

Entanglement measures for causally connected subregions and holography

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In this paper, we investigate entanglement for causally connected subregions A and B in quantum field theory and holography. Recent developments have established that a transition operator T_{AB} can be well-defined for such subregions, which is generally non-Hermitian. By employing the Schwinger-Keldysh formalism and the real-time replica method, we show how to construct T_{AB} and compute associated entanglement measures. In certain configurations, this leads to a notion of timelike entanglement entropy, for which we provide explicit quantum field theory computations and propose a holographic dual via analytic continuation from the Euclidean setup. Both analytical and numerical results are compared and found consistent. If entanglement between causally connected subregions is to be meaningful, it should admit well-defined extensions of other entanglement measures beyond the spacelike setting. With this motivation we study the analytic continuation of the twist correlators for reflected-entropy and propose a holographic geometric analog in complexified bulk geometry, providing evidence that entanglement measures can be extended to timelike regions.

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I. INTRODUCTION

Quantum entanglement plays a central role in understanding the holographic nature of gravity. Within the AdS/CFT correspondence, the Ryu-Takayanagi (RT) formula [1] and its covariant generalization, the Hubeny-Rangamani-Takayanagi (HRT) formula [2], establishes a profound connection between the entanglement entropy (EE) in the boundary theory and the extremal surfaces in the bulk. These foundational results provide a powerful framework for probing the microscopic degrees of freedom of quantum gravity via dual field theories.

Extensive studies [3–8] have significantly advanced our understanding of the holographic structure of gravitational systems. When quantum corrections are taken into account, the framework is generalized to the quantum extremal surface (QES) prescription [9], which plays a crucial role in addressing the black hole information paradox [10–14]. For a more comprehensive overview of recent progress in this field, one may refer to the review essay [15].

On the other hand, EE is only one among various useful quantities that characterize the entanglement between

subregions. In the case of mixed states, more refined entanglement measures are required to properly quantify quantum correlations beyond EE. From the holographic perspective, RT surfaces typically extend to the anti-de Sitter (AdS) boundary and hence yield divergent quantities. As such, they should be regarded as specific bulk geometric constructs that may correspond to well-defined boundary quantities under duality. Motivated by this, various bulk geometric quantities have been proposed as duals to other entanglement measures in the boundary theory.

One prominent example is the entanglement wedge cross section (EWCS), which is defined within the entanglement wedge bounded by the RT surfaces. EWCS is conjectured to be dual to the entanglement of purification [16,17] or the reflected entropy [18] in the boundary theory. Other entanglement measures that are widely studied in quantum field theories and may potentially admit holographic duals include entanglement negativity [19,20], Rényi entropy [21], and the entanglement spectrum [22,23], among others.

EE and other entanglement measures are typically defined on a given Cauchy surface for a subsystem A . The reduced density matrix is obtained by tracing out the complement $\bar{A}$: $\rho_A := \text{Tr}_{\bar{A}} \rho$, where ρ is the state of the system. In the framework of quantum field theories (QFTs), the Rényi entropy $S_n(\rho_A) = \frac{\log \text{Tr} \rho_A^n}{1-n}$ can be computed using the replica trick, and the EE is obtained as the von Neumann entropy of ρ_A at the limit $S_A(\rho_A) = \lim_{n \rightarrow 1} S_n(\rho_A)$. From the perspective of algebraic QFTs, the reduced density matrix ρ_A encodes information only within the causal

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domain of dependence of the region A . Therefore, EE and other entanglement measures primarily capture quantum correlations between spacelike-separated regions. We will refer to this EE as spacelike EE for comparison in what follows.

Time does not appear to play a central role in the conventional understanding of EE and the significance of the RT formula. EE becomes relevant for probing the thermalization dynamics of a system only when time evolution is explicitly considered, such as in quantum quenches [24]. However, in QFTs, time and space are intrinsically intertwined. This motivates the investigation of how temporal aspects can be incorporated into the concept of entanglement.

In [25], the authors introduce the timelike EE in Lorentzian spacetime, which may serve as an example of pseudoentropy [26]. In this setup, the entanglement is extended to subsystems of temporal duration. In $(1+1)$ -dimensional QFTs, this can be viewed as interchanging the roles of time and space coordinates. From the perspective of evaluating timelike EE, the key object is the correlation function of twist operators at timelike separations. Moreover, it is expected that timelike EE can also be computed holographically via the RT formula [25,27–32]. Exploring the holographic dual of timelike EE reveals novel features in the behavior of RT surfaces. Interestingly, timelike EE respects causality constraints and satisfies an exact sum rule that relates to spacelike EE [30,33].

Recently, the paper [34] introduced a new framework for understanding entanglement in time, which can be applied to arbitrary quantum systems. The central idea is to interpret the entanglement in time as quantum correlations between two subregions, A and B , which are separated by time. This perspective allows one to define a transition operator T_{AB} , which is generally non-Hermitian. The operator T_{AB} can be viewed as a natural generalization of the reduced density matrix ρ_{AB} when A and B are separated in space.

The construction of T_{AB} is applicable both to quantum systems with finite-dimensional Hilbert spaces and to quantum field theories. In certain cases, it reduces to the notion of timelike EE introduced in [25]. We will briefly review this concept in Sec. II. This paper continues the investigation of this framework from the perspective of QFTs and holography.

In our view, there are several motivations to consider the concept of entanglement in time:

- (i) Entanglement in time is a fundamental concept that can be applied to arbitrary quantum systems. It generalizes the notion of the density matrix to a more general object often referred to as the “space-time density matrix.” In this framework, the transition operator plays a role analogous to that of the density matrix and can be used to define entropy-related quantities, such as the von Neumann entropy. For convenience, we will refer to the entropy associated with the transition operator as the EE

for the transition operator, or simply as timelike EE, in contrast to the conventional spacelike EE. In a certain sense, the EE for the transition operator can be regarded as an example of the pseudoentropy introduced in [26];

- (ii) The transition operator T_{AB} can be constructed from the specification of subregions A and B , together with the quantum state of the system. As a result, entropy-related quantities derived from T_{AB} are, in principle, computable. Specifically, in QFTs, the conventional spacelike EE can be computed via state preparation using the Euclidean path integral and the replica method [35]. For the timelike case, the transition operator T_{AB} can be prepared using the Schwinger–Keldysh formalism [36–38], and the corresponding entropy-related quantities can be evaluated using the real-time replica method proposed in [39–41]. In this work, we focus on the evaluation of such quantities in QFTs. Furthermore, in certain cases, the computation can be reformulated in terms of correlation functions involving twist operators with timelike separation. This implies that timelike EE is intimately connected to Lorentzian correlation functions, which have not been extensively explored in previous studies;
- (iii) The transition operator T_{AB} encodes dynamical information and reflects the causal structure of the theory. As mentioned previously, the evaluation of entropy-related quantities associated with T_{AB} requires knowledge of Lorentzian correlators. These correlators exhibit a richer structure than their Euclidean counterparts, as they depend sensitively on the spacetime locations and temporal ordering of the operators. In [30,33], a relation between timelike and spacelike EE was derived for certain states by exploiting causality constraints. It is expected that this relation may hold more generally [42], suggesting a unified framework for timelike and spacelike entanglement. Notably, when A and B are taken to be Cauchy surfaces at different times, T_{AB} captures the full dynamical evolution of the system. In particular, if A and B correspond to the in-state ($t \rightarrow -\infty$) and out-state ($t \rightarrow +\infty$) Cauchy surfaces, T_{AB} is expected to be closely related to the scattering S -matrix;
- (iv) It is particularly intriguing to explore the holographic dual of timelike EE and related entanglement measures. Currently, there is no clear prescription for determining the corresponding bulk RT surface. Various proposals have been put forward, including the idea that the RT surface may extend into a complexified bulk geometry [29]. In this paper, we use a procedure to compute the holographic timelike EE by analytically continuing from its Euclidean counterpart [30]. Both numerical and analytical

methods will be employed in this analysis. Furthermore, we extend several well-established concepts from the spacelike case to the timelike setting, with a particular focus on the entanglement wedge and its cross section. As previously mentioned, these concepts play a crucial role in understanding subregion/subregion duality and the holographic duals of reflected entropy or entanglement of purification. We will present supporting evidence that analogous dualities also exist in the timelike case. In summary, timelike EE provides a promising avenue for extending real-time holography [43,44] and deepening our understanding of the dynamical aspects of holographic duality;

- (v) Lastly, the timelike EE can be measured in the laboratory for the spin chain system or quantum circuits. Some discussions can be found in [34,45].

In this paper, we focus primarily on the EE associated with the transition operator T_{AB} in QFTs. Our starting point is the preparation of T_{AB} using the Schwinger-Keldysh formalism. The real-time replica method, originally introduced in [39] to provide a derivation of the HRT formula, can also be used to evaluate spacelike EE. This framework proves particularly well suited to build and analyze the transition operator T_{AB} , allowing us to apply the replica method to compute its associated EE. We will examine several simple examples where the EE reduces to the computation of correlation functions of twist operators in Lorentzian spacetime.

We also investigate the holographic dual of timelike EE in several specific examples. By analytically continuing the Euclidean RT surface, we construct and evaluate its Lorentzian counterpart for timelike-separated regions. This method yields results that are in perfect agreement with the conformal field theory (CFT) calculations in AdS_3 . We further explore higher-dimensional strip geometries, where both numerical and analytical results are obtained, showing excellent consistency.

With the constructed complex extremal surfaces, we investigate the geometric structures in the bulk that arise from analytic continuation. Specifically, we construct a geometric surface analogous to the EWCS in the timelike domain. Our construction of this surface shares structural similarities with its spacelike counterpart, although notable differences emerge due to the complex nature of the saddle points. We present the precise geometric construction and evaluate its area in various examples. An important motivation for this investigation comes from the analytic behavior of boundary correlation functions. In particular, the formalism of reflected entropy for spacelike regions can be analytically continued into the Lorentzian regime, yielding well-defined functional results. This naturally motivates the search for a corresponding bulk geometric quantity that reproduces these values. The geometric analog of the cross section analyzed in this work serves this

purpose. We emphasize that the physical interpretation of the associated bulk region may differ significantly from the spacelike case. In particular, we do not expect that the subregion/subregion duality or bulk reconstruction theorems [8] can be directly extended to the timelike context without modification. The full holographic significance of these constructions therefore remains open for further investigation.

Finally, we conclude by outlining several open problems and promising directions for future exploration.

II. ENTANGLEMENT IN TIME

In this section, we will first review different views on entanglement in time. We will mainly focus on QFTs.

A. Views on entanglement in time

A promising definition of timelike EE was proposed in [25], where the roles of time and space coordinates are exchanged. Under this approach, timelike EE can be understood in a manner analogous to the spacelike case. This definition is particularly effective in $(1+1)$ -dimensional QFTs. However, in higher dimensions, where the number of spatial coordinates exceeds that of the temporal ones, extending this definition becomes subtle and problematic.

From the perspective of correlation functions, timelike EE can also be defined and computed via a Wick rotation of correlators involving twist operators. In this formulation, one analytically continues Euclidean correlators to the Lorentzian signature. Several examples of such calculations are presented in [28,33]. The reconstruction theorem in QFT ensures that correlators in Lorentzian spacetime can be obtained through appropriate analytic continuation procedures; see [46] for more details. Nevertheless, there are some unsatisfactory aspects of this approach. First, it appears difficult to generalize to quantum systems with finite-dimensional Hilbert spaces. Second, the physical interpretation of the Wick rotation remains obscure. In particular, it is still unclear which correlations within the system are captured by the timelike EE.

The recent work [34] provides a new perspective on entanglement in time, which is applicable to both finite-dimensional quantum systems and QFTs. In certain specific cases, the results for QFTs reduce to those obtained via analytic continuation of the correlators used to define timelike EE.

This idea can be viewed as a natural generalization of the concept of entanglement between two spacelike-separated subsystems. In QFTs, one can choose a Cauchy surface and define the Hilbert space of the theory accordingly. On this surface, two subregions A and B are selected. For a given state $|\psi\rangle$, the correlation functions between operators $\mathcal{O}_A$ and $\mathcal{O}_B$ localized in A and B , respectively, are defined as

$$\langle \mathcal{O}_A \mathcal{O}_B \rangle_\psi := \langle \psi | \mathcal{O}_A \mathcal{O}_B | \psi \rangle. \quad (1)$$

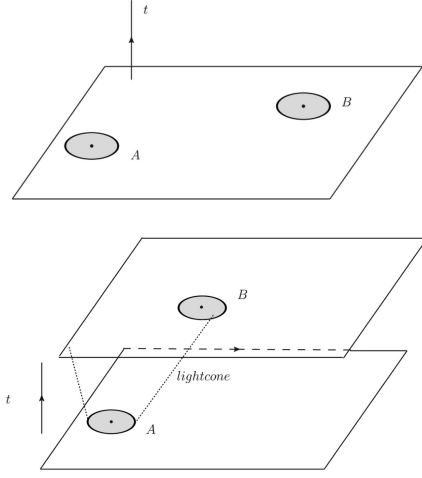


FIG. 1. An illustration of subsystems A and B : Top: A and B are causally disconnected; Bottom: A and B are causally connected.

The reduced density matrix $\rho_{AB} := \text{Tr}_{\overline{AB}} \rho$, where $\overline{AB}$ denotes the complement of AB , satisfies

$$\text{Tr}_{AB}(\rho_{AB} \mathcal{O}_A \mathcal{O}_B) = \langle \mathcal{O}_A \mathcal{O}_B \rangle_\psi. \quad (2)$$

According to the Wightman reconstruction theorem in QFTs ρ_{AB} can be fully determined by the set of correlation functions of all local operators $\mathcal{O}_A$ and $\mathcal{O}_B$. In other words, the reduced density matrix ρ_{AB} is completely specified by the correlation data within the subregions $\mathcal{O}_A$ and $\mathcal{O}_B$. According to microcausality $[\mathcal{O}_A, \mathcal{O}_B] = 0$. However, in general, the correlation function $\langle \mathcal{O}_A \mathcal{O}_B \rangle_\psi \neq 0$ does not vanish, reflecting the presence of quantum entanglement between two causally disconnected subsystems.

Entanglement is closely related to correlation functions between subsystems. In QFTs, correlation functions for causally connected subregions are also of fundamental importance. For instance, scattering processes involve correlation functions between timelike-separated operators. The definition of entanglement in time in [34] is a generalization of the above definition of ρ_{AB} to causally connected subregions A and B , see Fig. 1 for an illustration.

For causally connected subsystems A and B (figure on the right of Fig. 1) one could also evaluate the correlation functions in the state $|\psi\rangle$ (1). Similar to the spacelike case, we expect the existence of a transition operator $T_{AB}: \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$\langle \mathcal{O}_B(t) \mathcal{O}_A \rangle_\psi = \text{Tr}(T_{AB} \mathcal{O}_A \mathcal{O}_B), \quad (3)$$

where $\mathcal{O}_B(t) = U^\dagger \mathcal{O}_B U$. The precise definition of T_{AB} will be given later using the Schwinger-Keldysh formalism. In that construction, the contour ordering naturally reproduces the Wightman correlator $\langle \mathcal{O}_B(t) \mathcal{O}_A \rangle_\psi$ rather than $\langle \mathcal{O}_A \mathcal{O}_B(t) \rangle_\psi$. For this reason, relative to the convention of [34], the operator denoted here by T_{AB} corresponds to their

$T_{AB}^\dagger$. However, since the operators do not commute in general, i.e., $[\mathcal{O}_A, \mathcal{O}_B(t)] \neq 0$, the correlator $\langle \mathcal{O}_B(t) \mathcal{O}_A \rangle_\psi$ is generally complex-valued. This implies that the transition operator T_{AB} is non-Hermitian.

The transition operator T_{AB} can be viewed as a generalization of the reduced density matrix to more general configurations of subsystems. The usual reduced density matrix ρ_{AB} emerges as a special case when A and B are causally disconnected. Depending on the causal relations between subsystems A and B , different scenarios of interest arise:

- A and B are causally disconnected. In this case, the transition operator reduces to the standard reduced density matrix ρ_{AB} ;
- A and B represent the entire system on different Cauchy surfaces. In this case, the state at B corresponds to the final state resulting from the time evolution of the initial state at A , and thus T_{AB} encodes the dynamical information of the full system. If A and B correspond to the in-state ($t \rightarrow -\infty$) and out-state ($t \rightarrow +\infty$) Cauchy surfaces in a scattering process, then T_{AB} is expected to be related to the scattering S -matrix;
- A and B are causally connected subsystems, with B lying within the causal domain of A . This means that the information in B is determined by that in A . While this case resembles case 2, the transition operator T_{AB} is expected to encode not only the system's dynamics but also its nonlocal properties;
- A and B are causally connected subsystems, but only partially so. In this case, the information in B cannot be reconstructed solely from A . This broad class includes two representative situations. One is that only a portion of B lies within the causal domain of A . Another is that, although A and B are causally connected, their causal domains do not overlap at all. The latter is the scenario we will primarily focus on in this paper.

One may refer to Fig. 2 for an illustration of the four cases.

The above discussion has primarily focused on QFTs. However, the definition of T_{AB} is applicable to systems with finite-dimensional Hilbert spaces. The concept of entanglement in time is both fundamental and meaningful in any quantum mechanical system. Moreover, it may be possible to design experiments to measure entropy-related quantities associated with T_{AB} .

B. State preparation via the Schwinger-Keldysh formalism

We are particularly interested in how to understand and express the transition operator T_{AB} in QFTs. For ρ_{AB} the standard approach begins with preparing the density matrix $\rho = |\psi\rangle\langle\psi|$, followed by tracing out the complementary degrees of freedom to obtain ρ_{AB} via Euclidean path integral. However, in our case, the two subsystems A and B are timelike separated, and the Euclidean approach

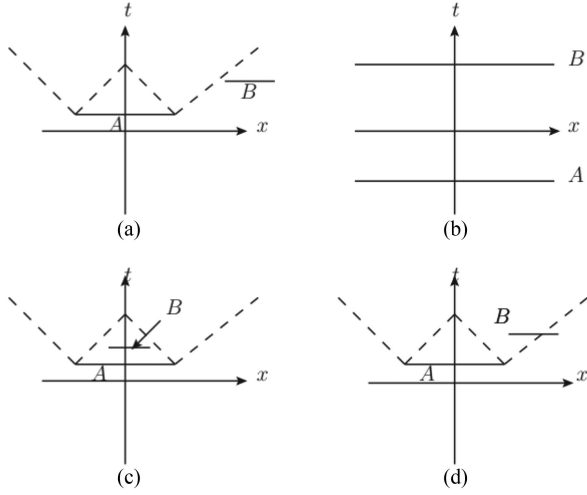


FIG. 2. An illustration of the four possible configurations of regions A and B : (a) A and B are causally disconnected; (b) A and B are Cauchy surfaces at different times; (c) A lies entirely within the causal domain of B ; (d) a schematic example in which only part of B is causally related to A .

must be extended to incorporate real-time evolution [34]. In [39], the authors propose using the Schwinger-Keldysh formalism to represent a general time-dependent density matrix $\rho(t)$. Within the Schwinger-Keldysh framework, both the reduced density matrix and the replica method can be generalized, leading to what is known as the real-time replica method. For spacelike-separated subsystems, this method reproduces the same results as the Euclidean replica approach [40,41].

In the following, we would like to show that the transition operator T_{AB} can also be constructed within the Schwinger-Keldysh formalism. Furthermore, we demonstrate how to compute the Rényi entropy associated with T_{AB} using the real-time replica method.

We begin by introducing the path integral representation of the time-evolved density matrix $\rho(t)$. Starting from an initial state ρ_0 at time t_0 , which can be prepared using a Euclidean path integral. The time-evolved density matrix in time $\rho(t)$ is given by

$$\rho(t) = U\rho_0 U^\dagger, \quad (4)$$

where $U := e^{iH(t-t_0)}$ is the time-evolution operator from t_0 to t . For simplicity, we focus on the case where the initial state is pure, i.e.,

$$\rho_0 = |\Psi_0\rangle\langle\Psi_0|, \quad \rho(t) = U|\Psi_0\rangle\langle\Psi_0|U^\dagger. \quad (5)$$

For mixed states, one can first purify the state and then proceed analogously.

To represent ket $|\Psi\rangle = U|\Psi_0\rangle$ and its conjugate $\langle\Psi| = \langle\Psi_0|U^\dagger$ in the path integral formalism, it is natural to employ a Schwinger-Keldysh contour, as pointed out

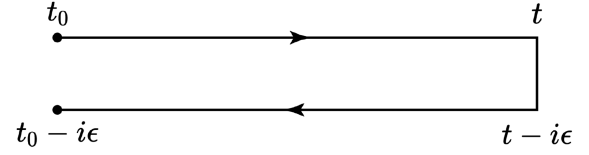


FIG. 3. A typical Keldysh contour.

in [39]. This contour evolves the ket state forward in time and the bra state backward, thus fully encoding the real-time dynamics. A typical Keldysh contour is illustrated in Fig. 3.

As a result, $\text{Tr}\rho(t)$ can be expressed as

$$\text{Tr}\rho(t) = \int [D\phi] e^{iS_{r;t_0}(\phi) - iS_{l;t_0,t}(\phi)}, \quad (6)$$

where $S_{r;t_0}$ is the action defined as $\int_{t_0}^t dt L[\phi]$. If we have $t_0 < t_1 < t$, the $\text{Tr}\rho(t_1)$ could be written as

$$\text{Tr}\rho(t_1) = \int [D\phi] e^{iS_{l;t_0,t_1}(\phi) + iS_{r;t_1,t}(\phi) - iS_{l;t_0,t}(\phi)}. \quad (7)$$

We could derive it from (4), the time evolution operator $U(t, t_1)$ and its Hermitian conjugate $U^\dagger(t, t_1)$ mutually cancel under the trace. It is more convenient to illustrate the path integral representation using diagrams, as shown in Fig. 4. In these diagrams, we include the spatial direction of the Keldysh contour to facilitate the definition of the reduced density matrix in the following discussion.

To evaluate the EE for a subsystem A on the Cauchy surface Σ_t , one needs to construct the reduced density matrix $\rho_A(t) = \text{Tr}_{\bar{A}}\rho(t)$. The full trace $\text{Tr}\rho(t)$ consists of a forward and a backward evolution, which are glued

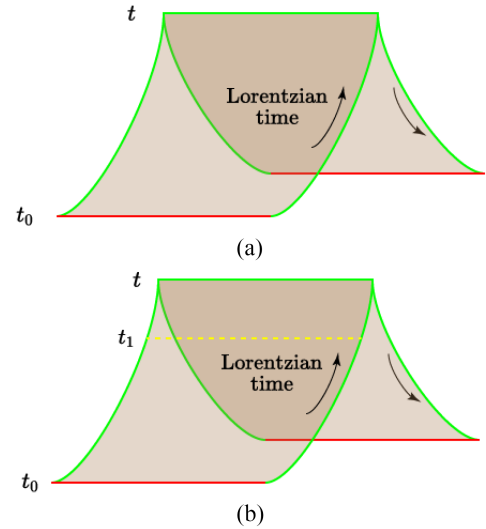


FIG. 4. The Schwinger-Keldysh formalism illustrating the path-integral representation of $\text{Tr}\rho$. (a) Calculate $\text{Tr}\rho(t)$ by (6). (b) Calculate $\text{Tr}\rho(t_1)$ by (7).

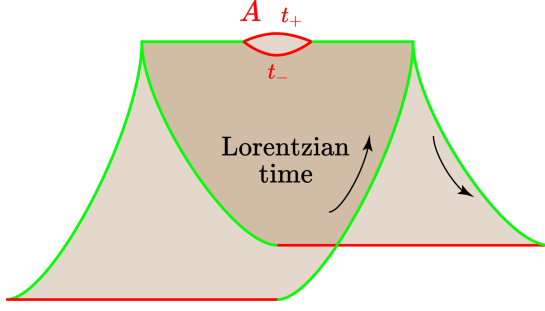


FIG. 5. The Schwinger-Keldysh formalism illustrating the path-integral representation of the reduced density matrix $\rho_A(t)$.

together along the Cauchy surface Σ_t . To obtain $\rho_A(t)$, the path integral is cut along the region A , and appropriate boundary conditions are imposed at the cut. The reduced density matrix can then be expressed as a path integral,

$$\rho_A = \int [D\Phi_R][D\Phi_L] e^{iS[\Phi_R] - iS[\Phi_L]} \delta(\Phi_{R,A}(t_-) - \Phi_A(t_-)) \times \delta(\Phi_{L,A}(t_+) - \Phi_A(t_+)), \quad (8)$$

where Φ_R and Φ_L denote the field configurations on the forward and backward segments of the contour in Fig. 5, respectively. The boundary conditions at subregion A are specified by $\Phi_A(t_-)$ and $\Phi_A(t_+)$, corresponding to the forward and backward branches. An illustration of the reduced density matrix $\rho_A(t)$ in Fig. 5.

C. Representation of T_{AB} via Schwinger-Keldysh formalism

In this section, we demonstrate that the transition operator T_{AB} can also be represented using the Schwinger-Keldysh formalism. We consider subsystems A and B located on Cauchy surfaces at times t_1 and t , respectively. The initial state $|\psi\rangle$ is prepared at $t \rightarrow -\infty$. In the context of this paper, A and B are taken to be semi-infinite intervals, which are appropriate for the computation of two-point correlation functions discussed later. Moreover, such semi-infinite intervals can be obtained as the limit of finite intervals when their lengths are extended to infinity, ensuring that our formalism naturally generalizes from the finite-interval case. In the previous section, we introduced the path integral representation of the reduced density matrix $\rho_A(t)$ as illustrated in Fig. 5. For the construction of T_{AB} , the path integral involves two cuts: one at time t on the forward segment, with appropriate boundary conditions imposed on subregion B ; and another at time t_1 , also on the forward segment, with boundary conditions specified on subregion A . The transition operator T_{AB} is then expressed in terms of the path integral as follows:

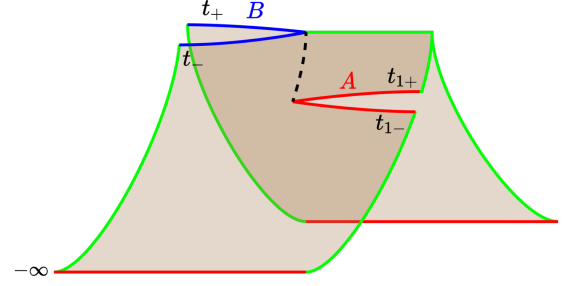


FIG. 6. The Schwinger-Keldysh formalism illustrating the path-integral representation of T_{AB} . This diagram shows the specific case for a two-point function, where the cuts for subsystems A and B are localized at times t_1 and t .

$$T_{AB} = \int [D\Phi_R][D\Phi_L] \times e^{iS_{t_1, -\infty}[\Phi_R]} \delta(\Phi_{R,A}(t_{1-}) - \Phi_A(t_{1-})) \times \delta(\Phi_{R,A}(t_{1+}) - \Phi_A(t_{1+})) e^{iS_{t,t_1}[\Phi_R]} \times \delta(\Phi_{R,B}(t_-) - \Phi_B(t_-)) \delta(\Phi_{L,B}(t_+) - \Phi_B(t_+)) \times e^{-iS_{-\infty,t}[\Phi_L]}, \quad (9)$$

where $\Phi_A(t_{1\pm})$ and $\Phi_B(t_{\pm})$ denote the boundary conditions in the cut. We show the presentation of T_{AB} in Fig. 6.

We can verify that the above path integral representation satisfies the basic properties of T_{AB} . According to the definition of the transition operator, inserting local operators $\mathcal{O}_A$ and $\mathcal{O}_B$ in subregions A and B , respectively, allows us to compute $\text{Tr}(T_{AB} \mathcal{O}_A \mathcal{O}_B)$, which reproduces the correlator $\langle \mathcal{O}_B(t) \mathcal{O}_A(t_1) \rangle_\psi$. This serves as a consistency check for the definition of T_{AB} via (3). Furthermore, T_{AB} also satisfies

$$\text{Tr}_A T_{AB} = \rho_B, \quad (10)$$

$$\text{Tr}_B T_{AB} = \rho_A. \quad (11)$$

The first relation involves taking the partial trace over region A . In the path integral representation, this corresponds to gluing along the cut at A in Fig. 7(a). As a result, the remaining path integral yields the reduced density matrix $\rho_B(t)$. The second relation is analogous; after gluing the cut at B , a cut remains at A on the Cauchy surface at time t_1 . The contributions from the path integral between $t_1 \rightarrow t$ on the forward segment and $t \rightarrow t_1$ on the backward segment cancel each other. Therefore, the final result $\text{Tr}_B T_{AB}$ is given by the reduced density matrix $\rho_A(t_1)$. We illustrate the partial trace operation diagrammatically in Fig. 7(b).

D. Real-time replica and timelike EE

In this subsection, what we compute directly via the Schwinger-Keldysh formalism is the entropy associated

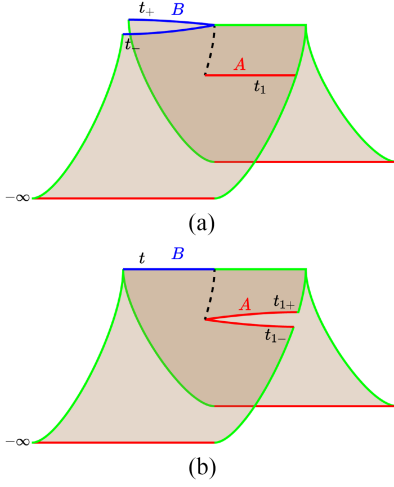


FIG. 7. The partial trace of T_{AB} . (a) $\text{Tr}_A T_{AB}$; obtained by gluing the cut along region A . (b) $\text{Tr}_B T_{AB}$; obtained by gluing the cut along region B . In this case, the resulting Schwinger–Keldysh formalism is equivalent to the geometry without the contribution over region A , as can be seen from (7).

with the transition operator T_{AB} for two subregions separated in time. We will focus on semi-infinite configurations, for which the real-time replica construction reduces to Lorentzian correlators of timelike-separated twist operators and can therefore be compared directly with previously proposed timelike entanglement entropy.

Recall the definition for pseudo-Rényi entropy and EE for T_{AB} ,

$$S_n(T_{AB}) = \frac{\text{Tr} T_{AB}^n}{1-n}, \quad (12)$$

$$S(T_{AB}) = \lim_{n \rightarrow 1} S_n(T_{AB}). \quad (13)$$

Given the transition operator T_{AB} we are now ready to compute the EE for T_{AB} , which requires understanding the product of T_{AB}^n via the replica method.

For our motivation in this paper, we would like to study the cases that would be related to the timelike EE evaluated by Wick rotation. We will mainly focus on some specific examples in $(1+1)$ D QFTs. The generalization to higher dimensions is straightforward.

1. Example I

Let us consider a simple example: $A \in [0, +\infty]$ at time t_1 and $B \in [-\infty, 0]$ at time t . The EE for T_{AB} is related to timelike EE for an interval $[0, t]$ on the time coordinate. To evaluate $\text{Tr}(T_{AB}^n)$ we should make an n -copy of the geometry and fields for T_{AB} with replica index $I = 1, 2, \dots, n$. $\text{Tr}(T_{AB}^n)$ involves identification along the cut with $\Phi_{I-} = \Phi_{(I+1)+}$ with $n+1 \rightarrow 1$. We draw a diagram in Fig. 8 to show the replica process. As a result, the final path integral can be considered as the insertion of

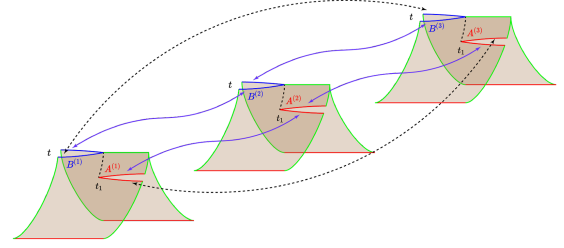


FIG. 8. Replica trick with a timelike interval for computing $\text{Tr}(T_{AB}^3)$.

local twist operator $\sigma_n(0, 0)$ and antitwist operator $\tilde{\sigma}_n(t, 0)$ in the n -copied theory. According to the Schwinger–Keldysh formalism, this can be translated to the two-point correlation function for the twist operator.

Thus, we have

$$\text{Tr}(T_{AB}^n) = \langle \tilde{\sigma}_n(t, 0) \sigma_n(0, 0) \rangle, \quad (14)$$

where the right-hand side can be computed via the analytic continuation of the two-point correlator in Euclidean theory, evaluated as follows:

$$\langle \tilde{\sigma}_n(t, 0) \sigma_n(0, 0) \rangle = \langle \tilde{\sigma}_n(\tau_1, 0) \sigma_n(\tau_2, 0) \rangle \Big|_{\substack{\tau_1 \rightarrow it + \epsilon_1 \\ \tau_2 \rightarrow i0 + \epsilon_2}}, \quad (15)$$

with $\epsilon_1 > \epsilon_2 > 0^1$. In $(1+1)$ -dimensional CFTs, the two-point correlation function is universal. Consequently, we obtain the Rényi entropy for T_{AB} as follows:

$$S_n(T_{AB}) = \frac{c}{6} \left(1 + \frac{1}{n} \right) \log \frac{t}{\delta} + \frac{c}{12} \left(1 + \frac{1}{n} \right) i\pi. \quad (16)$$

The EE is given by

$$S(T_{AB}) = \frac{c}{3} \log \frac{t}{\delta} + \frac{ic\pi}{6}. \quad (17)$$

2. Example II

Let us now consider a more involved example. We take four subregions; $A_1 \in (0, +\infty)$ at time t_1 , $A_2 \in (0, +\infty)$ at time t_3 , $B_1 \in (-\infty, 0)$ at time t_2 , and $B_2 \in (-\infty, 0)$ at time t_4 . We assume the order of time $t > t_4 > t_3 > t_2 > t_1 > -\infty$. For clarity, the configuration of the subregions is illustrated in Fig. 9.

We can now define and prepare the transition operator $T_{A_1 A_2 B_1 B_2}$ in the same way as in Example I. A diagram illustrating the path integral representation is shown in Fig. 10. The procedure for evaluating $\text{Tr}(T_{A_1 A_2 B_1 B_2})^n$ is nearly identical to that in Example I, so we will omit the details of the replica method here. After applying the replica trick, the problem reduces to evaluating the four-point,

¹The ordering of ϵ_i determines the correct operator ordering in the correlator.

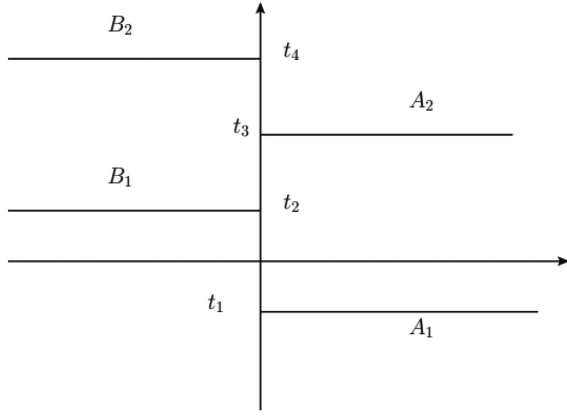


FIG. 9. An illustration of the four subregions considered in Sec. II D 2.

time-ordered correlation function of twist operators,

$$\text{Tr}(T_{A_1 A_2 B_1 B_2})^n = \langle \tilde{\sigma}_n(t_4, 0) \sigma_n(t_3, 0) \tilde{\sigma}_n(t_2, 0) \sigma_n(t_1, 0) \rangle, \quad (18)$$

which can be computed via analytical continuation of the four-point function in the Euclidean theory by taking $\tau = it$. We have

$$\begin{aligned} & \langle \tilde{\sigma}_n(t_4, 0) \sigma_n(t_3, 0) \tilde{\sigma}_n(t_2, 0) \sigma_n(t_1, 0) \rangle \\ &= \langle \tilde{\sigma}_n(\tau_4, 0) \sigma_n(\tau_3, 0) \tilde{\sigma}_n(\tau_2, 0) \sigma_n(\tau_1, 0) \rangle_{\tau_i \rightarrow i\tau_i + \epsilon_i}, \end{aligned} \quad (19)$$

with the order $\epsilon_4 > \epsilon_3 > \epsilon_2 > \epsilon_1 > 0$. Recall the definition of EE, we have

$$\begin{aligned} S(T_{A_1 A_2 B_1 B_2}) &= -\partial_n \langle \sigma_n(t_1, 0) \tilde{\sigma}_n(t_2, 0) \\ &\quad \times \sigma_n(t_3, 0) \tilde{\sigma}_n(t_4, 0) \rangle|_{n=1}. \end{aligned} \quad (20)$$

It is particularly interesting to define mutual information as

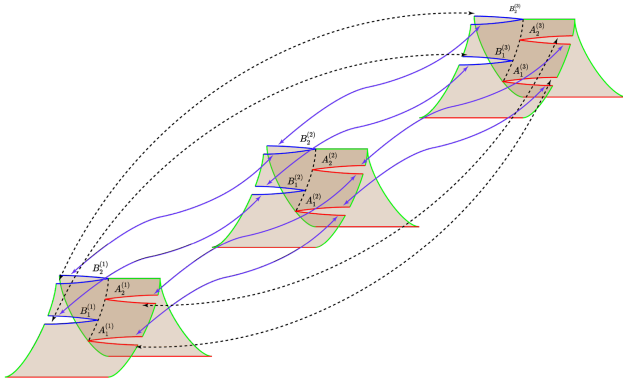


FIG. 10. Replica trick with two timelike intervals for computing $\text{Tr}(T_{A_1 A_2 B_1 B_2}^3)$.

$$\begin{aligned} I(A_1 B_1; A_2 B_2) &:= S(T_{A_1 B_1}) + S(T_{A_2 B_2}) \\ &\quad - S(T_{A_1 A_2 B_1 B_2}), \end{aligned} \quad (21)$$

which captures the correlation between the composite subsystems $A_1 B_1$ and $A_2 B_2$.

In $(1+1)$ D CFTs, there are scenarios in which we can explicitly compute $S(T_{A_1 A_2 B_1 B_2})$ and therefore $I(A_1 B_1; A_2 B_2)$. In the vacuum state, mutual information between two spacelike subregions is known to be conformally invariant. This suggests that $I(A_1 B_1; A_2 B_2)$ should depend only on the conformal cross-ratio. We provide further details on the evaluation of these quantities in Appendix A.

In this paper, we focus on holographic CFTs. The four-point function in (19) can be computed using the semi-classical approximation for conformal blocks. Detailed calculations and analytic continuation are presented in Appendix A. Our result is consistent with the definition of holographic timelike EE for two time intervals, as evaluated by holography in [28]. There exist two distinct phases of the EE, depending on the relative positions of the two time intervals $[t_1, t_2]$ and $[t_3, t_4]$. For case $\frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_3)(t_2 - t_4)} \sim 0$ we have

$$S(T_{A_1 A_2 B_1 B_2}) = \frac{c}{3} \log \frac{t_2 - t_1}{\delta} + \frac{c}{3} \log \frac{t_4 - t_3}{\delta} + \frac{i\pi c}{3}, \quad (22)$$

while for $\frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_3)(t_2 - t_4)} \sim 1$ we have

$$S(T_{A_1 A_2 B_1 B_2}) = \frac{c}{3} \log \frac{t_3 - t_2}{\delta} + \frac{c}{3} \log \frac{t_4 - t_1}{\delta} + \frac{i\pi c}{3}. \quad (23)$$

Note that the imaginary part remains constant and is twice that of the single-interval case (17). As a result, the mutual information is real in both phases. This result is consistent with previous studies on timelike EE for two intervals along the time direction. In the following sections, we will explore the holographic aspects of the EE for $T_{A_1 A_2 B_1 B_2}$. The phase transition of the EE corresponds to different bulk Ryu-Takayanagi surfaces.

III. RYU-TAKAYANAGI SURFACE FOR TIMELIKE EE

In [25], the holographic EE for a time interval in AdS_3 was computed as

$$S_A^{(T)} = \frac{c}{3} \log \frac{t_0}{\delta} + \frac{c}{6} i\pi, \quad (24)$$

where t_0 denotes the proper length of the timelike interval A , and δ is the UV cutoff.

This result contains both a real and an imaginary part. Although the logarithmic real part has a standard geometric interpretation in terms of the RT prescription, the imaginary

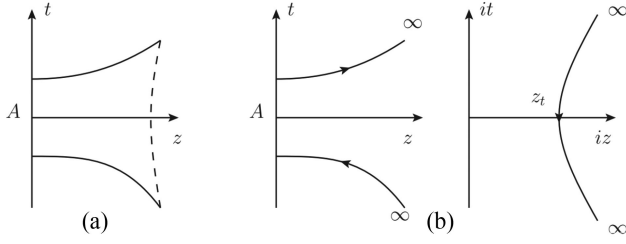


FIG. 11. (a) RT surface for a timelike interval A in AdS_3 spacetime consists of both spacelike and timelike geodesic segments. (b) Complexified RT surface passing through real and imaginary domains, connected via points at infinity.

contribution is more subtle. The imaginary term can be interpreted as arising from the timelike geodesic connecting the two infinities, as illustrated in Fig. 11(a) [25]. Another proposal is presented in [29], where the RT surface is extended to a complexified bulk.

In this construction, the embedding functions of the extremal surface are analytically continued into the complex domain. The continuation renders the surface complex-valued, representing an analytic saddle of the extremal-area functional. It acquires physical meaning only when evaluated on particular real slices of the complexified coordinate space—typically the real slice or the purely imaginary (Wick-rotated) slice. These slices are related by analytic continuation and provide different real projections of the same complex solution, allowing the surface to be represented in distinct domains.

This leads to a complexified RT surface that transitions between real and imaginary domains through points at complex infinity. As shown in Fig. 11(b), the surface initially follows the upper branch of a hyperbola in the real Lorentzian geometry. It then passes through a point at infinity, analytically continues in the imaginary time direction, and reaches a turning point z_t on the imaginary branch. After crossing into the complex domain, the surface reemerges through another point at infinity, entering the lower branch of the real geometry. Throughout this path, the surface remains smooth and continuous as it traverses different slices of the complexified coordinate space.

Because the RT surface is symmetric, it has a single turning point z_t on its imaginary branch that splits the surface into two identical halves. Consequently, the total area $\mathcal{A}$ of the RT surface anchored to region A can be written as

$$\mathcal{A} = 2 \int_{\delta}^{z_t} dz \mathcal{L}, \quad (25)$$

where $\mathcal{L}$ is the Lagrangian density along the surface. By the Ryu-Takayanagi prescription, the corresponding timelike EE is $S_A^{(T)} = \frac{\mathcal{A}}{4G}$, with G the cosmological constant. To proceed, we now derive the explicit form of $\mathcal{L}(z)$ and determine the location of the turning point z_t .

A. Procedure to construct the bulk RT surface for timelike subregions

There remains some ambiguity regarding how to determine the complex RT surface in the complexified geometry. In this section, we present a procedure for constructing the RT surface via analytical continuation from its Euclidean counterpart [30].

We consider the case of an AdS_{d+1} black hole, where the bulk metric takes the form of

$$ds^2 = \frac{1}{z^2} \left(-f(z) dt^2 + \frac{dz^2}{g(z)} + dx^2 + d\vec{y}^2 \right), \quad (26)$$

where $d\vec{y}^2 = \sum_{i=1}^{d-2} y_i^2$, and the event horizon is located at $z = z_H$ such that $f(z_H) = 0$. For a neutral (Schwarzschild) black hole, we take $f(z) = g(z) = 1 - \frac{z^d}{z_H^d}$. In the limit $z_H \rightarrow \infty$, the metric reduces to the Poincaré AdS case.

We define the boundary subregion A as a timelike strip with $x = 0$ and $t \in [0, t_0]$, spanning all directions in $\vec{y}$. The endpoints of this interval are located at $(t = 0, x = 0, \vec{y})$ and $(t = t_0, x = 0, \vec{y})$. For simplicity, we will label the two endpoints as $(0, 0)$ and $(t_0, 0)$, suppressing the transverse coordinates.

We analytically continue the spacetime (26) to Euclidean spacetime via $t \rightarrow -i\tau$, which yields the following:

$$ds^2 = \frac{1}{z^2} \left(f(z) d\tau^2 + \frac{dz^2}{g(z)} + dx^2 + d\vec{y}^2 \right). \quad (27)$$

The endpoints of region A are now located at $(0, 0)$ and $(\tau_0, 0)$. By symmetry, the corresponding RT surface can be parametrized as

$$\tau = \tau(z) \quad \text{and} \quad x = 0. \quad (28)$$

The induced metric on the RT surface then becomes

$$ds_{\text{ind}}^2 = \frac{1}{z^2} \left(f(z) \tau'(z)^2 + \frac{1}{g(z)} \right) dz^2 + \frac{1}{z^2} d\vec{y}^2. \quad (29)$$

Accordingly, the Lagrangian is

$$\mathcal{L} = \frac{1}{z^{d-1}} \sqrt{f(z) \tau'(z)^2 + \frac{1}{g(z)}}. \quad (30)$$

To proceed, we introduce a conserved quantity p_τ associated with translation invariance in τ , which is defined by

$$p_\tau := \frac{\partial \mathcal{L}}{\partial \tau'} = \frac{f(z) \tau'(z)}{z^{d-1} \sqrt{f(z) \tau'(z)^2 + g(z)^{-1}}}. \quad (31)$$

Solving for $\tau'(z)$ gives

$$\tau'(z)^2 = \frac{p_\tau^2}{f(z)g(z)[f(z)z^{2(1-d)} - p_\tau^2]}. \quad (32)$$

Substituting this into (30), the Lagrangian becomes

$$\mathcal{L} = \frac{1}{z^{2(d-1)}} \sqrt{\frac{f(z)}{g(z)[f(z)z^{2(1-d)} - p_\tau^2]}}. \quad (33)$$

At the turning point $z = z_\tau$ of the RT surface, we have $\tau'(z_\tau) \rightarrow \infty$. From (32), this divergence implies

$$p_\tau^2 = f(z_\tau)z_\tau^{2(1-d)}. \quad (34)$$

This condition allows us to eliminate p_τ in favor of z_τ , allowing the Lagrangian to be regarded as a function of z and z_τ , that is, $\mathcal{L} = \mathcal{L}(z, z_\tau)$.

To determine z_τ , we apply the boundary conditions $\tau(z=0) = 0$ and $\tau(z=z_\tau) = \frac{\tau_0}{2}$. This yields the integral constraint,

$$\int_0^{z_\tau} \tau'(z) dz = \frac{\tau_0}{2}. \quad (35)$$

Since $\tau'(z)$ can be expressed in terms of p_τ , and hence in terms of z_τ , the above relation allows one to solve for z_τ as a function of the Euclidean boundary separation τ_0 , that is, $z_\tau = z_\tau(\tau_0)$.

To obtain the physical surface in Lorentzian signature, we perform an analytic continuation back from Euclidean spacetime. An important subtlety arises here; both z_τ and p_τ , initially real in the Euclidean setting, become complex after continuation. In Lorentzian spacetime, multiple extremal surfaces may emerge depending on the choice of analytic branch. Instead of attempting to specify these surfaces directly, we adopt a constructive procedure. Starting from the Euclidean solution, we perform a Wick rotation by deforming $\tau \rightarrow i\tau$ in the complex plane. This method provides a canonical analytic continuation of z_τ and p_τ , thereby selecting a physically significant branch for the extremal surface.

Under this Wick rotation, the conserved quantity and the turning point transform as

$$p_t := -ip_\tau|_{\tau_0 \rightarrow i\tau_0}, \quad z_t := z_\tau|_{\tau_0 \rightarrow i\tau_0}. \quad (36)$$

Once z_t is determined, the area of the RT surface is given by

$$\mathcal{A}(0, 0; t_0, 0) = 2 \int_C dz \mathcal{L}(z, z_t), \quad (37)$$

where C denotes an integration contour in the complex z plane that connects the UV cutoff $z = \delta$ to the complex turning point $z = z_t$. The integrand $\mathcal{L}(z, z_t)$ is generally complex and features branch points due to the square-root

structure. To ensure a well-defined result, the contour C must be chosen such that it avoids all branch cuts in the complex plane.

B. Numerical method

Analytic solutions for extremal surfaces are rarely available in higher-dimensional AdS black hole spacetimes. Although exact results can be obtained in special cases such as AdS_3 or AdS_{d+1} in the Poincaré patch, generic black hole backgrounds typically necessitate perturbative or numerical techniques. In this section, we implement a numerical strategy based on the shooting method to determine the turning point z_t and evaluate the associated geometric quantities.

The key idea is to remake the original boundary value problem for the embedding function $t(z)$ into an initial value problem, which is more amenable to numerical integration. The required inputs include the interval length t_0 , the spacetime dimension d , the horizon location z_H , and the background functions $f(z)$ and $g(z)$. According to the boundary condition (35), the extremal surface satisfies $t(0) = 0$ and $t(z_t) = \frac{t_0}{2}$.

To proceed, we use the conservation relation (34) to express the conserved quantity p_t in terms of the unknown turning point z_t . Substituting this into the expression for $t'(z)$ yields

$$t'(z)^2 = \frac{p_t^2}{f(z)g(z)[f(z)z^{2(1-d)} + p_t^2]}. \quad (38)$$

With an initial guess $z_t^{(0)}$, the corresponding p_t is fixed, making $t'(z)$ a fully determined function. We then numerically integrate $t(z)$ from $z = 0$ with $t(0) = 0$ up to $z = z_t$, resulting in a numerical estimate $\bar{t}(z_t)$. The deviation from the required boundary value is quantified by $\epsilon \equiv \bar{t}(z_t) - \frac{t_0}{2}$.

The turning point is then refined iteratively using root-finding algorithms such as the Newton-Raphson or secant method, until the error satisfies $|\epsilon| < \text{tol}$, where tol is a prescribed numerical tolerance. The convergent value of z_t is taken as the physical turning point of the extremal surface.

Once z_t is fixed, physical observables, such as the area function $\mathcal{A}$ can be evaluated. By varying the interval size t_0 , one can extract the dependence of $\mathcal{A}(t_0)$ and other geometric quantities, enabling comparisons with analytical or perturbative results.

In the following sections, we apply this procedure to explicit examples and verify the consistency of our numerical results against known analytical limits.

C. Pure AdS

In the following examples, we consider Schwarzschild black holes with metric functions satisfying $f(z) = g(z) = 1 - \frac{z^d}{z_H^d}$.

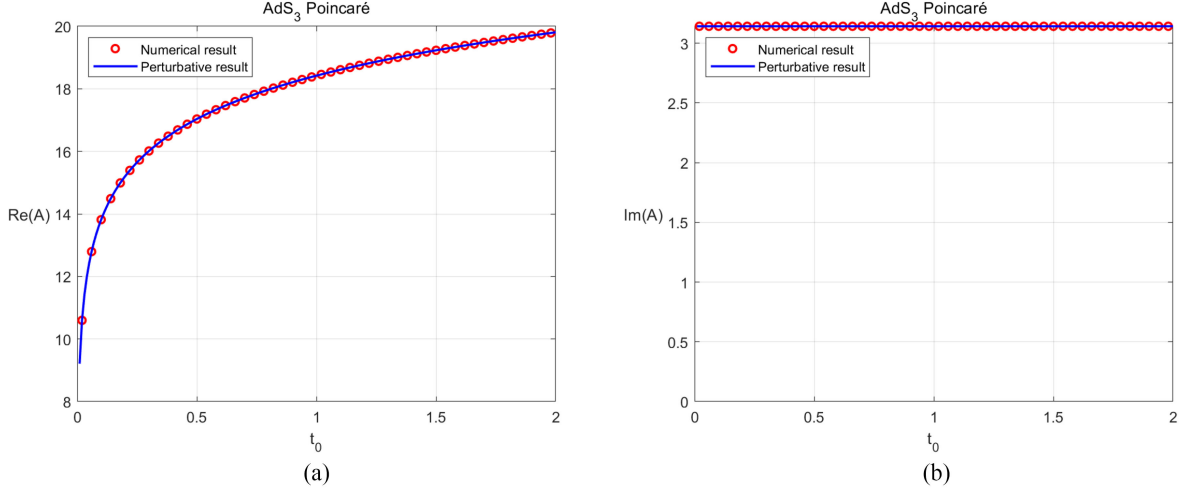


FIG. 12. Area $\mathcal{A}$ of the RT surface as a function of interval length t_0 in the timelike AdS_3 Poincaré case. The blue curve represents the analytical result (43), while the red circles are obtained numerically. Panel (a) shows the real component, and panel (b) displays the imaginary part. The interval t_0 is sampled over $(0,2)$ with step size 0.01, and we use a UV cutoff $\delta = 10^{-4}$.

In this subsection, we focus specifically on the Poincaré case taking the limit $z_H \rightarrow \infty$, under which the metric simplifies to $f(z) = g(z) = 1$.

As discussed previously, the RT surface area $\mathcal{A}$ for AdS_{d+1} in the Poincaré patch can be analytically calculated for arbitrary $d \geq 2$. Therefore, these cases serve as benchmarks to verify the accuracy of our numerical implementation.

1. The AdS_3 case

We begin with the Poincaré AdS_3 ($d = 2$) metric $ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + dx^2)$. Under $t \rightarrow -i\tau$, the Euclidean metric becomes $ds^2 = \frac{1}{z^2}(d\tau^2 + dz^2 + dx^2)$.

For a timelike interval between $(0,0)$ and $(\tau_0, 0)$, the extremal surface can be parametrized as $\tau = \tau(z)$ with $x = 0$. The corresponding Lagrangian reads $\mathcal{L} = \frac{1}{z} \sqrt{\tau'(z)^2 + 1}$.

The conserved quantity associated with τ is $p_\tau = \frac{\partial \mathcal{L}}{\partial \tau'} = \frac{\tau'}{z \sqrt{\tau'^2 + 1}}$, which can be inverted to produce

$$\tau'(z)^2 = \frac{p_\tau^2 z^2}{1 - p_\tau^2 z^2}. \quad (39)$$

Since the turning point satisfies $\tau'(z_\tau) \rightarrow \infty$, the denominator must vanish, resulting in

$$z_\tau = \frac{1}{p_\tau}. \quad (40)$$

Imposing the boundary condition $\int_0^{z_\tau} \tau'(z) dz = \frac{\tau_0}{2}$, and evaluating the integral analytically, we obtain

$$z_\tau = \frac{\tau_0}{2}. \quad (41)$$

Applying the Wick rotation $\tau_0 \rightarrow it_0$ then gives the Lorentzian turning point

$$z_t = z_\tau|_{\tau_0 \rightarrow it_0} = \frac{it_0}{2}. \quad (42)$$

Interestingly, the turning point becomes complex under this continuation. The area of the RT surface becomes

$$\begin{aligned} \mathcal{A}(0, 0; t_0, 0) &= 2 \int_\delta^{z_t} dz \mathcal{L} = 2 \int_\delta^{it_0/2} \frac{dz}{z \sqrt{1 + \left(\frac{2z}{t_0}\right)^2}} \\ &= 2 \log\left(\frac{t_0}{\delta}\right) + i\pi. \end{aligned} \quad (43)$$

This result consists of a real and an imaginary component, respectively, corresponding to the contributions from the segments of the RT surface in the real and imaginary regions of the complexified geometry, as illustrated in Fig. 11(b).

To verify the numerical implementation, we compare this with this analytical result. The numerical data are shown in Fig. 12, where red circles denote numerical values, and blue curves represent the analytic expression (43). As expected, the numerical and analytical results agree precisely, with the real part reproducing $2 \log(\frac{t_0}{\delta})$ and the imaginary part producing $i\pi$. This agreement validates the numerical method in the Poincaré AdS_3 case.

One can construct the RT surface in the global coordinate with the metric

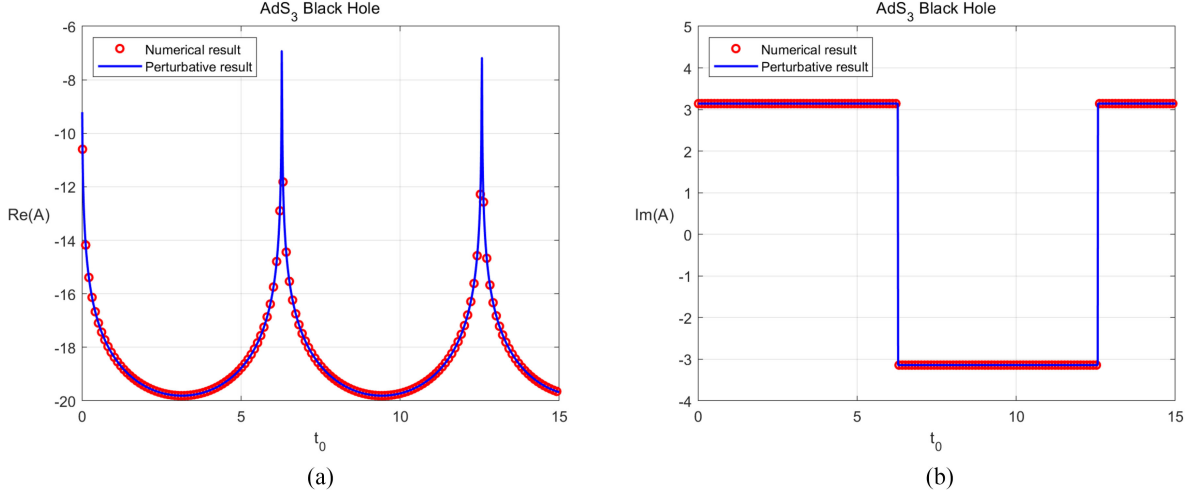


FIG. 13. Area $\mathcal{A}$ of the RT surface as a function of interval length t_0 in the timelike AdS_3 global coordinate. The blue curve shows the analytic result (46), while the red circles indicate numerical data. Panel (a) shows the real part of $\mathcal{A}$, and panel (b) presents the imaginary part. We set $\delta = 10^{-4}$ and sample $t_0 \in (0, 15)$ with step size 0.01.

$$ds^2 = -(1 + r^2)dt^2 + \frac{1}{1 + r^2}dr^2 + r^2d\theta^2, \quad (44)$$

where $0 < r < +\infty$, $0 \leq \theta < 2\pi$. CFT lives on the AdS boundary $r = r_\infty$, where r_∞ is the bulk cutoff. Consider the time interval $[0, t_0]$. By the same procedure we can find the turning point of the RT surface,

$$r_t = r_\tau|_{\tau_0 \rightarrow it_0} = i \csc\left(\frac{t_0}{2}\right), \quad (45)$$

which is imaginary. The area of RT surface is given by

$$\mathcal{A}(0, 0; t_0, 0) = 2 \int_C \sqrt{\frac{1}{r^2 - r_t^2}} = -2 \log\left(\frac{2 \sin\left(\frac{t_0}{2}\right)}{\delta}\right) + i\pi, \quad (46)$$

where the contour C is chosen as $r_\infty \rightarrow 0 \rightarrow r_t$ and $r_\infty = \frac{1}{\delta}$. As shown in Fig. 13, the numerical results (red circles) perfectly agree with the analytical expression (blue curves). In particular, the surface area exhibits a periodic structure with period 4π due to the dependence $\sin(\frac{t_0}{2})$. The real part shows a sequence of logarithmic divergences at $t_0 = 2n\pi$ ($n \in \mathbb{Z}$), corresponding to the vanishing of the sine function. Meanwhile, the imaginary part displays step-like jumps of $-2\pi i$ across each period, reflecting the branch cut behavior of the logarithm in the complexified geometry.

Although the global AdS_3 time coordinate is compactified with period 2π , the area function exhibits a 4π periodicity due to its $\sin(\frac{t_0}{2})$ dependence. This arises because $\sin(\frac{t_0}{2})$ is 4π -periodic, and its appearance inside a logarithm leads to branch cut jumps of $2\pi i$ across each 4π cycle. As a result, the complexified surface area encodes a

richer periodic structure than the underlying spacetime geometry.

We can also solve the RT surface $r(t)$. The results are shown in Appendix B 1, where we also show how to find the RT surface for an arbitrary interval in the global coordinate.

2. Higher-dimensional AdS_{d+1} case

We now generalize the analysis to higher-dimensional Poincaré AdS_{d+1} geometries. The analytic solution can be obtained for arbitrary d , and we numerically validate it in the AdS_4 ($d = 3$) case as a representative example.

The AdS_{d+1} Poincaré metric takes the form

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + dx^2 + d\vec{y}^2).$$

The corresponding Euclidean metric is given by

$$ds^2 = \frac{1}{z^2}(d\tau^2 + dz^2 + dx^2 + d\vec{y}^2).$$

For a timelike interval from $(0, 0)$ to $(\tau_0, 0)$, the extremal surface can be parametrized as $\tau = \tau(z)$ with $x = y_i = 0$. The corresponding Lagrangian becomes $\mathcal{L} = \frac{1}{z^{d-1}} \sqrt{1 + \tau'(z)^2}$.

The conserved constant associated with τ is $p_\tau = \frac{\tau'}{z^{d-1} \sqrt{\tau'^2 + 1}}$, which leads to

$$\tau'(z)^2 = \frac{p_\tau^2 z^{2(d-1)}}{1 - p_\tau^2 z^{2(d-1)}}. \quad (47)$$

The turning point z_τ corresponds to the divergence of $\tau'(z)$, giving

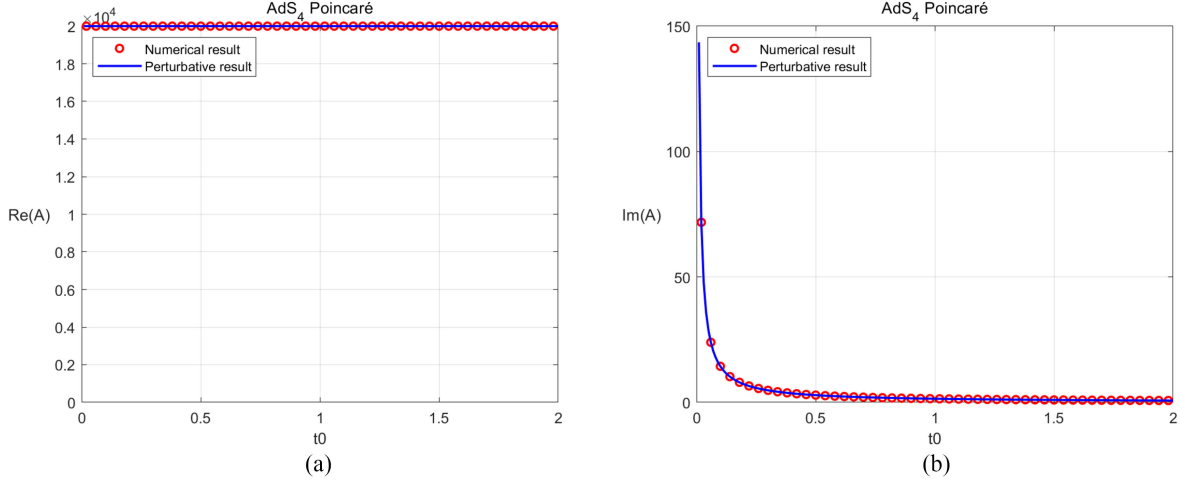


FIG. 14. Area $\mathcal{A}$ of the RT surface as a function of interval length t_0 in the timelike AdS_4 Poincaré case. The blue curve represents the analytical result (52), while the red circles are obtained numerically. Panel (a) shows the real component, and panel (b) displays the imaginary part. The interval t_0 is sampled over $(0,2)$ with step size 0.01, and we use a UV cutoff $\delta = 10^{-4}$.

$$z_\tau = \left(\frac{1}{p_\tau} \right)^{\frac{1}{d-1}}. \quad (48)$$

Imposing the boundary condition $\int_0^{z_\tau} \tau'(z) dz = \frac{\tau_0}{2}$, and evaluating the integral analytically, we find

$$z_\tau = \frac{\Gamma\left(\frac{1}{2(d-1)}\right)}{2\sqrt{\pi}\Gamma\left(\frac{d}{2(d-1)}\right)} \tau_0. \quad (49)$$

After Wick rotation $\tau_0 \rightarrow it_0$, the turning point becomes

$$z_t = \frac{i\Gamma\left(\frac{1}{2(d-1)}\right)}{2\sqrt{\pi}\Gamma\left(\frac{d}{2(d-1)}\right)} t_0, \quad (50)$$

which lies in the complex plane. The area of RT surface splits into real and imaginary contributions,

$$\begin{aligned} \mathcal{A}(0,0;t_0,0) &= 2 \int_\delta^{z_t} dz \mathcal{L} \\ &= \frac{2}{(d-2)\delta^{d-2}} + 2\kappa_d \frac{(-i)^d}{t_0^{d-2}}, \end{aligned} \quad (51)$$

with $\kappa_d = \frac{\pi^{\frac{d-1}{2}} 2^{d-2}}{d-2} \left(\frac{\Gamma\left(\frac{1}{2(d-1)}\right)}{\Gamma\left(\frac{d}{2(d-1)}\right)} \right)^{d-1}$.

As a concrete example, we take $d = 3$ (AdS_4), for which the area becomes

$$\mathcal{A}_4(0,0;t_0,0) = \frac{2}{\delta} + \frac{i8\pi^3}{t_0\Gamma\left(\frac{1}{4}\right)^4}. \quad (52)$$

To validate the result, we compare with the numerical data shown in Fig. 14, where the red circles represent the

numerical values and the blue curve denotes the analytical expression (52). As expected, the numerical and analytical results agree precisely: the real part reproduces the UV divergence $\frac{2}{\delta}$, while the imaginary part follows the inverse- t_0 behavior with coefficient proportional to $\frac{8\pi^3}{\Gamma\left(\frac{1}{4}\right)^4}$.

D. Examples of black hole

We now turn to the case of AdS black hole geometries. As discussed earlier, we consider metrics of the form $f(z) = g(z) = 1 - \frac{z^d}{z_H^d}$, where z_H denotes the position of the horizon.

In the AdS_3 Bañados–Teitelboim–Zanelli (BTZ) black hole, the extremal surface equations possess fully analytic solutions. However, for higher-dimensional AdS_{d+1} black holes, the equations become analytically intractable and require a perturbative treatment. As a result, the comparison between analytic and numerical results shows only approximate agreement in higher dimensions.

1. BTZ black hole

We now consider the AdS_3 BTZ black hole, whose metric takes the form,

$$ds^2 = \frac{1}{z^2} \left[-f(z) dt^2 + \frac{dz^2}{f(z)} + dx^2 \right],$$

where $f(z) = 1 - \frac{z^2}{z_H^2}$ and z_H denote the horizon radius. After Wick rotates $t \rightarrow -i\tau$, the Euclidean metric becomes

$$ds^2 = \frac{1}{z^2} \left[f(z) d\tau^2 + \frac{dz^2}{f(z)} + dx^2 \right].$$

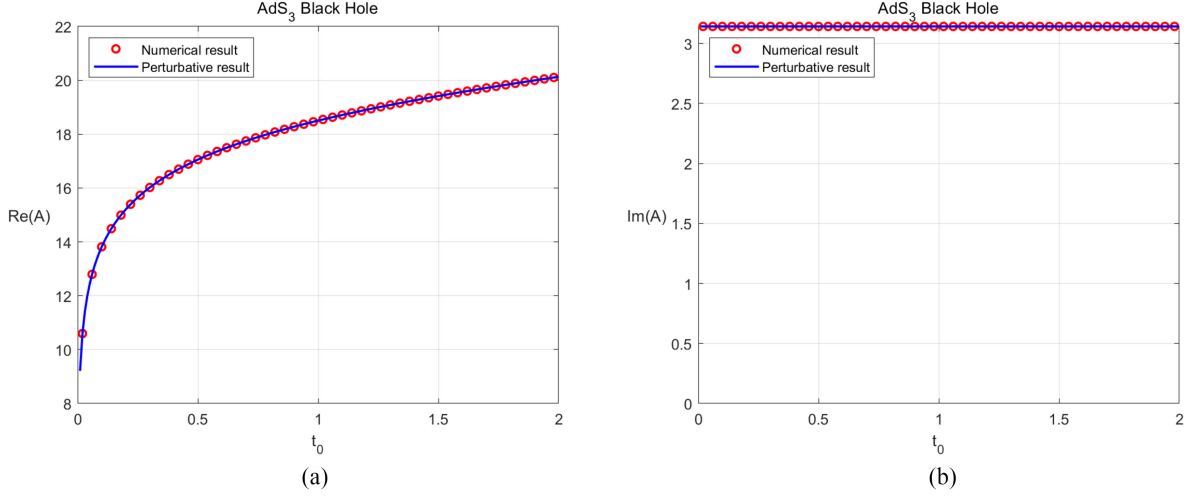


FIG. 15. Area $\mathcal{A}$ of the RT surface as a function of interval length t_0 for the timelike AdS_3 black hole case. The blue curve shows the analytical expression (57), while red circles denote the numerical results. Panel (a) displays the real component; panel (b) shows the imaginary part. The interval t_0 is sampled over (0,2) with spacing 0.01, and we use a UV cutoff $\delta = 10^{-4}$.

For an interval $(0, 0; \tau_0, 0)$, the RT surface is parametrized as $\tau = \tau(z)$ with $x = 0$, leading to Lagrangian $\mathcal{L} = \frac{1}{z} \sqrt{f(z)\tau'(z)^2 + \frac{1}{f(z)}}$. So, the conserved constant is $p_\tau = \frac{f(z)\tau'}{z\sqrt{f(z)\tau'^2 + f(z)^{-1}}}$, which leads to

$$\tau'(z)^2 = \frac{p_\tau^2}{f(z)^2[z^{-2}f(z) - p_\tau^2]}. \quad (53)$$

The turning point is defined by $\tau'(z_\tau) \rightarrow \infty$, which gives

$$z_\tau = \frac{z_H}{\sqrt{1 + (p_\tau z_H)^2}}. \quad (54)$$

Applying the boundary condition $\int_0^{z_\tau} \tau'(z) dz = \frac{\tau_0}{2}$, one finds the turning point

$$z_\tau = z_H \sin\left(\frac{\tau_0}{2z_H}\right). \quad (55)$$

After Wick rotation $\tau_0 \rightarrow it_0$, the turning point becomes complex,

$$z_t = z_\tau|_{\tau_0 \rightarrow it_0} = iz_H \sinh\left(\frac{t_0}{2z_H}\right). \quad (56)$$

The area function becomes

$$\begin{aligned} \mathcal{A}(0, 0; t_0, 0) &= 2 \int_\delta^{z_t} dz \mathcal{L} \\ &= 2 \int_\delta^{iz_H \sinh\left(\frac{t_0}{2z_H}\right)} \frac{dz}{z \sqrt{1 - \frac{z^2}{z_H^2} + p_\tau^2 z^2}} \\ &= 2 \ln\left(\frac{2z_H}{\delta} \sinh\left(\frac{t_0}{2z_H}\right)\right) + i\pi. \end{aligned} \quad (57)$$

The numerical results are shown in Fig. 15, where red circles denote numerical values and the blue curve represents the analytical expression (57). Both real and imaginary components are in strong agreement.

2. Higher-dimensional AdS_{d+1} black hole

This section provides a detailed analysis of higher-dimensional AdS_{d+1} black hole geometries. Since fully analytical solutions are generally inaccessible in this case, we adopt a perturbative approach in the small interval limit $t_0 \ll z_h$. Specifically, we expand the RT surface equations perturbatively around the pure AdS background and compute the leading corrections due to the black hole geometry.

The analytical expressions will then be compared with numerical solutions. To ensure the validity of the perturbative expansion, the numerical results presented in this section are restricted to the small interval regime with $t_0 \lesssim z_h = 1$.

We now turn to higher-dimensional AdS_{d+1} black hole geometries. The metric takes the form $ds^2 = \frac{1}{z^2}(-f(z) \times dt^2 + \frac{dz^2}{g(z)} + dx^2 + d\vec{y}^2)$, where $f(z) = g(z) = 1 - \frac{z^d}{z_H^d}$. For simplicity, we set $z_H = 1$ as follows. The corresponding

Euclidean metric is given by $ds^2 = \frac{1}{z^2}(f(z)d\tau^2 + \frac{dz^2}{g(z)} + dx^2 + d\bar{y}^2)$.

Talking into account a symmetric timelike interval $(0, 0; \tau_0, 0)$, we parametrize the extremal surface by $\tau = \tau(z)$ and $x = 0$. The associated Lagrangian becomes $\mathcal{L} = \frac{1}{z^{d-1}} \sqrt{(1-z^d)\tau'(z)^2 + \frac{1}{1-z^d}}$. So, the conserved constant is $p_\tau = \frac{(1-z^d)\tau'}{z^{d-1}\sqrt{(1-z^d)\tau'^2 + \frac{1}{1-z^d}}}$, which yields

$$\tau'(z)^2 = \frac{p_\tau^2}{(1-z^d)^2[(1-z^d)z^{2(1-d)} - p_\tau^2]}. \quad (58)$$

The turning point z_τ is defined by the divergence of $\tau'(z)$, leading to the condition

$$p_\tau^2 = (1-z_\tau^d)z_\tau^{2(1-d)}. \quad (59)$$

Substituting this into the area function, we obtain the following:

$$\begin{aligned} \mathcal{A}(0, 0; \tau_0, 0) &= 2 \int_\delta^{z_\tau} dz \frac{1}{z^{2(d-1)}} \\ &\times \sqrt{\frac{1}{(1-z^d)z^{2(1-d)} - (1-z_\tau^d)z_\tau^{2(1-d)}}}. \end{aligned} \quad (60)$$

This integral can be perturbatively evaluated in the small τ_0 regime, yielding

$$\begin{aligned} \mathcal{A}(0, 0; \tau_0, 0) &= \frac{2}{(d-2)\delta^{d-2}} - \frac{2\sqrt{\pi}\Gamma\left(\frac{d}{2(d-1)}\right)}{z_{\tau_0}^{d-2}(d-2)\Gamma\left(\frac{1}{2(d-1)}\right)} \\ &+ \frac{1}{2}\sqrt{\pi}z_{\tau_0}^2 \left[\frac{(d-3)\Gamma\left(\frac{d}{d-1}\right)}{(d-1)\Gamma\left(\frac{d+1}{2(d-1)}\right)} \right. \\ &\left. + \frac{2\Gamma\left(\frac{d}{2(d-1)}\right)}{(d-1)\Gamma\left(\frac{1}{2(d-1)}\right)} \right]. \end{aligned} \quad (61)$$

On the other hand, imposing the boundary condition $\int_0^{z_\tau} \tau'(z)dz = \frac{\tau_0}{2}$ allows us to solve for z_τ perturbatively,

$$z_{\tau_0} = \zeta_0 \tau_0 + \zeta_d \tau_0^{d+1}, \quad (62)$$

where

$$\begin{aligned} \zeta_0 &= \frac{\Gamma\left(\frac{1}{2(d-1)}\right)}{2\sqrt{\pi}\Gamma\left(\frac{d}{2(d-1)}\right)}, \\ \zeta_0^{d+1} &\left(2d-2 + \frac{(d-3)\Gamma\left(\frac{1}{2(d-1)}\right)\Gamma\left(\frac{d}{d-1}\right)}{\Gamma\left(\frac{1-3d}{2-2d}\right)\Gamma\left(\frac{d}{2(d-1)}\right)} \right) \\ \zeta_d &= -\frac{4(d-1)^2}{4(d-1)^2}. \end{aligned} \quad (63)$$

Finally, performing the Wick rotation $\tau_0 \rightarrow it_0$ gives the analytic expressions in Lorentzian signature, which we now proceed to evaluate for specific examples.

For the AdS_4 case ($d=3$), the perturbative expansion yields

$$\mathcal{A}(0, 0; t_0, 0) = \frac{2}{\delta} + i \frac{8\pi^3}{\Gamma\left(\frac{1}{4}\right)^4} \frac{1}{t_0}. \quad (64)$$

Figure 16 compares this analytical expression with numerical results. We observe excellent agreement in the small- t_0 regime, confirming the validity of the perturbative expansion. As t_0 increases toward the horizon scale $z_H = 1$, slight deviations begin to appear, although overall behavior remains qualitatively consistent and the approximation is still reasonably accurate.

Obviously, our result matches precisely the asymptotic behavior obtained in Fig. 4 of Ref. [29] as $\Delta t \rightarrow 0$, corresponding to the regularized area $\mathcal{A}_{\text{reg}}^{\text{v.c.}}$ of the vacuum-connected solution in their terminology.

As shown in [47], in higher-dimensional AdS_{d+1} spacetimes with Poincaré coordinates, the EE (excluding UV-divergent parts) depends on dimensional parity: it becomes purely imaginary for even-dimensional spacetimes AdS_{even} (d odd), while remaining strictly real for odd-dimensional spacetimes AdS_{odd} (d even). For higher-dimensional AdS_{d+1} black hole scenarios, the situation becomes more complex. Nevertheless, we still observe a clear distinction between odd and even dimensions. Therefore, in addition to the AdS_4 ($d=3$) case, we particularly examine the AdS_5 ($d=4$) scenario to highlight this contrast.

For the AdS_5 case ($d=4$), combining (61) and (62) yields,

$$\mathcal{A}(0, 0; t_0, 0) = \frac{1}{\delta^2} + 0.321 \frac{1}{t_0^2} - 0.508 t_0^2 + 0.505 t_0^6. \quad (65)$$

Here, we omit the explicit expressions for the Γ -function and present the coefficients numerically for clarity.

In contrast to the AdS_4 case, where the RT surface area develops an imaginary part, the area in AdS_5 remains strictly real at all computed orders, consistent with the parity behavior observed in the vacuum case. In Fig. 17(a), we compare the numerical results with the perturbative predictions. As in the AdS_4 case, the perturbative expansion

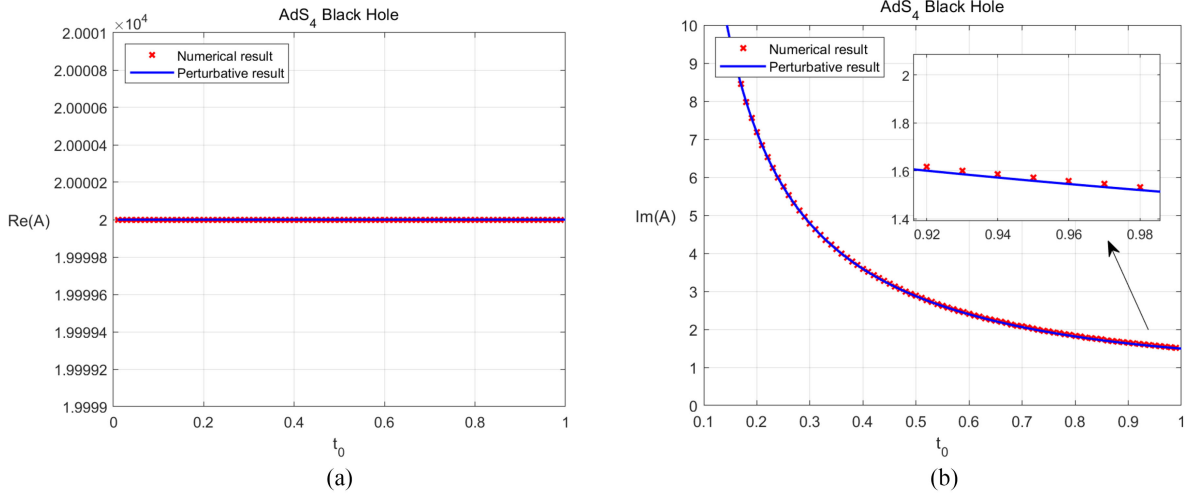


FIG. 16. Area $\mathcal{A}(0, 0; t_0, 0)$ for the timelike RT surface in the AdS₄ black hole background. The red crosses denote numerical results, while the blue curves represent the perturbative analytical result. Panel (a) shows the real part; panel (b) displays the imaginary part. We set $z_H = 1$ and $\delta = 10^{-4}$, with t_0 sampled over $(0, z_H = 1)$ in steps of 0.01.

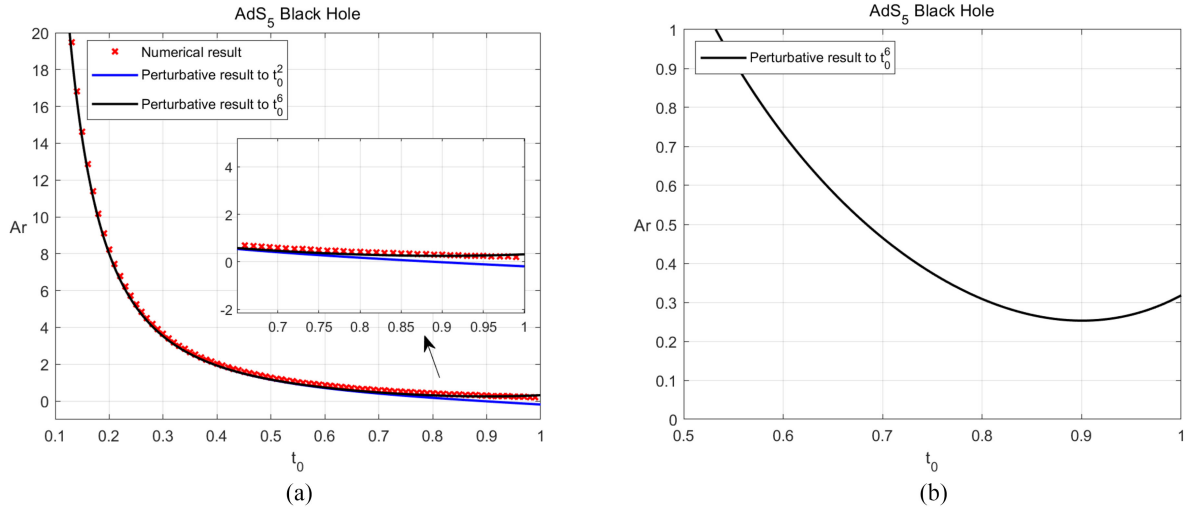


FIG. 17. Regularized area $\mathcal{A}_r = \mathcal{A}(0, 0; t_0, 0) - \frac{1}{\delta}$ as a function of interval length t_0 in the AdS₅ black hole case. Panel (a) compares the full numerical results (red circles) with the perturbative expressions truncated at order t_0^2 (blue curve) and t_0^6 (black curve). Panel (b) shows the analytic perturbative expansion up to t_0^6 only, with the vertical axis rescaled to highlight fine features such as local extrema. We set $z_H = R = 1$, sample t_0 over $(0, z_H = 1)$ with step size 0.01, and use a UV cutoff $\delta = 10^{-4}$.

agrees well with the numerics in the small $t_0 \ll z_H$ regime. Moreover, we observe that including higher-order terms in t_0 significantly improves agreement across a wider range of t_0 , with the t_0^6 truncation (black curve) closely following the numerical data up to $t_0 \sim z_H$. Interestingly, the perturbative expansion up to order t_0^6 shows a local extremum in Fig. 17(b), which is absent in the numerical data. We interpret this feature as a spurious artifact of the truncated expansion, and we expect it to disappear once higher-order terms are included, restoring the smooth monotonic behavior seen numerically.

IV. OTHER GEOMETRIC QUANTITY IN THE TIMELIKE REGION

In previous sections, we focus on the EE, which plays an important role as a measure of entanglement. We show that it can be properly defined via the transition operator T_{AB} , which is generally non-Hermitian. If the concept of entanglement in time is meaningful, it should be possible to define other entanglement measures for causally connected subregions. In the spacelike case, entanglement negativity and reflected entropy are two interesting measures that are well studied in both QFTs and holography.

The reflected entropy is associated with the entanglement wedge of the corresponding bulk region, and the EWCS [16,17] provides a natural holographic bulk counterpart [18].

However, the definition of these measures is generally more complicated than that of the EE. For two subregions A and B on a given Cauchy surface, a key step in the construction of the reflected entropy is the canonical purification of the reduced density matrix ρ_{AB} . It is standard to construct this purification since ρ_{AB} is a positive operator. But for causally connected subregions A and B , the associated transition operator T_{AB} is generally non-Hermitian. Thus, there are substantial conceptual and technical obstacles to generalizing the canonical purification to a non-Hermitian operator. In this paper, we will not try to find such an extension.

In QFTs, entanglement measures can be related to correlation functions of twist operators. In previous studies of timelike EE, one can define and compute the timelike EE via analytic continuation of twist operators. Motivated by this, we consider the analytic continuation of the reflected-entropy formalism to the timelike domain and examine the corresponding geometric structures in the bulk, specifically the analytic continuation of the cross section surface. In the following, we will show that it appears necessary to consider a complexified geometry in order to obtain a bulk description consistent with the QFT results. This can be viewed as support for the framework of complexified extremal surfaces proposed by Heller *et al.* [29].

A. Entanglement wedge and its cross section for spacelike subregions

The entanglement wedge is the bulk region that is dual to the reduced density matrix ρ_A . For two subregions A and B located on a common Cauchy surface Σ_t , the corresponding RT surface can exhibit two distinct phases; a connected phase and a disconnected phase. The definition of EWCS is based on the connected phase of the entanglement wedge and is believed to capture the quantum correlations between A and B . In this paper, we focus mainly on the $\text{AdS}_3/\text{CFT}_2$ correspondence.

For the spacelike case, consider two disjoint subsystems A and B on the boundary of AdS_3 . Let Γ_{AB} denote the RT surface associated with the union $A \cup B$. The entanglement wedge M_{AB} is then defined as the bulk region enclosed by the boundary subregions A , B , and the corresponding RT surface Γ_{AB} ,

$$\partial M_{AB} = A \cup B \cup \Gamma_{AB}. \quad (66)$$

Within M_{AB} , consider the minimal surface Σ_{AB} that divides M_{AB} into two subregions, M_A and M_B , such that $A \subset M_A$, $B \subset M_B$. This surface Σ_{AB} is called the EWCS. The setup is illustrated in Fig. 18, which helps visualize the definitions and geometric structure.

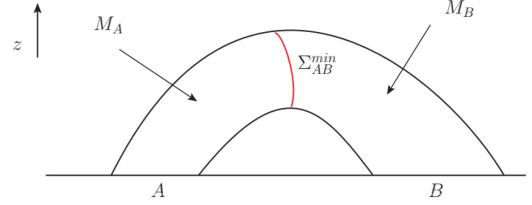


FIG. 18. Illustration of the entanglement wedge and its cross section for two subregions A and B . The RT surface is in the connected phase, and $\Sigma_{AB}^{\min}$ (red line) denotes the EWCS.

The EWCS $\Sigma_{AB}^{\min}$ is defined as the minimal area surface among all possible surfaces Σ_{AB} within the entanglement wedge that divides M_{AB} into two regions M_A and M_B , which contain A and B , respectively. Operationally, this reduces to finding the minimal surface within the entanglement wedge that partitions it appropriately. The EWCS is then defined by

$$E_W(A:B) = \frac{\text{Area}(\Sigma_{AB}^{\min})}{4G}. \quad (67)$$

This can be seen as an extension of the RT formula to bulk subregions.

B. Analytic continuation of the reflected entropy formalism

In the spatial domain, the reflected entropy is a measure of correlations between two subregions A and B , computed via the canonical purification. Holographically, it is dual to the area of the EWCS. While the full quantum information interpretation of such quantities in the timelike domain remains an open question, it is instructive to examine the behavior of the relevant correlation functions and their geometric duals under analytic continuation. In this subsection, we focus on the analytic continuation of the twist operator formalism to obtain the boundary quantities that our bulk analysis will seek to reproduce.

For $(1+1)D$ CFTs, the reflected entropy for spatial intervals $A = [x_1, x_2]$ and $B = [x_3, x_4]$ can be computed via the replica trick using twist operators [18],

$$\begin{aligned} S_R(A:B) &= \lim_{n \rightarrow 1} S_n(AA^*)_{\psi_m} \\ &= \lim_{n \rightarrow 1} \frac{1}{1-n} [\log \langle \sigma_{g_A}(0, x_1) \sigma_{g_A^{-1}}(0, x_2) \\ &\quad \times \sigma_{g_B}(0, x_3) \sigma_{g_B^{-1}}(0, x_4) \rangle \\ &\quad - n \log \langle \sigma_{g_m}(0, x_1) \sigma_{g_m^{-1}}(0, x_2) \\ &\quad \times \sigma_{g_m}(0, x_3) \sigma_{g_m^{-1}}(0, x_4) \rangle], \end{aligned} \quad (68)$$

where the twist operators σ_g implement cyclic permutations. The calculation reduces to evaluating a four-point function. By keeping the dominant conformal block, one

obtains the standard spatial result,

$$S_R(A:B) \simeq \frac{c}{6} \left(\log \left(\frac{1 + \sqrt{\eta}}{1 - \sqrt{\eta}} \right) + \log \left(\frac{1 + \sqrt{\bar{\eta}}}{1 - \sqrt{\bar{\eta}}} \right) \right), \quad (69)$$

where η and $\bar{\eta}$ are the conformal cross ratios.

We now proceed to formally extend this calculation to the timelike case via analytic continuation. In Euclidean signature, we write coordinates as $w = x + i\tau$. The transition to the Lorentzian timelike domain is achieved by the rotation $\tau_i \rightarrow i t_i + \epsilon_i$, with the ordering $\epsilon_4 > \epsilon_3 > \epsilon_2 > \epsilon_1 > 0$ to enforce the correct causal ordering for the intervals $A = [t_1, t_2]$ and $B = [t_3, t_4]$.

The analytic continuation of the quantity S_R , which we denote as $\tilde{S}_R(T_1: T_2)$, is formally defined by evaluating (69) with the timelike cross ratios,

$$\begin{aligned} \eta_t &= \frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_3)(t_2 - t_4)} + i\tilde{\epsilon}, \\ \bar{\eta}_t &= \frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_3)(t_2 - t_4)} - i\tilde{\epsilon}. \end{aligned} \quad (70)$$

We emphasize that we treat $\tilde{S}_R$ here primarily as a specific twist-operator correlation function continued to the timelike regime.

Choosing a symmetric configuration such that $-t_1 = t_4 = \Delta t_1$ and $-t_2 = t_3 = \Delta t_2$, the vacuum result becomes

$$\tilde{S}_R(T_1: T_2) = \frac{c}{3} \log \frac{\Delta t_1}{\Delta t_2}. \quad (71)$$

Similarly, for a thermal state with temperature β^{-1} , applying the conformal map $\xi = \exp(-\frac{2\pi w}{\beta})$ yields the analytically continued thermal result,

$$\tilde{S}_R(T_1: T_2) = \frac{c}{3} \log \frac{\tanh \frac{\pi \Delta t_1}{\beta}}{\tanh \frac{\pi \Delta t_2}{\beta}}. \quad (72)$$

Our main goal is to investigate the bulk geometric structures corresponding to these analytically continued quantities. In the standard spatial holography, S_R is dual to the EWCS. In the timelike context, finding a real-valued extremal surface that matches (71) or (72) is nontrivial because the bulk geometry is Lorentzian, and saddle points of the area functional may not exist in the real section.

To address this, we employ the complexified geodesic formalism discussed in Sec. III (and see also [29]). By allowing the embedding functions of the extremal surfaces to take values in the complex domain, we can locate the specific saddle points that reproduce the boundary correlation functions computed above. This defines a geometric analog of the cross section in the complexified bulk.

C. Analytic continuation of extremal surfaces in the timelike region

In the previous subsection, we derived the analytically continued boundary quantity $\tilde{S}_R$. In this section, our goal is to identify the corresponding geometric structure in the bulk. Specifically, we seek a surface within the complexified bulk geometry whose area satisfies the stationarity condition and reproduces the value of $\tilde{S}_R$. We refer to this surface as the geometric analog of the cross section in the timelike domain.

Since the EWCS is defined only after specifying the RT/HRT surface and the corresponding entanglement wedge, its timelike generalization must first be formulated at the level of the underlying extremal surfaces. In the spatial case, the EWCS is defined using real minimal surfaces in the bulk. For timelike boundary intervals, however, the relevant extremal surfaces are obtained by analytic continuation into the complexified bulk, as discussed in [29] and Sec. III. Accordingly, the timelike analog of the cross section should not be viewed as a naive Wick rotation of the ordinary real EWCS, but rather as a stationary surface defined with respect to the complexified RT geometry. This bulk construction is introduced to reproduce the analytically continued boundary quantity $\tilde{S}_R$ derived in the previous subsection.

Before defining the cross section analog, we must address the topological structure of the bulk regions bounded by these complexified surfaces. As demonstrated in Sec. IID 2 and Fig. 19, there are two distinct phases depending on the interval configurations.

In Phase I, the RT surfaces corresponding to the timelike intervals intersect or overlap in such a way that the region bounded by them is homotopic to two disconnected components [see Fig. 19(a)]. Geometrically, there is no connected “wedge” linking the two intervals through the bulk. Consequently, no nontrivial cross section surface exists that connects the two asymptotic boundaries. In this phase, the geometric analog of the EWCS vanishes, which

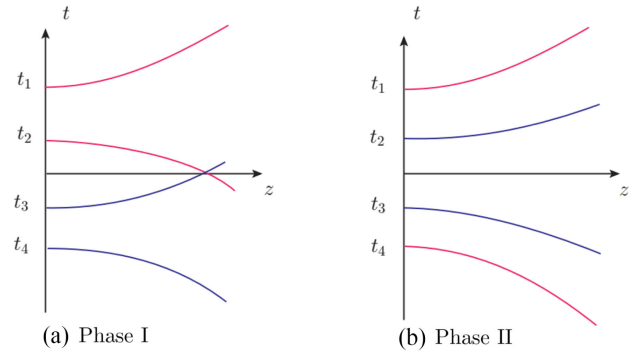


FIG. 19. Illustration of two different phases of the bulk RT surfaces, with only the Lorentzian parts shown. (a) corresponds to Phase I, matching the CFT result in (22); (b) corresponds to Phase II, matching (23).

naturally matches the vanishing of the logarithmic term in the boundary correlation function in this regime.

In Phase II, the topology changes. The bounding surfaces form a connected region $M_{T_1 T_2}$ [see Fig. 19(b)]. We formally define the boundary of this region as

$$\partial M_{T_1 T_2} = T_1 \cup T_2 \cup \Gamma_{T_1 T_2}, \quad (73)$$

where $\Gamma_{T_1 T_2}$ represents the union of the complexified RT surfaces. In this connected phase, it is topologically possible to define a surface $\Sigma_{T_1 T_2}$ that partitions $M_{T_1 T_2}$, analogous to the cross section in the spatial case.

We emphasize that we treat $M_{T_1 T_2}$ here strictly as a geometric construct defined by the complexified geodesics. We do not claim that it possesses all the information-theoretic properties (such as subregion duality or operator reconstruction [8]) associated with the spatial entanglement wedge, as the role of the modular Hamiltonian in the timelike domain remains an open question.

We now define the specific geometric condition for the surface $\Sigma_{T_1 T_2}$. In the spatial case, one minimizes the area. However, in our complexified framework, the area is generally a complex-valued function, rendering simple minimization ill-defined. Instead, we impose a saddle-point condition.

We parametrize the surface $\Sigma_{T_1 T_2}$ by its endpoints on the RT surfaces $\Gamma_{T_1 T_2}$. For explicit calculations in AdS_3 , these endpoints can be specified by complex parameters λ_1 and λ_2 . The generalized area is then a complex analytic function $\mathcal{A}_{T_1 T_2}(\lambda_1, \lambda_2)$.

The geometric analog of the EWCS is determined by the surface $\Sigma_{T_1 T_2}^s$ that satisfies the stationarity conditions

$$\partial_{\lambda_1} \mathcal{A}_{T_1 T_2}(\lambda_1, \lambda_2) = 0, \quad \partial_{\lambda_2} \mathcal{A}_{T_1 T_2}(\lambda_1, \lambda_2) = 0. \quad (74)$$

The value associated with this saddle point is

$$E_W(T_1 : T_2) = \frac{\text{Area}(\Sigma_{T_1 T_2}^s)}{4G}. \quad (75)$$

This condition generalizes the minimal area prescription: in the real domain, the stationary point of a positive functional corresponds to a minimum, whereas in the complex domain, it corresponds to a saddle point. A similar stationarity condition has been utilized in the context of timelike entropy in [28], suggesting it is the natural geometric prescription for analytically continued gravity duals.

1. Pure AdS_3 Poincare coordinate

Let us first consider the two time intervals in the vacuum state. The bulk metric in the Poincare coordinate is

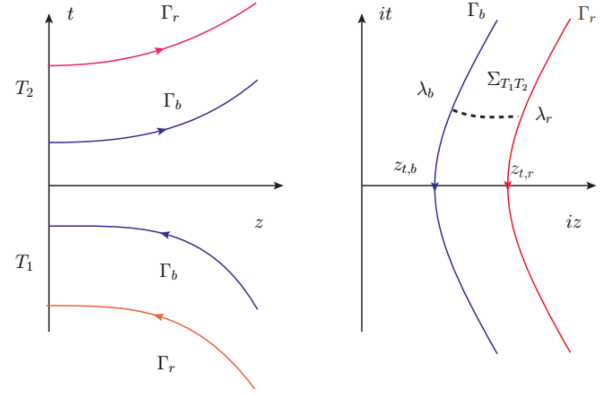


FIG. 20. Figure to show the connected phase for two time intervals T_1 and T_2 . $\Gamma_r \cup \Gamma_b$ is the RT surface for the two time intervals. λ_b and λ_r are the parameters on the RT surface. $\Sigma_{T_1 T_2}$ is the extreme surface in the entanglement wedge. $z_{t,b}$ and $z_{t,r}$ denotes the turning points of the RT surface in the complexified geometry.

$$ds^2 = \frac{-dt^2 + dz^2 + dx^2}{z^2}. \quad (76)$$

Without loss of generality we will choose symmetric intervals with $-t_1 = t_4 = \Delta t_1$ and $-t_2 = t_3 = \Delta t_2$. Analogous to the spatial case, the configuration of extremal surfaces for timelike intervals exhibits two phases depending on the ratio of Δt_1 and Δt_2 . We will consider the connected phase with $\frac{\Delta t_1}{\Delta t_2} > 3 + 2\sqrt{2}$. We can write down the equations for the geodesic lines,

$$\begin{aligned} \Gamma_r : z &= \sqrt{t^2 - \Delta t_1^2}, \\ \Gamma_b : z &= \sqrt{t^2 - \Delta t_2^2}, \end{aligned} \quad (77)$$

which are shown in Fig. 20. By using the complexified geodesic formalism, the above geodesic lines correspond to the saddle points relevant for the single-interval quantities. The imaginary parts are given by the geodesic line in the complex geometry.

In the connected phase, the configuration of the bounding surfaces is shown in Fig. 20. There exists a bulk region bounded by the extremal surfaces (blue and red lines in Fig. 20, denoted as $\Gamma_{T_1 T_2}$ here) and the boundary intervals $T_1 \cup T_2$.

Within this connected region, one can consider extreme surfaces $\Sigma_{T_1 T_2}$ ending on the bounding surface $\Gamma_r \cup \Gamma_b$. The parameters of the ending points are denoted λ_r and λ_b in Fig. 20. It should be noted that the extreme surface $\Sigma_{T_1 T_2}$ (and the connecting surface) may also be complex, and thus its area $\mathcal{A}(\lambda_r, \lambda_b) := \text{Area}(\Sigma_{T_1 T_2})$ may be complex valued. In the spatial case, the minimal cross section of the entanglement wedge is the geometric dual to the reflected entropy. In the timelike case, among all the extreme surfaces

$\Sigma_{T_1 T_2}$, we select the one satisfying the stationarity condition (74). We identify the value associated with this saddle point as the geometric analog of the cross section,

$$E_W(T_1, T_2) = \frac{\text{Area}(\Sigma_{T_1 T_2}^s)}{4G}. \quad (78)$$

For the case we consider here, we can choose the time t as the parameters of Γ_r and Γ_b , that is, $\lambda_r = t_r$ and $\lambda_b = t_b$. $\Sigma_{T_1 T_2}$ is given by the geodesic line $t = t(z)$ connecting $(t, z) = (t_r, \sqrt{t_r^2 - \Delta t_1^2})$ and $(t, z) = (t_b, \sqrt{t_b^2 - \Delta t_2^2})$. $\Sigma_{T_1 T_2}$ satisfies the relation,

$$\frac{t'(z)^2}{z^2(1 - t'(z)^2)} = p^2, \quad (79)$$

where p is the constant, which is given by

$$p^2 = 4(t_r - t_b)^2[-2\Delta t_1^2(2t_b(t_r - t_b) + \Delta t_2^2) + 4\Delta t_2^2 t_r(t_r - t_b) + \Delta t_1^4 + \Delta t_2^4]^{-1}.$$

With this equation we can obtain the area function $\mathcal{A}(t_r, t_b)$ of $\Sigma_{T_1 T_2}$. The expression of $\mathcal{A}(t_r, t_b)$ can be found in Appendix C 1. We can also show that

$$\partial_{t_r} \mathcal{A}(0, 0) = \partial_{t_b} \mathcal{A}(0, 0) = 0. \quad (80)$$

The saddle point is given by $t_r = t_b = 0$, which are the turning points of Γ_r and Γ_b on the complexified geometry. The geodesic line $\Sigma_{T_1 T_2}^s$ is given by the geodesic line connecting these two turning points. It is straightforward to calculate the length of the $\Sigma_{T_1 T_2}^s$,

$$E_W(T_1: T_2) = \frac{\mathcal{A}(0, 0)}{4G} = \frac{1}{4G} \log \frac{\Delta t_1}{\Delta t_2}. \quad (81)$$

In fact, in this example we obtain other solutions for Eq. (74). However, two of them correspond to extreme lines located at the AdS boundary, for which the length diverges. These are therefore not regarded as the relevant geometric saddle points. See Appendix C 1 for further details.

In this paper, we primarily focus on intervals along the time direction. For more general intervals, however, richer structures emerge. In fact, the condition (74) leads to two saddle points for the length between the two RT surfaces associated with a timelike interval; one lying in the Lorentzian manifold and the other in the complex domain. For two generic timelike intervals on the AdS₃ boundary, as shown in [48], the Lorentzian saddle yields the entanglement wedge cross section E_W as²

$$E_W = \frac{1}{8G} \log \frac{(\sqrt{\eta} - 1)(\sqrt{\bar{\eta}} + 1)}{(\sqrt{\eta} + 1)(\sqrt{\bar{\eta}} - 1)},$$

where η and $\bar{\eta}$ are the cross ratios. The saddle in the complex domain, on the other hand, reproduces the expression we derived earlier, and in the general case gives

$$E_W = \frac{1}{8G} \log \frac{(\sqrt{\eta} + 1)(\sqrt{\bar{\eta}} + 1)}{(\sqrt{\eta} - 1)(\sqrt{\bar{\eta}} - 1)}. \quad (82)$$

This expression is structurally identical to (69), indicating that the complex saddle correctly captures the analytic continuation of the boundary twist operator correlation function. It would be worthwhile to further investigate the geometric significance of the Lorentzian saddle.

2. BTZ black hole

Now let us consider a single interval in the thermal state, which is dual to the planar BTZ black hole,

$$ds^2 = \frac{-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2}{z^2}, \quad (83)$$

with $f(z) = 1 - \frac{z^2}{z_H^2}$. The black hole horizon is at $z = z_H$, which is related to the inverse temperature $\beta = 2\pi z_H$.

Consider that the timelike interval $T_1 = [-t, 0]$, T_2 is taken to be its complement. Analogous to the geometric construction of the cross section, we seek an extremal surface that partitions the bulk region associated with T_1 and T_2 . We consider the extreme surface $\Sigma_{T_1 T_2}$, which is anchored at the boundary of T_1 and T_2 , satisfying

$$\frac{f(z)^3 t'(z)^2}{(1 - f(z)^2 t'(z)^2) z^2} = p^2, \quad (84)$$

where p is a constant. Similar as in the spacelike case there exist two kinds of surface. One is the surface with $p \neq 0$ labeled by $\Sigma_{T_1 T_2}^{(1)}$, which starts from the boundary $z = \delta$ and ends on the future and past singularity. The other is the surface labeled by $\Sigma_{T_1 T_2}^{(2)}$, which corresponds to $p = 0$ or t being constant. See Fig. 21 for an illustration. For the geodesic line $\Sigma_{T_1 T_2}^{(2)}$, the length is given by

$$L^{(2)} = 2 \log \frac{\beta}{\delta\pi}. \quad (85)$$

For $\Sigma_{T_1 T_2}^{(1)}$ one could vary the ending points on the past and future singularities. One could obtain the stationary one $\Sigma_{T_1 T_2}^{(1),s}$, which is discussed in [28]. The length is

$$L^{(1)} = 2 \log \frac{\beta}{\pi\delta} \sinh\left(\frac{\pi t}{\beta}\right). \quad (86)$$

²We would like to thank the authors of [48] for pointing this out.

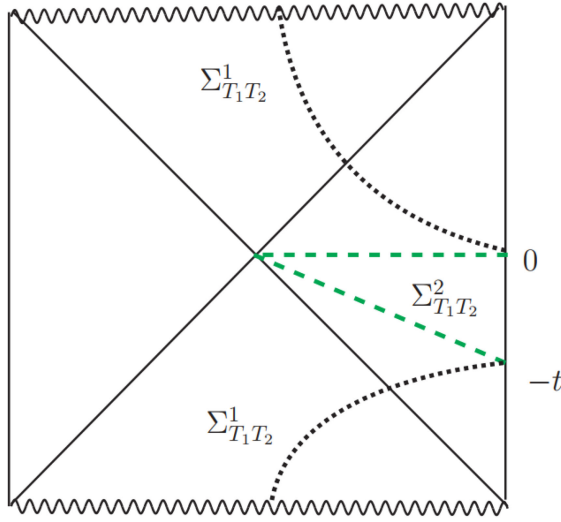


FIG. 21. The geometric configuration for the intervals $T_1 = [-t, 0]$ and $T_2 = (-\infty, -t] \cup [0, +\infty)$. Two types of geodesics are shown; $\Sigma_{T_1 T_2}^1$ (black dotted line) and $\Sigma_{T_1 T_2}^2$ (green dashed line).

Note that in this example, the area of the two surfaces are real and positive. Based on the saddle point analysis, the value corresponding to the geometric analog of the cross section in this case is

$$E_W(T_1 : T_2) = \frac{1}{4G} \min\{L^{(1)}, L^{(2)}\}. \quad (87)$$

Let us consider two intervals in planar BTZ black hole, to simplify the calculation, we also consider the symmetry time intervals $T_1 = [t_1, t_2]$ and $T_2 = [t_3, t_4]$ with $-t_1 = t_4 = \Delta t_1$ and $-t_2 = t_3 = \Delta t_2$.

We will consider the connected phase for the RT surfaces. We can write down the RT surface for two time intervals by using the result in Appendix B 2,

$$\begin{aligned} \Gamma_r : z_r &= z_H \sqrt{\frac{\cosh\left(\frac{2t_r}{z_H}\right) - \cosh\left(\frac{\Delta t_1}{2z_H}\right)}{1 + \cosh\left(\frac{2t_r}{z_H}\right)}}, \\ \Gamma_b : z_b &= z_H \sqrt{\frac{\cosh\left(\frac{2t_b}{z_H}\right) - \cosh\left(\frac{\Delta t_2}{2z_H}\right)}{1 + \cosh\left(\frac{2t_b}{z_H}\right)}}, \end{aligned} \quad (88)$$

where $\Gamma_r \cup \Gamma_b$ is the RT surface similar to the Poincaré case. We can choose the time t_r and t_b as parameters to evaluate the length of $\Sigma_{T_1 T_2}$. The stationary point of the length of $\Sigma_{T_1 T_2}$ gives the solution $t_r = t_b = 0$. The geometric analog of the EWCS is given by

$$E_W(T_1, T_2) = \frac{1}{4G} \log \left(\frac{\tanh\left(\frac{\Delta t_1}{2z_H}\right)}{\tanh\left(\frac{\Delta t_2}{2z_H}\right)} \right). \quad (89)$$

The calculation details can be found in Appendix C 2.

V. SUMMARY AND DISCUSSION

In this paper, we investigate the concept of entanglement in time in both QFTs and their holographic duals. Previous studies have primarily focused on quantum correlations between spacelike-separated subregions, leading to fruitful results that have significantly advanced our understanding of entanglement structures in QFTs and the holographic properties of gravity, particularly through the duality between entanglement measures and bulk geometry.

The idea of defining a “spacetime density matrix” provides a natural generalization of entanglement to subregions that are timelike separated [34]. This framework enables the definition of a transition operator T_{AB} , which is generally non-Hermitian in the timelike case and reduces to the standard reduced density matrix in the spacelike limit. The construction is fundamental and applicable to arbitrary quantum systems.

Our main focus is to further develop the notion of entanglement in time within QFTs. The transition operator T_{AB} can be prepared using the Schwinger-Keldysh formalism. This allows us to define entropy-related quantities, such as the pseudo-Rényi entropy and the von Neumann-like EE associated with T_{AB} . These quantities are computable using the real-time replica method, as demonstrated in Sec. II, thereby providing a concrete and well-defined notion of timelike EE.

In QFTs, evaluating these entropies is equivalent to computing correlation functions of twist operators with timelike separations. In this paper, we have considered several simple examples to illustrate this framework. However, the same setup is flexible enough to accommodate more general states. For instance, one could prepare the initial state as an excited state (see, e.g., [49–51]), and study the corresponding timelike EE during its dynamical evolution. It would be particularly interesting to explore the novel features that arise in such scenarios.

In this paper, we have primarily focused on the case where the subsystems A and B are semi-infinite, motivated by the goal of relating the EE of the transition operator T_{AB} to the timelike EE defined for time intervals in [25]. However, more general choices of subregions A and B are possible. The EE for T_{AB} in such cases can be expressed in terms of more general correlation functions of twist operators. We anticipate that new features and phenomena will emerge in these extended settings. As a simple extension of Example I in Sec. IID 1, one may consider both A and B to be the semi-infinite interval $[0, +\infty)$, located on different Cauchy surfaces. Analyzing such

general configurations would require a more detailed understanding of correlation functions in Lorentzian spacetime.

In Sec. III, we turn to the holographic dual description of timelike EE. At present, the holographic duality of timelike EE is not yet fully understood. However, several existing studies [25,27–30] offer valuable hints. In particular, since timelike EE can, in some cases, be computed via analytical continuation of correlation functions of twist operators, it is natural to expect that a corresponding holographic dual exists for the associated RT surface.

In this paper, we adopt this perspective and construct the corresponding RT surfaces in both AdS_3 and higher-dimensional examples. There is some recent progress in exploring holographic timelike EE in more general backgrounds; see, e.g., [52–53]. We compare analytical results with numerical computations and find excellent agreement in the cases considered. Furthermore, we observe novel features of timelike EE in AdS black hole backgrounds, especially in dimensions $d > 3$. For instance, in Sec. III D 2, we investigate the case of an AdS_5 black hole and find that the timelike EE is real-valued. However, it differs qualitatively from the spacelike EE. While spacelike EE captures thermal entropy and increases monotonically with the size of the subregion (e.g., a strip), the perturbative calculation of timelike EE may exhibit nonmonotonic behavior at finite order. Nonetheless, the full numerical results indicate that the true behavior is smooth and monotonic. In other dimensions, where the timelike EE is generally complex-valued, the real or imaginary parts similarly reflect smooth dependence on the interval size. Understanding these features may provide valuable insight into the physical meaning of timelike entanglement.

On the quantum field theory side, the quantity $\text{Tr}(T_{AB}^n)$ can be formulated via the Schwinger-Keldysh path integral. According to the real-time AdS/CFT correspondence, such a path integral should admit a bulk dual in terms of an appropriate gravitational configuration. Inspired by the Lewkowycz-Maldacena procedure to derive the RT formula in the Euclidean signature [54], and by the real-time replica method to prove the HRT formula in Lorentzian signature [39], it is plausible that there is a similar derivation for the holographic dual of timelike EE. This would not only support the validity of our proposal but also deepen our understanding of real-time holography in the AdS/CFT correspondence.

The entanglement wedge and its cross section are established concepts for characterizing entanglement between spacelike subregions. In this paper, we investigate the geometric structures that arise when these concepts are analyzed in the timelike context via analytic continuation. Since the configuration of extremal surfaces for timelike-separated regions involves complexified geometry, the construction of the associated bulk region requires a careful treatment of these complex saddle points. Consequently,

the geometric analog of the cross section is identified as the surface that satisfies the stationarity condition for the area functional within this complexified region, as discussed in Sec. IV C. We have highlighted the geometric similarities and distinctions between the spacelike and timelike scenarios. In Sec. IV, we presented explicit examples in AdS_3 , where we found that the area of this geometric saddle point remains positive. A primary motivation for this construction stems from the analytic continuation of the reflected entropy formalism. In the boundary theory, the correlation functions defining reflected entropy can be analytically continued into the Lorentzian regime, yielding well-defined results. This naturally suggests the existence of a corresponding bulk geometric quantity. As we have demonstrated, the saddle-point surface constructed in the complexified bulk provides a consistent geometric analog for this quantity, thereby offering nontrivial support for the validity of the complexified RT surface framework.

It is of interest to investigate whether the timelike entanglement wedge plays a similar role to its spacelike counterpart. We expect the answer to be negative. The reconstruction theorem established in [8] relies on the equivalence between bulk and boundary relative entropy [7]. However, for the transition operator, which is generally non-Hermitian, there is no well-defined notion of relative entropy. This fundamental distinction suggests that the timelike entanglement wedge differs substantially from the spacelike case. Nonetheless, certain expected structural properties can still be observed from the perspective of operator algebras. In [55–57], the author discusses the timelike tube theorem in curved spacetime. A timelike tube $\mathcal{E}(\gamma)$ associated with a timelike curve γ consists of all points that can be reached by deforming timelike curves with fixed endpoints, as illustrated in Fig. 22(a). In Sec. 3.4 of [57], a connection is drawn between the timelike tube theorem and the AdS/CFT reconstruction theorem.

Specifically, for a time interval T on the boundary CFT, one can identify a corresponding timelike tube $\mathcal{E}(T)$ in the bulk, as shown in Fig. 22(b). The timelike tube theorem then asserts

$$\mathcal{R}(T) = \mathcal{R}(\mathcal{E}(T)), \quad (90)$$

where $\mathcal{R}(\cdot)$ denotes the operator algebra associated with the specified region. Notably, the timelike tube $\mathcal{E}(T)$ in AdS coincides with the causal wedge of T . Hence, the timelike tube theorem (90) is equivalent to the causal wedge reconstruction theorem in the context of AdS/CFT [58–60].

In our study of timelike EE, we focus on a time interval T on the boundary CFT. From the bulk perspective, the corresponding RT surface does not lie within the causal wedge but instead extends into the deep IR region of the bulk. This suggests that the bulk geometry associated with the interval T probes regions beyond the causal wedge or the timelike tube defined by T . This observation also supports

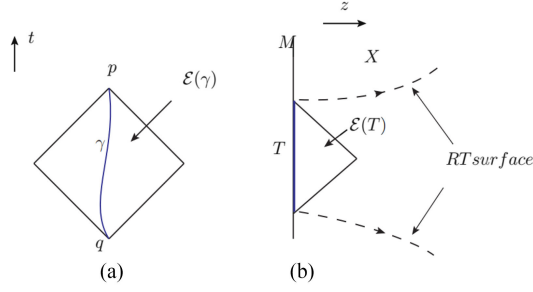


FIG. 22. Figure illustrating the concept of a timelike tube. The coordinates t and z represent the time and holographic directions, respectively. (a) The timelike tube is constructed by deforming the timelike curve γ while keeping the endpoints q and p fixed. (b) The timelike tube associated with a boundary time interval $T \subset M$ is shown. Here, X denotes the bulk spacetime, and $\mathcal{E}(T)$ represents the timelike tube corresponding to T in X . The RT surface associated with the interval T is indicated by the dashed line.

the viewpoint that timelike EE is more naturally defined between two subregions A and B , as discussed in Sec. IID 1, rather than solely in terms of a single time interval T . However, in our construction, the associated bulk region is bounded by the interval T and its corresponding RT surface. From this perspective, this timelike bulk region does not appear to serve the same role as its spacelike counterpart in the reconstruction theorem. It would be interesting to further investigate the geometric significance of these timelike bulk regions in holography.

In summary, the notion of entanglement in time broadens our understanding of the entanglement structure and the holographic aspects of gravity. Although many questions remain unresolved or not yet fully understood, there are strong indications that several key results established for the spacelike case can be extended to the timelike setting. This extension may offer new insights into the role of time in holography.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: MUTUAL INFORMATION

In this section we would like to introduce the details on the Example II discussed in Sec. IID 2. We need to evaluate the four-point functions of twist operators in Lorentzian spacetime (18), which can be obtained by analytical continuation of the Euclidean correlators [61]. In 2D Euclidean CFTs it is more convenient to use the coordinates $w = x + i\tau$ and $\bar{w} = x - i\tau$. In this section the twist operator will be denoted as $\sigma(\bar{w}, \bar{w})$.

Specifically, we are interested in the four-point function,

$$\langle \sigma_n(w_1, \bar{w}_1) \tilde{\sigma}_n(w_2, \bar{w}_2) \sigma_n(w_3, \bar{w}_3) \tilde{\sigma}_n(w_4, \bar{w}_4) \rangle.$$

By conformal transformation we have

$$\begin{aligned} & \langle \sigma_n(w_1, \bar{w}_1) \tilde{\sigma}_n(w_2, \bar{w}_2) \sigma_n(w_3, \bar{w}_3) \tilde{\sigma}_n(w_4, \bar{w}_4) \rangle \\ &= |w_1 - w_3|^{-4h_{\sigma_n}} |w_2 - w_4|^{-4h_{\sigma_n}} \\ & \times \langle \sigma_n(\infty) \tilde{\sigma}_n(1, 1) \sigma_n(\eta, \bar{\eta}) \tilde{\sigma}_n(0, 0) \rangle, \end{aligned} \quad (\text{A1})$$

where cross ratios η and $\bar{\eta}$ are given by

$$\eta = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)}, \quad \bar{\eta} = \frac{(\bar{w}_1 - \bar{w}_2)(\bar{w}_3 - \bar{w}_4)}{(\bar{w}_1 - \bar{w}_3)(\bar{w}_2 - \bar{w}_4)}. \quad (\text{A2})$$

The four-point functions in (A1) can be decomposed into conformal blocks,

$$\begin{aligned} & \langle \sigma_n(\infty) \tilde{\sigma}_n(1, 1) \sigma_n(\eta, \bar{\eta}) \tilde{\sigma}_n(0, 0) \rangle \\ &= \sum_p a_p \mathcal{F}(c, h_{\sigma_n}, h_p, \eta) \bar{\mathcal{F}}(c, \bar{h}_{\sigma_n}, \bar{h}_p, \bar{\eta}), \end{aligned} \quad (\text{A3})$$

where the summation is over the primary operator with conformal dimension h_p and $\bar{h}_p$, a_p are the operator product expansion coefficients. We can also define the timelike mutual information for subsystems $A_1 B_1$ and $A_2 B_2$,

$$I(A_1 B_1; A_2 B_2) = S(T_{A_1 B_1}) + S(T_{A_2 B_2}) - S(T_{A_1 B_1 A_2 B_2}).$$

In the vacuum state it is known that the mutual information in conformal invariant, that is it depends on only the cross ratios η and $\bar{\eta}$. One could directly check this by using the definition of mutual information and the results (A2).

In this paper, we mainly focus on the CFTs with large central charge, which are expected to have bulk dual. The domain contribution of the expansion (A3) comes from the primary operators with the lowest conformal dimensions.

Let us firstly focus on the EE. To evaluate EE we only need the correlator (A3) near $n \sim 1$. At large c limit, for $\eta \sim 0$, the domain contributions of the expansion of (A3) in the s-channel comes from the vacuum family $h_p = 0$, that is

$$\langle \sigma_n(\infty) \tilde{\sigma}_n(1, 1) \sigma_n(\eta, \bar{\eta}) \tilde{\sigma}_n(0, 0) \rangle \simeq e^{-f(h_{\sigma_n}, \eta) - \bar{f}(\bar{h}_{\sigma_n}, \bar{\eta})},$$

where f and $\bar{f}$ are functions of order $O(c)$. Almost $n \sim 1$ one could analytically obtain f and $\bar{f}$ by the monodromy method [61]. Keeping the leading order of $(n-1)$ we have

$$\begin{aligned} f(h_{\sigma_n}, \eta) &= -\frac{c(n-1)}{6} \log \eta + O((n-1)^2), \\ \bar{f}(\bar{h}_{\sigma_n}, \bar{\eta}) &= -\frac{c(n-1)}{6} \log \bar{\eta} + O((n-1)^2). \end{aligned} \quad (\text{A4})$$

While for $\eta \sim 1$ we can expand the conformal block by t-channel. In this case the function f and $\bar{f}$ near $n \sim 1$ can also be obtained analytically, the result is

$$\begin{aligned} f(h_{\sigma_n}, \eta) &= -\frac{c(n-1)}{6} \log(1-\eta) + O((n-1)^2), \\ \bar{f}(\bar{h}_{\sigma_n}, \bar{\eta}) &= -\frac{c(n-1)}{6} \log(1-\bar{\eta}) + O((n-1)^2). \end{aligned} \quad (\text{A5})$$

Taking above result into (A1) one could obtain the four-point correlation functions. For the spacelike case, it is straightforward to obtain the EE which is consistent with the holographic results.

Now we would like to discuss the case in the view of correlators of twist operators. Consider the EE for $T_{A_1 A_2 B_1 B_2}$ for the intervals on the time coordinates as shown in Fig. 9. The timelike EE can be obtained by the following analytical continuation:

$$\begin{aligned} S(T_{A_1 A_2 B_1 B_2}) &= -\partial_n \langle \tilde{\sigma}_n(w_4, \bar{w}_4) \sigma_n(w_3, \bar{w}_3) \\ &\quad \times \tilde{\sigma}_n(w_2, \bar{w}_2) \sigma_n(w_1, \bar{w}_1) \rangle \Big|_{\substack{\tau_i \rightarrow i\tau_i + \epsilon_i \\ x_i \rightarrow 0}}, \end{aligned}$$

with $\epsilon_4 > \epsilon_3 > \epsilon_2 > \epsilon_1 > 0$. With the analytical continuation we find the cross ratios become

$$\begin{aligned} \eta_t &:= \frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_3)(t_2 - t_4)} + i\tilde{\epsilon} \\ \bar{\eta}_t &:= \frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_3)(t_2 - t_4)} - i\tilde{\epsilon}, \end{aligned} \quad (\text{A6})$$

where $\tilde{\epsilon}$ are constant depending on ϵ_i and the coordinates t_i which vanishing at the limit $\epsilon_i \rightarrow 0$. We should note that both $\eta_t, \bar{\eta}_t$ and $1 - \eta_t, 1 - \bar{\eta}_t$ have a positive real part.

For the large c theory, we can also evaluate the four-point functions by the approximation (A4). For $\eta_t \sim 0$, we expect that the s-channel will give the dominant contributions to the correlator $\langle \sigma_n(\infty) \tilde{\sigma}_n(1, 1) \sigma_n(\eta_t, \bar{\eta}_t) \tilde{\sigma}_n(0, 0) \rangle$. Thus, according to (A4) the correlator $\langle \sigma_n(\infty) \tilde{\sigma}_n(1, 1) \sigma_n(\eta, \bar{\eta}) \tilde{\sigma}_n(0, 0) \rangle$ would be real. For $\eta_t \sim 1$, we expect that the t-channel would domain the contribution. According to (A5) this correlator will also be real. Thus, in both phases the imaginary part of the timelike EE of two intervals comes from the factor

$$\frac{1}{|w_1 - w_3|^{4h_{\sigma_n}} |w_2 - w_4|^{4h_{\sigma_n}}}$$

in (A1).

With some calculations, for $\eta_t \sim 0$ by using (A1) and (B7) we find

$$S(T_{A_1 A_2 B_1 B_2}) = \frac{c}{3} \log \frac{t_2 - t_1}{\delta} + \frac{c}{3} \log \frac{t_4 - t_3}{\delta} + \frac{i\pi c}{3}. \quad (\text{A7})$$

For $\eta_t \sim 1$ one could obtain

$$S(T_{A_1 A_2 B_1 B_2}) = \frac{c}{3} \log \frac{t_3 - t_2}{\delta} + \frac{c}{3} \log \frac{t_4 - t_1}{\delta} + \frac{i\pi c}{3}. \quad (\text{A8})$$

Note that the imaginary part of both phases is the constant $\frac{i\pi c}{3}$, which is double the case of the single interval. The result is also consistent with the holographic explanation of timelike EE in [28]. The mutual information can be evaluated using the results of $S(T_{A_1 B_1})$ and $S(T_{A_2 B_2})$. It is obvious that the mutual information is real.

APPENDIX B: RT SURFACE OF TIMELIKE INTERVAL IN COMPLEXIFIED METRIC

Prior to evaluating the geometric analog of the cross section, we must first determine the configuration of the associated extremal surfaces. We employ the complex geometric interpretation to derive their explicit forms.

1. Global metric

We begin by considering an interval on the boundary of global AdS_3 . The endpoints are located at $(t_1, r_\infty, \theta_1)$ and $(t_2, r_\infty, \theta_2)$, with $t_2 > t_1$. The Lorentzian metric for global AdS_3 is given by

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\theta^2. \quad (\text{B1})$$

By performing a Wick rotation to Euclidean time, $t \rightarrow -i\tau$, the metric becomes

$$ds^2 = (1 + r^2)d\tau^2 + \frac{dr^2}{1 + r^2} + r^2 d\theta^2. \quad (\text{B2})$$

To find the RT surface anchored to this interval, we parametrize it using the radial coordinate r , such that $\tau = \tau(r)$ and $\theta = \theta(r)$. The Euler-Lagrangian equations yield two conserved quantities, p_τ and p_θ ,

$$p_\tau = \frac{(1+r^2)^{3/2}\tau'}{\sqrt{1+(1+r^2)^2\tau'^2+(1+r^2)r^2\theta'^2}},$$

$$p_\theta = \frac{\sqrt{1+r^2}\theta'}{\sqrt{1+(1+r^2)^2\tau'^2+(1+r^2)r^2\theta'^2}}. \quad (\text{B3})$$

In the complex geometry interpretation, we assume that the surface has a turning point at a complex radial coordinate $r = r_\tau$, where the derivatives $\tau'(r)$ and $\theta'(r)$ diverge. This leads to the following set of conditions:

$$p_\theta^2 - r_\tau^2 + p_\tau^2 r_\tau^2 + p_\theta^2 r_\tau^2 - r_\tau^4 = 0,$$

$$\int_{r_\infty}^{r_\tau} \tau' dr = \frac{\tau_1 - \tau_2}{2},$$

$$\int_{r_\infty}^{r_\tau} \theta' dr = \frac{\theta_2 + \theta_1}{2}. \quad (\text{B4})$$

Solving this system of equations yields

$$p_\tau = \frac{\sinh(\tau_2 - \tau_1)}{\cos(\theta_2 + \theta_1) - \cosh(\tau_2 - \tau_1)},$$

$$p_\theta = \frac{\sin(\theta_2 + \theta_1)}{\cos(\theta_2 + \theta_1) - \cosh(\tau_2 - \tau_1)},$$

$$r_\tau = \sqrt{2} \frac{i \cos\left(\frac{\theta_2 + \theta_1}{2}\right)}{\sqrt{\cos(\theta_2 + \theta_1) - \cosh(\tau_2 - \tau_1)}}. \quad (\text{B5})$$

For a purely timelike interval, we can, without loss of generality, set the angular positions be equal $\theta_1 = \theta_2 = \theta_0 = 0$ and choose a symmetric time interval $-t_1 = t_2 = \Delta t$. After rotating back to the Lorentzian signature, the profile of the RT surface is described by the relation

$$r(t) = \frac{1}{|\cos(\Delta t)| \sqrt{\tan^2(t) - \tan^2(\Delta t)}}. \quad (\text{B6})$$

2. RT surface in the BTZ black hole

We now apply the same procedure to a timelike interval in the BTZ black hole background. The Lorentzian BTZ metric is given by

$$ds^2 = \frac{1}{z^2} \left(-f(z) dt^2 + \frac{dz^2}{f(z)} + dx^2 \right), \quad (\text{B7})$$

where $f(z) = 1 - \frac{z^2}{z_H^2}$ and z_H are the positions of the event horizon. Wick rotating $t \rightarrow -it$ gives the Euclidean metric

$$ds^2 = \frac{1}{z^2} \left(f(z) d\tau^2 + \frac{dz^2}{f(z)} + dx^2 \right). \quad (\text{B8})$$

We consider a timelike interval on the boundary with endpoints at (τ_1, x_1) and (τ_2, x_2) . To determine the RT surface anchored to these points, we parametrize it by the holographic coordinate z , such that $\tau = \tau(z)$ and $x = x(z)$. The Euler-Lagrange equations again yield two conserved quantities, p_τ and p_x ,

$$p_\tau = \frac{f(z)\tau'}{z^2 \sqrt{f(z)\tau'^2 + x'^2 + \frac{1}{f(z)}}},$$

$$p_x = \frac{x'}{z^2 \sqrt{f(z)\tau'^2 + x'^2 + \frac{1}{f(z)}}}. \quad (\text{B9})$$

Assuming the surface has a unique turning point at $z = z_\tau$, where both τ' and x' diverge, we impose the following conditions:

$$p_\tau^2 z_\tau^4 + z_H^2 - z_\tau^2 [1 + z_H^2 (p_\tau^2 + p_x^2)] = 0,$$

$$\int_0^{z_\tau} \tau'(z) dz = \frac{\tau_2 - \tau_1}{2},$$

$$\int_0^{z_\tau} x'(z) dz = \frac{x_2 - x_1}{2}. \quad (\text{B10})$$

Solving this system provides explicit expressions for the conserved quantities and the turning point,

$$p_\tau = \frac{\sin\left(\frac{\tau_2 - \tau_1}{z_H}\right)}{z_H \left[\cos\left(\frac{\tau_2 - \tau_1}{z_H}\right) - \cosh\left(\frac{x_2 - x_1}{z_H}\right) \right]}, \quad (\text{B11})$$

$$p_x = \frac{\sinh\left(\frac{x_2 - x_1}{z_H}\right)}{z_H \left[\cosh\left(\frac{x_2 - x_1}{z_H}\right) - \cos\left(\frac{\tau_2 - \tau_1}{z_H}\right) \right]}, \quad (\text{B12})$$

$$z_\tau = \sqrt{\frac{z_H \left[\cosh\left(\frac{x_2 - x_1}{z_H}\right) - \cos\left(\frac{\tau_2 - \tau_1}{z_H}\right) \right]}{1 + \cosh\left(\frac{x_2 - x_1}{z_H}\right)}}. \quad (\text{B13})$$

To describe the RT surface in the original Lorentzian spacetime, we Wick rotate back via $\tau \rightarrow it$. For a symmetric, purely timelike interval defined by $-t_1 = t_2 = \Delta t$ and $x_1 = x_2 = 0$, the profile of the RT surface is

$$z(t) = z_H \sqrt{\frac{\cosh\left(\frac{2t}{z_H}\right) - \cosh\left(\frac{2\Delta t}{z_H}\right)}{1 + \cosh\left(\frac{2t}{z_H}\right)}}. \quad (\text{B14})$$

This RT surface has a complex-valued turning point, consistent with the analytic continuation framework discussed in the main text. Its shape is analogous to that of

extremal surfaces in pure AdS, and its near-boundary divergence can be regularized by introducing a cutoff at $z = \delta$.

APPENDIX C: CALCULATION OF THE GEOMETRIC SADDLE POINT

In this appendix, we provide the detailed derivation of the geometric analog of the cross section and discuss the procedure for determining the relevant saddle point.

1. Poincaré coordinate

We consider two timelike symmetric intervals A and B with endpoints specified by $-t_1 = t_4 = \Delta t_1$ and $-t_2 = t_3 = \Delta t_2$ in the Poincaré patch of AdS_3 . We denote the corresponding RT surfaces by Γ_r for interval A and Γ_b for interval B . The geometric analog of the cross section, denoted as $\Sigma_{T_1 T_2}$, is defined as the geodesic connecting the point $(t_r, z_r) = (t_r, \sqrt{t_r^2 - \Delta t_1^2})$ on Γ_r and $(t_b, z_b) = (t_b, \sqrt{t_b^2 - \Delta t_2^2})$ on Γ_b .

The length of the $\Sigma_{T_1 T_2}$ is given by

$$\begin{aligned} \text{Area}(\Sigma_{T_1 T_2}) &= \mathcal{A}(t_r, t_b) = \int_{z_b}^{z_r} \frac{1}{z} \sqrt{1 - \left(\frac{dt}{dz}\right)^2} dz \\ &= \log \left(\frac{z}{1 + \sqrt{p^2 z^2 + 1}} \right) \Big|_{z_b}^{z_r}, \end{aligned}$$

where the derivative of the geodesic satisfies $t'(z) = \frac{dt}{dz} = \frac{pz}{\sqrt{1+p^2 z^2}}$ which follows from the relation $\frac{t'(z)^2}{z^2(1-t'(z)^2)} = p^2$. Parameter p^2 is determined by the boundary conditions and is explicitly given by

$$\begin{aligned} p^2 &= 4(t_r - t_b)^2 [-2\Delta t_1^2 (2t_b(t_r - t_b) + \Delta t_2^2) \\ &\quad + 4\Delta t_2^2 t_r(t_r - t_b) + \Delta t_1^4 + \Delta t_2^4]^{-1}. \end{aligned}$$

Substituting z_r, z_b and p^2 into the expression for the area, we obtain

$$\mathcal{A}(t_r, t_b) = \log \left(\frac{\sqrt{t_r^2 - \Delta t_1^2} (1 + \sqrt{1 + p^2 (t_b^2 - \Delta t_2^2)})}{\sqrt{t_b^2 - \Delta t_2^2} (1 + \sqrt{1 + p^2 (t_r^2 - \Delta t_1^2)})} \right). \quad (\text{C1})$$

The saddle points are determined by solving the extremality conditions

$$\partial_{t_r} \mathcal{A}(t_r, t_b) = 0, \quad \partial_{t_b} \mathcal{A}(t_r, t_b) = 0.$$

The solutions include

$$(t_r, t_b) = (0, 0), \quad (\pm \Delta t_1, \pm \Delta t_2), \quad (\pm \Delta t_1, \mp \Delta t_2). \quad (\text{C2})$$

In the spatial setting, one typically selects the minimal-area surface. In the present context of analytic continuation, we select the solution $(t_r, t_b) = (0, 0)$. This corresponds to the geodesic connecting the turning points of the complexified RT surfaces (as shown in Fig. 20), which is the natural geometric choice for the saddle point in the bulk. The other solutions correspond to geodesics ending at the asymptotic boundary (where the metric diverges), and thus are not relevant for the construction of the interior cross section. Therefore, the area associated with the relevant saddle point is

$$\mathcal{A}(0, 0) = \log \frac{\Delta t_1}{\Delta t_2}. \quad (\text{C3})$$

2. BTZ black hole

The thermal state in a $(1+1)$ -dimensional CFT is dual to the planar BTZ black hole, whose metric is given by

$$ds^2 = \frac{-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2}{z^2}, \quad (\text{C4})$$

where $f(z) = 1 - \frac{z^2}{z_H^2}$, and $z = z_H$ denotes the location of the black hole horizon.

We again consider two symmetric timelike intervals A and B with endpoints $-t_1 = t_4 = \Delta t_1$ and $-t_2 = t_3 = \Delta t_2$. The RT surfaces corresponding to these intervals, denoted by Γ_r and Γ_b , have bulk points given by

$$\begin{aligned} z_r &= z_H \sqrt{\frac{\cosh\left(\frac{2t_r}{z_H}\right) - \cosh\left(\frac{2\Delta t_1}{z_H}\right)}{1 + \cosh\left(\frac{2t_r}{z_H}\right)}}, \\ z_b &= z_H \sqrt{\frac{\cosh\left(\frac{2t_b}{z_H}\right) - \cosh\left(\frac{2\Delta t_2}{z_H}\right)}{1 + \cosh\left(\frac{2t_b}{z_H}\right)}}. \end{aligned} \quad (\text{C5})$$

To compute the length of the geometric analog of the cross section connecting these points, we embed the BTZ black hole into $\mathbb{R}^{2,2}$ with the coordinates,

$$\begin{aligned}
X_0 &= \frac{z_H}{z} \sqrt{f(z)} \sinh\left(\frac{t}{z_H}\right), \\
X_1 &= \frac{z_H}{z} \cosh\left(\frac{x}{z_H}\right), \\
X_2 &= \frac{z_H}{z} \sinh\left(\frac{x}{z_H}\right), \\
X_3 &= \frac{z_H}{z} \sqrt{f(z)} \cosh\left(\frac{t}{z_H}\right). \tag{C6}
\end{aligned}$$

The embedding-space inner product between points $X^{(r)}$ and $X^{(b)}$ is

$$\begin{aligned}
X^{(r)} \cdot X^{(b)} &= -X_0^{(r)} X_0^{(b)} + X_1^{(r)} X_1^{(b)} \\
&\quad + X_2^{(r)} X_2^{(b)} - X_3^{(r)} X_3^{(b)}. \tag{C7}
\end{aligned}$$

The geodesic distance is therefore

$$d = \text{arccosh}(-X^{(r)} \cdot X^{(b)}). \tag{C8}$$

Thus, the length of the $\Sigma_{T_1 T_2}$ is

$$\begin{aligned}
\text{Area}(\Sigma_{T_1 T_2}) &= \mathcal{A}(t_r, t_b) \\
&= \text{arccosh} \left[\frac{2 \cosh\left(\frac{t_r + t_b}{z_H}\right) \cosh\left(\frac{\Delta t_1}{z_H}\right) \cosh\left(\frac{\Delta t_2}{z_H}\right) - 2 \cosh\left(\frac{t_r}{z_H}\right) \cosh\left(\frac{t_b}{z_H}\right)}{\sqrt{\cosh\left(\frac{2t_r}{z_H}\right) - \cosh\left(\frac{2\Delta t_1}{z_H}\right)} \sqrt{\cosh\left(\frac{2t_b}{z_H}\right) - \cosh\left(\frac{2\Delta t_2}{z_H}\right)}} \right]. \tag{C9}
\end{aligned}$$

At the saddle point $(t_r, t_b) = (0, 0)$, we obtain the area associated with the relevant geometric saddle,

$$\begin{aligned}
\mathcal{A}(0, 0) &= \text{arccosh} \left[\coth\left(\frac{\Delta t_1}{z_H}\right) \coth\left(\frac{\Delta t_2}{z_H}\right) - \frac{1}{\sinh\left(\frac{\Delta t_1}{z_H}\right) \sinh\left(\frac{\Delta t_2}{z_H}\right)} \right] \\
&= \log \left(\frac{\tanh\left(\frac{\Delta t_1}{2z_H}\right)}{\tanh\left(\frac{\Delta t_2}{2z_H}\right)} \right). \tag{C10}
\end{aligned}$$

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- [1] S. Ryu and T. Takayanagi, *Phys. Rev. Lett.* **96**, 181602 (2006).
 - [2] V. E. Hubeny, M. Rangamani, and T. Takayanagi, *J. High Energy Phys.* **07** (2007) 062.
 - [3] M. Van Raamsdonk, *Gen. Relativ. Gravit.* **42**, 2323 (2010).
 - [4] B. Swingle, *Phys. Rev. D* **86**, 065007 (2012).
 - [5] J. Maldacena and L. Susskind, *Fortschr. Phys.* **61**, 781 (2013).
 - [6] A. Almheiri, X. Dong, and D. Harlow, *J. High Energy Phys.* **04** (2015) 163.
 - [7] D. L. Jafferis, A. Lewkowycz, J. Maldacena, and S. J. Suh, *J. High Energy Phys.* **06** (2016) 004.
 - [8] X. Dong, D. Harlow, and A. C. Wall, *Phys. Rev. Lett.* **117**, 021601 (2016).
 - [9] N. Engelhardt and A. C. Wall, *J. High Energy Phys.* **01** (2015) 073.
 - [10] G. Penington, *J. High Energy Phys.* **09** (2020) 002.
 - [11] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, *J. High Energy Phys.* **12** (2019) 063.
 - [12] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, *J. High Energy Phys.* **03** (2022) 205.
 - [13] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, *J. High Energy Phys.* **05** (2020) 013.
 - [14] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, *Rev. Mod. Phys.* **93**, 035002 (2021).
 - [15] T. Takayanagi, *Phys. Rev. Lett.* **134**, 240001 (2025).
 - [16] T. Takayanagi and K. Umemoto, *Nat. Phys.* **14**, 573 (2018).
 - [17] P. Nguyen, T. Devakul, M. G. Halbasch, M. P. Zaletel, and B. Swingle, *J. High Energy Phys.* **01** (2018) 098.
 - [18] S. Dutta and T. Faulkner, *J. High Energy Phys.* **03** (2021) 178.
 - [19] P. Calabrese, J. Cardy, and E. Tonni, *Phys. Rev. Lett.* **109**, 130502 (2012).
 - [20] P. Calabrese, J. Cardy, and E. Tonni, *J. Stat. Mech.* (2013) P02008.

- [21] X. Dong, *Nat. Commun.* **7**, 12472 (2016).
- [22] H. Li and F. Haldane, *Phys. Rev. Lett.* **101**, 010504 (2008).
- [23] P. Calabrese and A. Lefevre, *Phys. Rev. A* **78**, 032329 (2008).
- [24] P. Calabrese and J. Cardy, *J. Stat. Mech.* (2016) P064003.
- [25] K. Doi, J. Harper, A. Mollabashi, T. Takayanagi, and Y. Taki, *Phys. Rev. Lett.* **130**, 031601 (2023).
- [26] Y. Nakata, T. Takayanagi, Y. Taki, K. Tamaoka, and Z. Wei, *Phys. Rev. D* **103**, 026005 (2021).
- [27] Z. Li, Z.-Q. Xiao, and R.-Q. Yang, *J. High Energy Phys.* **04** (2023) 004.
- [28] K. Doi, J. Harper, A. Mollabashi, T. Takayanagi, and Y. Taki, *J. High Energy Phys.* **05** (2023) 052.
- [29] M. P. Heller, F. Ori, and A. Serantes, *Phys. Rev. Lett.* **134**, 131601 (2025).
- [30] W.-z. Guo and J. Xu, *Phys. Rev. D* **112**, L101901 (2025).
- [31] C. Nunez and D. Roychowdhury, *Phys. Rev. D* **112**, L081902 (2025).
- [32] M. P. Heller, F. Ori, and A. Serantes, *Phys. Rev. X* **15**, 041022 (2025).
- [33] W.-z. Guo, S. He, and Y.-X. Zhang, *Phys. Rev. D* **112**, 086020 (2025).
- [34] A. Milekhin, Z. Adamska, and J. Preskill, *arXiv:2502.12240*.
- [35] P. Calabrese and J. L. Cardy, *J. Stat. Mech.* (2004) P06002.
- [36] J. S. Schwinger, *J. Math. Phys. (N.Y.)* **2**, 407 (1961).
- [37] R. P. Feynman and F. L. Vernon, Jr., *Ann. Phys. (N.Y.)* **24**, 118 (1963).
- [38] L. V. Keldysh, *Sov. Phys. JETP* **20**, 1018 (1965).
- [39] X. Dong, A. Lewkowycz, and M. Rangamani, *J. High Energy Phys.* **11** (2016) 028.
- [40] S. Colin-Ellerin, X. Dong, D. Marolf, M. Rangamani, and Z. Wang, *J. High Energy Phys.* **05** (2021) 117.
- [41] S. Colin-Ellerin, X. Dong, D. Marolf, M. Rangamani, and Z. Wang, *J. High Energy Phys.* **08** (2021) 171.
- [42] W.-z. Guo, *Int. J. Mod. Phys. D* **34**, 2544006 (2025).
- [43] K. Skenderis and B. C. van Rees, *Phys. Rev. Lett.* **101**, 081601 (2008).
- [44] K. Skenderis and B. C. van Rees, *J. High Energy Phys.* **05** (2009) 085.
- [45] A. Bou-Comas, C. R. Marimón, J. T. Schneider, S. Carignano, and L. Tagliacozzo, *arXiv:2409.05517*.
- [46] T. Hartman, S. Jain, and S. Kundu, *J. High Energy Phys.* **05** (2016) 099.
- [47] J. Xu and W.-z. Guo, *J. High Energy Phys.* **02** (2025) 094.
- [48] Q. Wen, M. Xu, and H. Zhong, *SciPost Phys.* **18**, 204 (2025).
- [49] P. Calabrese and J. L. Cardy, *Phys. Rev. Lett.* **96**, 136801 (2006).
- [50] P. Calabrese and J. Cardy, *J. Stat. Mech.* (2007) P10004.
- [51] S. He, T. Numasawa, T. Takayanagi, and K. Watanabe, *Phys. Rev. D* **90**, 041701 (2014).
- [52] M. Afrasiar, J. K. Basak, and D. Giataganas, *J. High Energy Phys.* **07** (2024) 243.
- [53] C. Nunez and D. Roychowdhury, *J. High Energy Phys.* **11** (2025) 100.
- [54] A. Lewkowycz and J. Maldacena, *J. High Energy Phys.* **08** (2013) 090.
- [55] A. Strohmaier and E. Witten, *Ann. Henri Poincare* **25**, 4543 (2024).
- [56] A. Strohmaier and E. Witten, *Commun. Math. Phys.* **405**, 153 (2024).
- [57] E. Witten, *Proc. Symp. Pure Math.* **107**, 247 (2024).
- [58] A. Hamilton, D. N. Kabat, G. Lifschytz, and D. A. Lowe, *Phys. Rev. D* **74**, 066009 (2006).
- [59] A. Hamilton, D. N. Kabat, G. Lifschytz, and D. A. Lowe, *Phys. Rev. D* **75**, 106001 (2007); **75**, 129902(E) (2007).
- [60] D. Kabat, G. Lifschytz, and D. A. Lowe, *Phys. Rev. D* **83**, 106009 (2011).
- [61] T. Hartman, *arXiv:1303.6955*.