

Long-distance reconstruction of QED corrections to the hadronic vacuum polarization for the muon $g - 2$

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(Received 15 September 2025; accepted 4 February 2026; published 4 March 2026)

The long-distance contribution of QED corrections to the hadronic vacuum polarization is particularly challenging to compute in lattice QCD + QED. Currently, it is one of the limiting factors toward matching the precision of the recent result by the Fermilab E989 experiment for the muon $g - 2$. In this work, we present a method for obtaining high-precision results for this contribution by reconstructing exclusive finite-volume state contributions. We find relations between the pion-photon contributions of individual diagrams and demonstrate the reconstruction method with lattice QCD + QED data at a single lattice spacing of $a^{-1} \approx 1.73$ GeV and $m_\pi \approx 275$ MeV.

DOI: [10.1103/77sf-5cg9](https://doi.org/10.1103/77sf-5cg9)

I. INTRODUCTION

In the recent white paper of the Muon $g - 2$ Theory Initiative [1], lattice QCD + QED calculations have for the first time been used to provide the standard model result for the hadronic vacuum polarization contribution to the muon $g - 2$. This has been made possible by a concerted effort of the lattice community over the past decade to subdivide the calculation into individual contributions with specific challenges that are then addressed in a targeted approach. The calculations are typically organized as an expansion around an isospin symmetric QCD calculation, and corrections are added to leading order in the fine-structure constant $\alpha = e^2/(4\pi)$ as well as the mass difference of up- and down-quarks. The currently available results [2–6] unfortunately do not compute the long-distance part of the QED corrections from first principles but instead either truncate at some distance or replace the long-distance part with a model such as the phenomenological model of Ref. [7].

For the isospin symmetric QCD contribution, the challenges of large statistical noise and finite-volume effects at long distances have recently been successfully addressed by a reconstruction of the long-distance part from a dedicated study of exclusive finite-volume states [6,8], completing an effort started many years ago that even included a study of four-pion states [9]. In the current paper, we extend this idea to QCD + QED calculations by

providing a method to reconstruct the long-distance part of the QED corrections to the hadronic vacuum polarization.

In Sec. II, we describe the method, and we provide a first numerical demonstration in Sec. III for a single lattice spacing of $a^{-1} = 1.7312(28)$ GeV, $m_\pi = 274.8(2.5)$ MeV, and $m_K = 530.1(3.1)$ MeV with $m_\pi L = 3.8$ on a $24^3 \times 48 \times 24$ Mobius domain wall ensemble with Iwasaki gauge action [8,10]. We use the QED_L [11] formulation, but our results directly translate to other QED regulators with a well-defined transfer matrix such as QED_r [12].

II. METHODOLOGY

A. QCD + QED correlation functions

Let us consider the spectral representation of the light vector-vector correlator $G(t)$ in Euclidean QCD + QED at second order in the electromagnetic charge e ,

$$\begin{aligned} G(t) &\equiv \frac{1}{3} \sum_i \langle V_i(t, \vec{p} = 0) V_i^\dagger(t = 0, \vec{p} = 0) \rangle \\ &= G^{(0)}(t) + e^2 G^{(2)}(t), \end{aligned} \quad (1)$$

with sum over spatial i , and

$$\begin{aligned} V_j(t, \vec{p}) &= \frac{1}{\sqrt{L^3}} \sum_{\vec{x}} e^{i\vec{x} \cdot \vec{p}} \left(i \frac{2}{3} \bar{u}(t, \vec{x}) \gamma_j u(t, \vec{x}) \right. \\ &\quad \left. - i \frac{1}{3} \bar{d}(t, \vec{x}) \gamma_j d(t, \vec{x}) \right), \end{aligned} \quad (2)$$

where u and d are the up- and down-quark fields and L^3 is the spatial volume. Performing the Wick contractions gives

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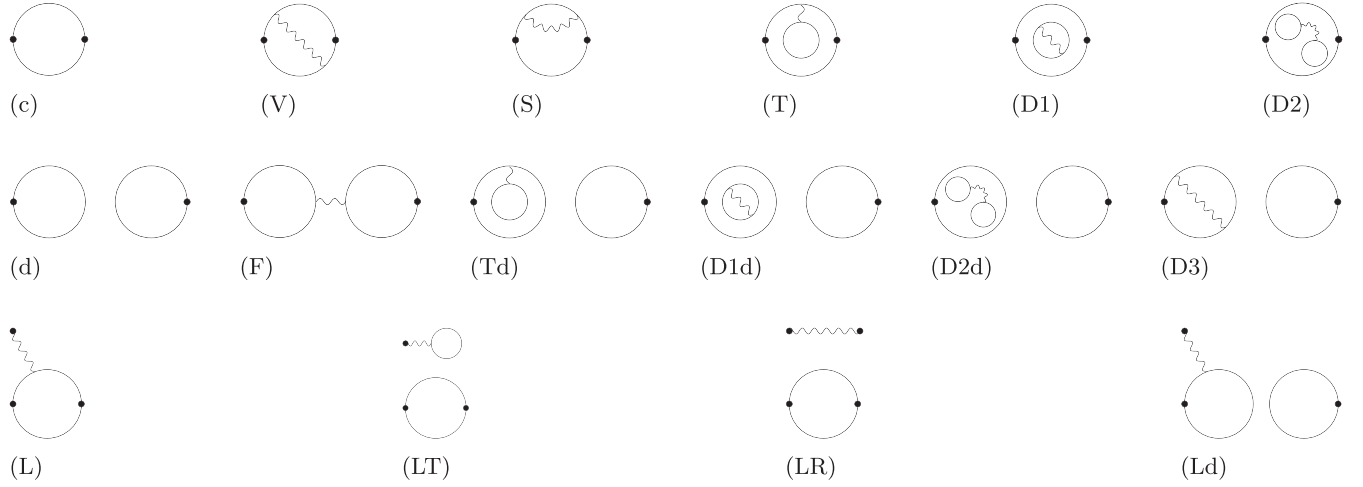


FIG. 1. QCD + QED diagrams considered in this work. Diagrams related by symmetry are identified by their common topology. The black dots represent the insertion of an external operator. The diagrams in the first two rows are generated by two external electromagnetic vector currents. The diagrams in the third row are generated by Wick contractions of the pion-photon operator with the vector operator (L, LT, Ld) and with itself (LR) according to Eqs. (28)–(32) and (42).

$$G = \frac{5}{9}(c) - \frac{1}{9}(d) + e^2 \left(-\frac{17}{81}((V) + 2(S)) + \frac{25}{81}(F) + \frac{14}{81}((T) + (D3)) - \frac{10}{81}(Td) \right. \\ \left. + \frac{25}{162}(D1) - \frac{5}{162}((D1d) + (D2)) + \frac{1}{162}(D2d) \right) \quad (3)$$

with diagrams defined in Fig. 1. Symmetry factors, signs related to fermion loops, and the $-ie$ charge factors are kept explicit throughout this paper. Suppressing the coordinates, diagram (c) therefore stands for $\text{Tr}[D^{-1}\gamma_i D^{-1}\gamma_i]$ for a fixed value of spatial i . Since the vector currents transform in the three-dimensional T_1^u representation of the octahedral symmetry group, each fixed value of spatial i gives the same result. We can therefore equivalently consider the diagrams to represent the average over the spatial i .

For sufficiently long temporal extents, the correlators can be written in terms of sums over all eigenstates n of the finite-volume Hamiltonian

$$G(t) = \sum_n c_n e^{-E_n t} = \sum_n c_n^{(0)} e^{-E_n^{(0)} t} + e^2 \sum_n (c_n^{(2)} - t E_n^{(2)} c_n^{(0)}) e^{-E_n^{(0)} t}, \quad (4)$$

with $c_n = c_n^{(0)} + e^2 c_n^{(2)} = |\langle 0 | V_i | n \rangle|^2$ and $E_n = E_n^{(0)} + e^2 E_n^{(2)}$. Linear terms in e are absent due to the charge-conjugation symmetry of the theory. The sum over n is over all states of the QCD + QED system, i.e., it includes $\pi^+ \pi^-$ states, a ρ state, as well as states with a photon such as $\pi^0 \gamma$. States including a photon do not contribute in leading order and, therefore, must have $c_n^{(0)} = 0$ in this expression. The contribution of the $\pi^0 \gamma$ state in the second-order diagrams in e therefore can be written simply as $e^2 c_{\pi\gamma}^{(2)} e^{-E_{\pi\gamma}^{(0)} t}$. States

that are already present in pure QCD such as the ρ or $\pi\pi$ states receive a correction to both their amplitude and their energy.

The external vector currents need to be renormalized by multiplying appropriate factors of the vector current renormalization factor $Z_V = Z_V^{\text{QCD}} Z_V^{\text{QED}}$. If conserved vector currents are used in the lattice calculation, we have $Z_V^{\text{QCD}} = 1$. It is useful to write $Z_V^{\text{QED}} = 1 + e^2 Z_V^{\text{QED},(2)}$ such that at second order in e the renormalized correlator is

$$G^{\text{ren}}(t) = Z_V^2 G(t) = (Z_V^{\text{QCD}})^2 \sum_n c_n^{(0)} e^{-E_n^{(0)} t} \\ + e^2 (Z_V^{\text{QCD}})^2 \sum_n \left((c_n^{(2)} + 2Z_V^{\text{QED},(2)} c_n^{(0)}) - t E_n^{(2)} c_n^{(0)} \right) e^{-E_n^{(0)} t}. \quad (5)$$

Two additional powers of Z_V^{QCD} for the e^2 term need to be included if local currents are used to couple to the photons in the action. We adopt this choice in this work and, therefore, additional photon tadpole diagrams are absent in Fig. 1. The amplitudes $c_n^{(2)}$ for states that are already present in pure QCD are therefore modified by this renormalization procedure, however, the energy corrections $E_n^{(2)}$ and the new state amplitudes such as $c_{\pi\gamma}^{(2)}$ are well defined even without the inclusion of Z_V^{QED} . Note that additional diagrams to account for shifts of the quark

masses need to be included as well to complete the renormalization of the theory, but this is not needed for the long-distance reconstruction considered in this work.

In order to reconstruct the long-distance behavior of $G^{(2)}$, we need to determine $c_n^{(2)}$ and $E_n^{(0)}$ for the new states including a photon, as well as $c_n^{(0)}$, $c_n^{(2)}$, $E_n^{(0)}$, and $E_n^{(2)}$ for the states already present in pure QCD. This can be achieved by studying a correlation matrix

$$C_{ij} = \langle O_i O_j^\dagger \rangle, \quad (6)$$

where one of the operators needs to be V_i and the other operators serve the purpose to project to the states n . In the pure QCD case such a correlation matrix is then used to solve a generalized eigenvalue problem (GEVP)

$$C(t) = V \text{Diag}(e^{-E_1 t}, \dots, e^{-E_n t}) V^\dagger, \quad (7)$$

$$C(t) \vec{v}_n = e^{-E_n(t-t_0)} C(t_0) \vec{v}_n, \quad (8)$$

with $V_{\text{in}} = \langle 0 | O_i | n \rangle$. This is usually studied by considering

$$C(t) C(t_0)^{-1} = V \text{Diag}(e^{-E_1(t-t_0)}, \dots, e^{-E_n(t-t_0)}) V^{-1} \quad (9)$$

and solving for the eigenvalues and eigenvectors of this matrix for sufficiently large t and t_0 .

In the case of QCD + QED calculations, however,

$$V = V^{(0)} + e V^{(1)} + e^2 V^{(2)}, \quad (10)$$

and a perturbative expansion in e using a wise power-counting scheme for the operators O_i may be more efficient. To simplify the discussion, we consider only a two-operator system, where the first operator $O_1 = V_i$, which serves the purpose to illustrate the pure QCD sector, and $O_2 = O_{\pi\gamma}$, which should create the lowest-lying pion-photon state. Our method extends without loss of generality to the general case with additional multipion and pion-photon operators. In practice, our choice of heavier pion mass for the numerical demonstration in Sec. III makes this two-operator setup already very efficient for the long-distance reconstruction of $G^{(2)}$. We discuss the construction of the pion-photon operators in Sec. II B.

For sufficiently heavy pion mass the long-distance part of the correlator is dominated by two states $n = \rho$ and $n = \pi\gamma$, such that the correlation matrix becomes

$$C(t) = \begin{pmatrix} V_{1,\rho} & V_{1,\pi\gamma} \\ V_{2,\rho} & V_{2,\pi\gamma} \end{pmatrix} \begin{pmatrix} e^{-E_\rho t} & 0 \\ 0 & e^{-E_{\pi\gamma} t} \end{pmatrix} \begin{pmatrix} V_{1,\rho}^* & V_{2,\rho}^* \\ V_{1,\pi\gamma}^* & V_{2,\pi\gamma}^* \end{pmatrix} \quad (11)$$

$$= \begin{pmatrix} |V_{1,\rho}|^2 e^{-E_\rho t} + |V_{1,\pi\gamma}|^2 e^{-E_{\pi\gamma} t} & V_{1,\rho} V_{2,\rho}^* e^{-E_\rho t} + V_{1,\pi\gamma} V_{2,\pi\gamma}^* e^{-E_{\pi\gamma} t} \\ V_{2,\rho} V_{1,\rho}^* e^{-E_\rho t} + V_{2,\pi\gamma} V_{1,\pi\gamma}^* e^{-E_{\pi\gamma} t} & |V_{2,\rho}|^2 e^{-E_\rho t} + |V_{2,\pi\gamma}|^2 e^{-E_{\pi\gamma} t} \end{pmatrix}. \quad (12)$$

For physical pion mass, two-pion operators and corresponding states need to be added to complete the low-lying spectrum which dominates the long-distance part of C . For the numerical demonstration provided in Sec. III, the lowest two states are the ρ and the $\pi\gamma$ state with smallest relative momentum allowed by the finite volume. We note that, while the vector-vector correlator C_{11} gives rise to the familiar diagrams in the first two rows of Fig. 1, the

contraction of the operator $O_{\pi\gamma}$ with the vector operator, i.e., C_{12} and C_{21} , and with itself, i.e., C_{22} , generates the diagrams in the third row of Fig. 1, which have not been studied in the context of isospin breaking corrections to the hadronic vacuum polarization before. It is convenient to adopt a power-counting in which $O_{\pi\gamma}$ is counted as order e , e.g., by multiplying the photon operator with an explicit power of e . We then find

$$V = \begin{pmatrix} V_{1,\rho}^{(0)} + e^2 V_{1,\rho}^{(2)} & e V_{1,\pi\gamma}^{(1)} \\ e^2 V_{2,\rho}^{(2)} & e V_{2,\pi\gamma}^{(1)} \end{pmatrix} \quad (13)$$

and, therefore,

$$C(t) = \begin{pmatrix} |V_{1,\rho}^{(0)}|^2 e^{-E_\rho^{(0)} t} & 0 \\ 0 & 0 \end{pmatrix} + e^2 \begin{pmatrix} (2\text{Re}(V_{1,\rho}^{(0)} V_{1,\rho}^{(2)*}) - t |V_{1,\rho}^{(0)}|^2 E_\rho^{(2)}) e^{-E_\rho^{(0)} t} + |V_{1,\pi\gamma}^{(1)}|^2 e^{-E_{\pi\gamma}^{(0)} t} & V_{1,\rho}^{(0)} V_{2,\rho}^{(2)*} e^{-E_\rho^{(0)} t} + V_{1,\pi\gamma}^{(1)} V_{2,\pi\gamma}^{(1)*} e^{-E_{\pi\gamma}^{(0)} t} \\ V_{2,\rho}^{(2)} V_{1,\rho}^{(0)*} e^{-E_\rho^{(0)} t} + V_{2,\pi\gamma}^{(1)} V_{1,\pi\gamma}^{(1)*} e^{-E_{\pi\gamma}^{(0)} t} & |V_{2,\pi\gamma}^{(1)}|^2 e^{-E_{\pi\gamma}^{(0)} t} \end{pmatrix}. \quad (14)$$

In the present work, we first determine $E_{\pi\gamma}^{(0)}$ and $|V_{2,\pi\gamma}^{(1)}|$ from a fit to C_{22} before determining $|V_{1,\pi\gamma}^{(1)}|$ from a fit to C_{21} . With this knowledge, we subtract the $\pi\gamma$ contribution from C_{11} and fit the subtracted correlator to obtain $\text{Re}(V_{1,\rho}^{(0)}V_{1,\rho}^{2,*})$ and $E_\rho^{(2)}$. Combined with $V_{1,\rho}^{(0)}$ and $E_\rho^{(0)}$, which can be obtained from a leading-order GEVP study [8], we can then reconstruct both the new $\pi\gamma$ state contribution as well as the QED corrections to the ρ state.

Our method also allows for a separation of the pion-photon contribution or more generally of the new states including a photon such that they can be computed independently from the remainder of the inclusive QED corrections. This also makes it possible to subtract the QED_L pion-photon contribution and to add it back from an infinite-volume QED calculation.

B. Pion-photon operators

In this section, we discuss the construction of the pion-photon operator for general relative momentum between the pion and photon for the total system at rest. The construction of a real photon interpolation operator is most straightforward in the Coulomb gauge in which unphysical longitudinal contributions do not propagate. For ease of combination with the usual QCD + QED calculation of the hadronic vacuum polarization correlators, we adopt the Feynman gauge in this calculation and benefit from gauge invariance removing the propagation of additional longitudinal states in $G^{(2)}$. We need to prepare a transverse-polarized photon with momentum $\vec{p}$ combined with a pion state with momentum $-\vec{p}$ and combine the operators such as to create a three-dimensional operator $O_{\pi\gamma,i}$ that transforms in the T_1^u irreducible representation of the octahedral group.

We first define a charged pion operator

$$O_\pi(t, \vec{p}) = \frac{i}{\sqrt{L^3}} \sum_{\vec{x}} e^{i\vec{x}\vec{p}} \bar{u}(x) \gamma_5 d(x), \quad (15)$$

and defer the discussion of the general case to Sec. II C. The photon operator is given by

$$A_i(t, \vec{p}) = \frac{1}{\sqrt{L^3}} \sum_{\vec{x}} e^{i\vec{x}\vec{p}} \tilde{A}_i(x) \quad (16)$$

with photon field $\tilde{A}_i(x)$ of the QED action. We project to transversal photons by considering

$$A_j^T(t, \vec{p}) = (\delta_{jm} - \hat{p}_j \hat{p}_m) A_m(t, \vec{p}), \quad \hat{p} = \frac{\vec{p}}{|\vec{p}|}. \quad (17)$$

Let $H(\vec{p})$ be the orbit under the chiral octahedral group with 24 elements of a representative momentum vector $\vec{p}$ (e.g., choose the representative with the first nonzero

component positive). Then there is a unique combination of the operators defined above that transforms in T_1^u of the full octahedral group,

$$O_{\pi\gamma, \vec{p}, i}(t) = \frac{e}{2} \sum_{\vec{q} \in H(\vec{p})} \left([\hat{q} \times \vec{A}^T(t, \vec{q})]_i O_\pi(t, -\vec{q}) - [\hat{q} \times \vec{A}^T(t, -\vec{q})]_i O_\pi(t, +\vec{q}) \right) \quad (18)$$

$$= \frac{e}{2} \sum_{\vec{q} \in H(\vec{p})} \left([\hat{q} \times \vec{A}(t, \vec{q})]_i O_\pi(t, -\vec{q}) - [\hat{q} \times \vec{A}(t, -\vec{q})]_i O_\pi(t, +\vec{q}) \right). \quad (19)$$

We note that the projection to the transversal photon is redundant for this operator. For the smallest allowed momentum we simply find

$$O_{\pi\gamma, 100, i}(t) = \frac{e}{2} \sum_{j,k} \varepsilon_{ijk} (O_\pi(t, \vec{p} = |\vec{p}| \hat{e}_k) \times A_j(t, \vec{p} = -|\vec{p}| \hat{e}_k) - O_\pi(t, \vec{p} = -|\vec{p}| \hat{e}_k) \times A_j(t, \vec{p} = |\vec{p}| \hat{e}_k)) \quad (20)$$

with $|\vec{p}| = 2\pi/L$ and unit vector $\hat{e}_k$ in the k direction. We use this operator for $O_{\pi\gamma}$ in Sec. III.

C. Isospin decomposition and relations between diagrams

We are interested in the long-distance behavior of the correlator $\langle V_i V_j^\dagger \rangle$ for vector currents V_i . For physical quark charges $q_u = 2/3$ and $q_d = -1/3$, performing the Wick contraction generates the diagrams in the two upper rows of Fig. 1. As outlined in Sec. II A, to obtain the coefficients $c_n^{(2)}$ and eigenenergies $E_n^{(0)}$, we consider an extended operator basis with $O_i \in \{V_i, O_{\pi\gamma}\}$, where $O_{\pi\gamma}$ is defined in Eq. (18). Wick contractions of $O_{\pi\gamma}$ with V_i and with itself generate the diagrams in the third row of Fig. 1.

In order to disentangle the contribution of individual diagrams it is, however, instructive to additionally consider a general isospin decomposition and charge configurations. This will allow us to isolate the pion-photon contributions of subsets of diagrams and identify relations between the pion-photon contributions of individual diagrams. We define

$$O_{\pi(I_3=-1)} = i\bar{u}\gamma_5 d, \quad O_{\pi(I_3=0)} = \frac{i}{\sqrt{2}} (\bar{u}\gamma_5 u - \bar{d}\gamma_5 d), \quad O_{\pi(I_3=1)} = i\bar{d}\gamma_5 u, \quad (21)$$

$$V_{I=1, I_3=-1}^j = i\bar{u}\gamma_j d, \quad V_{I=1, I_3=0}^j = \frac{i}{\sqrt{2}} (\bar{u}\gamma_j u - \bar{d}\gamma_j d), \quad V_{I=1, I_3=1}^j = i\bar{d}\gamma_j u, \quad (22)$$

$$V_{I=0}^j = \frac{i}{\sqrt{2}} (\bar{u} \gamma_j u + \bar{d} \gamma_j d). \quad (23)$$

We first consider isospin $I = 1$ and different charge assignments for the quark QED charges. We begin our investigation by keeping the sum of charges constant,

$$\langle V_{I=1, I_3=-1} V_{I=1, I_3=-1}^\dagger \rangle_{q_u=1, q_d=1} = (c) + e^2((D1) - 2(D2) + 4(T) - 2(S) - (V)), \quad (24)$$

$$\langle V_{I=1, I_3=0} V_{I=1, I_3=0}^\dagger \rangle_{q_u=1, q_d=1} = (c) + e^2((D1) - 2(D2) + 4(T) - 2(S) - (V)), \quad (25)$$

$$\langle V_{I=1, I_3=-1} V_{I=1, I_3=-1}^\dagger \rangle_{q_u=2, q_d=0} = (c) + e^2(2(D1) - 2(D2) + 4(T) - 4(S)), \quad (26)$$

$$\langle V_{I=1, I_3=0} V_{I=1, I_3=0}^\dagger \rangle_{q_u=2, q_d=0} = (c) + e^2(2(D1) - 2(D2) + 4(T) - 4(S) + 2(F) - 2(V)). \quad (27)$$

In this case, the correlators with the pion-photon operator obtain the same contribution from each diagram for the different cases of quark charges, i.e.,

$$\langle O_{\pi(I_3=-1), \gamma} V_{I=1, I_3=-1}^\dagger \rangle_{q_u=1, q_d=1} = e^2(-2i(LT) + 2i(L)), \quad (28)$$

$$\langle O_{\pi(I_3=0), \gamma} V_{I=1, I_3=0}^\dagger \rangle_{q_u=1, q_d=1} = e^2(-2i(LT) + 2i(L)), \quad (29)$$

$$\langle O_{\pi(I_3=-1), \gamma} V_{I=1, I_3=-1}^\dagger \rangle_{q_u=2, q_d=0} = e^2(-2i(LT) + 2i(L)), \quad (30)$$

$$\langle O_{\pi(I_3=0), \gamma} V_{I=1, I_3=0}^\dagger \rangle_{q_u=2, q_d=0} = e^2(-2i(LT) + 2i(L)), \quad (31)$$

$$\langle O_{\pi(I_3=-1), \gamma} O_{\pi(I_3=-1), \gamma}^\dagger \rangle = e^2(LR). \quad (32)$$

From these findings, we can determine relations for the pion-photon contribution, which depends only on the overlap factor $|V_{1, \pi\gamma}^{(1)}|^2$ and the pion-photon energy $E_{\pi\gamma}^{(0)}$, determined by the pion-photon operator correlators, as described at the end of Sec. II A. Since the correlators in Eqs. (28)–(31) are unchanged by imposing these quark charges, this implies that the pion-photon contribution is exactly the same in Eqs. (24)–(27) and obeys the relations

$$(F) = (V), \quad 2(S) - (D1) = (V). \quad (33)$$

We emphasize that these relations are derived using first principles in QCD + QED. We note that similar and compatible relations for the sum of connected and the sum of disconnected contributions were already found by the Mainz collaboration using a phenomenological model [7]. Based on these relations, we could construct correlators in which the pion-photon contribution is absent, as has been done for the hadronic light-by-light contribution's pion pole [13]. In the present work, however, it is our intention to reconstruct this contribution instead by studying the enlarged operator basis.

If we let the charges sum to zero instead, the off-diagonal correlators vanish,

$$\langle O_{\pi(I_3=-1), \gamma} V_{I=1, I_3=-1}^\dagger \rangle_{q_u=1, q_d=-1} = 0, \quad (34)$$

$$\langle O_{\pi(I_3=0), \gamma} V_{I=1, I_3=0}^\dagger \rangle_{q_u=1, q_d=-1} = 0, \quad (35)$$

so that $V_{1, \pi\gamma}^{(1)} = 0$, and

$$\langle V_{I=1, I_3=-1} V_{I=1, I_3=-1}^\dagger \rangle_{q_u=1, q_d=-1} = (c) + e^2((D1) - 2(S) + (V)), \quad (36)$$

$$\langle V_{I=1, I_3=0} V_{I=1, I_3=0}^\dagger \rangle_{q_u=1, q_d=-1} = (c) + e^2((D1) - 2(S) + 2(F) - (V)). \quad (37)$$

The pion-photon contributions need to cancel in these linear combinations of diagrams, and using Eq. (33), we see that this is indeed the case.

Next, we consider the $I = 0$ case for which an isospin symmetric charge assignment yields

$$\begin{aligned} \langle V_{I=0} V_{I=0}^\dagger \rangle_{q_u=1, q_d=1} &= (c) - 2(d) + e^2(4(T) - 8(Td) \\ &\quad - 2(D1d) + 4(D3) + 4(D2d) \\ &\quad - 2(D2) + (D1) \\ &\quad + 2(F) - (V) - 2(S)), \end{aligned} \quad (38)$$

$$\langle O_{\pi(I_3=0), \gamma} V_{I=0}^\dagger \rangle_{q_u=1, q_d=1} = 0, \quad (39)$$

and the pion-photon contribution in the linear combination of diagrams in Eq. (38) needs to vanish.

Combined with Eq. (33) the pion-photon relations can be written as

$$\begin{aligned} (F) &= (V), \quad 2(S) - (D1) = (V), \\ 2(T) - 4(Td) - (D1d) + 2(D3) + 2(D2d) - (D2) &= 0. \end{aligned} \quad (40)$$

If we collect the pion-photon contribution from diagrams (V), (S), (D1), and (F) in the total hadronic vacuum polarization contribution given in Eq. (3), we find

$$-e^2 \frac{9}{162} ((V) + 2(S)). \quad (41)$$

We observe that the pion-photon contribution is strongly suppressed due to a cancellation between the (F), (D1), (S), and (V) diagrams. To obtain the total contribution of the pion-photon states created by the physical $V_i^\dagger$, we need

$$\langle O_{\pi(I_3=0), \gamma} V^\dagger \rangle = e^2 \left(-\frac{\sqrt{2}}{6} i((LT) + (Ld)) + \frac{\sqrt{2}}{3} i(L) \right). \quad (42)$$

In the following analysis, we will focus only on the dominant diagrams (V), (S), and (F).

III. RESULTS

In this section, we provide a numerical demonstration of our approach. We use data at a single lattice spacing of $a^{-1} = 1.7312(28)$ GeV, $m_\pi = 274.8(2.5)$ MeV, and $m_K = 530.1(3.1)$ MeV with $m_\pi L = 3.8$ on a $24^3 \times 48 \times 24$ Mobius domain wall ensemble with Iwasaki gauge action. This corresponds to “ensemble 4” of Refs. [8,10]. For this pion mass, the lowest two states are the ρ and the $\pi\gamma$ states with lowest relative momentum.

We proceed as outlined in Sec. II, by successively calculating the effective masses $E_n^{(i)}$ and overlap factors $|V_{j,n}^{(i)}|$ of the correlation matrix Eq. (14). First, we calculate the diagram (LR) that corresponds to the correlation function C_{22} of two pion-photon operators in Eq. (32) using stochastic propagators for the pion and the exact QED_L photon propagator. Since at leading order the interaction between the pion and photon is absent, this correlator can be written as

$$C_{22}(t) = C_{22}^\pi(t) C_{22}^\gamma(t) \quad (43)$$

with

$$\begin{aligned} C_{22}^\pi(t) &= c_\pi (e^{-E_\pi t} + e^{-E_\pi(T-t)}), \\ C_{22}^\gamma(t) &= c_\gamma (e^{-E_\gamma t} + e^{-E_\gamma(T-t)}) \end{aligned} \quad (44)$$

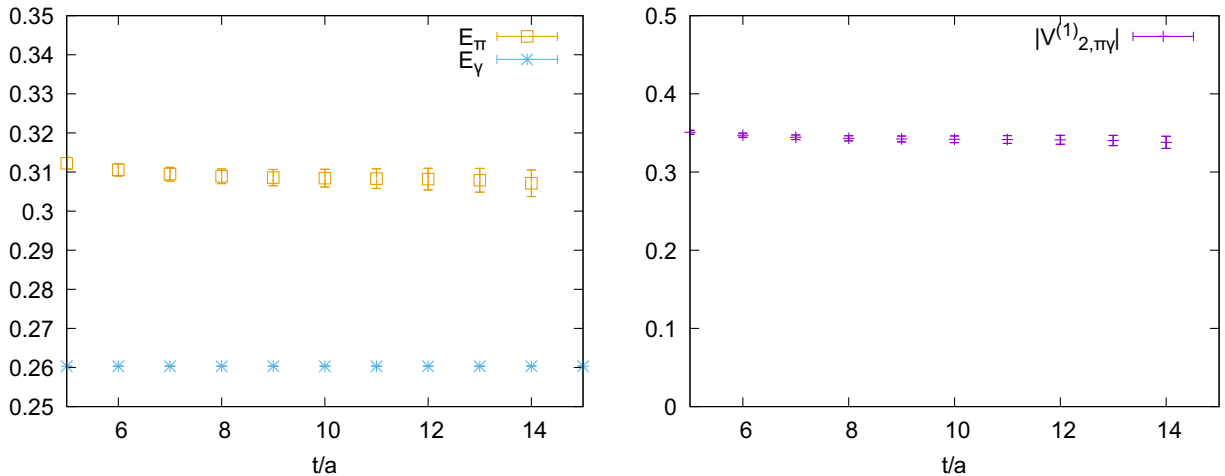


FIG. 2. Analysis of diagram (LR). The left plot shows the effective masses separately obtained for the photon and pion factors of the diagram. The right plot shows the fitted normalization of the pion-photon state with fit range $[t, 24]$.

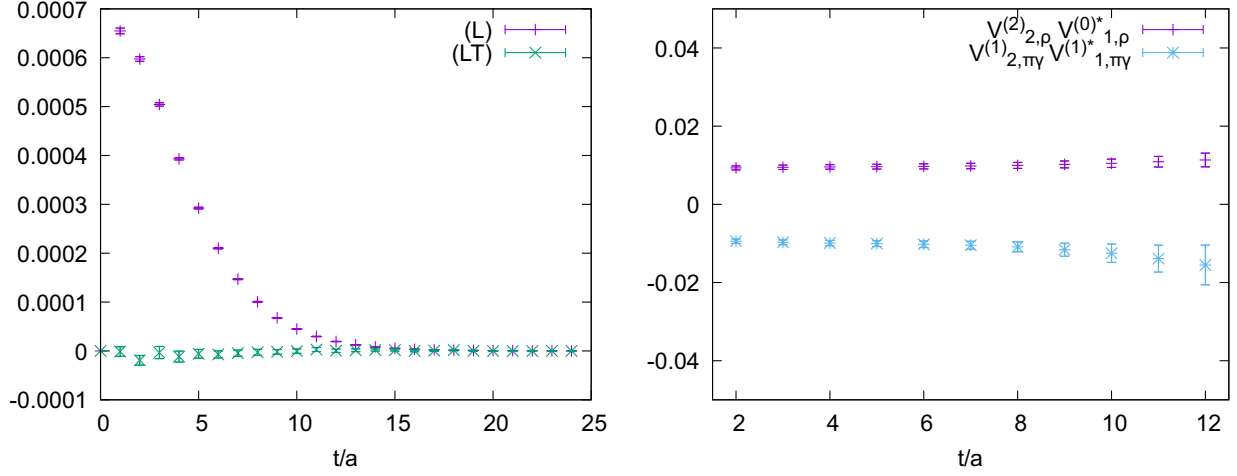


FIG. 3. Analysis of diagrams (L) and (LT). The diagram (LT) is numerically negligible compared to (L), as shown in the left plot. On the right side, we show the fitted overlap factors to the ρ and $\pi\gamma$ states with fit range $[t, 24]$.

for sufficiently large times with temporal extent T . We therefore find

$$|V_{2,\pi\gamma}^{(1)}|^2 = c_\gamma c_\pi \quad (45)$$

and

$$E_{\pi\gamma}^{(0)} = E_\pi + E_\gamma. \quad (46)$$

This factored approach simplifies the analysis of around-the-world contributions compared to directly fitting to $C_{22}(t)$ due to the absence of the combined forward-backward

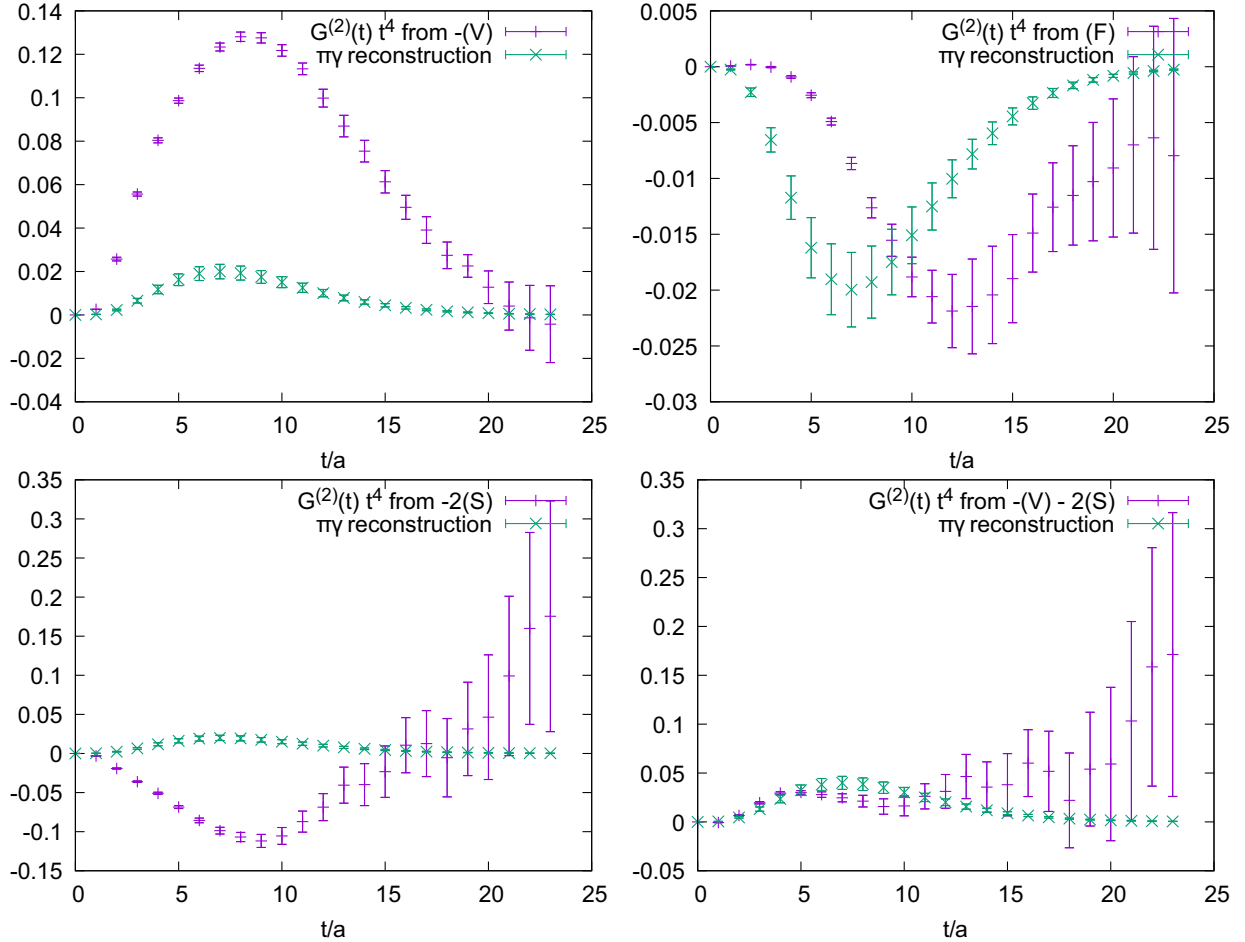


FIG. 4. Study of t^4 integrand and the $\pi\gamma$ reconstruction for diagrams (V), (S), and (F).

propagating cross terms between C_{22}^π and C_{22}^γ . These cross terms are otherwise difficult to separate due to the similar energies of the pion and photon in our volume. We show the fit results in Fig. 2 by varying the fit range $[t, 24]$ to the correlators as a function of t . In the following, we use the

values for $E_{\pi\gamma}^{(0)}$ and $|V_{2,\pi\gamma}^{(1)}|$ by fitting the correlator in the range $[10, 24]$.

With these values fixed, our next goal is to obtain $|V_{1,\pi\gamma}^{(1)}|$. To this end, we calculate the correlator for diagrams (LT) and (L) corresponding to, e.g., Eqs. (28)–(32). We

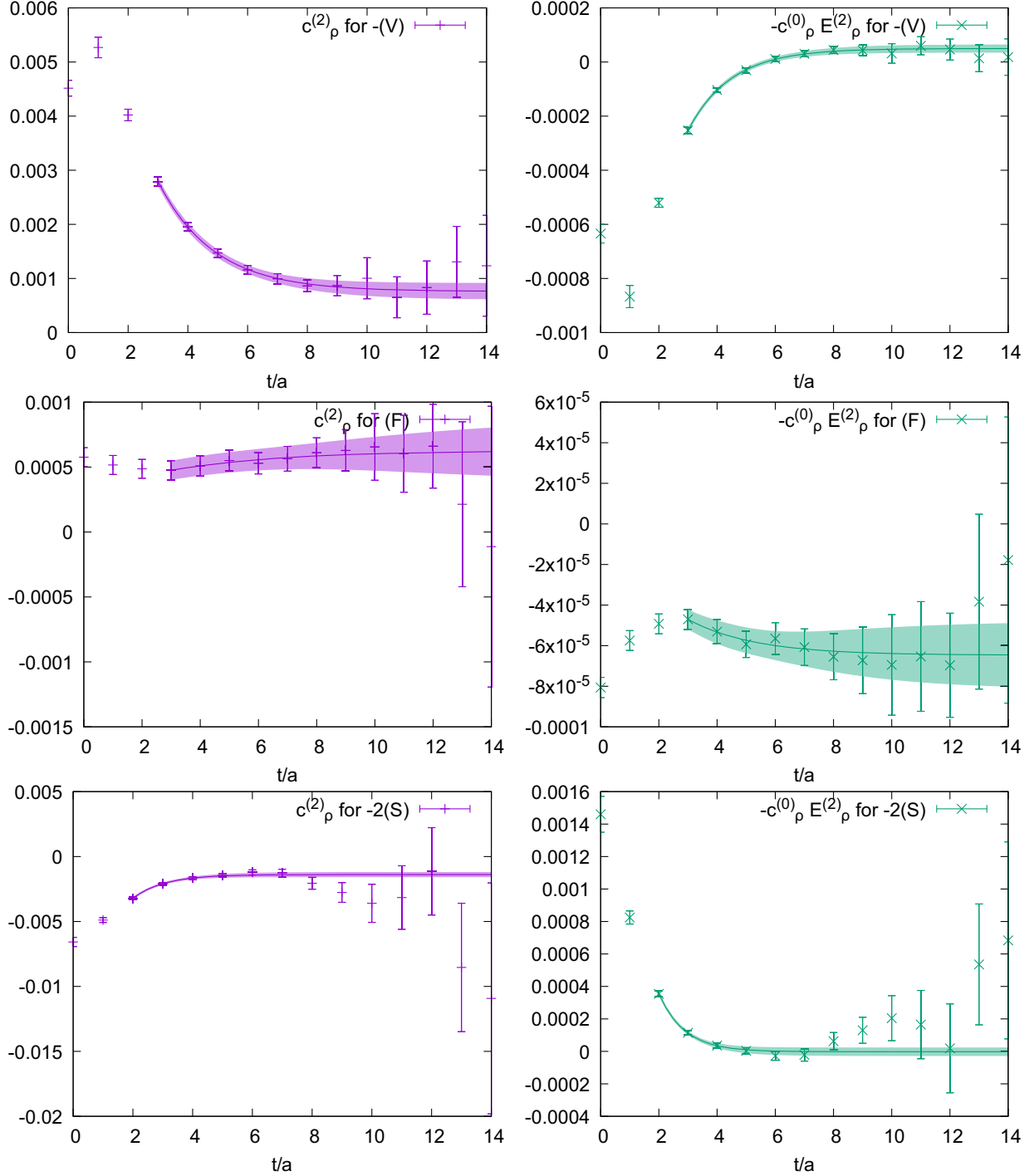


FIG. 5. Fits for the QED corrections to ρ parameters. We fit the correlator C_{11} after subtraction of the $\pi\gamma$ contribution to Eq. (14) with fit range $[t, 24]$, where t is given on the horizontal axis of the plots. We show the chosen fit ranges by the filled bands in the plots and observe good agreement with the fit form. Changing the fit range to a slightly larger value yields consistent results with slightly larger statistical uncertainties.

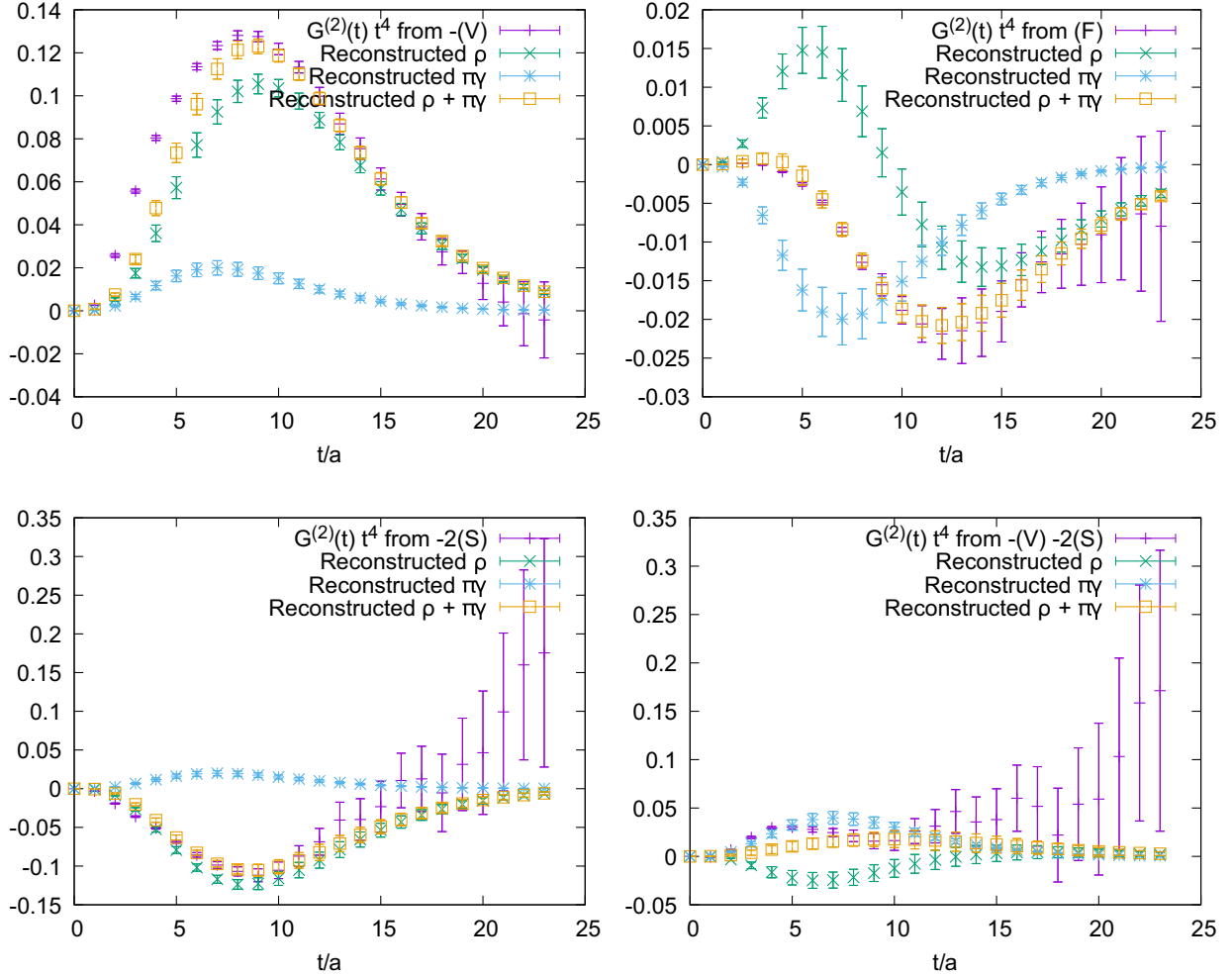


FIG. 6. Full reconstruction for diagrams (V), (S), and (F).

determine $|V_{1,\pi\gamma}^{(1)}|$ from a fit to C_{21} , as shown in the right panel of Fig. 3. The left panel of this figure shows the contribution of the individual diagrams to C_{21} . We note that diagram (LT) is numerically negligible compared to (L). Note that diagram (Ld) does not contribute to Eqs. (28)–(32). For this fit, we use the fit range [6, 24].

With the knowledge of $|V_{1,\pi\gamma}^{(1)}|$ and $E_{\pi\gamma}^{(0)}$, it is straightforward to construct the pion-photon contribution to the vector-vector correlator, i.e., the term $|V_{1,\pi\gamma}^{(1)}|^2 e^{-E_{\pi\gamma}^{(0)} t}$ in C_{11} . Using the relations of Eq. (40), we reconstruct the pion-photon contribution to individual diagrams (V), (S), and (F) assuming dominance over diagram (D1) for now. Neglecting (D1) is furthermore justified by the fact that in the phenomenological model used by the Mainz group [7] the pion-photon contribution to this diagram is exactly zero. In Fig. 4, we compare this reconstruction of the pion-photon contribution to the correlators of diagrams (V), (S), and (F), where the latter are computed from stochastic all-to-all quark propagators. Here, the correlator is plotted with a simple t^4 kernel, which behaves similarly to the time-momentum representation kernel [14] for the muon $g-2$,

in the integration region that we investigated in this work. We observe that the pion-photon contribution alone does not capture the integrand for diagrams (V), (S), and (F) due to the missing ρ contribution in C_{11} . We note, however, that the combination $(V) + 2(S)$, which also appears for the hadronic vacuum polarization Eq. (3), is well approximated by the pion-photon state.

Since we do not perform a calculation of the QED corrections to the vector current renormalization Z_V , as outlined in Sec. II in this work, this comparison is incomplete. As shown in Eq. (5), the renormalization affects the contribution of the states that are already present in pure QCD, i.e., the ρ state in our case. The $\pi\gamma$ contribution is not affected by the lack of complete renormalization.

With the data obtained so far, we subtract the $\pi\gamma$ contribution from C_{11} and fit the subtracted correlator to obtain $c_\rho^{(2)} = 2\text{Re}(V_{1,\rho}^{(0)} V_{1,\rho}^{(2),*})$ and $c_\rho^{(0)} E_\rho^{(2)} = |V_{1,\rho}^{(0)}|^2 E_\rho^{(2)}$. In this fit, we also use the information for $V_{1,\rho}^{(0)}$ and $E_\rho^{(0)}$ from the leading-order GEVP study [8] for this ensemble. We show the values resulting from these fits in Fig. 5 as a

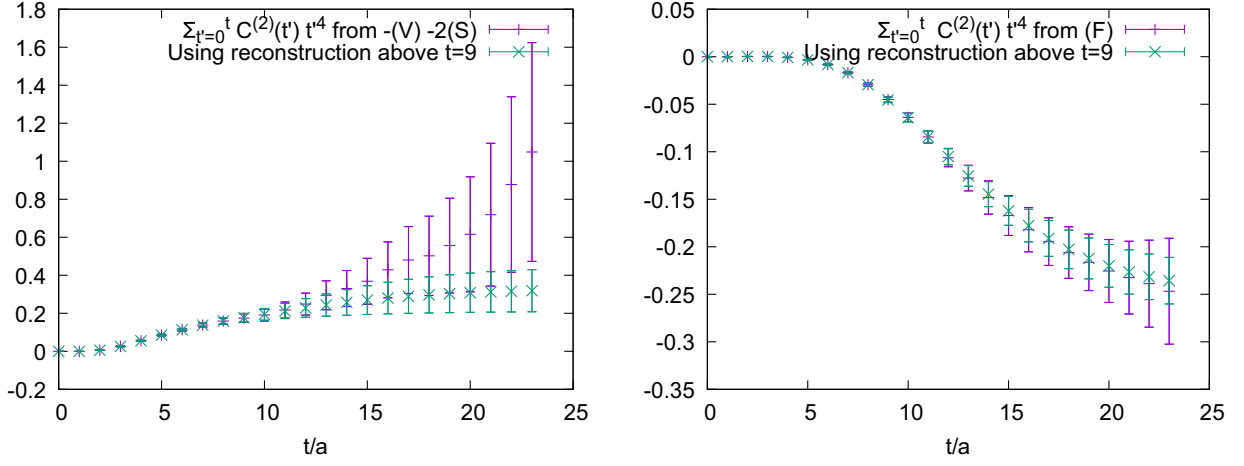


FIG. 7. Partial sum of the $G^{(2)}(t)t^4$ integrand that closely resembles the time-momentum kernel for the hadronic vacuum polarization contribution to the muon $g - 2$.

function of the lower end t of the fit range $[t, 24]$. We then perform a fit to a constant plus a single exponential to the determinations of Fig. 5, accounting amongst others for the higher-lying two-pion states. The fits perform well even for small time slices and allow us to reconstruct both the ρ and $\pi\gamma$ contributions to the individual diagrams. We show the results for this reconstruction in Fig. 6.

We observe that with the inclusion of the ρ contribution, the integrands for diagrams (V) and (S) are well approximated by the reconstructed correlator above time slice $t = 9$. For diagram (F), the agreement is good even in the region of small t/a , for which we see a strong cancellation between the pion-photon and ρ contribution.

With these findings, we conclude that the vector-vector correlator in the long-distance region for diagrams (V), (S), and (F) can be well approximated by the two lowest-lying states, for the ensemble 4, studied in this work. This yields a significant suppression of the statistical error in the tail that one encounters typically by calculating the vector-vector correlator directly, as shown in Fig. 6.

In Fig. 7, we show the partial sum of the $G^{(2)}(t)t^4$ integrand which closely resembles the desired time-momentum integrand for the QED corrections to the hadronic vacuum polarization contribution to the muon $g - 2$ [14]. We observe a reduction of statistical noise by more than a factor of 5 for the noisy diagram (S). We note that for a reconstruction at physical pion mass, it is necessary to include two pion states, which lie significantly below the ρ state, and possibly also higher-momentum operators.

IV. CONCLUSION

In this paper, we have described a method for a high-precision first-principles calculation of the long-distance tail of the QED corrections to the hadronic vacuum polarization contribution for the muon $g - 2$. We found relations between the pion-photon contributions of individual diagrams which allows for the dominant diagrams

(V), (S), and (F) to be analyzed individually after accounting for the contribution of (D1). We also observed a substantial suppression factor of the pion-photon contribution to the hadronic vacuum polarization because of a cancelation of these individual contributions. Our numerical results at approximately twice the physical pion mass demonstrate the potential for noise reduction of our approach. For the noisy diagram (S), the reduction of uncertainty was observed to be greater than a factor of 5. In future work, we will extend our studies to different photon regulators, including a photon in infinite volume. We will also include our existing dataset of multipion operators and physical pion mass ensembles. Finally, we will include the subleading diagrams that we have neglected so far.

ACKNOWLEDGMENTS

We thank our colleagues of the RBC and UKQCD collaborations for many valuable discussions and joint efforts over the years. The authors gratefully acknowledge the Gauss Centre for Supercomputing e.V. [15] for funding this project by providing computing time on the GCS Supercomputer JUWELS at Jülich Supercomputing Centre (JSC). We gratefully acknowledge disk and tape storage provided by USQCD and by the University of Regensburg with support from the DFG. The lattice data analyzed in this project were generated using GPT [16] and Grid [17–19].

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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