϕ field, there are no boundary conditions to be imposed at the location of the plates. So we can integrate it out to get a reduced action for the σ fields and then do the integration over them. The propagator for ϕ has the usual form

$$\langle \phi(x) \phi(y) \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{\omega^2 + k_3^2}, \quad (4)$$

where $\omega^2 = k_1^2 + k_2^2 + k_4^2$. Integration over the ϕ 's will require

$$\begin{aligned} K_{00}(\vec{x}, \vec{x}') &= K_{RR}(\vec{x}, \vec{x}') = \langle \phi(\vec{x}, 0) \phi(\vec{x}', 0) \rangle \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega} e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')}, \\ K_{0R}(\vec{x}, \vec{x}') &= K_{R0}(\vec{x}, \vec{x}') = \langle \phi(\vec{x}, 0) \phi(\vec{x}', R) \rangle \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega} e^{-\omega R} e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')}. \end{aligned} \quad (5)$$

Integrating out ϕ , we get

$$\begin{aligned} Z &= (\det(-\partial^2/2\pi))^{-\frac{1}{2}} \int [D\sigma] \exp\left(-\frac{1}{2} \int \sigma_1 K_{00} \sigma_1 \right. \\ &\quad \left. - \frac{1}{2} \int \sigma_2 K_{RR} \sigma_2 - \int \sigma_1 K_{0R} \sigma_2\right). \end{aligned} \quad (6)$$

The determinant is evaluated with modes over all of space, so it does not depend on R . Hence it is irrelevant for the Casimir energy. Consider now integrating out σ_1 . This will give another determinant (of K_{00}), which is also independent of R . The result for the partition function then takes the form

$$\begin{aligned} Z &= \mathcal{N} \int [D\sigma_2] \exp\left(-\frac{1}{2} \int \sigma_2 (K_{RR} - K_{R0} K_{00}^{-1} K_{0R}) \sigma_2\right) \\ &= \mathcal{N}' [\det(K_{RR} - K_{R0} K_{00}^{-1} K_{0R})]^{-\frac{1}{2}}. \end{aligned} \quad (7)$$

The normalization factors are independent of R . The R -dependent part of the partition function is thus given by

$$-\log Z = \frac{1}{2} \text{Tr} \log(1 - K_{RR}^{-1} K_{R0} K_{00}^{-1} K_{0R}). \quad (8)$$

Notice that K_{RR}^{-1} and K_{00}^{-1} are separately on the plates at $x_3 = R$ and $x_3 = 0$, respectively, so they do not involve the separation R and we find

$$K_{00}^{-1} = K_{RR}^{-1} = \int \frac{d^3 k}{(2\pi)^3} (2\omega) e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')}. \quad (9)$$

It is then easy to see that the R -dependent part of the partition function takes the form

$$\begin{aligned} -\log Z &= \frac{1}{2} \int d^3 x \int \frac{d^3 k}{(2\pi)^3} \log(1 - e^{-2\omega R}) \\ &= -\frac{1}{4\pi^2} \int d^3 x \sum \int_0^\infty d\omega \omega^2 \frac{e^{-2\omega R n}}{n} \\ &= -\frac{1}{16\pi^2} \frac{1}{R^3} \zeta(4) \int d^3 x = -\frac{\pi^2}{1440 R^3} \mathcal{A} T. \end{aligned} \quad (10)$$

The integral of $d^3 x$ gives the expected area factor $\mathcal{A}$ and the length of the time direction T . The latter gets removed in

the free energy, so that the Casimir energy per unit area is given as

$$\frac{\mathcal{E}}{\mathcal{A}} = -\frac{\pi^2}{1440 R^3}. \quad (11)$$

This is exactly the answer we get if we consider just the free action but use a mode expansion for the fields ϕ which obey Dirichlet (vanishing) conditions on the plates. The advantage of this calculational procedure is that the correlation functions in (5) are evaluated in free space with no plates.

It is well known that the partition function in the presence of a heavy charged particle is given by the functional integral of e^{-S} with the insertion of a Wilson line. The two terms $e^{i \int \sigma_1 \phi}$ and $e^{i \int \sigma_2 \phi}$ can be viewed as the analogs, corresponding to the insertion of the plates. When we consider the plates coupled to a gauge field, we need the corresponding boundary action which captures the required boundary conditions. This action is of the form

$$S_b = 2i \int C^a F^a = i \int \left\{ \begin{array}{ll} C^a e^{ij} F_{ij}^a & 2 + 1 \text{ dim,} \\ C_\mu^a e^{\mu\nu\alpha} F_{\nu\alpha}^a & 3 + 1 \text{ dim,} \end{array} \right. \quad (12)$$

where C is a scalar in $2 + 1$ dimensions and is a one-form in $3 + 1$ dimensions. It is easy to see that setting the variation of S_b with respect to C to zero gives the vanishing of F on the plates. Thus in $3 + 1$ dimensions, with the plate normal to the third direction, for example, we find $F_{14} = F_{24} = F_{12} = 0$, i.e., the vanishing of the tangential components of the electric field and the normal component of the magnetic field. These are the chromoelectric boundary conditions. We will comment on other boundary conditions later.

Notice that this action is invariant under the gauge transformations $A \rightarrow gAg^{-1} - dg g^{-1}$, $C \rightarrow gCg^{-1}$. Further, in $3 + 1$ dimensions, it is invariant under $C \rightarrow C + (d + A)f$, by virtue of the Bianchi identity on F . This is a new gauge symmetry for C and it shows that we have only two independent components for C , since one component can be removed (or related to the other two) by a suitable choice of f . This would suggest that we should expect only two independent boundary conditions on F upon carrying out the variation with respect to C . This is indeed correct and is in complete agreement with the fact that the Bianchi identity connects F_{14} , F_{24} and F_{12} . This also implies that the integration over C has to be done using a measure for C which is invariant under this new gauge symmetry $C_\mu \rightarrow C_\mu + (\partial + A)_\mu f$, i.e., a volume element for the space of C 's modulo this gauge transformation.

One of the important consequences of the interactions in the Yang-Mills theory will be the dynamical generation of a propagator mass for the gluon. As mentioned in the Introduction, we expect that, for the range of R we consider, a suitable approximation for the Casimir energy will be to

neglect all further interactions once we take account of the dynamically generated mass. In other words, we expect that we can use a massive free propagator for the gauge field and also take $F^a \approx dA^a$ in the boundary action (12). With this simplification, integrating out the A 's, we get $Z = \mathcal{N} \int [DC] e^{-\mathcal{S}_{\text{eff}}}$ with

$$\begin{aligned} \mathcal{S}_{\text{eff}} = & \frac{1}{2} \int (\partial C_1)^{a\mu} \langle A_\mu^a(\vec{x}, 0) A_\nu^b(\vec{x}', 0) \rangle (\partial C_1)^{b\nu} \\ & + \frac{1}{2} \int (\partial C_2)^{a\mu} \langle A_\mu^a(\vec{x}, R) A_\nu^b(\vec{x}', R) \rangle (\partial C_2)^{b\nu} \\ & + \int (\partial C_1)^{a\mu} \langle A_\mu^a(\vec{x}, 0) A_\nu^b(\vec{x}', R) \rangle (\partial C_2)^{b\nu}, \end{aligned} \quad (13)$$

where

$$\begin{aligned} (\partial C_1)^{a\mu} &= \epsilon^{\mu\alpha\beta} (\partial_\alpha C_\beta^a - \partial_\beta C_\alpha^a)(\vec{x}, 0), \\ (\partial C_2)^{a\mu} &= \epsilon^{\mu\alpha\beta} (\partial_\alpha C_\beta^a - \partial_\beta C_\alpha^a)(\vec{x}, R). \end{aligned} \quad (14)$$

Thus we need the propagators for A , with zero separation in x_3 and with a separation of R . Because of the gauge symmetry of C , we only need the gauge-invariant part of the propagators for A .

In $2+1$ dimensions, we get the same action as in (13) but C is a scalar, so (14) become

$$(\partial C_1)^{a\mu} = \epsilon^{\mu\alpha} \partial_\alpha C^a(\vec{x}, 0), \quad (\partial C_2)^{a\mu} = \epsilon^{\mu\alpha} \partial_\alpha C^a(\vec{x}, R) \quad (15)$$

with all indices α, μ taking only two values corresponding to the tangential directions on the plates.

III. CASIMIR ENERGY IN $2+1$ DIMENSIONS

We now turn to the details of the calculation of the Casimir energy, starting with the $(2+1)$ -dimensional case. This is basically a review of our previous analytical calculation of the Casimir energy in three dimensions [7] using the gauge-invariant Hamiltonian methods.

In this Hamiltonian analysis, we start with the gauge $A_0 = 0$ and the spatial components of the gauge potentials were parametrized as [4,6]

$$\begin{aligned} A_z &= \frac{1}{2} (A_1 + iA_2) = -\partial M M^{-1}, \\ A_{\bar{z}} &= \frac{1}{2} (A_1 - iA_2) = M^{\dagger-1} \bar{\partial} M^{\dagger}. \end{aligned} \quad (16)$$

Here we use complex coordinates $z = x_1 - ix_2$, $\bar{z} = x_1 + ix_2$ with $\partial = \frac{1}{2}(\partial_1 + i\partial_2)$, $\bar{\partial} = \frac{1}{2}(\partial_1 - i\partial_2)$, and M is an element of the complexified group $G^{\mathbb{C}}$. Thus M is an $SL(N, \mathbb{C})$ -matrix if the gauge transformations take values in $SU(N)$. Gauge transformations act on M via $M \rightarrow M^g = gM$, where g is an element of the group G , say, for example, $SU(N)$.

Wave functions are functionals of the gauge-invariant combination $H = M^{\dagger} M$. The inner product for the wave functions is given as

$$\langle 1|2 \rangle = \int d\mu(H) \exp[2c_A S_{\text{WZW}}(H)] \Psi_1^* \Psi_2. \quad (17)$$

Here S_{WZW} is the Wess-Zumino-Witten (WZW) action given by

$$\begin{aligned} S_{\text{WZW}}(H) = & \frac{1}{2\pi} \int \text{Tr}(\partial H \bar{\partial} H^{-1}) \\ & + \frac{i}{12\pi} \int \epsilon^{\mu\nu\alpha} \text{Tr}(H^{-1} \partial_\mu H H^{-1} \partial_\nu H H^{-1} \partial_\alpha H). \end{aligned} \quad (18)$$

Also c_A is the value of the quadratic Casimir operator for the adjoint representation; it is equal to N for $SU(N)$. Further, $d\mu(H)$ in (17) is the Haar measure for H which takes values in $SL(N, \mathbb{C})/SU(N)$.

The Hamiltonian and other observables can be expressed in terms of the current J of the WZW action, namely, $J = \frac{2}{e} \partial H H^{-1}$. It has the form $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, with

$$\begin{aligned} \mathcal{H}_0 &= m \int_z J_a(\vec{z}) \frac{\delta}{\delta J_a(\vec{z})} + \frac{2}{\pi} \int_{z,w} \frac{1}{(z-w)^2} \frac{\delta}{\delta J_a(\vec{w})} \frac{\delta}{\delta J_a(\vec{z})} \\ &+ \frac{1}{2} \int_x : \bar{\partial} J^a(x) \bar{\partial} J^a(x) :, \\ \mathcal{H}_1 &= i e f_{abc} \int_{z,w} \frac{J^c(\vec{w})}{\pi(z-w)} \frac{\delta}{\delta J_a(\vec{w})} \frac{\delta}{\delta J_b(\vec{z})}, \end{aligned} \quad (19)$$

where $m = e^2 c_A / 2\pi$. Regularization issues have been discussed in the cited references.

Standard perturbation theory (in the Hamiltonian formulation) corresponds to expanding $H = \exp(t_a \varphi^a)$ in powers of the Hermitian field φ^a . Further, we should also expand in powers of $m = e^2 c_A / 2\pi$. In our approach, we developed an expansion scheme where we solve the Schrödinger equation keeping all terms in $\mathcal{H}_0$ at the lowest order, treating $\mathcal{H}_1$ as a perturbation. We also keep the term involving m even at the lowest order. Formally, we keep m and e as independent parameters in keeping track of different orders in solving the Schrödinger equation, only setting $m = e^2 c_A / 2\pi$ at the end. We may note that if we further expand H in terms of φ^a , our scheme would correspond to a partially resummed version of the usual perturbation expansion.

The lowest order result for the vacuum wave function was given in [4] and gave the string tension for a Wilson loop in the representation R as $\sigma_R = e^4 c_A c_R / 4\pi$, c_R being the quadratic Casimir value for the representation R . Corrections to this formula were considered in [5], taking the expansion to the next higher order (which still involves

an infinity of correction terms) and found that the corrections were small, of the order of -0.03% to -2.8% . The resulting values for the string tension agree well with the lattice estimates [14,15].¹

For the present purpose of obtaining the two-point function for A_μ , it is useful to write the Hamiltonian in a different form [17]. We can absorb the factor $e^{2c_A S_{\text{wzw}}}$ in (17) into the definition of the wave function by writing $\Psi = e^{-c_A S_{\text{wzw}}} \Phi$. However, the Hamiltonian acting on Φ will now be given by $\mathcal{H} \rightarrow e^{c_A S_{\text{wzw}}} \mathcal{H} e^{-c_A S_{\text{wzw}}}$. We then expand H as $H = \exp(t_a \varphi^a) \approx 1 + t_a \varphi^a + \dots$. This “small φ ” expansion is suitable for the intermediate kinematic regime of interest; it corresponds to a (resummed) perturbation theory. The Hamiltonian is then

$$\mathcal{H} = \frac{1}{2} \int \left[-\frac{\delta^2}{\delta \phi^2} + \phi(-\nabla^2 + m^2)\phi + \dots \right], \quad (20)$$

where $\phi_a(\vec{k}) = \sqrt{c_A k \bar{k} / (2\pi m)} \varphi_a(\vec{k})$. This is clearly the Hamiltonian for a field of mass m with the corresponding vacuum wave function

$$\Phi_0 \approx \exp \left[-\frac{1}{2} \int \phi^a \sqrt{m^2 - \nabla^2} \phi^a \right]. \quad (21)$$

[Since we absorbed the factor $e^{2c_A S_{\text{wzw}}(H)}$ into the wave functions, the inner product for the Φ 's will involve just the Haar measure, as is clear from (17).] We can view $\mathcal{H}$ in (20) as arising from the action

$$S = \int d^3x \frac{1}{2} [\dot{\phi}^a \dot{\phi}^a - (\nabla \phi^a)(\nabla \phi^a) - m^2 \phi^a \phi^a] + \dots \quad (22)$$

This shows that, effectively, we have $\dim G$ copies of a massive scalar field, the mass being $m = e^2 c_A / (2\pi)$.

Notice that for small φ , for the gauge-invariant part of the potential we have $A_i^a \approx -\frac{i}{2} \epsilon_{ij} \partial_j \varphi^a$. What we have shown is that φ^a , modulo the multiplicative factor $\sqrt{c_A k \bar{k} / (2\pi m)}$, is described by a massive scalar field. It is possible to use this to construct the two-point function for A_μ and use it in (13) to obtain the boundary action and obtain the Casimir energy by integrating over the C 's. However, in this case, the problem is simpler, since the boundary condition of vanishing tangential electric field is

$$\epsilon_{ij} n_i F_{0j}^a = 0, \quad (23)$$

where n_i is the unit vector normal to the plate. In terms of φ^a , this becomes

¹For considerations on glueball masses in this approach, see [16].

$$n_i \epsilon_{ij} \epsilon_{jk} \partial_k \dot{\varphi}^a = -n_i \partial_i \dot{\varphi}^a = 0. \quad (24)$$

This is equivalent to imposing Neumann boundary conditions on the scalar field φ^a . The field ϕ^a will obey the same condition. We can then, in a straightforward way, expand ϕ^a in terms of modes obeying the Neumann condition and calculate the Casimir energy. This was carried out in [7]; we will therefore just quote the result. The Casimir energy per unit length is given by

$$\frac{\mathcal{E}}{L} = -\frac{\dim G}{16\pi R^2} [2mR \text{Li}_2(e^{-2mR}) + \text{Li}_3(e^{-2mR})], \quad (25)$$

where $\text{Li}_s(w)$ is the polylogarithm function

$$\text{Li}_s(w) = \sum_{n=1}^{\infty} \frac{w^n}{n^s}. \quad (26)$$

For ease of comparison with the lattice-based numerical calculations, it is useful to express this in terms of the string tension σ_F for the fundamental representation. For $SU(N)$, we have

$$\begin{aligned} \sigma_F &= e^4 \frac{c_A c_F}{4\pi} = e^4 \frac{N^2 - 1}{8\pi}, \\ m &= \frac{e^2 c_A}{2\pi} = \sqrt{\frac{c_A}{\pi c_F}} \sqrt{\sigma_F} = \sqrt{\frac{2N^2}{\pi(N^2 - 1)}} \sqrt{\sigma_F}, \end{aligned} \quad (27)$$

where we used $c_A = N$, $c_F = (N^2 - 1)/(2N)$. In terms of σ_F , the Casimir energy per unit length is thus

$$\begin{aligned} \frac{\mathcal{E}}{L\sigma_F} &= -\frac{\dim G}{16\pi} \left[\frac{2\sqrt{c_A/\pi c_F}}{x} \text{Li}_2(e^{-2\sqrt{c_A/\pi c_F} x}) \right. \\ &\quad \left. + \frac{1}{x^2} \text{Li}_3(e^{-2\sqrt{c_A/\pi c_F} x}) \right], \end{aligned} \quad (28)$$

where $x = R\sqrt{\sigma_F}$.

For very small distances, $R \ll m^{-1}$, the Casimir energy (28) is that of $\dim G$ massless scalar fields [7],

$$\frac{\mathcal{E}}{L\sigma_F} \rightarrow -\dim G \frac{\zeta(3)}{16\pi x^2}. \quad (29)$$

The boundary condition (23) or (24) corresponds to the case where both wires are perfect electric conductors (PEC). In [7], the Casimir energy was calculated for other boundary conditions as well. For example, the case where the wires are perfect magnetic conductors (PMC), i.e., with the magnetic field vanishing on the wires, is realized by the field ϕ^a obeying Dirichlet conditions. The Casimir energy for scalar fields obeying Dirichlet or Neumann conditions on both plates is the same (25), so we can conclude that the Casimir energy for PMC conditions on both wires is the

same as for PEC, i.e., $\mathcal{E}_{\text{PMC}} = \mathcal{E}_{\text{PEC}}$. For the case of one wire with the PEC condition and the other with the PMC condition, the Casimir energy is positive, corresponding to a repulsive Casimir force [7]. More general boundary conditions as in [12,13] can also be considered. However, since there are no available data on these from lattice simulations, we do not quote these results here. But it is worth pointing out that lattice simulations with boundary conditions other than the perfect electrical conductor will be very interesting, providing additional points of comparison.

IV. CASIMIR ENERGY IN 3+1 DIMENSIONS

The key ingredient which facilitated the Hamiltonian analysis in 2 + 1 dimensions was the parametrization of the gauge potentials as in (16). A simple argument for this uses the fact that we can think of the spatial manifold as the large radius limit of a two-sphere $S^2 = SU(2)/U(1)$. Since the representation matrices or Wigner $\mathcal{D}$ functions of $SU(2)$ constitute a complete set, we can then obtain the parametrization (16) [6,18]. The complex structure of S^2 is also crucial for this result. The manifold $\mathbb{CP}^2 = SU(3)/U(2)$ is a suitable choice for a similar strategy in four dimensions. Since this has the required number of spacetime dimensions, the appropriate approach would be to use a Euclidean functional integral, rather than a Hamiltonian analysis. The parametrization of the gauge fields is of the form [10]

$$\begin{aligned} A_i &= -\nabla_i M M^{-1} - M a_i M^{-1}, \\ \bar{A}_{\bar{i}} &= M^{\dagger-1} \bar{\nabla}_{\bar{i}} M^{\dagger} + M^{\dagger-1} \bar{a}_{\bar{i}} M^{\dagger}, \\ a_i &= -g_{ii} \epsilon^{i\bar{j}} (\bar{\nabla}_{\bar{j}} \chi + [H^{-1} \nabla_j H, \chi]), \\ \bar{a}_{\bar{i}} &= -g_{i\bar{i}} \epsilon^{ij} (\nabla_j \chi^{\dagger} + [-\nabla_j H H^{-1}, \chi^{\dagger}]) = a_i^{\dagger}. \end{aligned} \quad (30)$$

M is again a complex matrix taking values in the complexification of the group G . χ is a complex matrix field, but it is not a scalar; rather, $\chi \epsilon^{i\bar{j}}$ is a rank-2 antisymmetric tensor under $U(2)$. [While there is only a single complex field χ , it is not a scalar—it transforms nontrivially under the “hypercharge” $U(1) \in U(2) \sim SU(2) \times U(1)$; see [10].] The gauge-invariant variables are $H = M^{\dagger} M$ and χ and $\chi^{\dagger}$.

Since gauge transformations act as before, namely, $M \rightarrow M^g = gM$, it is possible to factor out the gauge degrees of freedom and define a functional measure of integration on the gauge-orbit space. However, unlike the lower dimensional case, the (functional) Jacobians for the change of variables $A_i, \bar{A}_{\bar{i}} \rightarrow H, \chi, \chi^{\dagger}$ cannot be calculated exactly. We have calculated the leading terms in a derivative expansion of the fields. The key result is that there is a WZW action with a finite coefficient and the fields a_i and $\bar{a}_{\bar{i}}$ acquire mass [10]. However, unlike the lower dimensional case, this mass is an input parameter. Recall that to define the quantum Yang-Mills theory in four dimensions we need an input parameter with the dimension of mass.

In perturbative calculations, this is usually taken as Λ_{QCD} . In our nonperturbative setup, the dimensional transmutation is obtained via the mass parameter for a_i and $\bar{a}_{\bar{i}}$. If we were to calculate the one-loop corrections, then we could obtain Λ_{QCD} as a function of this mass parameter but we have not completed this yet.

We can simplify the parametrization (30) in the large radius limit of $\mathbb{CP}^2$. For our purpose it is sufficient to approximate M as $M \approx 1 + i\theta - \varphi_3$. The Hermitian and Cartesian components of the fields then take the form

$$A_4^a = \partial_4 \theta^a + \nabla \cdot \varphi^a, \quad A_i^a = \partial_i \theta^a - \partial_4 \varphi_i^a - \epsilon_{ijk} \partial_j \varphi_k^a. \quad (32)$$

Evidently, up to linear order of gauge transformations, the gauge degrees of freedom are θ^a ; in addition, we have three sets of fields φ_i^a , $i = 1, 2, 3$, with χ in (31) given by $\chi = -\varphi_1 - i\varphi_2$.

In order to use our theory for Casimir energy calculations in $\mathbb{R}^4$, we consider the large radius limit of $\mathbb{CP}^2$, effectively reproducing flat space. However, recall that the large radius limit of $\mathbb{CP}^2$ is not S^4 but a compactified version of $\mathbb{C}^2$, so that we do not obtain full rotational symmetry. We can covariantize our results in complex space by integrating over the placements of $\mathbb{C}^2$ in $\mathbb{R}^4$. This recovers the full rotational symmetries, leading to results relevant to standard $\mathbb{R}^4$. (This is similar to what is done with twistors where one eventually integrates over all local complex structures on S^4 .) The fields φ_i^a acquire dynamically generated masses as explained above. Under covariantization, all φ_i^a have the same mass (which depends on the mass parameter for a_i , $\bar{a}_{\bar{i}}$ and the finite coefficient of the WZW action). The action for the φ fields becomes

$$\mathcal{S} = \frac{1}{2} \int [(\Box \varphi_i^a)^2 + m^2 (\partial \varphi_i^a)^2]. \quad (33)$$

For more details leading to this result, see the Appendix.

In terms of gauge fields, up to quadratic order in φ_i^a , the action (33) can be written as

$$\mathcal{S} = \int \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{4} F_{\mu\nu} \frac{1}{(-\Box)} F^{\mu\nu} + \dots \right]. \quad (34)$$

As mentioned in Sec. II, we need the gauge-invariant part of the two-point function for A 's. Given the parametrization of the fields as in (32), it is convenient in this case to take the plates to be orthogonal to the x_4 direction, so that we have A_i , $i = 1, 2, 3$, in the boundary action. [The mapping (1234) $\rightarrow$ (4123) will connect these results back to the case in Sec. II.] Further, θ is irrelevant for our purpose, so we need

$$\begin{aligned} \langle A_i^a(x) A_j^b(y) \rangle &= \langle (\partial_4 \varphi_i^a + \epsilon_{ipq} \partial_p \varphi_q^a)(x) (\partial_4 \varphi_j^b \\ &\quad + \epsilon_{jrs} \partial_r \varphi_s^b)(y) \rangle. \end{aligned} \quad (35)$$

For this correlation function, the basic result we need from the action (33) is

$$\begin{aligned}\langle \varphi_i^a(x) \varphi_j^b(y) \rangle &= \delta^{ab} \delta_{ij} G(x, y) \\ &= \delta^{ab} \delta_{ij} \int \frac{d^4 k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{k^2(k^2 + m^2)}.\end{aligned}\quad (36)$$

Using this result it is easy to see that

$$\begin{aligned}\langle A_i^a(x) A_j^b(y) \rangle &= \delta^{ab} [-\partial_4^2 \delta_{ij} - \epsilon_{ipj} \partial_p \partial_4 - \epsilon_{jpi} \partial_p \partial_4 \\ &\quad - (\delta_{ij} \delta_{pr} - \delta_{ip} \delta_{jr}) \partial_p \partial_r] G(x, y) \\ &= \delta^{ab} [-\square \delta_{ij} + \partial_i \partial_j] G(x, y).\end{aligned}\quad (37)$$

This can now be used in (13) to obtain the effective boundary action as

$$\begin{aligned}\mathcal{S}_{\text{eff}} &= \frac{1}{2} \int (C_1^T)^{ai}(x) K_{00}(x, y) (C_1^T)^{ai}(y) \\ &\quad + \frac{1}{2} \int (C_2^T)^{ai}(x) K_{\text{RR}}(x, y) (C_2^T)^{ai}(y) \\ &\quad + \int (C_1^T)^{ai}(x) K_{0\text{R}}(x, y) (C_2^T)^{ai}(y).\end{aligned}\quad (38)$$

In carrying out this simplification, we note that we can separate C^{ai} into longitudinal (proportional to ∂^i of some function) and transverse components. The longitudinal component does not appear in $\mathcal{S}_{\text{eff}}$ because (37) shows that the correlation function is transverse. For the transverse components, the $\partial_i \partial_j$ part in (37) gives zero. An integration by parts transfers the derivatives to the correlator from the C 's. This gives the expression (38). We have used the superscript T on C to emphasize that it is only the transverse component which appears here. Further the two-point correlation functions in (38), after k_4 -integration, are given by

$$\begin{aligned}K_{00}(x, y) &= [\nabla^2 \square G(x, y)]_{x_4=y_4=0} = \int \frac{d^3 k}{(2\pi)^3} \frac{\vec{k}^2}{2\omega_k} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})}, \\ K_{\text{RR}}(x, y) &= [\nabla^2 \square G(x, y)]_{x_4=y_4=R} = K_{00}(x, y), \\ K_{0\text{R}}(x, y) &= [\nabla^2 \square G(x, y)]_{x_4=0, y_4=R} \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{\vec{k}^2 e^{-\omega_k R}}{2\omega_k} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})},\end{aligned}\quad (39)$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$.

The integration over $(C_1^T)^{ai}$ is now straightforward. Because of the transversality condition, there are only two independent fields for each of C_1^T and C_2^T . To be explicit, one could even decompose these fields as

$$(C_1^T)^{ai} = \int d^3 x [i\psi_1^a(q) e_{(1)}^i(q) + \psi_2(q) e_{(2)}^i(q)] e^{-i\vec{q} \cdot \vec{x}}, \quad (40)$$

where ψ_1 and ψ_2 are independent fields and $e_{(1)}^i(q)$ and $e_{(2)}^i(q)$ are two polarization vectors transverse to $\vec{q}$. One can write a similar expansion for $(C_2^T)^{ai}$ as well. Integrating over the C 's we find the R -dependent terms in $\log Z$ to be given by

$$-\log Z = \dim G \text{Tr} \log(1 - K_{\text{RR}}^{-1} K_{\text{R0}} K_{00}^{-1} K_{0\text{R}}). \quad (41)$$

There is no factor of $\frac{1}{2}$ as in (8) since there are two polarizations which contribute. Using (39) we find

$$\begin{aligned}-\log Z &= \dim G \int d^3 x \frac{d^3 k}{(2\pi)^3} \log(1 - e^{-\omega_k R}) \\ &= -\dim G \int d^3 x \frac{d^3 k}{(2\pi)^3} \sum_n \frac{e^{-n\omega_k R}}{n} \\ &= -\dim G \int d^3 x \sum_n \frac{m^3}{8\pi^2} \int_0^\infty ds (\cosh 3s \\ &\quad - \cosh s) \frac{e^{-2mRn \cosh s}}{n},\end{aligned}\quad (42)$$

where we have used the substitution $|\vec{k}| = m \sinh s$ and carried out the angular integrations. We can now use the integral representation of the modified Bessel function

$$K_\nu(z) = \int_0^\infty ds e^{-z \cosh s} \cosh(\nu s) \quad (43)$$

to simplify (42) as

$$\begin{aligned}-\log Z &= -\dim G \int d^3 x \sum_n \frac{m^3}{8\pi^2 n} (K_3(2mRn) \\ &\quad - K_1(2mRn)) \\ &= -\dim G \int d^3 x \frac{m^2}{4\pi^2 R} \sum_n \frac{K_2(2mRn)}{n^2},\end{aligned}\quad (44)$$

where we have also used the identity

$$K_3(z) - K_1(z) = \frac{4}{z} K_2(z). \quad (45)$$

The Casimir energy per unit area of the plates is thus given by

$$\frac{\mathcal{E}}{\mathcal{A}(\sigma_F)^{\frac{3}{2}}} = -C \dim G \frac{(\tilde{m})^2}{4\pi^2 x} \sum_n \frac{K_2(2\tilde{m}xn)}{n^2}, \quad (46)$$

where $x = R\sqrt{\sigma_F}$ and $\tilde{m} = m/\sqrt{\sigma_F}$, with σ_F being the string tension for the fundamental representation. We have also included a prefactor C . This is actually 1 for us, but we include it to facilitate comparisons and comments later.

The final result is equivalent to what we would find from the exchange of 2 dim G massive scalar fields between the plates. (For the calculation of the energy for massive scalars, see also [11,19].)

In the limit of small $Rm = \tilde{m}x$ (i.e., at short distances $x \ll \tilde{m}^{-1}$ or in the limit of $m \rightarrow 0$), the modified Bessel function can be approximated by

$$K_2(2\tilde{m}xn) \sim \frac{1}{2\tilde{m}^2 n^2 x^2} + \dots \quad (47)$$

Using this, the energy formula (46) becomes

$$\frac{\mathcal{E}}{\mathcal{A}(\sigma_F)^{\frac{3}{2}}} = -C \dim G \frac{1}{8\pi^2 x^3} \sum_n \frac{1}{n^4} = -C \dim G \frac{\pi^2}{720x^3}. \quad (48)$$

This agrees (with $C = 1$) with the result for 2 dim G massless scalar fields. (We have normalized the result such that $C = 1$ corresponds to two polarizations.)

The result (46) is also what was used as an ansatz in [2] in analyzing their lattice data. However, we emphasize that, for us, it is not an ansatz; it follows from our gauge-invariant analysis for the 4d Yang-Mills (YM) theory.

It is also interesting to consider other types of boundary conditions on the fields. The conditions we used, i.e., the vanishing of the components of $F_{\mu\nu}^a$ tangential to the plates, correspond to a perfect (chromo)electric conductor (PEC). The vanishing of the dual components, namely, $F_{14}^a = F_{24}^a = F_{34}^a = 0$ (or $F_{34}^a = F_{13}^a = F_{23}^a = 0$ in the choice of axes used in Sec. II) is another possibility, corresponding to the perfect (chromo)magnetic conductor (PMC). For this case, one might consider the boundary action

$$S_b = i \int C_i^a F_{i4}^a. \quad (49)$$

In the PEC case, the Bianchi identity among the components of F in the boundary term in (12) led to a new gauge symmetry $C \rightarrow C + \partial f$, thus effectively reducing C to two polarizations. If we use (49) though, all three polarizations of C will in principle contribute. The Casimir energy will thus be $\frac{3}{2}$ times the result for the PEC conditions [20,21]. A similar derivation in $2 + 1$ dimensions would imply that the Casimir energy for PMC conditions would be 2 times the result for the PEC conditions, which would be contradictory to the results we found in Sec. III. The Hamiltonian analysis in Sec. III showed that the Casimir energy for both PEC and PMC conditions is the same.

We believe that the resolution of this issue has to do with the nature of mass m . The mass used here is dynamically generated and therefore we do not expect it to be a hard mass, although one can use the action (33) for a certain kinematic regime. At high energies or short distances, we should also expect the theory to reduce to a theory of

massless gluons, by virtue of asymptotic freedom. But a hard mass, however small, should have three polarizations just by representation theory for the Poincaré symmetry. Therefore, with a hard mass, generally we will not obtain a smooth limit to the zero mass case at short distances. [The PEC case does correspond to two polarizations in the short-distance limit, as is clear from (48).] The dynamical mass on the other hand will go to zero at high energies, so it is possible to get a result that agrees with two contributing polarizations. The likely scenario is that even if we use the hard mass as in (33) for the kinematic regime relevant to the Casimir energy simulations, we should have two polarizations contributing.

The result for the Casimir energy, for massless electrodynamics is the same for both PMC and PEC boundary conditions and results from the contribution of two polarizations; see [22,23]. This indicates that (49) is not the right boundary action to be used for the PMC conditions in the case of dynamical mass and has to be modified for consistency with the short-distance limit. One way to do that is to notice that if we have a mode expansion of the fields where $F_{i4} = 0$, by the Bianchi identity, these give $\partial_4 F_{12} = \partial_4 F_{23} = \partial_4 F_{31} = 0$. By the same Bianchi identities, these are equivalent to $\epsilon^{ijk} \partial_j F_{4k} = 0$. Therefore, an alternate, perhaps better, possibility is to use the boundary action

$$S_b = i \int C_i^a \epsilon^{ijk} \partial_j F_{4k}^a = i \int C_i^a \epsilon^{ijk} \partial_j F_{\mu k}^a n^\mu \quad (50)$$

as the action relevant to the PMC conditions.² (Here n^μ is the unit vector normal to the plate.) In this case, we see that we again have the gauge symmetry $C \rightarrow C + \partial f$; thus C has only two polarizations and the result of calculating with (50) gives the same result for the Casimir energy as the PEC conditions. An equivalent way would be to use the original boundary action (49) but impose by hand a transversality condition for C .

Again, lattice simulations with PMC boundary conditions would be very useful in establishing and clarifying the relation for the Casimir energy for PEC and PMC boundary conditions in the truly nonperturbative setting of Yang-Mills theory.

V. COMPARISON WITH NUMERICAL CALCULATIONS

We will now compare our analytical formulae (valid for any gauge group) with the results for the numerical calculation via lattice simulations of the Casimir energy

²Another choice would be

$$S_b = i \int C_i^a \epsilon^{ijk} \partial_4 F_{jk}^a = i \int C_i^a \epsilon^{ijk} \partial_\mu F_{jk}^a n^\mu. \quad (51)$$

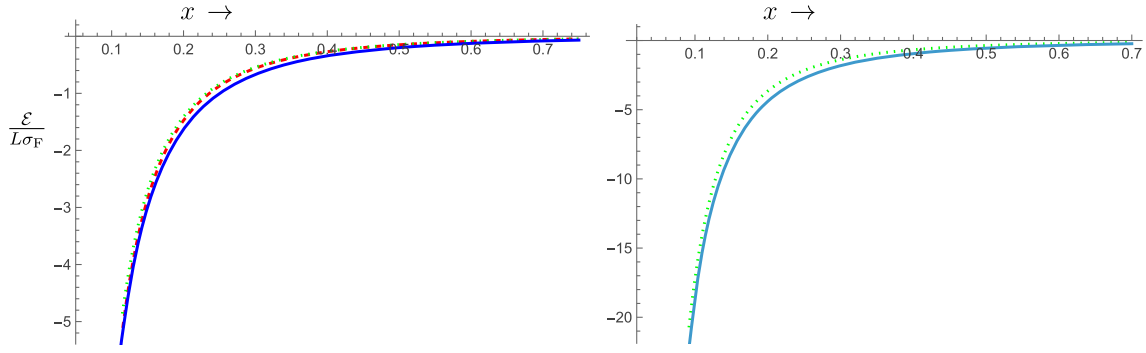


FIG. 1. Comparison of lattice-based results from [1] (red, dashed line) and from [3] (green, dotted line) with our formula (28) for $SU(2)$ on the left and $SU(3)$ on the right.

for the non-Abelian gauge theory. The lattice simulations were performed for perfect chromoelectric boundary conditions (PEC case). Results are available for the groups $SU(2)$ and $SU(3)$ in both three (i.e., $2+1$) and four (i.e., $3+1$) dimensions. We will consider each dimension separately.

A. Three dimensions

The first lattice calculation for the Casimir energy for $SU(2)$ was given in [1]. The authors fit their results to the formula

$$\frac{\mathcal{E}}{L\sigma_F} = -\dim G \frac{\zeta(3)}{16\pi} x^{-\nu} e^{-M_{\text{Cas}}x/\sqrt{\sigma_F}}. \quad (52)$$

The best-fit values of the parameters were $\nu = 2.05$ and $M_{\text{Cas}} = 1.38\sqrt{\sigma_F}$. More recently, there has been another calculation for $SU(2)$ [3]. The authors again fit their results to a formula of the form (52), with $\nu = 2.022$ and $M_{\text{Cas}} = 1.38\sqrt{\sigma_F}$.

In our view, and as we have already argued in [7], there is stronger motivation for our formula (28) as it follows from a Hamiltonian analysis based on first principles which has also produced a prediction for the string tension in close agreement with lattice-based measurements. Further, in (28), there is a *predicted* value for the mass parameter as well; see (27). We may also note that there is a difference in interpretation. For us, the relevant degrees of freedom are the gluons with a propagator mass, we are not considering new excitations.

The comparison of these cases is given in Fig. 1. The red dashed line shows the formula (52) for $\nu = 2.05$ and $M_{\text{Cas}} = 1.38\sqrt{\sigma_F}$, corresponding to [1]; the green dotted line is for $\nu = 2.022$ and $M_{\text{Cas}} = 1.38\sqrt{\sigma_F}$ corresponding to [3]. The blue solid line is our result (28), with no adjustable parameters. We note that the mass which controls the exponential decay in our formula (28) is $2m = 2\sqrt{c_A/\pi c_F}\sqrt{\sigma_F} \approx 1.84\sqrt{\sigma_F}$ for $SU(2)$, from (27).

The lattice-based estimate of the energy has been obtained for the case $SU(3)$ in [3]. The authors again fit the results to the formula (52), the best-fit values being $\nu = 2.02$, $M_{\text{Cas}} = 1.51\sqrt{\sigma_F}$. We show this graph in Fig. 1 on the right side. Our result (28) is again shown in the blue solid line. In this case, we note that $2m = \sqrt{9/\pi}\sqrt{\sigma_F} \approx 1.69\sqrt{\sigma_F}$.

In both cases, we note that the result from the Hamiltonian analysis is very close to the lattice-based numerical calculations, with no adjustable parameters.

Another feature about the formula (52) *vis-à-vis* our result (28) which is worthy of comment is the following. Notice that best-fit value of M_{Cas} in units of $\sqrt{\sigma_F}$ is higher for $SU(3)$ compared to $SU(2)$. In the same units, the lowest-lying glueball masses are 4.716 for $SU(2)$ and 4.330 for $SU(3)$ [24]. It is a bit peculiar that while the glueball mass decreases as we go from $SU(2)$ to $SU(3)$, M_{Cas} increases. This would be hard to understand if any excitation corresponding to M_{Cas} is part of the glueball. On the other hand, for us, the value of $2m$ decreases from 1.84 to 1.69 as we go from $SU(2)$ to $SU(3)$ in consonance with the behavior of glueball masses.

B. Four dimensions

The situation in four dimensions is more involved. The Casimir energies have been calculated by lattice simulations for $SU(2)$ in [3] and for $SU(3)$ by [2] and by [3]. For $SU(2)$, the authors of [3] fit their results to a formula of the form

$$\frac{\mathcal{E}}{\mathcal{A}(\sigma_F)^{\frac{3}{2}}} = -\dim G \frac{\zeta(4)}{16\pi^2} x^{-\nu} e^{-M_{\text{Cas}}x/\sqrt{\sigma_F}}. \quad (53)$$

The best-fit choice of parameters is $\nu = 3.09$ and $M_{\text{Cas}} = 0.38\sqrt{\sigma_F}$ for $SU(2)$.

We consider the values given by the function (53) for the range $0.2 \leq x \leq 1$ and fit to our formula (46). The best-fit values are $C = 1.126$ and $m = 1.067\sqrt{\sigma_F}$ (i.e., $\tilde{m} = 1.067$). However, our calculation gave $C = 1$. So in Fig. 2, we show the curve corresponding to (53) with $\nu = 3.09$,

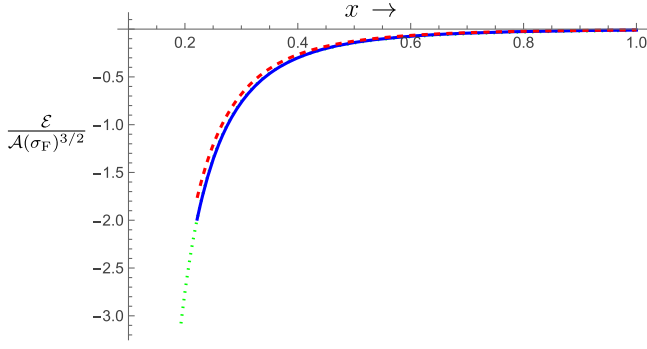


FIG. 2. Comparison of lattice-based results for $SU(2)$ in four dimensions from [3] (green, dotted line) with our formula (46) for $SU(2)$, with $m = 1.067\sqrt{\sigma_F}$ and $C = 1.126$ (red, dashed line) and $C = 1$ (blue, solid line).

$M_{\text{Cas}} = 0.38\sqrt{\sigma_F}$ (corresponding to lattice data) and our formula (46) with $C = 1.126$ (red, dashed line), $C = 1$ (blue, solid line) and $m = 1.067\sqrt{\sigma_F}$. Notice that all three curves are essentially identical.

It is also possible to find a fit for a larger range, $0.1 \leq x \leq 1$, with somewhat different parameters. But we have chosen the range $0.2 \leq x \leq 1$ for the graph of our formula shown in Fig. 2 since the lattice data for [2] are in this range. We note however that, as mentioned earlier, we expect the analysis to hold only for a certain range, where the dynamical mass generation effects are important, but the degrees of freedom are still gluons rather than glueballs, but it is difficult to know *a priori* what this range should be.

As mentioned, for $SU(3)$ there are calculations from [2,3]. The authors of [2] fit the results to a formula of the same form as we have argued for, namely (46), but with a prefactor $C \neq 1$. Although this is basically the formula we also have, we emphasize that the merit of our gauge-invariant analysis is really to justify the use of a formula of this form. The best-fit values for [2] give $C = 5.6$ and $m = 1.0\sqrt{\sigma_F}$. The authors of [3] fit to the formula (53), the best-fit values for $SU(3)$ being $\nu = 3.002$ and $M_{\text{Cas}} = 0.06\sqrt{\sigma_F}$.

In Fig. 3, we show three curves, the red dashed curve for [2] [from (46) with $C = 5.6$, $m = 1$], the green dotted curve for [3] [from (53) with $\nu = 3.002$, $M_{\text{Cas}} = 0.06$] and the blue solid curve for our formula (46) with $C = 1$ and $m = 1.0\sqrt{\sigma_F}$.

The key point here is that there is some disagreement between the two sets of lattice data. If we use the mass value from [2] and set $C = 1$, our result is somewhere between the two lattice results, but given the discrepancy between the two lattice results, it is difficult to make a definite conclusion. Obviously if we allow for a prefactor C which is not necessarily 1, it is possible to fit our results to the data from [2], with $C = 5.6$ and $m = 1.0\sqrt{\sigma_F}$, as is done in [2], since the formulas are identical. We can similarly fit our formula (46) to the lattice data from [3] for $SU(3)$ by choosing $C = 0.5$ and $m = 0.38\sqrt{\sigma_F}$.

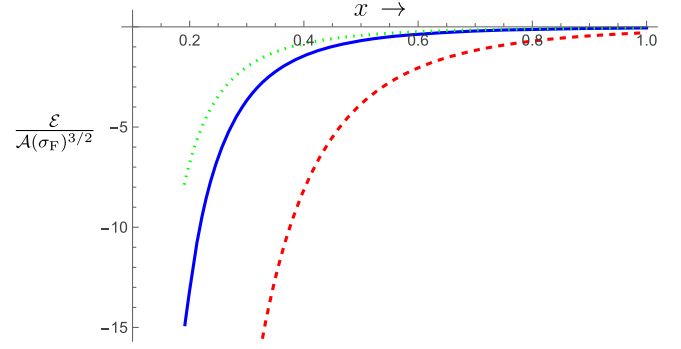


FIG. 3. Comparison of lattice-based results for $SU(3)$ in four dimensions from [2] (red, dashed line) and from [3] (green, dotted line) with our formula (46) for $SU(3)$, with $m = 1.0\sqrt{\sigma_F}$ and $C = 1$ (blue, solid line).

Another comparison one might consider is the following. There is by now evidence, based on lattice calculations and arguments based on Schwinger-Dyson equations, that the gluon propagator acquires a mass [25–27]. In the latest analyses, for example, the value obtained for $SU(3)$ is $m \approx 686 \text{ MeV} \approx 1.41\sqrt{\sigma_F}$, where we use $\sqrt{\sigma_F} \approx 0.485 \text{ GeV}$ [27]. If we use this value and calculate the energy according to our formula (46), with $C = 1$, the result is again between the two sets of lattice data, see Fig. 4. Again, an unambiguous set of lattice data is needed before we can conclude anything about possible compatibility with other methods of calculation.

A comment on the prefactor C is appropriate at this stage. The energy is given by a determinant which arises from a functional integration, so it is easy to see that C is related to the number of degrees of freedom which can contribute. Our result (28) corresponds to one field degree of freedom for each direction in the Lie algebra (or one polarization) and result (46) corresponds, with $C = 1$, to two polarizations. This is in agreement with general expectations for the gauge theory in $2+1$ and $3+1$ dimensions. In principle, C cannot therefore be a parameter for which we seek a best-fit value. Nevertheless, there is some variance possible in this parameter.

For example, we know from lattice simulations that the number of degrees of freedom is somewhat less than the value corresponding to a gas of free gluons even at very high temperatures, where we expect a deconfined gluon plasma. For a review with updated results, see [28]. Another factor is that, even though we consider a regime where we expect the energy to be given by the exchange of massive gluons, in principle, one can also have some contribution from glueballs. Generally, such contributions will be of the form $\sum_{\alpha} C_{\alpha} e^{-2m_{\alpha}x}$, where C_{α} are some constants and m_{α} is the mass of the glueball labeled by α . (The factor of 2 in the exponent is because we need two transits between the plates to complete a loop for the virtual particle, as explained in [7]. It is also clear from the worldline method of calculating the Casimir energy.) While

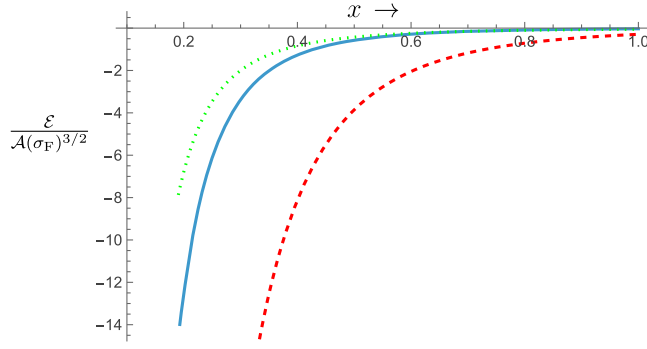


FIG. 4. Comparison of lattice-based results for $SU(3)$ in four dimensions from [2] (red, dashed line) and from [3] (green, dotted line) with our formula (46) for $SU(3)$, with $m = 1.41\sqrt{\sigma_F}$ and $C = 1$ (blue, solid line).

a proper calculation of such terms will require information about how glueballs can couple to the plates and what boundary conditions should be imposed on the glueball fields, we can estimate the importance of such terms by comparison with glueball masses, which are typically higher than the value of the dynamical mass m for the gluons. So each such contribution is exponentially small compared to our formulas (28) and (46), even though there could be many such terms.

The lowest-lying glueball in $2 + 1$ dimensions has a mass 4.716 for $SU(2)$ and 4.330 for $SU(3)$ (in units of $\sqrt{\sigma_F}$), while the corresponding values in $3 + 1$ dimensions are 3.781 and 3.405 [29]. In $2 + 1$ dimensions, the values for the dynamical mass of the gluon are $0.921\sqrt{\sigma_F}$ for $SU(2)$ and $0.846\sqrt{\sigma_F}$ for $SU(3)$, as seen from (27). Since the glueball mass is much higher, we might expect that any contribution from the glueballs will be very small in $2 + 1$ dimensions, so that we can expect good agreement with (28). In $3 + 1$ dimensions, the mass for the gluon [for $SU(3)$] was in the range 1 to 1.41 in units of $\sqrt{\sigma_F}$. The glueball masses are closer to this value compared to $2 + 1$ dimensions, so it is possible that there is some extra contribution to the energy, beyond (46). This might explain why a better fit is obtained in the four-dimensional case if we use a value for C not equal to 1. Even so, we think it should not deviate too much from 1. Once again, an unambiguous set of lattice data is needed to reach any definite conclusion.

Finally, we comment on the use of the formula (53). If we set $M_{\text{Cas}} = 0$, formula (52) (with $\nu = 2$) correctly reduces to the Casimir energy due to one massless field (for each direction in the Lie algebra), in agreement with the expectation of one physical polarization for the gauge field in $2 + 1$ dimensions. However, even though setting $M_{\text{Cas}} = 0$ in (53) (with $\nu = 3$) also gives the correct form of the energy for $\dim G$ massless fields in $3 + 1$ dimensions, it still corresponds to only one polarization, not two, as expected. This may be a point of concern regarding the use of the formula (53) to obtain a fit to the lattice data.

VI. CONCLUSIONS

We have calculated the Casimir energy for the standard arrangement of two parallel plates for non-Abelian gauge fields in $2 + 1$ and $3 + 1$ dimensions. In both cases, the result corresponds to that of $\dim G$ massive scalar fields, with the appropriate number of polarizations (one in $2 + 1$ and two in $3 + 1$ dimensions). These modes are the gauge-invariant versions of the gluons with their propagators having a dynamically generated mass. We also compared the analytic formulae, with the available results based on lattice simulations, all for the case of chromoelectric (PEC) boundary conditions. The agreement in $2 + 1$ dimensions is excellent. We emphasize that our formula in $2 + 1$ dimensions has no adjustable parameters. The two quantities needed, namely the string tension and the propagator mass of the gluon, both follow from our Hamiltonian analysis and are numbers calculated purely based on group theory. It is important that we do have an excellent agreement, for both $SU(2)$ and $SU(3)$ for which lattice data are available, without any adjustable parameters.

In $3 + 1$ dimensions, the propagator mass is a parameter which defines the theory. It is the parameter of dimensional transmutation, the analog of Λ_{QCD} in our gauge-invariant analysis. Here too the agreement with lattice data is good, but there seem to be differences in the lattice data from two different groups, so an unambiguous conclusion is not possible at this stage.

Note added. Recently, a paper discussing analytic calculations of the Casimir energy was posted on the arXiv [21]. The authors consider the Curci-Ferrari (CF) model which describes a Yang-Mills theory with massive gluons in $2 + 1$ and $3 + 1$ dimensions. There is soft breaking of the BRST (Becchi-Rouet-Stora-Tyutin) symmetry due to the mass term, but it is argued that the CF model is an acceptable description for YM theory with dynamical generation of mass. Formulas similar to our Eqs. (25) and (46) are obtained and comparison with lattice data is done with the prefactor C and m as fitting parameters. We mention a couple of points worth emphasizing in this regard. Our analysis in both $2 + 1$ and $3 + 1$ dimensions are fully gauge invariant. The mass term, even in $3 + 1$ dimensions, is gauge invariant, although the truncation of the theory to the quadratic terms of the action is all that is effective for the Casimir energy. Further, as mentioned above, we do not have adjustable parameters in $2 + 1$ dimensions.

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DATA AVAILABILITY

While no data was created in the manuscript, we compare our analytic results with the numerical work of [1–3], which are cited in the relevant figures.

APPENDIX: COVARIANTIZING THE THEORY ON $\mathbb{CP}^2$

In Sec. IV we have seen that by considering the 4d non-Abelian gauge theory on $\mathbb{CP}^2 = SU(3)/U(2)$ we can use the completeness of the Wigner $\mathcal{D}$ functions on $SU(3)$ to find a general parametrization of the gauge fields as in (30). The complex gauge fields on $\mathbb{CP}^2$ are parametrized in terms of matrix variables $(M, M^\dagger, \chi, \chi^\dagger)$ on which gauge transformations act homogeneously ($M \rightarrow M^g = gM$), allowing for a straightforward separation of gauge degrees of freedom. For a more detailed account see [10].

While complexification in 4d proves useful for a manifestly gauge-invariant description, the resulting theory is ultimately a theory in complex space rather than in $\mathbb{R}^4$. Imposing a complex structure in 4d Euclidean space breaks the full rotational symmetry of Euclidean space [reducing the rotational group from $O(4)$ to $U(2)$]. We can restore this symmetry by integrating over all the possible complex structures in $\mathbb{R}^4$. To this end, we employ a twistor space construction, where the complex projective space $\mathbb{CP}^3$ can be viewed as a bundle of local complex structures over S^4 , i.e., locally, $\mathbb{CP}^3 \sim S^4 \times \mathbb{CP}^1$, where $\mathbb{CP}^1$ parametrizes the space of local complex structures. In particular, we will link our gauge theory in the large volume limit of $\mathbb{CP}^2$ to a theory on S^4 with a particular complex structure given by a point on $\mathbb{CP}^1$. Integrating over the $\mathbb{CP}^1$ fiber—equivalent to integrating over all possible complex structures—restores Euclidean covariance.

An explicit realization of $\mathbb{CP}^3 \sim S^4 \times \mathbb{CP}^1$ is given in twistor theory, where the homogeneous coordinates on $\mathbb{CP}^3$, $Z = (Z_1, Z_2, Z_3, Z_4)$, with the identification $Z \sim \lambda Z$, $\lambda \in \mathbb{C} - \{0\}$, can be written in terms of a pair of two-component spinors w and π in the following way:

$$Z = (w_1, w_2, \pi_1, \pi_2). \quad (\text{A1})$$

Here, $\pi = (\pi_1, \pi_2) \neq (0, 0)$ are homogenous coordinates on $\mathbb{CP}^1$, defined up to an overall scaling $\pi \sim \lambda \pi$, with $\lambda \in \mathbb{C} - \{0\}$. The coordinates on S^4 , x^μ , $\mu = 1, 2, 3, 4$, are introduced via the incidence relations—relations linking spacetime to twistor space:

$$w = (x^4 - i\vec{\sigma} \cdot \vec{x})\pi, \quad (\text{A2})$$

where $w = (w_1, w_2)$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

The basic idea is to take w and $\bar{w}$ as coordinatizing our theory on $\mathbb{CP}^2$ in the large radius limit, effectively

$\mathbb{CP}^2 \rightarrow \mathbb{C}^2$.³ A point on the $\mathbb{CP}^1$ fiber corresponds to a particular complex combination of the real coordinates x^μ —for example, $\pi = (0, i)$ yields $w_1 = x^1 - ix^2$, $w_2 = -x^3 + ix^4$. Integrating over the $\mathbb{CP}^1$ coordinates will restore Euclidean covariance.

Following the incidence relations in (A2), it is straightforward to relate the complex gauge fields, $A_i, \bar{A}_{\bar{i}}$, $i, \bar{i} = 1, 2$, to the real ones, A_μ , $\mu = 1, 2, 3, 4$, as follows:

$$\begin{aligned} A_i &= \frac{1}{2\bar{\pi} \cdot \pi} (\bar{\pi}_i A_4(x) + i(\bar{\pi} \sigma_a)_i A_a(x)), \\ \bar{A}_{\bar{i}} &= \frac{1}{2\bar{\pi} \cdot \pi} (\pi_{\bar{i}} A_4(x) - i(\sigma_a \pi)_{\bar{i}} A_a(x)), \end{aligned} \quad (\text{A3})$$

where $a = 1, 2, 3$. We can further parametrize the real gauge fields as follows:

$$A_4 = -i(\partial_4 \theta + \partial_a \varphi_a), \quad A_a = -i(\partial_a \theta - \partial_4 \varphi_a - \epsilon_{abc} \partial_b \varphi_c), \quad (\text{A4})$$

where the gauge fields A_μ are chosen to be anti-Hermitian, $A_\mu^\dagger = -A_\mu$, for Hermitian fields $\theta^\dagger = \theta$ and $\varphi^\dagger = \varphi$.

Since the complex gauge fields A_i and $\bar{A}_{\bar{i}}$ are, in general, functions of both x^μ and $\mathbb{CP}^1$ coordinates, we expect the same to be the case for the matrix parameters $M, M^\dagger, \chi$, and $\chi^\dagger$. From (A4) and the parametrization in (30), the complex matrix parameters to first order in the real fields θ, φ are given by

$$\begin{aligned} M &= 1 + i\theta + \frac{\bar{\pi} \sigma_a \pi}{\bar{\pi} \cdot \pi} \varphi_a + \dots, \\ \chi &= \frac{\epsilon^{ij} \bar{\pi}_i (\bar{\pi} \sigma_a)_j}{\bar{\pi} \cdot \pi} \varphi_a + \dots, \end{aligned} \quad (\text{A5})$$

where the ellipses indicate terms with more powers of θ and φ .

We now have everything we need to covariantize the action of the complex theory on $\mathbb{C}^2$. As detailed in [10] the Jacobian determinant for the change of variables $A_i, \bar{A}_{\bar{i}} \rightarrow H, \chi, \chi^\dagger$ contributes an effective action of the form

$$\Gamma = CS_{\text{wzw}}(H) - \mu_{\text{ren}}^2 \int \text{Tr} \bar{a} \cdot H a H^{-1} + \Gamma_{\log \epsilon} + \dots, \quad (\text{A6})$$

where the ellipsis includes terms of higher scaling dimension that are subdominant in the low-energy regime. The coefficient C is a finite coefficient and μ_{ren}^2 is the finite

³For us the fields will vanish at infinity, and the difference between the large radius limits of $\mathbb{CP}^2$, S^4 and $\mathbb{R}^4$ will not be important. Thus w and $\bar{w}$ can be taken to coordinatize $\mathbb{C}^2$, or S^4 (with the one point removed), when we take the radius to be large.

renormalized value of mass for the $a_i, \bar{a}_i$ fields. The log-divergent terms $\Gamma_{\log \epsilon}$ are, in general, complicated and take the form

$$\begin{aligned} \Gamma_{\log \epsilon} = & \frac{\log \epsilon}{12} \int \text{Tr}[(\bar{\nabla}(\nabla H H^{-1}))^2 + (\bar{a} H a H^{-1})^2 \\ & + g^{\bar{i}i} g^{\bar{j}j} [\bar{a}_{\bar{i}}, H a_i H^{-1}] \bar{\nabla}_{\bar{j}}(\nabla_j H H^{-1}) \\ & - g^{\bar{i}i} g^{\bar{j}j} (\bar{\nabla}_{\bar{i}}(\nabla_j H H^{-1}) [\bar{a}_{\bar{j}}, H a_j H^{-1}] \\ & + \bar{\nabla}_{\bar{i}} \bar{a}_{\bar{j}} \mathcal{D}_i (H a_j H^{-1}))], \end{aligned} \quad (\text{A7})$$

where $\mathcal{D}$ is a covariant derivative with $-\nabla H H^{-1}$ as its connection and ϵ is the UV cutoff. The log divergence needs to be renormalized through the introduction of counterterms.

Overall, the effective action for the Yang-Mills theory in terms of the complex matrix parameters is given by

$$\mathcal{S} = -\Gamma + S_{\text{YM}}, \quad (\text{A8})$$

where $S_{\text{YM}} = -\frac{1}{2g^2} \int F_{\mu\nu}^2$ is the standard Yang-Mills action.

To covariantize this action we first reexpress the complex matrix parameters in terms of real fields through (A5), and then we integrate over the $\mathbb{CP}^1$ variables. To show this by example, let us covariantize the WZW action up to quadratic order in φ fields. First, we write the WZW action on $\mathbb{CP}^3$ in the following way:

$$\begin{aligned} S_{\text{WZW}}(H) = & \frac{1}{2\pi} \int_{\mathbb{CP}^1} d\mu(\pi, \bar{\pi}) \int \frac{d^2 w d^2 \bar{w}}{(\bar{\pi} \cdot \pi)^2} \text{Tr} \partial H \cdot \bar{\partial} H^{-1} \\ & - \frac{1}{2\pi} \int_{\mathbb{CP}^1, w} dt g^{\bar{i}i} \text{Tr} H^{-1} \partial_i H [H^{-1} \partial_i H, H^{-1} \bar{\partial}_{\bar{i}} H], \end{aligned} \quad (\text{A9})$$

where $d\mu(\pi, \bar{\pi})$ is the volume element on $\mathbb{CP}^1$. To quadratic order in φ 's only the first term is relevant. Rewriting H in terms of φ fields,

$$\begin{aligned} S_{\text{WZW}} = & -\frac{1}{2\pi} \int_{\mathbb{CP}^1} \frac{\bar{\pi} \sigma_a \pi}{\bar{\pi} \cdot \pi} \frac{\bar{\pi} \sigma_b \pi}{\bar{\pi} \cdot \pi} \int d^4 x \text{Tr}(\partial_\mu \varphi_a \partial_\mu \varphi_b) + \theta(\varphi^3) \\ = & -\frac{1}{6\pi} \int d^4 x \text{Tr}(\partial_\mu \varphi_a \partial_\mu \varphi_a) + \theta(\varphi^3). \end{aligned} \quad (\text{A10})$$

Integrating over the $\mathbb{CP}^1$ coordinates in the other terms in $\mathcal{S}$ leads to

$$\mathcal{S} = \frac{1}{2g^2} \int d^4 x [(\Box \varphi_a^\alpha)^2 + m^2 (\partial \varphi_a^\alpha)^2] + \theta(\varphi^3), \quad (\text{A11})$$

where g is the renormalized coupling constant and $m^2 = \frac{c_A g^2}{3} (C/\pi + \mu_{\text{ren}}^2)$ with c_A being the value of the quadratic Casimir operator in the adjoint representation of the gauge group. φ_a^α are the Lie algebra components of the fields $\varphi_a = t_a \varphi_a^\alpha$.

Upon covariantization, the mass term for the a_i fields combines with the WZW term producing a single mass term for the real fields φ . Up to quadratic order in φ , $\Gamma_{\log \epsilon}$ has the same form as S_{YM} ; thus, it simply contributes to the renormalization of g^2 . We can redefine our fields as $\varphi \rightarrow g\varphi$, resulting in

$$\mathcal{S} = \frac{1}{2} \int [(\Box \varphi_a^\alpha)^2 + m^2 (\partial \varphi_a^\alpha)^2]. \quad (\text{A12})$$

In this appendix, we have used α for the color index and a, b , and c for the spatial vector components, so as to avoid confusion with i and $\bar{i}$ used for the complex components and coordinates. To be in conformity with the text, we will now switch to using a as the color index and i, j , and k for the spatial vector indices. The action is thus

$$\mathcal{S} = \frac{1}{2} \int [(\Box \varphi_i^a)^2 + m^2 (\partial \varphi_i^a)^2]. \quad (\text{A13})$$

This is the action (33) used in the text.

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