

Order statistics for multijet events

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We show that rank-ordered jet rapidity distributions provide simple probes of high-energy QCD dynamics at the LHC. Comparing the Balitsky-Fadin-Kuraev-Lipatov-based Monte Carlo (BFKLEX) to standard event generators based on collinear factorization and Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) parton showers (PYTHIA8, HERWIG7) in inclusive dijet topologies at 13 TeV, we find clear shape differences. While inclusive distributions can be very similar, rank-ordered spectra reveal that BFKLEX populates the rapidity interval more democratically, whereas DGLAP showers exhibit stronger edge enhancement and central concentration. We quantify the separation using the total-variation and one-dimensional Wasserstein distances between normalized histograms. The separation persists when applying the harder selection $p_{T,j} > 60$ GeV and $H_{T2} > 250$ GeV. Rank-ordered rapidity distributions thus compress exclusive multijet information into simple histograms, offering a new handle on high-energy radiation patterns measurable with existing data.

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I. INTRODUCTION

Understanding the emergent properties of QCD in the high-energy (small- x) regime is an important goal of the LHC program. When the center-of-mass energy of the hard scattering is much larger than any other hard scale, the perturbative series develops enhanced terms $\alpha_s^n \ln^n s$ which are resummed in the Balitsky-Fadin-Kuraev-Lipatov (BFKL) framework [1–6]. Mueller-Navelet (MN) dijets—two high- p_T jets produced with a large rapidity separation and additional radiation treated inclusively—have long been proposed [7] as a promising laboratory to look for BFKL dynamics. More generally, inclusive dijet topologies with extra radiation offer a broad and experimentally accessible setting in which to probe high-energy QCD.

Over the last decade, several LHC measurements of these quantities have been found to be consistent both with next-to-leading-logarithmic-approximation BFKL calculations

and with fixed-order QCD supplemented by Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP)-based parton showers. In hindsight this is not surprising: most of the standard Mueller-Navelet observables are either fully inclusive, or depend only on low-dimensional projections (such as single-jet distributions or a single azimuthal angle) of a complicated multi-jet final state. Very different underlying point processes in rapidity can share the same inclusive jet density dN_{jets}/dy and similar azimuthal moments. What is missing, therefore, is an observable that is still as simple as a one-dimensional histogram, but that is built to retain genuinely *exclusive* information about how jets populate the interjet rapidity interval.

In this work we propose exactly such a class of observables. We revisit the multiperipheral picture of high-energy scattering, in which produced particles are strongly ordered in rapidity and only weakly correlated in transverse momentum, and recast it in the language of *order statistics*. Concretely, we treat the set of jet rapidities in an event as a finite sample drawn from an effective parent density in rapidity space, and study the distributions of the ordered rapidities (most backward, second-most backward, ..., most forward). This order-statistics view makes precise a simple but powerful statement: if two dynamics induce

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different “parent” rapidity patterns, they *must* differ in at least some of the rank-ordered distributions, even when their inclusive dN_{jets}/dy happen to look similar.

Our main result is that rank-ordered rapidity distributions expose robust shape differences between a BFKL-based generator and standard DGLAP showers [8–12] under realistic LHC conditions. These differences appear already at $p_{T,j} \simeq 20$ GeV, survive modest variations of jet radius and parton distribution functions (PDFs), and persist when restricting to events with sizable rapidity spans. Because the observables are normalized one-dimensional histograms, they are simple to measure with existing MN selections and lend themselves to straightforward comparison between data and theory.

II. ORDER STATISTICS FOR JET RAPIDITIES

Consider an event with N reconstructed jets in a rapidity interval $I = [y_{\min}, y_{\max}]$. Let $Y_1, \dots, Y_N$ denote their rapidities, and define the ordered rapidities

$$Y_{(1)} < Y_{(2)} < \dots < Y_{(N)}, \quad (1)$$

so that $Y_{(1)}$ and $Y_{(N)}$ are the most backward and most forward jets, respectively.

The usual inclusive jet rapidity density dN_{jets}/dy is a *marginal*: it counts jets in rapidity bins, summed over all multiplicities and over all multi-jet configurations. As such, it does not retain information about how jets are distributed *within* a given event. Rank-ordered rapidities $Y_{(1)}, \dots, Y_{(N)}$, by contrast, probe the joint rapidity structure of the event in a way that is still summarized by one-dimensional histograms.

A simple toy example makes this loss of information explicit. Consider events with exactly two jets in a symmetric rapidity window. In “Model A,” each jet is placed independently and uniformly in the window. In “Model B,” the jets are always produced in perfectly symmetric pairs at $y = \pm U$, with U uniform in the forward half of the detector. By construction, both models have the same single-jet distribution dN_{jets}/dy : every rapidity bin is equally populated on average. Yet their ordered configurations differ dramatically: the most forward and most backward jets in Model B tend to lie close to the edges and span a large rapidity interval, whereas in Model A they are typically closer together. The two models are therefore indistinguishable at the level of dN_{jets}/dy but are trivially distinguished by the distributions of $Y_{(1)}$, $Y_{(2)}$ and their difference $Y_{(2)} - Y_{(1)}$.

Rank-ordered observables such as $Y_{(1)}, Y_{(2)}, \dots, Y_{(N)}$ thus provide simple one-dimensional probes of the *joint* rapidity structure of the event. They compress multiparticle exclusive information into a handful of normalized histograms that are as easy to measure and compare as dN_{jets}/dy , but remain sensitive to correlations and production patterns that the inclusive density averages away.

This can be formalized in an idealized independent and identically distributed (IID) limit.

Suppose the single-jet rapidities are approximately IID draws from a parent density $f(y)$ on I with cumulative distribution $F(y)$. The density of the ℓ th order statistic $Y_{(\ell)}$ is the textbook result

$$f_{(\ell)}(y) = \frac{N!}{(\ell-1)!(N-\ell)!} [F(y)]^{\ell-1} [1-F(y)]^{N-\ell} f(y), \quad \ell = 1, \dots, N. \quad (2)$$

Once the parent law $f(y)$ is fixed, the *shape* of each ranked rapidity spectrum is fully determined. For a uniform parent on I (a “rapidity plateau,”) the ordered rapidities map to beta distributions in an affine variable $x \in [0, 1]$, with outer ranks piling up toward the edges and inner ranks remaining broad. Any departure from a plateau—for example, an edge-enhanced or centrally peaked parent—induces characteristic, rank-dependent distortions of these shapes.

In an LHC analysis, jets are only reconstructed inside a finite window $\mathcal{W} = [-4.7, 4.7]$. If the produced parent density on the real line is $f_{\text{prod}}(y)$, the *observable* parent within $\mathcal{W}$ is the truncated law

$$f_{\text{obs}}(y) = \frac{f_{\text{prod}}(y)}{\int_{y_{\min}}^{y_{\max}} f_{\text{prod}}(u) du}, \quad y \in \mathcal{W}, \quad (3)$$

and Eq. (2) holds with $f \rightarrow f_{\text{obs}}$ and $F \rightarrow F_{\text{obs}}$ for events with exactly N tagged jets in $\mathcal{W}$. A transverse-momentum threshold $p_{T,\min}$ affects f_{obs} only through the rapidity dependence of the pass probability $\Pr(p_T > p_{T,\min} | Y = y)$.

From this perspective, rank-ordered rapidity histograms are order-statistics projections of the effective parent law realized by a given dynamics. In a multiperipheral/BFKL scenario one expects a relatively broad, plateaulike parent, whereas DGLAP showers with coherence and recoil generically generate narrower parents.

III. MONTE CARLO SETUP AND OBSERVABLES

We study proton-proton collisions at 13 TeV in inclusive dijet topologies. Jets are reconstructed with the anti- k_T algorithm, as implemented in the FASTJET package [13], with radius parameter $R = 0.4$, and are required to lie in $|y| < 4.7$ with $p_{T,j} > 20$ GeV.

This low-threshold selection is used as the baseline because it maximizes the available multijet phase space and illustrates the potential gain of dedicated forward-dijet or Mueller-Navelet analyses with lowered jet thresholds. Later in the manuscript we apply the same rank-ordered observables to a harder selection, $p_{T,j} > 60$ GeV and $H_{T2} > 250$ GeV. All jets passing these cuts are ordered by rapidity.

We analyze two families of normalized histograms:

(i) *Fixed multiplicity* $N = 3$. We consider events with exactly three jets and measure the rapidity distributions of $\text{Jet1} \equiv Y_{(1)}$, $\text{Jet2} \equiv Y_{(2)}$, $\text{Jet3} \equiv Y_{(3)}$. (ii) *Across multiplicities*. Over all multiplicities $N \geq 2$, we consider the *most backward* (MB), *most forward* (MF), and the *second-most backward* (SMB) and *second-most forward* (SMF) jets, defined as $Y_{(1)}$, $Y_{(N)}$, $Y_{(2)}$, and $Y_{(N-1)}$, respectively, whenever the corresponding rank exists. In all cases the histograms are normalized to unit area, so only shapes matter.

For the dynamics, we compare three generators:

- (1) BFKLEX [14–18], a BFKL-based Monte Carlo that generates gluon ladders ordered in rapidity but not in transverse momentum, convoluted with proton PDFs and clustered into jets.
- (2) PYTHIA8 [19], representing a standard DGLAP parton shower with p_T ordering, multiple parton interactions (MPI), and Lund string hadronization.
- (3) HERWIG7 [20,21], with an angular-ordered DGLAP shower, MPI, and cluster hadronization.

Unless otherwise specified, default tunes and PDF sets are used. The BFKLEX events are generated at leading logarithmic (LL) accuracy. For the DGLAP side we compare to LL parton showers as implemented in PYTHIA8 and HERWIG7. Including fixed-order NLO matrix elements for low jet multiplicities (e.g., via POWHEG dijet samples matched to showers) would improve the description of the first few jets but would treat higher multiplicities inhomogeneously, which is undesirable in an analysis that explicitly mixes events from all jet multiplicities. For our purposes, a comparison between LL BFKL evolution and standard LL DGLAP showers is therefore the most transparent approach. Since we investigate the structure of radiation across the entire rapidity interval, rather than just the hardest emissions, the distinct ordering criteria of the evolution equations (BFKL diffusion vs DGLAP collinear ordering) are the dominant shaping mechanisms, rendering fixed-order corrections subleading for these shape observables.

We have verified that turning hadronization on and off in PYTHIA8 and HERWIG7 has a negligible impact on the *shape-normalized* rank distributions. We also find that, in all three generators, the rank shapes are stable under variations of the jet radius $R \in \{0.4, 0.5, 0.6\}$ and under replacing the default proton PDF with alternative sets. These variations affect overall rates but do not qualitatively change the normalized rank shapes. A full theory-uncertainty assessment, including correlated scale, PDF, tune, hadronization/MPI and selection variations, is beyond the scope of this article and is required for a precision comparison to data.

IV. RESULTS

To quantify the separation of two unit-normalized rank histograms we use two complementary shape distances. Let N_{bin} be the number of bins in the histograms and $p_i = h_i \Delta y_i$ and $q_i = g_i \Delta y_i$ be the bin probabilities, with $\sum_i p_i = \sum_i q_i = 1$. The total-variation distance [22],

$$D_{\text{TV}}(p, q) = \frac{1}{2} \sum_{i=1}^{N_{\text{bin}}} |p_i - q_i|, \quad (4)$$

measures the fraction of probability weight that must be reassigned among bins. It is dimensionless and satisfies $0 \leq D_{\text{TV}} \leq 1$.

Since rapidity is ordered, we also use the one-dimensional Wasserstein distance [22]

$$W_1(p, q) = \int dy |F_p(y) - F_q(y)|, \quad (5)$$

where F_p and F_q are the cumulative distributions. In binned form we evaluate

$$W_1(p, q) \simeq \sum_{i=1}^{N_{\text{bin}}-1} \left| \sum_{j=1}^i (p_j - q_j) \right| \Delta y_{i,i+1}. \quad (6)$$

This is the one-dimensional analog of the earth-mover distance used for collider energy flows [23]. It has units of rapidity and measures how far probability weight must be transported. Thus D_{TV} quantifies the reshuffled probability weight, while W_1 measures the associated transport in rapidity.

The quantities D_{TV} and W_1 are used here as shape-distance measures between normalized predictions, not as goodness-of-fit tests or exclusion criteria. A comparison to data should use the full experimental covariance matrix together with the theory variations appropriate to each generator. The purpose of the present work is to quantify the separation between the predicted shapes and to test whether this separation survives a harder, experimentally motivated jet selection.

We first show results for the baseline selection $p_{T,j} > 20$ GeV (Figs. 1–3), and then for the harder selection $p_{T,j} > 60$ GeV and $H_{T2} > 250$ GeV (Figs. 4 and 5).

In the following figures the quoted values refer to the PYTHIA8-BFKLEX comparison. The smallest values of D_{TV} and W_1 are found for the MB/MF ranks across multiplicities, while the larger values occur for the fixed- $N = 3$ outer ranks and for SMB/SMF. This confirms quantitatively the hierarchy seen by eye. The lower panels show the ratios of the normalized PYTHIA8 distributions to the corresponding BFKLEX and HERWIG7 results.

Figure 1 shows the 13 TeV results for fixed multiplicity $N = 3$. The largest shape differences between BFKLEX and the DGLAP showers are observed for the outer ranks, Jet1 and Jet3: BFKLEX populates the backward and forward edges more democratically, while PYTHIA8 and HERWIG7 show comparatively stronger edge enhancement. For the central jet (Jet2) the differences are smaller, but remain systematic: BFKLEX yields a broader, more plateaulike distribution, whereas the DGLAP showers favor a more concentrated midrapidity region.

Figures 2 and 3 display the across-multiplicity results at 13 TeV. The MB and MF distributions are again very

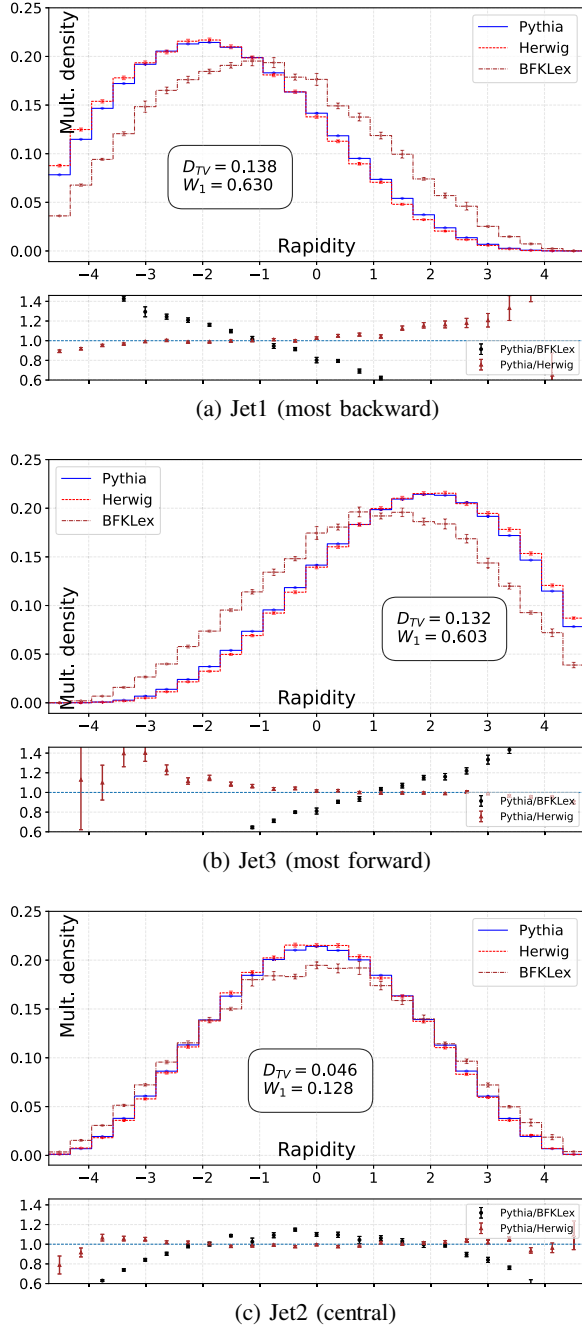


FIG. 1. 13 TeV, fixed multiplicity $N = 3$. Normalized rapidity distributions for Jet1, Jet2, and Jet3 ($p_{T,j} > 20$ GeV), ordered by rapidity.

similar between BFKLEX, PYTHIA8 and HERWIG7, as expected for the extremal ranks that are most constrained by the tagging and acceptance. In contrast, the SMB and SMF distributions show the clearest separation: BFKLEX produces broader distributions, with clear left and right skewness whereas the DGLAP showers yield distributions with more pronounced central weight.

To assess the impact of a harder selection on our observables, we applied a lower jet cutoff of $p_{T,j} > 60$ GeV and

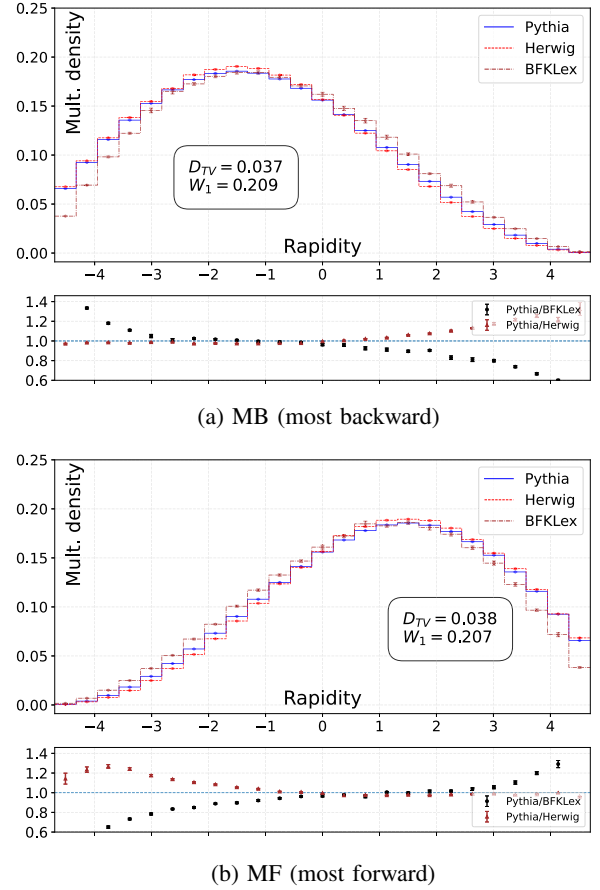


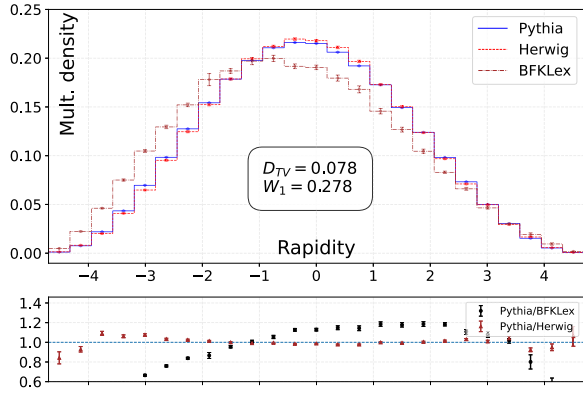
FIG. 2. 13 TeV, across multiplicities. Normalized rapidity distributions for the MB and MF jets ($p_{T,j} > 20$ GeV).

additionally demanded that the sum of the transverse momenta of the two leading jets be $H_{T2} > 250$ GeV. This choice is motivated by the harder jet thresholds used in recent ATLAS multijet-ratio measurements [24].

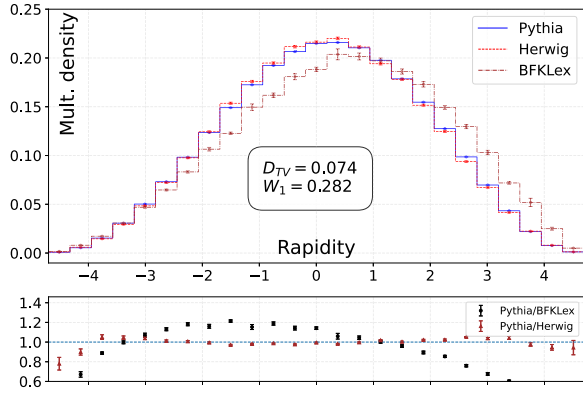
As can be seen in Figs. 4 and 5, the harder selection does not wash out the separation. The same ranks that were most discriminating in the original setup remain separated. For example, for SMB and SMF the PYTHIA8–BFKLex total-variation distances increase from $D_{TV} = 0.078, 0.074$ at the baseline selection to $D_{TV} = 0.175, 0.180$ under the harder cuts, while the corresponding Wasserstein distances increase from $W_1 = 0.278, 0.282$ to $W_1 = 0.424, 0.446$. The fixed- $N = 3$ outer ranks also remain well separated, with $D_{TV} = 0.187, 0.178$ and $W_1 = 0.557, 0.527$ for Jet1 and Jet3.

V. DISCUSSION AND OUTLOOK

Rank-ordered rapidity distributions are standard objects in mathematical statistics (order statistics) but have not, to our knowledge, been used as primary observables in collider QCD. Here we have shown that, when applied to multijet final states in inclusive dijet topologies, they provide simple and theoretically interpretable probes of the underlying rapidity production law.



(a) SMB (second-most backward)



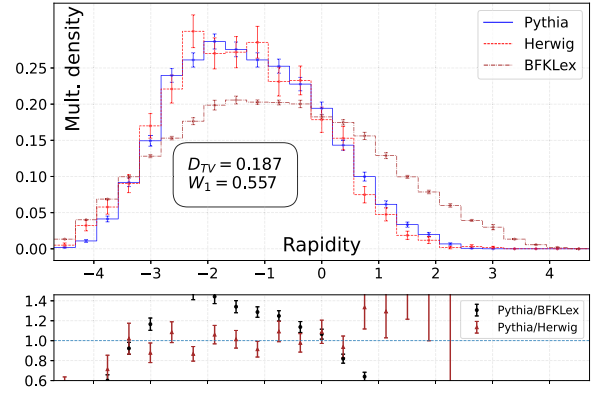
(b) SMF (second-most forward)

FIG. 3. 13 TeV, across multiplicities. Normalized rapidity distributions for the SMB and SMF jets ($p_{T,j} > 20$ GeV).

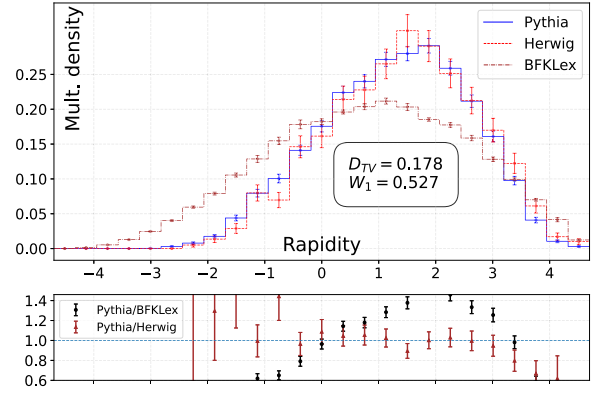
Within an approximate IID picture, the mapping in Eq. (2) guarantees that the shapes of the rank distributions are direct images of the effective parent density realized by a given dynamics. In practice, effects such as global recoil, color coherence, transverse-momentum thresholds, and acceptance boundaries simply deform this effective parent. Our Monte Carlo study suggests that rank shapes are largely insensitive to nonperturbative details (MPI, hadronization) and to standard variations of jet radius and PDFs, but respond strongly to differences between BFKL-like and DGLAP-like radiation patterns.

A natural next step would be to extend the present analysis to next-to-leading logarithmic (NLL) BFKL accuracy, including NLO jet vertices. This is technically demanding, and lies beyond the scope of this first study.

Preliminary checks at the level of the Green's function in related work suggest that going from LL to NLL mainly reshuffles the overall normalizations and energy dependence and does not qualitatively alter the pattern by which high-energy radiation fills rapidity space. Furthermore, because the rank-ordered observables proposed here are shape normalized to unit area, they are less sensitive to the large K factors that typically differentiate LL from NLL cross sections. They are instead sensitive to the diffusive



(a) Jet1 (most backward)



(b) Jet3 (most forward)

FIG. 4. 13 TeV, harder selection. Normalized Jet1 and Jet3 rapidity distributions for $p_{T,j} > 60$ GeV and $H_{T2} > 250$ GeV.

properties of the evolution kernel, which persist at NLL. While we cannot yet make a similarly firm statement about the full NLO impact factors, we do not expect the rank-ordered rapidity patterns found here to disappear at NLL: they are a direct imprint of the underlying high-energy radiation mechanism rather than a peculiarity of a specific logarithmic truncation. A detailed NLL study is therefore well motivated, but is a substantial project in its own right.

The distances D_{TV} and W_1 quantify the separation between normalized predictions, but they are not exclusion criteria. A precision confrontation with data will require the experimental covariance matrix and dedicated theory variations for each generator, which are beyond the scope of this article.

From a broader perspective, the main message of this work is conceptual as much as it is technical. Interpreting the rapidity distributions as order statistics of an effective parent rapidity density provides a remarkably simple way to expose differences in multijet radiation patterns that are invisible in more conventional one-dimensional observables. In our Monte Carlo study, a BFKL-based generator and DGLAP showers produce very similar shapes for the outer ranks (MB, MF) across multiplicities, yet differ clearly in two ways: (i) in the inner ranks (SMB, SMF)

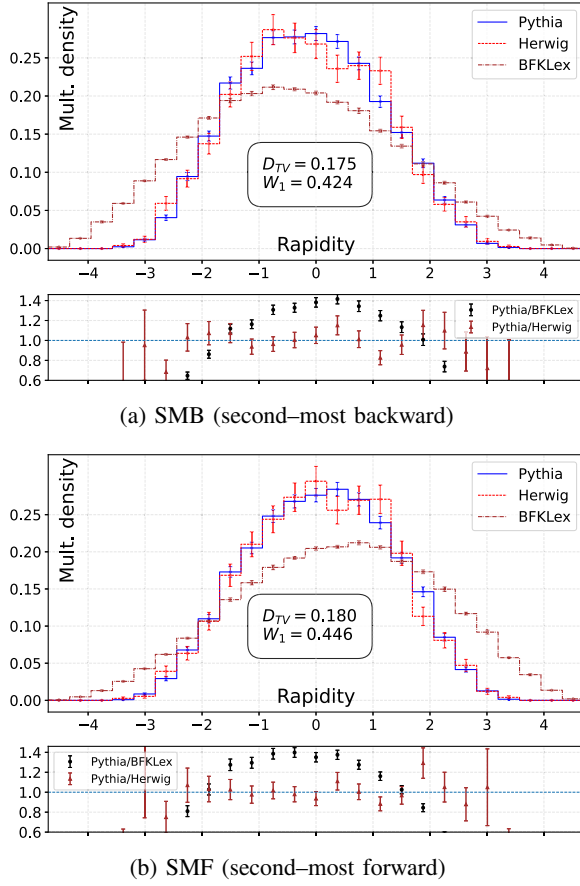


FIG. 5. 13 TeV, harder selection. Normalized SMB and SMF rapidity distributions for $p_{T,j} > 60$ GeV and $H_{T2} > 250$ GeV.

across multiplicities, and (ii) in fixed $N = 3$ events, most prominently for the outer ranks (Jet1, Jet3), with smaller but systematic differences for Jet2. This is precisely the kind of “hidden” information that rank-ordered observables are designed to retain.

Because these observables are just normalized one-dimensional histograms of standard jets, they can be

measured in existing inclusive dijet samples, including but not limited to classic Mueller-Navelet selections with large rapidity separation between the two extremal jets. In particular, they can reuse the same triggers and event samples as current azimuthal-decorrelation and dijet measurements at the LHC. We therefore view the present work as a proof of concept for a wider program: using order statistics as a systematic language to turn multiperipheral and BFKL intuition about “how rapidity space is filled” into concrete, data-driven tests. It will be particularly interesting to confront the predictions shown here with Run 2 and Run 3 measurements, to refine the comparison with alternative high-energy frameworks, and to explore extensions such as joint distributions of multiple ranks and lower p_T thresholds with track-based jets.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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