

Columbia plot based on the symmetry-improved Cornwall-Jackiw-Tomboulis formalism in the linear sigma model

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We study the Columbia plot for the chiral phase transition in the framework of a three-flavor linear sigma model based on the Cornwall-Jackiw-Tomboulis (CJT) formalism. The conventional CJT approach with the Hartree truncation suffers from artificial chiral breaking, leading to the violation of the Nambu-Goldstone theorem and the (anomalous) chiral Ward-Takahashi identities. We apply the symmetry-improved CJT formalism to resolve this issue. We observe a first-order phase transition and a tricritical point in the light-quark mass regime, which is relatively independent of the size of the sigma meson, in contrast to the conventional CJT approach. The tricritical point, found on the m_s axis, is at $m_s^{\text{tri}}/m_s^{\text{phys}} = 0.175$ with m_s^{phys} being the physical strange quark mass in real-life QCD. The critical pion mass in the three-flavor symmetric limit, on the second-order boundary, is measured at $m_\pi \sim 52.4$ MeV, with the critical temperature $T_c \sim 51.7$ MeV.

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I. INTRODUCTION

Quantum chromodynamics (QCD) exhibits a rich phase structure at high temperature associated with chiral symmetry restoration and deconfinement. A deep understanding of the chiral phase transition is of importance to explore the origin of mass and matter. The study along this line has attracted a growing interest in not only the current terrestrial experiments, but also astrophysical and cosmological observations; heavy-ion collisions [1–4], neutron star interiors [5–7], and the thermal history of the early Universe [8,9].

In the general QCD parameter space, the chiral phase transition can be characterized by the lightest quark mass $m_u = m_d \equiv m_l$ and the strange quark mass m_s . The parameter space is called the Columbia plot [10,11]. The physical point at which QCD of the Standard Model is located, thus, is merely a single point in the continuous

quark mass parameter space. Going off the physical point therefore brings a broader and deeper view of how the chiral phase transition, hence the origin of mass and matter, is realized in real-life QCD.

The phase structure of the Columbia plot for the small m_l regime is still an open question. It is typically classified into three “scenarios” [12–14]:

- (i) the *tricritical point* at $m_s = m_s^{\text{tri}}$ on the m_s -axis with $m_u = m_d = 0$ is present, at which point the first order regime is closed and the phase transition becomes second order, which ever persists toward the two-flavor chiral limit, $m_s \rightarrow \infty$, reflecting the universality class of the $O(4)$ theory with no $U(1)$ axial symmetry restoration [11,12,15–17];
- (ii) the tricritical point is absent and $U(1)$ axial symmetry is restored; hence, that is not in the $O(4)$ universality class, so that the first-order regime can be observed till the two-flavor chiral limit [16,17]; and
- (iii) no first-order regime emerges and the second-order regime shows up only on the m_s axis [12,18,19].

Upon the current status of the lattice simulations, for $2 + 1$ flavor QCD with small m_l and large m_s , either scenario i) or ii) tends to be favored [20–28]. The functional renormalization group (fRG) approach to QCD has also realized scenario i) for the two-flavor [29] and $2 + 1$ -flavor cases [30]. In the three-flavor symmetric case with $m_l = m_s$,

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Scenario iii) has been supported [17,17–19,31,32]. The first-order signal has not yet been observed even for a small pion mass, down to $m_\pi \lesssim 50$ MeV with highly improved staggered fermions [17,31] and $m_\pi \lesssim 110$ MeV with improved Wilson fermions [32]. A recent analysis using the Dyson-Schwinger equation has also supported a second-order phase transition for three massless quark flavors [12], namely, implying no first-order regime in the small-quark mass regime.

Alternatively to lattice QCD studies, low-energy effective field theories (LEFTs) can also provide a complementary view of the investigation on the chiral phase structure on the Columbia plot [15,33–47]. These theories qualitatively capture the relevant infrared dynamics through effective degrees of freedom (light mesons and quarks), keeping the global symmetries such as the chiral symmetry and its violation as present in QCD. The linear sigma model (LSM) is one candidate description as such, which serves as a particularly valuable framework for studying chiral phase transitions.

In employing the LSM, one notices one remark made in the literature [15], showing that fluctuations like mesonic loop corrections substantially modify the result in the mean-field approximation projected onto the Columbia plot. To properly incorporate the potentially nonperturbative mesonic loop corrections, one can use the Cornwall-Jackiw-Tomboulis (CJT) method in two-particle (2PI) irreducible effective action formulations. Plugging this formalism into an LSM, one will solve the gap equation for two-point functions of meson fields, which involve resummed quantum-dressed propagators. Accordingly, the ground state in the CJT formalism will be determined at the stationary point for not only the chiral-order parameter but also the meson propagators. This formalism would thus provide an alternative approach to beyond-perturbation study the Columbia plot based on a different systematics compared to other nonperturbative methods, such as the fRG and Dyson-Schwinger equations, and also lattice simulations.

However, it has also been pointed out that the CJT method inevitably suffers from the difficulty that the Nambu-Goldstone (NG) theorem or the threshold property are violated [33,35,48]. This is because a nonsystematic truncation of the 2PI digrams, e.g., often referred to as the Hartree approximation, is required to apply in practically evaluating the CJT effective action at the two-loop level; hence a nonchiral invariant set of the two-loop corrections generates artificial chiral breaking, which feeds back to the violent deviation from the low-energy theorem and the NG theorem in the chiral limit as well. This issue is therefore crucial at around the critical temperature of the chiral phase transition. In fact, the violation of the NG theorem is reflected in generating the massive NG boson at finite temperature even in the case without the explicit symmetry breaking (e.g., the chiral limit for the QCD case). In the

literature [48], it has been clarified that the loopwise truncation deviates the Ward-Takahashi identities (WTIs) of the one-particle irreducible (1PI) from those of 2PI actions, with the former protecting the NG theorem at any order of truncation.

Thus, the so-called symmetry-improved (SI) CJT formalism has been proposed with the enforcement of the 1PI WTIs to preserve the original global symmetry [$O(N)$ symmetry as a reference model] [48]. An application of this original SICJT formalism to $O(4)$ LSM has been performed in [49], where the two-flavor chiral phase transition has been addressed in the chiral limit with the exact NG boson even at finite temperature.

In this paper, we apply the SICJT formalism to a three-flavor LSM and explore the chiral phase structure in the small-quark mass regime of the Columbia plot. We observe a first-order regime and a tricritical point consistently with Scenario i) above. This is a definitive conclusion fairly insensitive to the size of the sigma meson mass, in contrast to the conventional CJT approach in [33,35].

This paper is organized as follows. In Sec. II, we introduce a three-flavor LSM and summarize what is necessary to address the chiral phase transition in the CJT formalism at zero and finite temperature. This section is essentially just a review of the literature [33,35,50]. In Sec. III, applying the argument in [48] to the present LSM case, we discuss the violation of the NG theorem by demonstrating the discrepancy between the WTIs associated with the 1PI and the 2PI formalisms. Then, we introduce the SICJT formulation established in [48]. In Sec. IV, we show our results on the chiral phase transitions and projection onto the Columbia plot, and discuss the related physical consequences, in comparison with the results based on the conventional CJT approach. We also present the critical exponents around several critical points in the Columbia plot evaluated from the SICJT formalism. The conclusion of this paper is provided in Sec. V. In Appendixes A, B, C, and D, we also present supplementary materials for more details relevant to the present work: tree-level formulas in the LSM (A); the CJT formalism (B); the WTIs with and without explicit symmetry breaking (C); benchmarked temperature dependences for the chiral phase transitions at certain critical points in the Columbia plot (D).

II. AN LSM AND CJT FORMALISM

We begin by introducing a phenomenologically successful three-flavor LSM proposed in [51] and analyzed at finite temperature in [50] within the conventional CJT formalism [52]. This LSM can, in particular, reproduce what is called the inverse mass hierarchy between $a_0(950)$ and $\kappa(700) = K_0^*(700)$ spectra, due to the introduction of the explicit-chiral breaking contribution related to the $U(1)$ -axial anomaly-induced term [see Eq. (8)] [51]. Some of the detailed formulas can be found in Appendixes A and B.

The central building block of the model is the LSM field Φ , which transforms bifundamentally under the full $U(3)_L \times U(3)_R$ symmetry as

$$\Phi \rightarrow g_L \cdot \Phi \cdot g_R^\dagger, \quad (1)$$

with $g_{L,R} \in U(3)_{L,R}$. Throughout the present paper, we work in Euclidean spacetime. The model Lagrangian is given as

$$\mathcal{L}_{\text{LSM}} = \text{tr}[\partial_\mu \Phi^\dagger \partial_\mu \Phi] + V(\Phi, \Phi^\dagger). \quad (2)$$

The mesonic field Φ is parametrized as $\Phi = \phi_a T^a = (\sigma_a + i\pi_a)T^a$ for $a = 0, \dots, 8$, in which σ_a and π_a denote the scalar and pseudoscalar fields, respectively. $T^a = \lambda^a/2$ represents generators of the $U(3)$ group, in which $\lambda^{a=1,\dots,8}$ are the Gell-Mann matrices, and $\lambda^0 = \sqrt{2/3}\mathbf{1}_{3 \times 3}$. The mesonic potential $V(\Phi, \Phi^\dagger)$ in Eq. (2) is given by

$$V(\Phi, \Phi^\dagger) = V_0 + V_{\text{anom}} + V_{\text{SB}} + V_{\text{SB-anom}}. \quad (3)$$

The first term V_0 in Eq. (3) corresponds to the $U(3)_L \times U(3)_R$ invariant part. This potential term is, up to mass dimension four, given as

$$V_0 = \mu^2 \text{tr}[\Phi^\dagger \Phi] + \lambda_1 \text{tr}[(\Phi^\dagger \Phi)^2] + \lambda_2 (\text{tr}[\Phi^\dagger \Phi])^2. \quad (4)$$

The second term V_{anom} in Eq. (3) arises from the QCD instanton contribution associated with the $U(1)$ axial anomaly. It takes the determinant form, the so-called Kobayashi-Maskawa-'t Hooft (KMT) term [53–56],

$$V_{\text{anom}} = -B(\det[\Phi] + \det[\Phi^\dagger]). \quad (5)$$

The third term V_{SB} in Eq. (3) is the leading-order explicit chiral-breaking term tied with the low-energy theorem,

$$V_{\text{SB}} = -c \text{tr}[\mathcal{M} \Phi^\dagger + \mathcal{M}^\dagger \Phi]. \quad (6)$$

Here $\mathcal{M}$ is the spurion field, which transforms in the same way as Φ in Eq. (1),

$$\mathcal{M} \rightarrow g_L \cdot \mathcal{M} \cdot g_R^\dagger, \quad (7)$$

and takes the vacuum expectation value (VEV) $\mathcal{M} \equiv \text{diag}\{m_u, m_d, m_s\}$ to incorporate the proper explicit-chiral breaking into mesons arising from the current quark masses present in QCD.

The fourth term $V_{\text{SB-anom}}$ in Eq. (3) is characteristic to the present LSM,

$$V_{\text{SB-anom}} = -kc[\epsilon_{abc}\epsilon^{def}\mathcal{M}_d^a\Phi_e^b\Phi_f^c + \text{H.c.}]. \quad (8)$$

This is the axial-anomaly induced flavor-breaking term, which controls the inverse mass hierarchy for a_0 and K_0^* masses [51].

The dynamical chiral symmetry breaking is monitored by the chiral order parameter, i.e., the VEV, or the thermal average of Φ , which is parametrized as

$$\langle \Phi \rangle \equiv \bar{\Phi} = \bar{\sigma}_a T^a, \quad (9)$$

without loss of generality. The isospin symmetry is adopted as $m_u = m_d \equiv m_l \neq m_s$, such that the VEVs of the σ fields read

$$\bar{\sigma}_a T^a = \bar{\sigma}_0 T^0 + \bar{\sigma}_8 T^8 = \text{diag}\{\bar{\Phi}_1, \bar{\Phi}_1, \bar{\Phi}_3\}, \quad (10)$$

together with the following relations:

$$\begin{aligned} \bar{\Phi}_1 &= \frac{1}{\sqrt{6}}\bar{\sigma}_0 + \frac{1}{2\sqrt{3}}\bar{\sigma}_8, \\ \bar{\Phi}_8 &= \frac{1}{\sqrt{6}}\bar{\sigma}_0 - \frac{1}{\sqrt{3}}\bar{\sigma}_8. \end{aligned} \quad (11)$$

At tree level, the stationary condition reads

$$\left. \frac{\delta S_{\text{LSM}}}{\delta \Phi_1} \right|_{\Phi=\bar{\Phi}} = 0, \quad \left. \frac{\delta S_{\text{LSM}}}{\delta \Phi_3} \right|_{\Phi=\bar{\Phi}} = 0, \quad (12)$$

where $S_{\text{LSM}} = \int_x \mathcal{L}_{\text{LSM}}$ is the Euclidean action, with the shorthand notation $\int_x \equiv \int d\tau \int d^3x$. The meson mass matrices are given by

$$\begin{aligned} [m_S^2(\bar{\sigma})]^{ab} &= \left. \frac{\partial^2 V_{\text{LSM}}}{\partial \sigma_a \partial \sigma_b} \right|_{\sigma_a=\bar{\sigma}_a}, \\ [m_P^2(\bar{\sigma})]^{ab} &= \left. \frac{\partial^2 V_{\text{LSM}}}{\partial \pi_a \partial \pi_b} \right|_{\sigma_a=\bar{\sigma}_a}. \end{aligned} \quad (13)$$

The explicit expressions and derivations for those formulas can be found in Appendix A. For the scalar mesons, $[m_S^2(\bar{\sigma})]^{11}$ are $[m_S^2(\bar{\sigma})]^{44}$, respectively, identified as the squared masses for $a_0(980)$, $m_{a_0}^2$, and $\kappa(700) = K_0^*(700)$, m_K^2 . The $f_0(500)$ and $f_0(980)$ masses arise as the mass eigenvalues from the isospin-singlet scalar mass matrix through a mixing angle θ_S^0 defined with an orthogonal transformation matrix in Eq. (A14). Similarly to the scalar mesons, $[m_P^2(\bar{\sigma})]^{11} = m_\pi^2$ and $[m_P^2(\bar{\sigma})]^{44} = m_K^2$, respectively. The η' and η masses are read as the mass eigenvalues for the isospin-singlet pseudoscalar-mass matrix with a mixing angle θ_P^0 , as in Eq. (A19).

To evaluate the quantum and thermal corrections arising in the present LSM at finite temperature T , we apply the CJT formalism [52,57]. We start from the general loop expansion of the 2PI effective potential for the LSM with N_f flavors, which goes like

$$\begin{aligned}
 V_{\text{CJT}}[\sigma, S, P] &= V(\sigma) + \frac{1}{2} \int_k \text{tr}[\log S^{-1}(k) + \log P^{-1}(k)] \\
 &+ \frac{1}{2} \int_k \text{tr}[\bar{S}^{-1}(k; \sigma) S(k) + \bar{P}^{-1}(k; \sigma) P(k) - 2 \cdot \mathbf{1}_{\text{adj}}] \\
 &+ V_2[\sigma, S, P], \tag{14}
 \end{aligned}$$

where σ is a general background field; the trace is taken in the flavor space; $V(\sigma)$ is the tree-level potential defined in Eq. (3), and $V_2[\sigma, S, P]$ is the sum of all 2PI diagrams, which are constructed by the full-meson propagators $\mathcal{S}$ and $\mathcal{P}$; $\mathbf{1}_{\text{adj}}$ is the unit matrix in the adjoint representation of $SU(3)$ having the size of $N_f^2 \times N_f^2$; the full scalar (pseudoscalar) propagators are denoted as S (P), which are treated as independent variables in the present framework. In Eq. (14), we have abbreviated the loop momentum integral as $\int_k f(k) \equiv T \sum_n \int d^3k / (2\pi)^3 f(\omega_n, \mathbf{k})$ with ω_n being the bosonic Matsubara frequency. In Eq. (14), $\bar{S}^{-1}(k; \sigma)$ and $\bar{P}^{-1}(k; \sigma)$ are the tree-level mesonic inverse propagators induced from the tree-level action, which reads

$$\begin{aligned}
 [\bar{S}^{-1}(k; \sigma)]^{ab} &= k^2 \delta^{ab} + [m_S^2(\sigma)]^{ab}, \\
 [\bar{P}^{-1}(k; \sigma)]^{ab} &= k^2 \delta^{ab} + [m_P^2(\sigma)]^{ab}. \tag{15}
 \end{aligned}$$

In the present work, we adopt the Hartree approximation, by which the sum of the 2PI diagram terms V_2 in Eq. (14) is truncated at the two-loop level as follows:

$$\begin{aligned}
 V_2[S, P] &= \mathcal{F}^{abcd} \left[\int_k S_{ab}(k) \int_q S_{cd}(q) \right. \\
 &+ \left. \int_k P_{ab}(k) \int_q P_{cd}(q) \right] \\
 &+ 2\mathcal{H}^{abcd} \int_k S_{ab}(k) \int_q P_{cd}(q), \tag{16}
 \end{aligned}$$

where $\mathcal{F}^{abcd}$ and $\mathcal{H}^{abcd}$ are associated with the four-meson couplings λ_1 and λ_2 in Eq. (4), and the explicit expressions are given in Eq. (B8). The corresponding Feynman diagrams are illustrated in Fig. 1. In the current setup, the daisy-type diagrams are resummed systematically by solving the gap equation of the meson propagators [see also Eq. (20)].

The nonperturbative VEVs (or the thermal averages) of σ_a and the full scalar and pseudoscalar meson propagators S_{ab} and P_{ab} are determined by the stationary condition for V_{CJT} at $\sigma = \bar{\sigma}$, $S = \mathcal{S}$, and $P = \mathcal{P}$ as

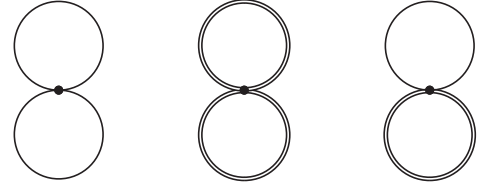


FIG. 1. Feynman diagrams in the Hartree approximation contributing to V_2 in Eq. (16). The solid lines represent the scalar meson propagators S_{ab} , and the doubled solid lines the pseudoscalar meson propagators P_{ab} . The black blob stands for the bare vertices, which are denoted in the text as $\mathcal{F}^{abcd}$ and $\mathcal{H}^{abcd}$, respectively.

$$\begin{aligned}
 \left. \frac{\partial V_{\text{CJT}}[\sigma, S, P]}{\partial \sigma_a} \right|_{\sigma=\bar{\sigma}, S=\mathcal{S}, P=\mathcal{P}} &= 0, \\
 \left. \frac{\delta V_{\text{CJT}}[\sigma, S, P]}{\delta S_{ab}} \right|_{\sigma=\bar{\sigma}, S=\mathcal{S}, P=\mathcal{P}} &= 0, \\
 \left. \frac{\delta V_{\text{CJT}}[\sigma, S, P]}{\delta P_{ab}} \right|_{\sigma=\bar{\sigma}, S=\mathcal{S}, P=\mathcal{P}} &= 0. \tag{17}
 \end{aligned}$$

See also Eq. (B5). In the Hartree approximation, as evident from Fig. 1, the loop corrections are all contact such that no external momentum dependence is renormalized from the tree-level one for meson-two point functions. Therefore, we parametrize the nontrivial solution for (inverse) meson propagators, $\mathcal{S}_{ab}$ and $\mathcal{P}_{ab}$, only with momentum-independent dynamical mass terms as follows:

$$\begin{aligned}
 [\mathcal{S}^{-1}(k)]^{ab} &= k^2 \delta^{ab} + [M_S^2]^{ab}, \\
 [\mathcal{P}^{-1}(k)]^{ab} &= k^2 \delta^{ab} + [M_P^2]^{ab}. \tag{18}
 \end{aligned}$$

Then, the stationary condition in Eq. (17) is cast into the form,

$$\begin{aligned}
 0 &= \frac{\partial V(\bar{\sigma})}{\partial \bar{\sigma}_0} - 3\mathcal{G}^{0bc} \left[\int_k \mathcal{S}_{cb}(k) - \int_k \mathcal{P}_{cb}(k) \right] \\
 &+ 4\mathcal{F}^{0bcd} \bar{\sigma}_b \int_k \mathcal{S}_{dc}(k) + 4\mathcal{H}^{0bcd} \bar{\sigma}_b \int_k \mathcal{P}_{dc}(k), \\
 0 &= \frac{\partial V(\bar{\sigma})}{\partial \bar{\sigma}_8} - 3\mathcal{G}^{8bc} \left[\int_k \mathcal{S}_{cb}(k) - \int_k \mathcal{P}_{cb}(k) \right] \\
 &+ 4\mathcal{F}^{8bcd} \bar{\sigma}_b \int_k \mathcal{S}_{dc}(k) + 4\mathcal{H}^{8bcd} \bar{\sigma}_b \int_k \mathcal{P}_{dc}(k), \tag{19}
 \end{aligned}$$

and

$$\begin{aligned}
 [M_S^2]^{ab} &= [m_S^2(\bar{\sigma})]^{ab} + 4\mathcal{F}^{abcd} \int_k \mathcal{S}_{cd}(k) + 4\mathcal{H}^{abcd} \int_k \mathcal{P}_{cd}(k), \\
 [M_P^2]^{ab} &= [m_P^2(\bar{\sigma})]^{ab} + 4\mathcal{F}^{abcd} \int_k \mathcal{P}_{cd}(k) + 4\mathcal{H}^{abcd} \int_k \mathcal{S}_{cd}(k). \tag{20}
 \end{aligned}$$

The detailed expression of $\mathcal{G}^{abc}$, which has been introduced in Eq. (19) is given in Eq. (B26), which is associated with the $U(1)$ axial anomaly, i.e., the KMT determinant term proportional to B in Eq. (5).

III. (ANOMALOUS) WTIS, NG THEOREM, AND SYMMETRY IMPROVEMENT

In this section, following [48], we review the intrinsic issue of the 2PI formalism on the violation of the NG theorem and the (anomalous) WTIs. For readability, we will pick up only the resultant key equations and the detailed derivations of the WTIs are referred to as in Appendix C.

Consider an action $S[\phi]$ having a symmetry associated with the infinitesimal transformation of a set of general fields ϕ_a ($a = 1, \dots$) as $\phi_a \rightarrow \phi_a + i\epsilon\delta_\epsilon\phi_a$, where $\delta_\epsilon\phi_a$ is the corresponding generator in the field space. Momentarily, no explicit breaking is assumed for the original symmetry. Through the symmetry transformation of the generating functional (with the source J^a), we have the following WTI

$$J^a \cdot \langle \delta_\epsilon \phi_a \rangle_J = 0. \quad (21)$$

If the generator belongs to the linear symmetry representation in the corresponding Lie group, such as $\delta_\epsilon\phi_a = d^b_a\phi_b$ with d being an arbitrary coefficient matrix, we then get the following WTI for the 1PI effective action Γ :

$$\frac{\delta\Gamma}{\delta(\phi_{\text{cl}})_a} \cdot \delta_\epsilon(\phi_{\text{cl}})_a = 0, \quad (22)$$

where $(\phi_{\text{cl}})_a \equiv \langle \phi_a \rangle_J$ is the VEV including the source field J .

In the case of the chiral- $SU(3)$ transformation given in Eq. (1), we find

$$\delta^\alpha \sigma_a = d^{ab}_a \pi_b, \quad \delta^\alpha \pi_a = -d^{ab}_a \sigma_b, \quad (23)$$

Following Eq. (22), in the case without explicit breaking, i.e., in the three-flavor chiral limit, the WTIs for these chiral rotations are then read off as

$$\frac{\delta\Gamma[\sigma_{\text{cl}}, \pi_{\text{cl}}]}{\delta(\sigma_{\text{cl}})_a} \cdot d^{ab}_a (\pi_{\text{cl}})_b + \frac{\delta\Gamma[\sigma_{\text{cl}}, \pi_{\text{cl}}]}{\delta(\pi_{\text{cl}})_a} \cdot d^{ab}_a (\sigma_{\text{cl}})_b = 0. \quad (24)$$

Passing through a couple of algebraic computations (see Appendix C), Eq. (24) can finally be reduced to the following simple relations:

$$M_\pi^2 \bar{\Phi}_1 = 0, \quad M_K^2 (\bar{\Phi}_1 + \bar{\Phi}_3) = 0, \quad (25)$$

where we have also used Eq. (11). The M_π and M_K denote the full nonperturbative pion and kaon masses arising from the full propagator P_{ab} . Equation (25) manifests itself that in the chiral limit, the spontaneously chiral broken phase

with $\bar{\Phi}_1 \neq 0$ or $\bar{\Phi}_3 \neq 0$ yields $M_\pi = 0$ and $M_K = 0$, precisely reflecting the NG theorem. The chiral-limit WTI only for the two-flavor chiral rotation has been discussed in [49], which has been currently extended by including the M_K term in Eq. (25).

The above formulations can be further generalized to the case with explicit chiral symmetry breaking supplied from the spurion field for the current quark masses, $\mathcal{M}$. The action S , corresponding to the LSM Lagrangian (2), is $SU(3)$ chiral invariant by including the (infinitesimal) transformation of $\mathcal{M}$ given in Eq. (7) together with Eq. (23). The WTIs (22) are then generalized into the so-called anomalous chiral WTIs,¹

$$\left\langle \frac{\delta S}{\delta \mathcal{M}} \right\rangle_J \cdot \delta_\epsilon \mathcal{M} + \left\langle \frac{\delta S}{\delta \mathcal{M}^\dagger} \right\rangle_J \cdot \delta_\epsilon \mathcal{M}^\dagger + J^a \cdot \langle \delta_\epsilon \phi_a \rangle_J = 0. \quad (26)$$

After somewhat lengthy calculations (see Appendix C), we see that Eq. (26) gives a naturally extended formula in the presence of the current quark masses for M_π and M_K from the chiral limit case in Eq. (25),

$$M_\pi^2 \bar{\Phi}_1 = cm_l(1 + 2k\bar{\Phi}_3), \\ M_K^2 (\bar{\Phi}_1 + \bar{\Phi}_3) = (cm_l + cm_s)(1 + 2k\bar{\Phi}_1). \quad (27)$$

To see the more physical consequences of the constraint relations in Eq. (27), we shall read the light and strange quark condensates, defined as the response of the light and strange quark masses, from the LSM Lagrangian in Eq. (2) as

$$\langle \bar{\ell}\ell \rangle = \left\langle \frac{\partial \mathcal{L}_{\text{LSM}}}{\partial m_l} \right\rangle = -2c(\bar{\Phi}_1 + 2k\bar{\Phi}_1\bar{\Phi}_3), \\ \langle \bar{s}s \rangle = \left\langle \frac{\partial \mathcal{L}_{\text{LSM}}}{\partial m_s} \right\rangle = -2c(\bar{\Phi}_3 + 2k\bar{\Phi}_1^2). \quad (28)$$

We also read the pion and kaon decay constants f_π and f_K , which arise as the overlap amplitudes between the corresponding axial vector currents and pseudoscalars, as

$$f_\pi = 2\bar{\Phi}_1, \\ f_K = \bar{\Phi}_1 + \bar{\Phi}_3. \quad (29)$$

From these, we find that two constraint relations in Eq. (27) take the

$$f_\pi^2 m_\pi^2 = -2m_l \langle \bar{\ell}\ell \rangle, \\ f_K^2 m_K^2 = -\frac{m_l + m_s}{2} (\langle \bar{\ell}\ell \rangle + \langle \bar{s}s \rangle), \quad (30)$$

¹Another type of anomalous chiral WTIs would be available in Refs. [58,59].

which are precisely the Gell-Mann-Oakes-Renner (GMOR) relations. Thus, the low-energy theorem is intact even in the presence of the intrinsically flavor-violating k term [50,51].

The *threshold property* of the (pseudo) NG bosons ($\bar{\Phi}_{1,3} \rightarrow 0$) is thus described by the anomalous WTIs in 1PI formalism. The masses of the NG modes are constrained to zero in the chiral broken phase ($\bar{\Phi}_1 \neq 0$, $\bar{\Phi}_3 \neq 0$), or proportional to the explicit breaking current masses. Since the relations (25) or (27) hold before performing the loop expansion in the 1PI effective action, we expect that the threshold property holds even when we truncate the 1PI effective action loopwise. Hereafter, we shall call both the anomalous chiral WTIs and the chiral-limit WTIs simply the WTIs, otherwise specialized.

Examining the WTIs derived in the 2PI formalism, however, we realize that the situation would be drastically changed. In the chiral limit, the WTIs for the 2PI effective action read

$$0 = \frac{\delta\Gamma[\phi_{\text{cl}}, \Delta]}{\delta(\phi_{\text{cl}})_a} \cdot d_a^b(\phi_{\text{cl}})_b + \frac{\delta\Gamma[\phi_{\text{cl}}, \Delta]}{\delta\Delta_{ab}} \cdot (d_b^c\Delta_{ac} + d_a^c\Delta_{cb}), \quad (31)$$

which leads to the second derivatives,

$$0 = \frac{\delta^2\Gamma[\phi_{\text{cl}}, \Delta]}{\delta(\phi_{\text{cl}})_c\delta(\phi_{\text{cl}})_a} \cdot d_a^b(\phi_{\text{cl}})_b + \frac{\delta\Gamma[\phi_{\text{cl}}, \Delta]}{\delta(\phi_{\text{cl}})_a} \cdot d_a^c + \frac{\delta^2\Gamma[\phi_{\text{cl}}, \Delta]}{\delta(\phi_{\text{cl}})_c\delta\Delta_{ab}} \cdot (d_a^c\Delta_{ac} + d_a^c\Delta_{cb}). \quad (32)$$

Due to the third term appearing in the second line of Eq. (32), this identity does not provide a direct constraint to prevent the NG boson from being massive. If the 1PI and 2PI effective actions are evaluated exactly without any truncation, the solutions of ϕ_{cl}^a and the propagators Δ_{ab} should be the same since they come from the same sourceless generating functional. In this case, the WTIs in the 1PI formalism [e.g., Eq. (26) without the $\mathcal{M}$ term] and the one in the 2PI formalism (31) should be satisfied simultaneously at the solution to the stationary condition. However, truncating both effective actions at the loop level ($\Gamma_{\text{tr}}^{1\text{PI}}$ and $\Gamma_{\text{tr}}^{2\text{PI}}$), we find that they will follow different WTIs. Then, since the 2PI WTI does not directly constrain the NG boson mass, the threshold property or the NG theorem would be violated. The schematic sketch of the above scenario is shown in Fig. 2. This issue would be crucial for the phase transition of the system, since the massless nature is directly connected to the scale-invariance of the correlators.

To resolve this issue, we employ the SICJT formalism [48], for which the strategy is to replace the stationary condition in Eq. (19) by the generalized GMOR relation in Eq. (27) derived from the WTIs of the 1PI formalism, while

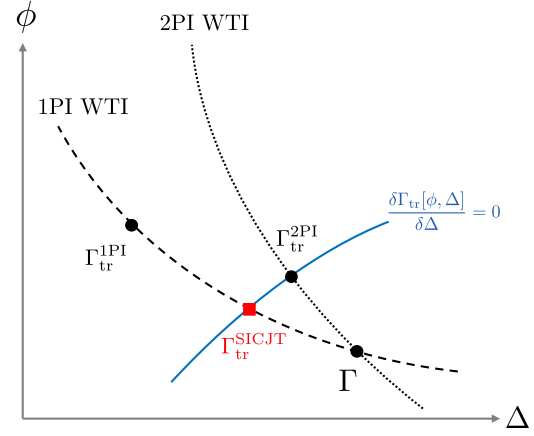


FIG. 2. Schematic sketch of the SICJT formalism. The figure is drawn in the (ϕ, Δ) plane where ϕ is the sourced field VEV and Δ denotes the sourced propagator, describing the variables of the effective actions. The black blobs (from right to left) represent the solution obtained from the full-quantum action Γ , the truncated 1PI effective action $\Gamma_{\text{tr}}^{1\text{PI}}$, and the truncated 2PI effective action $\Gamma_{\text{tr}}^{2\text{PI}}$. The black-dashed and -dotted curves, respectively, denote the constraint trajectories from the 1PI or the 2PI formalism, and the blue solid line denotes the gap equation of the propagators Δ with arbitrary ϕ . The red square mark points to the solution realized by the SICJT formalism.

keeping the second constraint with the gap equations for meson masses in Eq. (20). The solution found in this procedure is denoted as $\Gamma_{\text{tr}}^{\text{SICJT}}$ in Fig. 2. This completes the current formulation, and we shall apply this method to study the Columbia plot.

IV. NUMERICAL ANALYSIS

In this section, we show our numerical results on the T dependences for the chiral order parameters, masses of scalar and pseudoscalar meson nonets, by solving the gap equations of the meson propagators in Eq. (20) coupled with the generalized GMOR relations in Eq. (27) based on the SICJT formalism. The analysis is performed both at the physical point and in the chiral limit. We then discuss the Columbia plot and make a comparison with the results based on the conventional CJT formalism, which follows by solving the gap equations coupled to the stationary conditions in Eq. (19), which significantly breaks the NG theorem and threshold property fixed as in the generalized GMOR relations in Eq. (27). The criticalities at the tricritical point and the second-order phase boundary, as well as the critical pion mass in the three-flavor symmetric limit, are also addressed in depth.

A. Numerical implements

Before presenting our results, we briefly summarize the numerical strategy applied in the current work. To solve the coupled equations, we shall adapt the iteration method and

decompose the problem into three layers to gain better numerical stability. Given a set of test values for the VEVs $\bar{\Phi}_{1,3}$ and full meson masses $M_{S,P}$, we fix the values of the mixing angles $\theta_{S,P}$ through Eq. (B22), by using the Newtonian iteration method with $\theta_{S,P}^0$ in Eqs. (A15) and (A20) used as initial conditions. Then we solve the gap equations (20) using the fixed-point iteration method with the tree-level masses $m_{S,P}$, which can be found in Appendix A 1, used as the initial conditions. Finally, we solve either the stationary condition in Eq. (19) or the GMOR relations in Eq. (27) by using the Newtonian method with tests of different initial guesses, to search for different phases at a given temperature.

B. Parameter setup at physical point

We take two different sets of model parameters, characterized by a different $f_0(500)$ mass as an input, to examine the sensitivity of the size of the $f_0(500)$ mass to the order of phase transition or the phase structure of the Columbia plots, as was discussed earlier [33,35] based on the conventional CJT formalism.

1. Parameter set I

The first set of parameters is chosen following the literature [50,51] as

$$\begin{aligned}\mu^2 &= 1.02 \times 10^4 \text{ MeV}^2, & \lambda_1 &= 11.8, & \lambda_2 &= 20.4, \\ cm_l^{\text{phys}} &= 6.11 \times 10^5 \text{ MeV}^3, \\ cm_s^{\text{phys}} &= 198 \times 10^5 \text{ MeV}^3, \\ B &= 3.85 \times 10^3 \text{ MeV}, & k &= 3.40 \times 10^{-3} \text{ MeV}^{-1},\end{aligned}\quad (33)$$

which provides the pion and kaon decay constants as $f_\pi = 92.1 \text{ MeV}$ and $f_K = 109.8 \text{ MeV}$ in Eq. (29), and the meson mass spectra

$$\begin{aligned}m[f_0(500)] &= 672.4 \text{ MeV}, & m[f_0(980)] &= 990.4 \text{ MeV}, \\ m_{a_0} &= 937.6 \text{ MeV}, & m_K &= 863.4 \text{ MeV}, \\ m_{\eta'} &= 958.2 \text{ MeV}, & m_\eta &= 552.9 \text{ MeV}, \\ m_\pi &= 137.9 \text{ MeV}, & m_K &= 494.1 \text{ MeV}.\end{aligned}\quad (34)$$

The mixing angles at vacuum are

$$\theta_S^0 = 44.3^\circ, \quad \theta_P^0 = 6.80^\circ. \quad (35)$$

2. Parameter set II

We re-fit the parameters following the χ^2 -fit procedure as performed in Ref. [51], with replacing the $f_0(500)$ mass by $m[f_0(500)] = 800 \text{ MeV}$. Then we get the following best-fit values for the model parameters,

$$\begin{aligned}\mu^2 &= -1.79 \times 10^5 \text{ MeV}^2, & \lambda_1 &= 30.9, & \lambda_2 &= 22.2, \\ cm_l^{\text{phys}} &= 6.11 \times 10^5 \text{ MeV}^3, \\ cm_s^{\text{phys}} &= 198 \times 10^5 \text{ MeV}^3, \\ B &= 3.86 \times 10^3 \text{ MeV}, & k &= 1.51 \times 10^{-3} \text{ MeV}^{-1},\end{aligned}\quad (36)$$

which provides the pion and kaon decay constants as $f_\pi = 92.1 \text{ MeV}$ and $f_K = 109.8 \text{ MeV}$, and the meson mass spectra

$$\begin{aligned}m[f_0(500)] &= 797.1 \text{ MeV}, & m[f_0(980)] &= 1151 \text{ MeV}, \\ m_{a_0} &= 943.1 \text{ MeV}, & m_K &= 966.5 \text{ MeV}, \\ m_{\eta'} &= 897.0 \text{ MeV}, & m_\eta &= 513.9 \text{ MeV}, \\ m_\pi &= 125.8 \text{ MeV}, & m_K &= 460.1 \text{ MeV}.\end{aligned}\quad (37)$$

The mixing angles in the vacuum are

$$\theta_S^0 = 45.0^\circ, \quad \theta_P^0 = 0.291^\circ. \quad (38)$$

Notice that the physical observables obviously deviate from the inputs, which indicates that the current parameter space may not fully cover the input observables.

C. Chiral phase transition at the physical point

In Fig. 3 we show the T dependences on $\bar{\Phi}_{1,3}$, and the scalar and pseudoscalar meson masses M_S and M_P at the physical point $(m_l, m_s) = (m_l^{\text{phys}}, m_s^{\text{phys}})$ with the parameter set I provided in Sec. IV B 1. In the figure, the upper panel refers to the conventional CJT method, while the lower panel corresponds to the result from the SICJT method. Comparing both methods, we find the following qualitative characteristics:

- (i) The pseudocritical temperature of the light quark condensate, T_{pc} , which is defined by the temperature with the maximum thermal susceptibility $\chi_T = -\partial \langle \bar{\ell} \ell \rangle_T / \partial T$, is around $\sim 185 \text{ MeV}$ in the SICJT formalism, which is reduced compared to the one estimated from the conventional CJT formalism, $T_{\text{pc}} \sim 215 \text{ MeV}$. Those values of T_{pc} are anyhow higher than the ones obtained from other nonperturbative approaches, such as fRG (e.g., Ref. [60]), Dyson-Schwinger equations (e.g., Ref. [61]) or lattice QCD (e.g., Ref. [62]). This higher T_{pc} can be understood by the lack of dynamical quark degrees of freedom. The extended studies including quarks would be performed in the future; and
- (ii) The pion mass M_π shows relatively smoother T dependence for $T \lesssim T_{\text{pc}}$ in the SICJT formalism, compared to the conventional CJT case, before the chiral restoration happens. This is due to the realization of the proper threshold property and

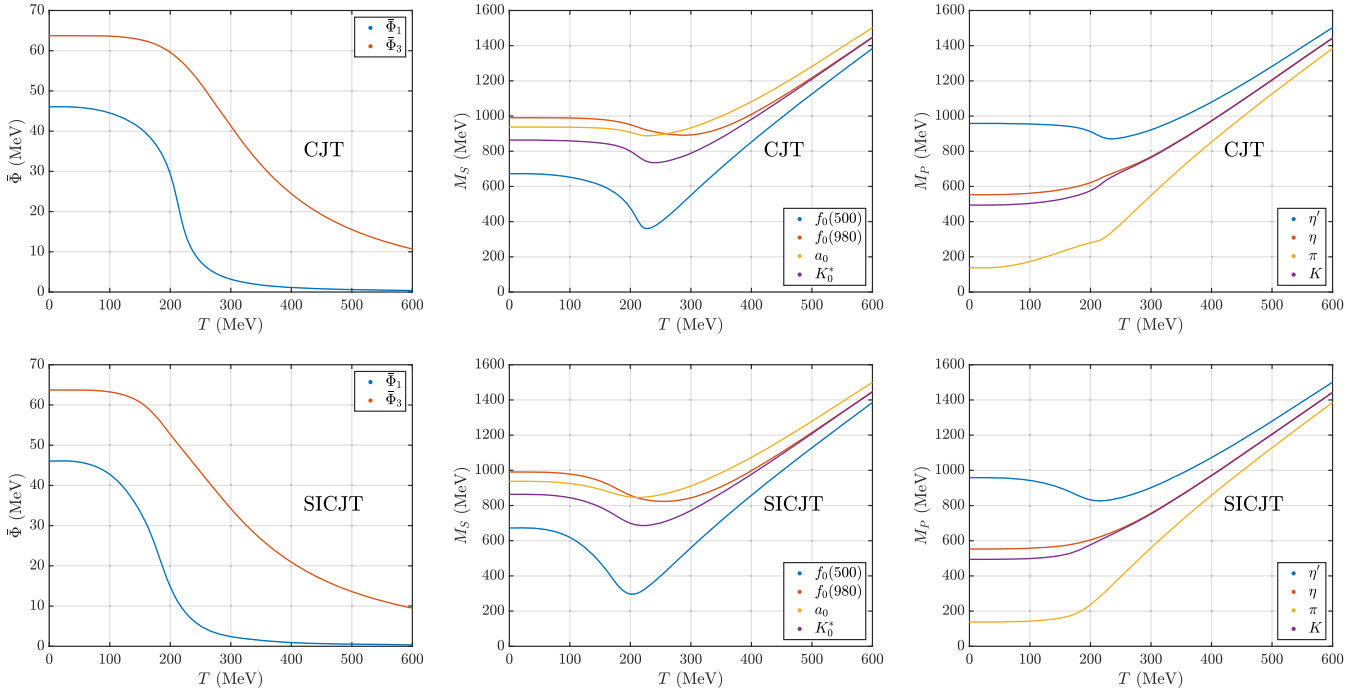


FIG. 3. Plots of T dependences on $\bar{\Phi}_{1,3}$ (left panel) and masses of scalar (middle) and pseudoscalar mesons (right), at the physical point, in the cases of the conventional CJT (upper panels) and SICJT (lower panels) formalisms applied to the present LSM with the parameter set I in Sec. IV B 1.

the manifest NG theorem: the introduction of loop corrections in the conventional CJT method breaks the low-energy theorem, i.e., the GMOR relations, such that (pseudo-) NG boson mass spectra can develop artificially and unphysically with finite T .

Both cases show the degeneracy of the meson mass spectra at $T > T_{pc}$, which precisely reflects the effective restoration of the chiral symmetry due to the reduction of the size of the light quark condensate. The parameter set II in Sec. IV B 2 yields essentially the same results.

D. Chiral phase transition in chiral limit

We also explore the chiral phase transitions in the chiral limit at $(m_l, m_s) = (0, 0)$ for both the conventional CJT and SICJT formalisms. See Fig. 4, where the plots have been created by taking the parameter set I in Sec. IV B 1, but with $m_l = m_s = 0$.

In the conventional CJT formalism, we have examined all stationary points, including the local minimum as well as the local maximum. For the latter point, those maximum states connect the vanishing point of the chiral broken phase and the emerging point of the chiral symmetric phase. In the SICJT case, the tachyonic modes with negative curvatures (corresponding to the local maximum states) show up, which can be reflected as the smoother dropping blue curve in the bottom-right panel of Fig. 4: around the phase transition temperature $T_c \sim 25$ MeV, the $f_0(500)$ mass shows the trend to be vanishing. This brings further

numerical complexity in evaluating the meson loops, but makes milder effects on the scope of the current work; thus, we drop the local maximum state in evaluating the chiral order parameter in the SICJT case. For the mass spectra in both cases, all meson masses are merged into one, denoted by the purple line after the chiral restoration happens, and linked with the unified mass spectra in the false vacuum, which is marked by the red-dashed line in the right-bottom panel of Fig. 4.

We find the following characteristics for the phase transition in the chiral limit.

- (i) From the left panels of both CJT and SICJT cases, we find that the chiral order parameter discontinuously changes with temperature around some critical values, and the local maximum state is clearly seen in the conventional CJT case. Those behaviors of the chiral order parameters indicate the orders of the phase transitions in both cases to be of the first order. The phase transition takes place at the critical temperature $T_c \sim 25$ MeV in the case of the SICJT formalism (the bottom-left panel of Fig. 4). This T_c is much smaller than the one measured based on the conventional CJT formalism, which yields $T_c \sim 97$ MeV (see the top-left panel). This reduction can be understood by the proper reflection of the NG theorem in the chiral limit even at finite T : thermal loop corrections arising from lighter mesons make the critical temperature lower. The presence of the first-order phase transition in both two formalisms is

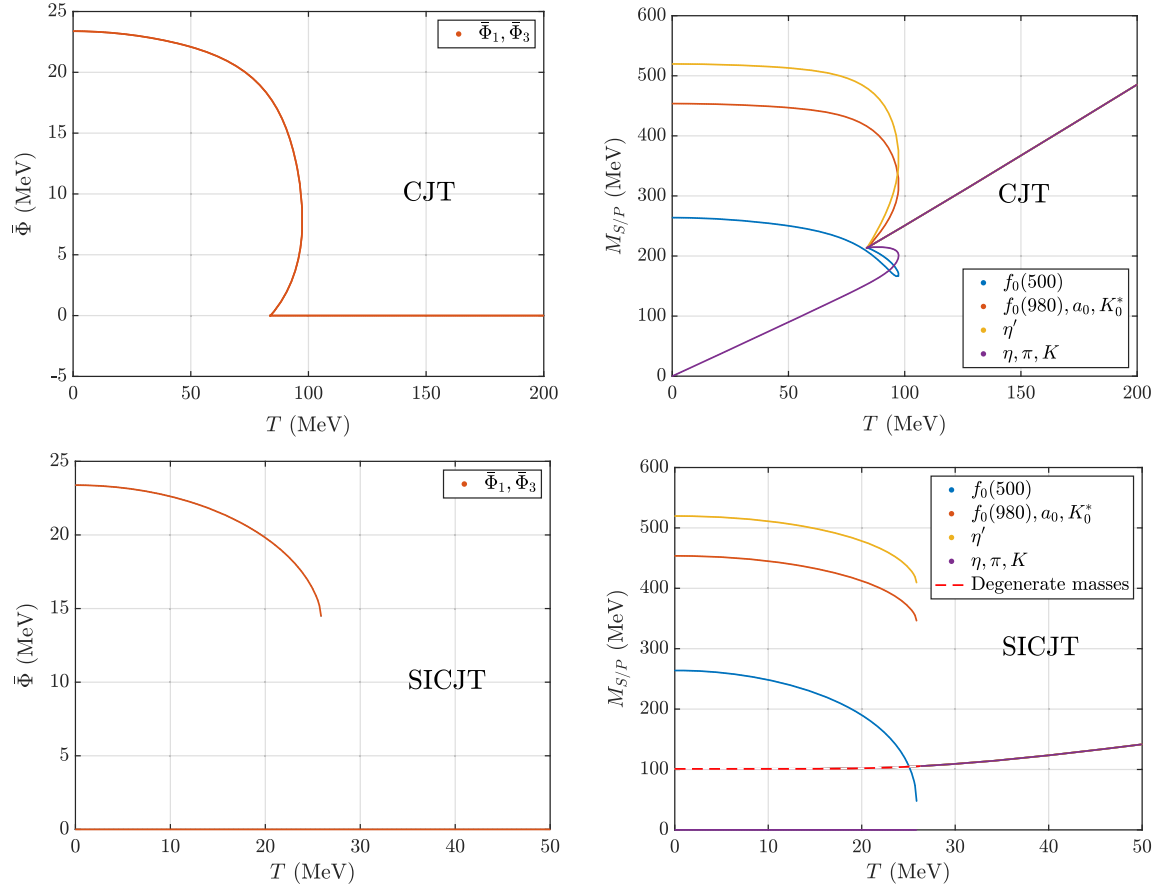


FIG. 4. Same as Fig. 3, for the chiral limit case, with the plots of the scalar and pseudoscalar meson masses combined into a single one, which has been summarized in the right panel. The red-dashed line in the bottom-right panel denotes the degenerate masses of all mesons in the chirally symmetric phase below the critical temperature.

essentially due to the non-restoration of the $U(1)$ axial anomaly, which is currently encoded in the KMT determinant [the B term in Eq. (5)] providing a negatively large cubic potential along the chiral-order parameter direction $\bar{\Phi}_1$ (See also the parameter sets I and II in Secs. IV B 1 and IV B 2).

- (ii) The masses of the pion M_π and the kaon M_K are fixed to vanish in the SICJT formalism below the exact chiral restoration (see the bottom-right panel of Fig. 4), while they start to develop in the CJT case at finite T , even before the chiral restoration happens (see the top-right panel). The former feature is closely related to the realization of the proper threshold property and the NG theorem, which is violated in the CJT formalism. More remarkably, at $T = T_c$ in the chiral limit for the SICJT formalism, the $f_0(500)$ mass goes to zero, reflecting the chiral partner of the massless pion.

Recently, based on highly improved staggered quarks on lattices with temporal extent $N_\tau = 8$ [17]—which is an extension of the study with $N_\tau = 6$ [31]—the three-flavor lattice QCD study has reported that no evidence of the first-order signal is observed for $m_\pi \gtrsim 80$ MeV. The critical

temperature in the three-flavor chiral limit has also been estimated via the finite-size scaling analysis, to give $T_c = 98_{-6}^{+3}$ MeV, based on the data and statistical averaging over different fit ansatz with m_π in a range of $80 \text{ MeV} \lesssim m_\pi \lesssim 140 \text{ MeV}$. The present model with the SICJT formalism thus tends to give a lower T_c than the lattice result, which, however, may only be of limited relevance because of different model setups and systematics.

E. Columbia plots

In Fig. 5, we make the Columbia plots based on the SICJT formalism (bottom panels) in comparison with the conventional CJT case (top panels). The parameter sets I and II in Secs. IV B 1 and IV B 2 have been reflected, respectively, in the left and right panels.²

²The present analysis has been limited by the large size of m_s up to $2m_s^{\text{phys}}$, above which a part of the k -term proportional to m_s [Eq. (8)] will cause an instability in the system, because of the artificial nondecoupling effect of the strange quark in the k term. This issue needs to be fixed in a separate publication.

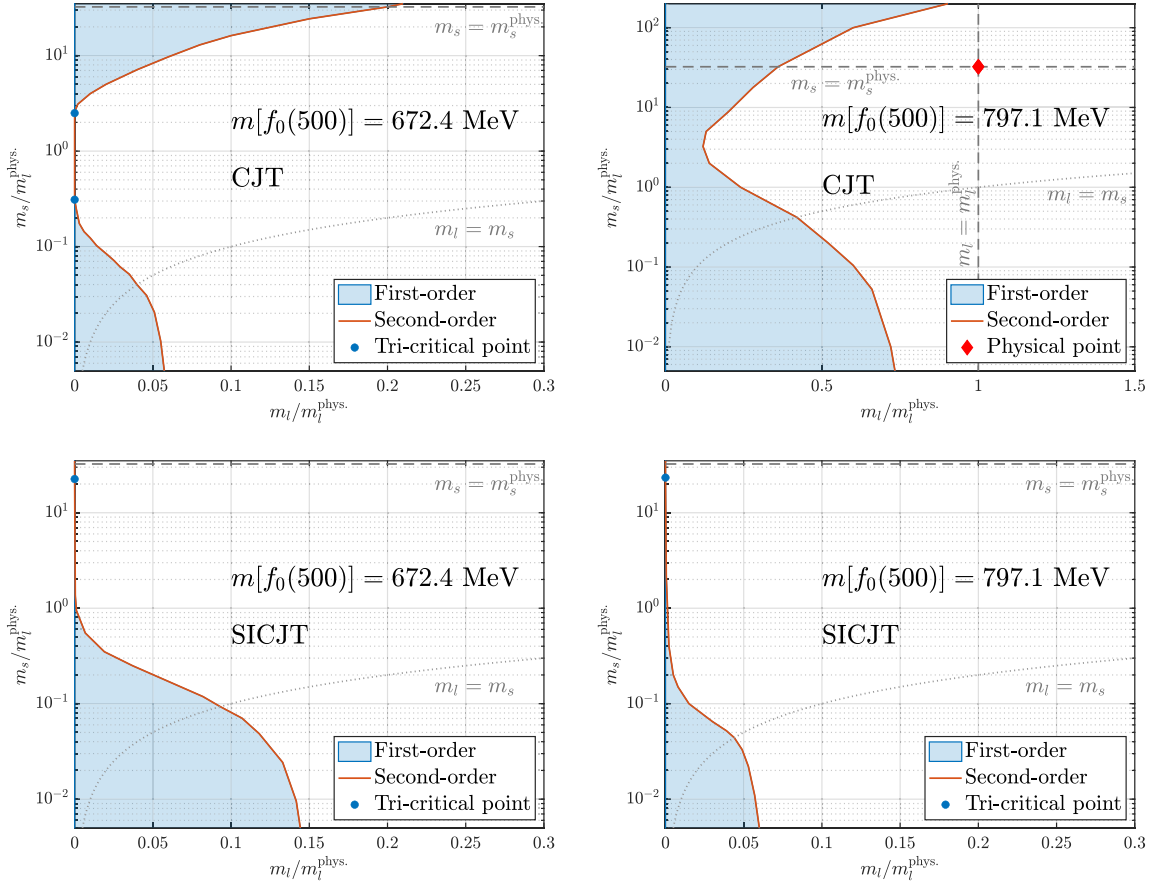


FIG. 5. Conventional CJT result projected onto the Columbia plots with the parameter sets I in Sec. IV B 1 (top-left panels) and II in Sec. IV B 2 (top-right panel), in comparison with the SICJT results (bottom-left and -right panels) on the same parameter setup. The orange solid curves denote the second-order phase boundaries, the blue-shaded areas stand for the first-order domains, and blank areas represent the crossover regions. The three-flavor symmetric limit (the case with $m_l = m_s$) has been drawn by the dotted curves, and the red-diamond marker corresponds to the physical point where $(m_c, m_s) = (m_l^{\text{phys}}, m_s^{\text{phys}})$. The m_l and m_s axes have been normalized by m_l^{phys} .

In the literature [33,35], the Columbia plot has already been discussed based on a three-flavor LSM with the conventional CJT formalism, which is similar to the present model except for the k term [Eq. (8)], where the input observables to fix the model parameters are somewhat, but not substantially different from those used in Secs. IV B 1 and IV B 2. In the top panels of Fig. 5, we have reproduced the resulting phase structure fully consistent with the one observed in the references, including the overall characteristics: the first-order regime is present, and two tricritical points emerge at around $m_s = \mathcal{O}(0.1 m_s^{\text{phys}})$ and $\gtrsim \mathcal{O}(m_s^{\text{phys}})$ for lighter $f_0(500)$, while those two points get merged to be gone as $m_{f_0(500)}$ is heavier and heavier, still keeping the presence of the first-order regime.

We find that the SICJT formalism removes the artificial first-order regime and gives the *symmetry-improved* Columbia plot (bottom panels in Fig. 5): no significant $m_{f_0(500)}$ dependence is seen, and the first-order regime as

well as the single tricritical point are stably present. This also implies no existence of the extra first-order regime which has been observed in the conventional CJT formalism. Those features have been measured due to the proper realization of the low-energy theorem at finite T , which protects the pion to be light enough and safely suppresses the other heavier meson contributions in the thermal bath, including those from $f_0(500)$, up to $T \lesssim T_{(p)c}$. It has been claimed in Refs. [33,35] that the existence of the extra first-order regime in the conventional CJT formalism is tied with the Hartree-level approximation. Our present result supports another possibility that working on the Hartree-level approximation is not the main reason for generating the extra first-order regime; rather, the violation of the NG theorem may be the reason.

We also still see from Fig. 5 that in the SICJT case, the first-order regime gets shrunk when $m_{f_0(500)}$ gets larger. This is simply due to the generation of more stringent

convex negative curvature [by the negative μ^2 term in Eq. (4) with the sign choices given in Secs. IV B 1 and IV B 2]. On more physics grounds, the generation of heavier $f_0(500)$ tends to require dynamical chiral symmetry breaking essentially triggered by the negative μ^2 term, while in the lighter $f_0(500)$ case, the $U(1)$ axial anomaly essentially triggers the dynamical chiral symmetry breaking even with a positive μ^2 term.

When taking a particular look at the three-flavor symmetric limit where $m_l = m_s$, we find the critical quark mass on the second-order boundary (red curves in Fig. 5) to be $m_l^c = m_s^c = 0.0931m_l^{\text{phys}}$, [e.g., for the parameter set I case, in Sec. IV B 1, with $m[f_0(500)] = 672.4$ MeV (bottom-left panel)]. This quark mass value is translated into the critical pion mass at the vacuum as $m_\pi^c = 52.4$ MeV. The chiral restoration is observed at $T_c \sim 51.6$ MeV.

F. Critical exponents

In this section, we present our result on the criticality of the light quark condensate in Eq. (28) evaluated from the SICJT formalism. The scaling behavior of the light quark condensate $\langle \bar{\ell}\ell \rangle$ with respect to the temperature T and the current quark mass m_l in the vicinity of the criticality is characterized by the critical exponents β and δ . Suppose the critical point of the second-order phase transition is located at $(T, m_l) = (T^c, m_l^c)$ with the critical current mass for the strange quark, m_s^c , we define the following dimensionless variables as

$$t \equiv \frac{T - T^c}{T^c}, \quad \tilde{m}_l \equiv \frac{m_l - m_l^c}{m_l^{\text{phys}}}, \quad (39)$$

with the following scaling function

$$\langle \bar{\ell}\ell \rangle(t, \tilde{m}_l) - \langle \bar{\ell}\ell \rangle(0, 0) = f_l(t, \tilde{m}_l), \quad (40)$$

where $f_l(t, \tilde{m}_l)$ is a dimensionful function and satisfies $f_l(0, 0) = 0$. The critical exponents of the thermal and the external source response are then given by

$$f_l(t \rightarrow 0, 0) \propto (-t)^\beta, \quad f_l(0, \tilde{m}_l \rightarrow 0) \propto \tilde{m}_l^{1/\delta}. \quad (41)$$

We summarize the method and related details of the evaluation of the critical exponents in the Appendix E.

Based on the symmetry argument, we expect the universality classes on the second-order line in the Columbia plot as follows. For the critical points located at the boundary dividing the first-order and the crossover areas (except for the tricritical point), the chiral symmetry is explicitly broken with the presence of three massive quarks. In this case, the chiral symmetry is reduced to remain as the discretized one, i.e., the Z_2 symmetry. We shall expect the Z_2 universality class on this boundary, described by the three-dimensional Ising model [63]. As to the critical points lying on the massless light flavor axis (the y axis in the

Columbia plot above the tricritical point), the criticalities are expected to be described by the $O(4)$ universality classes [64], since the massless two-flavor theory possesses the $SU(2)_L \times SU(2)_R \sim O(4)$ symmetry. Furthermore, if one assumes that the $U(1)_A$ symmetry is restored around the critical temperature of the chiral phase transition, the $U(2) \times U(2)$ universality class would also be possible to be found [65,66]. Regarding the tricritical point, which connects the line of the first-order and second-order regions along the y axis in the Columbia plot, the Ginzburg-Landau description of the free energy around this point is read as the following tricritical σ^6 theory:

$$\mathcal{F}(\sigma) = a_2(T, m_s)\sigma^2 + a_4(T, m_s)\sigma^4 + a_6\sigma^6 + \dots, \quad (42)$$

where $\sigma \sim \langle \bar{\ell}\ell \rangle$ denotes the light quark condensate, respectively. The tricritical point is located where $a_4(T, m_s^{\text{tri}}) = 0$, leading to the interplay between the first- and second-order phase transitions. The upper critical dimension of this tricritical σ^6 theory is $d_c = 3$, which is the same as the spatial dimension concerned in the present work, indicating that fluctuations become irrelevant and merely referring to mean-field exponents is validated. Thus, we shall expect the mean-field universality class at the tricritical point in the Columbia plot.

In Table I, we list the critical exponents evaluated at several critical points, as well as the aforementioned universality classes. For the critical exponents at the point where $m_s^c = 0$, the results fit well with the Z_2 universality class. For the flavor-symmetric point ($m_l^c = m_s^c = 0.0931m_l^{\text{phys}}$) and the tricritical point ($m_l^{\text{tri}} = 0$, $m_s^{\text{tri}} = 0.696m_s^{\text{phys}}$), the values of β are relatively close to the ones expected from the Z_2 and $O(4)$ universality classes, while the δ starts to deviate from those benchmark values. For the values obtained at the critical point on the y axis above the tricritical point ($m_l^c = 0$, $m_s^c = m_s^{\text{phys}}$), our result does not meet any universality class mentioned above.

Given the result of the criticality in the present work, we find that the current framework of the analysis based on the SICJT formulation may not quantitatively capture the critical exponents. We speculate that the observed deviation from the expected universality class may originate from three sources. First, the truncation scheme adopted in this work corresponds to the Hartree-Fock level, which incorporates only momentum-independent corrections to the meson propagators. Because mesonic fluctuations play a central role near criticality, it would be valuable to explore higher-order effects, such as wave-function (field-renormalization) corrections to the mesons. Second, the inclusion of the axial-anomaly-induced flavor-breaking term in Eq. (8) introduces an additional source of the chiral symmetry breaking. In this setup, the current quark masses no longer couple linearly to the vacuum expectation value of the mesonic field, thereby modifying the system's response

TABLE I. List of the critical exponents at several critical points estimated from the present LSM based on the SICJT formalism with the parameter set I in Sec. IV B 1. The resulting numbers are compared to those from typical statistical models, referred to as the critical points when specified: $O(4)$, Z_2 , and 3D XY-universality classes. Also shown are the corresponding pion mass m_π in the vacuum and the critical temperatures T_c observed in the present SICJT formalism (see also the bottom-left panel in Fig. 5).

Model	β	δ	m_π (MeV)	T_c (MeV)
SICJT: ($m_l^c = 0.1474m_l^{\text{phys}}$, $m_s^c = 0$)	0.3235	4.7907	65.4	51.6
SICJT: ($m_l^c = m_s^c = 0.0931m_l^{\text{phys}}$)	0.3222	3.2128	52.1	51.6
SICJT: ($m_l^{\text{tri}} = 0$, $m_s^{\text{tri}} = 0.696m_s^{\text{phys}}$)	0.2842	7.4597	0.0	113.5
SICJT: ($m_l^c = 0$, $m_s^c = m_s^{\text{phys}}$)	0.6532	3.7805	0.0	125.4
$O(4)$ universality class [64]	0.38	4.82	...	...
Z_2 universality class [63]	0.326	4.78	...	...
$U(2) \times U(2)$ universality class [65,66]	0.388	4.789	...	...
3D XY universality class [67]	0.3490	4.7798	...	...
Mean-field universality class	0.25	5	...	...

to external sources. Finally, the evaluation of the gap equations for meson masses in the concave region is also crucial near critical points. The treatment of this region influences the convergence of the solutions and can alter the resulting universality class. Thus, the present setup and analysis can be used for qualitative studies, such as the analysis of the transition temperature or the meson-mass spectra properties; however, it needs further improvement in several aspects to be capable of accurately evaluating the QCD criticality.

V. CONCLUSION

In this work, we have investigated the structure of the Columbia plot within a three-flavor LSM monitoring the underlying QCD, which exhibits the global chiral $U(3)_L \times U(3)_R$ symmetry, explicitly broken by current quark masses and by the $U(1)$ axial anomaly. We have included both leading-order anomaly terms and anomaly-induced flavor-breaking interactions, motivated by the phenomenological success of the model shown in the literature [51]. Our analysis was carried out within the CJT framework for the 2PI effective action, considering both the conventional and the symmetry-improved formulations, the latter being dubbed the SICJT [48].

A primary focus of the present work is to address the known issues in truncated 2PI approaches, particularly the violation of the NG theorem and the threshold property in the chiral broken phase. These violations become significant near the criticality of the chiral phase transition. By enforcing the WTIs for the 1PI effective action through the SICJT formalism, we have refined the conventional CJT approach on the three-flavor LSM and analyzed the consequences for the chiral phase structure and the projection onto the Columbia plot.

Our analysis has clarified that an enhanced first-order region, which has been detected in the conventional CJT method with the standard Hartree-level approximation, is merely an artifact due to the lack of the manifest NG and

low-energy theorems at finite temperature. This result suggests that a consistent incorporation of mesonic fluctuations that preserves the chiral symmetry is crucial for properly capturing the chiral phase structure, even within relatively simple truncation schemes.

Still, our current approach should potentially be applicable as a tool for studying other qualitative behaviors of critical phenomena, such as the search for the critical endpoint in QCD-like theories and their LEFTs. Beyond mean-field and large- N approximations, the SICJT approach would sufficiently incorporate mesonic fluctuation contributions through the self-consistent resummation of meson two-point functions by respecting the necessary symmetry constraints. Thus, our present work would be a definite step forward in improving the reliability of exploring meson mass spectra near the chiral critical temperature.

The symmetry-improved 2PI formalism would also offer a useful framework for exploring the QCD phase diagram in terms of LEFTs, particularly for applications at some critical point where the proper treatment of soft modes and collective excitations is essential. Other possible and simple-minded outlooks may also be involved: extending the current framework by including vector and axialvector mesons; incorporation of the coupling to Polyakov loops; finite chemical potential; the convolution with other non-perturbative methods such as the fRG; the studies of the electroweak phase transition and/or dark chiral phase transitions including physics beyond the Standard Model. In that case, in a way consistent with the (anomalous) WTIs, the SICJT formalism would help perform a proper resummation of 2PI diagram series, improving the radius of convergence like the conventional daisy resummation, and the validity of the conventional perturbative formulations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: THE THREE-FLAVOR LSM AT TREE LEVEL

In this appendix, we summarize the tree-level formulas in the three-flavor LSM employed in the present paper.

1. The vacuum constraint by the stationary condition

At tree level, the potential (3) along the vacuum direction specialized only by the $(\bar{\Phi}_1, \bar{\Phi}_3)$ -background fields reads

$$V(\bar{\Phi}) = \mu^2(2\bar{\Phi}_1^2 + \bar{\Phi}_3^2) + \lambda_1(2\bar{\Phi}_1^4 + \bar{\Phi}_3^4) + \lambda_2(2\bar{\Phi}_1^2 + \bar{\Phi}_3^2)^2 - 2B\bar{\Phi}_1^2\bar{\Phi}_3 - 2(2cm_l\bar{\Phi}_1 + cm_s\bar{\Phi}_3) - 4k(cm_s\bar{\Phi}_1^2 + 2cm_l\bar{\Phi}_1\bar{\Phi}_3). \quad (\text{A1})$$

The slopes of the classical action $S_{\text{LSM}} = \int d^4V(\bar{\Phi})$ are then given as

$$\begin{aligned} \left. \frac{\delta S_{\text{LSM}}}{\delta \bar{\Phi}_1} \right|_{\Phi=\bar{\Phi}} &= 4(\mu^2\bar{\Phi}_1 + 2\lambda_1\bar{\Phi}_1^3 + 4\lambda_2\bar{\Phi}_1^3 + 2\lambda_2\bar{\Phi}_1\bar{\Phi}_3^2 - B\bar{\Phi}_1\bar{\Phi}_3 - cm_l - 2kcm_l\bar{\Phi}_3 - 2kcm_s\bar{\Phi}_1), \\ \left. \frac{\delta S_{\text{LSM}}}{\delta \bar{\Phi}_3} \right|_{\Phi=\bar{\Phi}} &= 2(\mu^2\bar{\Phi}_3 + 2\lambda_1\bar{\Phi}_3^3 + 4\lambda_2\bar{\Phi}_1^2\bar{\Phi}_3 - B\bar{\Phi}_1^2 - cm_s - 4kcm_l\bar{\Phi}_1). \end{aligned} \quad (\text{A2})$$

Alternatively, using Eq. (11), the tree-level potential $V(\bar{\Phi})$ in terms of the $(\bar{\sigma}_0, \bar{\sigma}_8)$ -background fields reads

$$\begin{aligned} V(\bar{\sigma}) &= \frac{1}{2}\mu^2(\bar{\sigma}_0^2 + \bar{\sigma}_8^2) + \lambda_1 \left[\frac{1}{12}\bar{\sigma}_0^4 + \frac{1}{2}\bar{\sigma}_0^2\bar{\sigma}_8^2 - \frac{1}{3\sqrt{2}}\bar{\sigma}_0\bar{\sigma}_8^3 + \frac{1}{8}\bar{\sigma}_8^4 \right] + \frac{\lambda_2}{4}(\bar{\sigma}_0^2 + \bar{\sigma}_8^2)^2 \\ &\quad - B \left[\frac{1}{3\sqrt{6}}\bar{\sigma}_0^3 - \frac{1}{2\sqrt{6}}\bar{\sigma}_0\bar{\sigma}_8^2 - \frac{1}{6\sqrt{3}}\bar{\sigma}_8^3 \right] - \sqrt{\frac{2}{3}}(2cm_l + cm_s)\bar{\sigma}_0 - \frac{2}{\sqrt{3}}(cm_l - cm_s)\bar{\sigma}_8 \\ &\quad - k \left[\frac{cm_l}{3}(4\bar{\sigma}_0^2 - 2\sqrt{2}\bar{\sigma}_0\bar{\sigma}_8 - 4\bar{\sigma}_8^2) + \frac{cm_s}{3}(2\bar{\sigma}_0^2 + 2\sqrt{2}\bar{\sigma}_0\bar{\sigma}_8 + \bar{\sigma}_8^2) \right]. \end{aligned} \quad (\text{A3})$$

The corresponding slopes of the classical action are then given as

$$\begin{aligned} \left. \frac{\delta S_{\text{LSM}}}{\delta \sigma_0} \right|_{\sigma_a=\bar{\sigma}_a} &= \mu^2\bar{\sigma}_0 + \lambda_1 \left[\frac{1}{3}\bar{\sigma}_0^3 + \bar{\sigma}_0\bar{\sigma}_8^2 - \frac{1}{3\sqrt{2}}\bar{\sigma}_8^3 \right] + \lambda_2(\bar{\sigma}_0^2 + \bar{\sigma}_8^2)\bar{\sigma}_0 - B \left[\frac{1}{\sqrt{6}}\bar{\sigma}_0^2 - \frac{1}{2\sqrt{6}}\bar{\sigma}_8^2 \right] \\ &\quad - \sqrt{\frac{2}{3}}(2cm_l + cm_s) - k \left[\frac{cm_l}{3}(8\bar{\sigma}_0 - 2\sqrt{2}\bar{\sigma}_8) + \frac{cm_s}{3}(4\bar{\sigma}_0 + 2\sqrt{2}\bar{\sigma}_8) \right], \\ \left. \frac{\delta S_{\text{LSM}}}{\delta \sigma_8} \right|_{\sigma_a=\bar{\sigma}_a} &= \mu^2\bar{\sigma}_8 + \lambda_1 \left[\bar{\sigma}_0^2\bar{\sigma}_8 - \frac{1}{\sqrt{2}}\bar{\sigma}_0\bar{\sigma}_8^2 + \frac{1}{2}\bar{\sigma}_8^3 \right] + \lambda_2(\bar{\sigma}_0^2 + \bar{\sigma}_8^2)\bar{\sigma}_8 + B \left[\frac{1}{\sqrt{6}}\bar{\sigma}_0\bar{\sigma}_8 + \frac{1}{2\sqrt{3}}\bar{\sigma}_8^2 \right] \\ &\quad - \frac{2}{\sqrt{3}}(cm_l - cm_s) - k \left[\frac{cm_l}{3}(-2\sqrt{2}\bar{\sigma}_0 - 8\bar{\sigma}_8) + \frac{cm_s}{3}(2\sqrt{2}\bar{\sigma}_0 + 2\bar{\sigma}_8) \right]. \end{aligned} \quad (\text{A4})$$

Note that those two sets of slopes in Eqs. (A2) and (A4) are related to each other through the linear transformation arising from the chain rule as

$$\begin{pmatrix} \frac{\delta S_{\text{LSM}}}{\delta \Phi_1} \\ \frac{\delta S_{\text{LSM}}}{\delta \Phi_3} \end{pmatrix} = \begin{pmatrix} 2\sqrt{\frac{2}{3}} & \frac{2}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{\delta S_{\text{LSM}}}{\delta \sigma_0} \\ \frac{\delta S_{\text{LSM}}}{\delta \sigma_8} \end{pmatrix}. \quad (\text{A5})$$

which is equivalent to

$$\left. \frac{\delta S_{\text{LSM}}}{\delta \sigma_0} \right|_{\sigma_a = \bar{\sigma}_a} = 0, \quad \left. \frac{\delta S_{\text{LSM}}}{\delta \sigma_8} \right|_{\sigma_a = \bar{\sigma}_a} = 0 \quad (\text{A7})$$

At the classical level, the equilibrium criteria, or the stationary conditions, read

$$\left. \frac{\delta S_{\text{LSM}}}{\delta \Phi_1} \right|_{\Phi = \bar{\Phi}} = 0, \quad \left. \frac{\delta S_{\text{LSM}}}{\delta \Phi_3} \right|_{\Phi = \bar{\Phi}} = 0, \quad (\text{A6})$$

through Eq. (A5).

If we adopt the $SU(3)_V$ flavor symmetry, i.e., $m_l = m_s \equiv m$, we find that the slope of the σ_8 direction in Eq. (A5) becomes

$$\left. \frac{\delta S_{\text{LSM}}}{\delta \sigma_8} \right|_{\sigma_a = \bar{\sigma}_a} = \bar{\sigma}_8 \left\{ \mu^2 + \lambda_1 \left[\bar{\sigma}_0^2 - \frac{1}{\sqrt{2}} \bar{\sigma}_0 \bar{\sigma}_8 + \frac{1}{2} \bar{\sigma}_8^2 \right] + \lambda_2 (\bar{\sigma}_0^2 + \bar{\sigma}_8^2) + B \left[\frac{1}{\sqrt{6}} \bar{\sigma}_0 + \frac{1}{2\sqrt{3}} \bar{\sigma}_8 \right] + 2kcm \right\}. \quad (\text{A8})$$

The stationary condition along this σ_8 direction thus includes the trivial solution $\bar{\sigma}_8 = 0$, i.e., $\bar{\Phi}_1 = \bar{\Phi}_3$, which precisely reflects the $SU(3)_V$ symmetry. This flavor symmetric solution can also be realized when we rearrange Eq. (A2) as follows:

$$\begin{aligned} \frac{1}{4} \left. \frac{\delta S_{\text{LSM}}}{\delta \Phi_1} \right|_{\Phi = \bar{\Phi}} - \frac{1}{2} \left. \frac{\delta S_{\text{LSM}}}{\delta \Phi_3} \right|_{\Phi = \bar{\Phi}} &= (\bar{\Phi}_1 - \bar{\Phi}_3) [\mu^2 + 2\lambda_1 (\bar{\Phi}_1^2 + \bar{\Phi}_1 \bar{\Phi}_3 + \bar{\Phi}_3^2) \\ &\quad + \lambda_2 (4\bar{\Phi}_1^2 + 2\bar{\Phi}_3^2) + B\bar{\Phi}_1 + 2kcm_l] - (cm_l - cm_s)(1 - 2k\bar{\Phi}_1). \end{aligned} \quad (\text{A9})$$

We thus find the trivial solution $\bar{\Phi}_1 - \bar{\Phi}_3 = 0$ when $m_l = m_s$.

2. Curvature masses

The LSM field Φ includes the scalar and pseudoscalar meson fields as $\Phi = \phi_a T^a = (\sigma_a + i\pi_a) T^a$ for $a = 0, \dots, 8$. The Hessian matrix with respect to those meson fields is found by taking the second functional derivative for $S_{\text{LSM}} = \int_x \mathcal{L}_{\text{LSM}}$ with $\mathcal{L}_{\text{LSM}}$ in Eq. (2). It is in four-dimensional momentum space given as

$$\begin{aligned} [\bar{S}^{-1}(k, k'; \bar{\sigma})]^{ab} &= \frac{\delta^2 S_{\text{LSM}}}{\delta \sigma_a(-k') \delta \sigma_b(k)} = (2\pi)^4 \delta^{(4)}(k - k') (k^2 \delta^{ab} + [m_S^2(\bar{\sigma})]^{ab}) \\ &\equiv (2\pi)^4 \delta^{(4)}(k - k') [\bar{S}^{-1}(k; \bar{\sigma})]^{ab}, \\ [\bar{P}^{-1}(k, k'; \bar{\sigma})]^{ab} &= \frac{\delta^2 S_{\text{LSM}}}{\delta \pi_a(-k') \delta \pi_b(k)} = (2\pi)^4 \delta^{(4)}(k - k') (k^2 \delta^{ab} + [m_P^2(\bar{\sigma})]^{ab}) \\ &\equiv (2\pi)^4 \delta^{(4)}(k - k') [\bar{P}^{-1}(k; \bar{\sigma})]^{ab}, \end{aligned} \quad (\text{A10})$$

where we have defined the tree-level curvature masses as

$$[m_S^2(\bar{\sigma})]^{ab} = \left. \frac{\partial^2 V_{\text{LSM}}}{\partial \sigma_a \partial \sigma_b} \right|_{\sigma_a = \bar{\sigma}_a}, \quad [m_P^2(\bar{\sigma})]^{ab} = \left. \frac{\partial^2 V_{\text{LSM}}}{\partial \pi_a \partial \pi_b} \right|_{\sigma_a = \bar{\sigma}_a}. \quad (\text{A11})$$

Notice that the mixing matrix elements such as $\frac{\delta^2 S_{\text{LSM}}}{\delta \pi_a(-k') \delta \sigma_b(k)}$ vanish because of the parity invariance. From Eq. (A11), the matrix elements for the square of the nonet scalar meson masses read

$$\begin{aligned}
[m_S^2(\bar{\sigma})]^{00} &= \mu^2 + \lambda_1(\bar{\sigma}_0^2 + \bar{\sigma}_8^2) + \lambda_2(3\bar{\sigma}_0^2 + \bar{\sigma}_8^2) - B\sqrt{\frac{2}{3}}\bar{\sigma}_0 - \frac{4}{3}k(2cm_l + cm_s), \\
[m_S^2(\bar{\sigma})]^{88} &= \mu^2 + \lambda_1\left[\bar{\sigma}_0^2 - \sqrt{2}\bar{\sigma}_0\bar{\sigma}_8 + \frac{3}{2}\bar{\sigma}_8^2\right] + \lambda_2(\bar{\sigma}_0^2 + 3\bar{\sigma}_8^2) + \frac{B}{\sqrt{3}}\left[\frac{1}{\sqrt{2}}\bar{\sigma}_0 + \bar{\sigma}_8\right] + \frac{2}{3}k(4cm_l - cm_s), \\
[m_S^2(\bar{\sigma})]^{08} &= [m_S^2(\bar{\sigma})]^{80} = \lambda_1\left[2\bar{\sigma}_0\bar{\sigma}_8 - \frac{1}{\sqrt{2}}\bar{\sigma}_8^2\right] + 2\lambda_2\bar{\sigma}_0\bar{\sigma}_8 + \frac{B}{\sqrt{6}}\bar{\sigma}_8 + \frac{2\sqrt{2}}{3}k(cm_l - cm_s), \\
[m_S^2(\bar{\sigma})]^{11} &= [m_S^2(\bar{\sigma})]^{22} = [m_S^2(\bar{\sigma})]^{33} \\
&= \mu^2 + \lambda_1\left[\bar{\sigma}_0^2 + \sqrt{2}\bar{\sigma}_0\bar{\sigma}_8 + \frac{1}{2}\bar{\sigma}_8^2\right] + \lambda_2(\bar{\sigma}_0^2 + \bar{\sigma}_8^2) + \frac{B}{\sqrt{3}}\left[\frac{1}{\sqrt{2}}\bar{\sigma}_0 - \bar{\sigma}_8\right] + 2kcm_s, \\
[m_S^2(\bar{\sigma})]^{44} &= [m_S^2(\bar{\sigma})]^{55} = [m_S^2(\bar{\sigma})]^{66} = [m_S^2(\bar{\sigma})]^{77} \\
&= \mu^2 + \lambda_1\left[\bar{\sigma}_0^2 - \frac{1}{\sqrt{2}}\bar{\sigma}_0\bar{\sigma}_8 + \frac{1}{2}\bar{\sigma}_8^2\right] + \lambda_2(\bar{\sigma}_0^2 + \bar{\sigma}_8^2) + \frac{B}{\sqrt{3}}\left[\frac{1}{\sqrt{2}}\bar{\sigma}_0 + \frac{1}{2}\bar{\sigma}_8\right] + 2kcm_l, \\
(\text{Others}) &= 0.
\end{aligned} \tag{A12}$$

The matrix elements $[m_S^2(\bar{\sigma})]^{11}$ and $[m_S^2(\bar{\sigma})]^{44}$ are identified as the square of the $a_0(980)$ mass, $m_{a_0}^2$, and the $\kappa(700) = K_0^*(700)$ mass, m_κ^2 , respectively. The mixed (σ_0, σ_8) sector can be diagonalized through an orthogonal transformation

$$\begin{aligned}
\tilde{\sigma}_i &= (O_S^{-1})_i^a \sigma_a, \\
[\tilde{m}_S^2(\bar{\sigma})]_{(i)} \delta^{ij} &= (O_S^{-1})_i^a [m_S^2(\bar{\sigma})]^{ab} (O_S)_b^j,
\end{aligned} \tag{A13}$$

where $[\tilde{m}_S^2(\bar{\sigma})]_{(i)}$ denotes the diagonalized scalar mass matrix, and the orthogonal transformation matrix O_S is given as

$$O_S = \begin{pmatrix} \cos \theta_S^0 & -\sin \theta_S^0 \\ \sin \theta_S^0 & \cos \theta_S^0 \end{pmatrix}, \tag{A14}$$

with the mixing angle,

$$\theta_S^0 = \frac{1}{2} \arctan \left[\frac{2[m_S^2(\bar{\sigma})]^{08}}{[m_S^2(\bar{\sigma})]^{00} - [m_S^2(\bar{\sigma})]^{88}} \right]. \tag{A15}$$

Note that in the three-flavor symmetric limit where $\bar{\sigma}_8 = 0$ and $m_l = m_s = m$, the off diagonal element $[m_S^2(\bar{\sigma})]^{08}$ vanishes. The $f_0(500)$ and the $f_0(980)$ masses are identified through the lowest and highest eigenvalues, respectively, as

$$\begin{aligned}
m^2[f_0(500)] &= [\tilde{m}_S^2(\bar{\sigma})]_{(0)} = [m_S^2(\bar{\sigma})]^{00} \cos^2 \theta_S^0 + [m_S^2(\bar{\sigma})]^{88} \sin^2 \theta_S^0 + 2[m_S^2(\bar{\sigma})]^{08} \cos \theta_S^0 \sin \theta_S^0, \\
m^2[f_0(980)] &= [\tilde{m}_S^2(\bar{\sigma})]_{(8)} = [m_S^2(\bar{\sigma})]^{00} \sin^2 \theta_S^0 + [m_S^2(\bar{\sigma})]^{88} \cos^2 \theta_S^0 - 2[m_S^2(\bar{\sigma})]^{08} \cos \theta_S^0 \sin \theta_S^0.
\end{aligned} \tag{A16}$$

Similarly, from Eq. (A11), the matrix elements for the square of the nonet pseudoscalar meson masses are read off as

$$\begin{aligned}
[m_P^2(\bar{\sigma})]^{00} &= \mu^2 + \frac{\lambda_1}{3}(\bar{\sigma}_0^2 + \bar{\sigma}_8^2) + \lambda_2(\bar{\sigma}_0^2 + \bar{\sigma}_8^2) + B\sqrt{\frac{2}{3}}\bar{\sigma}_0 + \frac{4}{3}k(2cm_l + cm_s), \\
[m_P^2(\bar{\sigma})]^{88} &= \mu^2 + \lambda_1\left[\frac{1}{3}\bar{\sigma}_0^2 - \frac{\sqrt{2}}{3}\bar{\sigma}_0\bar{\sigma}_8 + \frac{1}{2}\bar{\sigma}_8^2\right] + \lambda_2(\bar{\sigma}_0^2 + \bar{\sigma}_8^2) - \frac{B}{\sqrt{3}}\left[\frac{1}{\sqrt{2}}\bar{\sigma}_0 + \bar{\sigma}_8\right] - \frac{2}{3}k(4cm_l - cm_s), \\
[m_P^2(\bar{\sigma})]^{08} &= [m_P^2(\bar{\sigma})]^{80} = \frac{\lambda_1}{3}\left[2\bar{\sigma}_0\bar{\sigma}_8 - \frac{1}{\sqrt{2}}\bar{\sigma}_8^2\right] - \frac{B}{\sqrt{6}}\bar{\sigma}_8 - \frac{2\sqrt{2}}{3}k(cm_l - cm_s), \\
[m_P^2(\bar{\sigma})]^{11} &= [m_P^2(\bar{\sigma})]^{22} = [m_P^2(\bar{\sigma})]^{33} \\
&= \mu^2 + \frac{\lambda_1}{3}\left[\bar{\sigma}_0^2 + \sqrt{2}\bar{\sigma}_0\bar{\sigma}_8 + \frac{1}{2}\bar{\sigma}_8^2\right] + \lambda_2(\bar{\sigma}_0^2 + \bar{\sigma}_8^2) - \frac{B}{\sqrt{3}}\left[\frac{1}{\sqrt{2}}\bar{\sigma}_0 - \bar{\sigma}_8\right] - 2kcm_s, \\
[m_P^2(\bar{\sigma})]^{44} &= [m_P^2(\bar{\sigma})]^{55} = [m_P^2(\bar{\sigma})]^{66} = [m_P^2(\bar{\sigma})]^{77} \\
&= \mu^2 + \frac{\lambda_1}{3}\left[\bar{\sigma}_0^2 - \frac{1}{\sqrt{2}}\bar{\sigma}_0\bar{\sigma}_8 + \frac{1}{2}\bar{\sigma}_8^2\right] + \lambda_2(\bar{\sigma}_0^2 + \bar{\sigma}_8^2) - \frac{B}{\sqrt{3}}\left[\frac{1}{\sqrt{2}}\bar{\sigma}_0 + \frac{1}{2}\bar{\sigma}_8\right] - 2kcm_l, \\
(\text{Others}) &= 0.
\end{aligned} \tag{A17}$$

The matrix elements $[m_P^2(\bar{\sigma})]^{11}$ and $[m_P^2(\bar{\sigma})]^{44}$ are identified as the squares of the pion mass, m_π^2 , and the kaon mass m_K^2 , respectively. The mixed (π_0, π_8) sector can be diagonalized through an orthogonal transformation, similarly to the (σ_0, σ_8) sector in Eq. (A13) as

$$\begin{aligned} \tilde{\pi}_i &= (O_P^{-1})_i^a \pi_a, \\ [\tilde{m}_P^2(\bar{\sigma})]_{(i)} \delta^{ij} &= (O_P^{-1})_i^a [m_P^2(\bar{\sigma})]^{ab} (O_P)_b^j, \end{aligned} \quad (\text{A18})$$

where $[\tilde{m}_P^2(\bar{\sigma})]_{(i)}$ denotes the diagonalized pseudoscalar meson mass matrix, and the transformation matrix O_P reads

$$O_S = \begin{pmatrix} \cos \theta_P^0 & -\sin \theta_P^0 \\ \sin \theta_P^0 & \cos \theta_P^0 \end{pmatrix}. \quad (\text{A19})$$

with the mixing angle,

$$\theta_P^0 = \frac{1}{2} \arctan \left[\frac{2[m_P^2(\bar{\sigma})]^{08}}{[m_P^2(\bar{\sigma})]^{00} - [m_P^2(\bar{\sigma})]^{88}} \right]. \quad (\text{A20})$$

The off diagonal element $[m_P^2(\bar{\sigma})]^{08}$ vanishes in the three-flavor symmetric limit where $\bar{\sigma}_8 = 0$. Then, the η' and the η masses are identified through the eigenvalues

$$\begin{aligned} m_{\eta'}^2 &= [\tilde{m}_P^2(\bar{\sigma})]_{(0)} = [m_P^2(\bar{\sigma})]^{00} \cos^2 \theta_P^0 + [m_P^2(\bar{\sigma})]^{88} \sin^2 \theta_P^0 \\ &\quad + 2[m_P^2(\bar{\sigma})]^{08} \cos \theta_P^0 \sin \theta_P^0, \\ m_\eta^2 &= [\tilde{m}_P^2(\bar{\sigma})]_{(8)} = [m_P^2(\bar{\sigma})]^{00} \sin^2 \theta_P^0 + [m_P^2(\bar{\sigma})]^{88} \cos^2 \theta_P^0 \\ &\quad - 2[m_P^2(\bar{\sigma})]^{08} \cos \theta_P^0 \sin \theta_P^0. \end{aligned} \quad (\text{A21})$$

APPENDIX B: CJT FORMALISM AT FINITE TEMPERATURE

In this appendix, we summarize the formulations in the conventional CJT formalism applied to the present LSM at finite temperature.

1. CJT formalism and Hartree approximation

We begin by recasting the CJT effective potential given by Eq. (14), i.e.,

$$\begin{aligned} V_{\text{CJT}}[\sigma, S, P] &= V(\sigma) + \frac{1}{2} \int_k \text{tr} [\log S^{-1}(k) + \log P^{-1}(k)] \\ &\quad + \frac{1}{2} \int_k \text{tr} [\bar{S}^{-1}(k; \sigma) S(k) \\ &\quad + \bar{P}^{-1}(k; \sigma) P(k) - 2] + V_2[\sigma, S, P]. \end{aligned} \quad (\text{B1})$$

Here, the shorthand notation of the momentum integral is defined as

$$\int_k f(k) = T \sum_{n=-\infty}^{\infty} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} f(\omega_n, \mathbf{k}), \quad (\text{B2})$$

with $\omega_n = 2n\pi T$ being the bosonic Matsubara frequency. The tree-level mesonic propagators $\bar{S}(k; \bar{\sigma})$ and $\bar{P}(k; \bar{\sigma})$ are defined as

$$\begin{aligned} [\bar{S}^{-1}(k; \sigma)]^{ab} &= k^2 \delta^{ab} + [m_S^2(\sigma)]^{ab}, \\ [\bar{P}^{-1}(k; \sigma)]^{ab} &= k^2 \delta^{ab} + [m_P^2(\sigma)]^{ab}, \end{aligned} \quad (\text{B3})$$

where $m_S^2(\bar{\sigma})$ and $m_P^2(\bar{\sigma})$ are the mesonic curvature masses that are read off from Eqs. (A12) and (A17).

The VEV of σ and the propagators S and $\bar{P}$ are determined the stationary conditions and the gap equations

$$\begin{aligned} \left. \frac{\partial V_{\text{CJT}}[\sigma, S, P]}{\partial \sigma_a} \right|_{\sigma=\bar{\sigma}, S=S, P=P} &= 0, \\ \left. \frac{\delta V_{\text{CJT}}[\sigma, S, P]}{\delta S_{ab}} \right|_{\sigma=\bar{\sigma}, S=S, P=P} &= 0, \\ \left. \frac{\delta V_{\text{CJT}}[\sigma, S, P]}{\delta P_{ab}} \right|_{\sigma=\bar{\sigma}, S=S, P=P} &= 0. \end{aligned} \quad (\text{B4})$$

The latter two gap equations can be expressed as

$$\begin{aligned} [S^{-1}(k)]^{ab} &= [\bar{S}^{-1}(k; \sigma)]^{ab} + \Sigma^{ab}, \\ [P^{-1}(k)]^{ab} &= [\bar{P}^{-1}(k; \sigma)]^{ab} + \Pi^{ab}, \end{aligned} \quad (\text{B5})$$

where

$$\begin{aligned} \Sigma^{ab} &= 2 \left. \frac{\delta V_2[\sigma, S, P]}{\delta S_{ba}} \right|_{\sigma=\bar{\sigma}, S=S, P=P}, \\ \Pi^{ab} &= 2 \left. \frac{\delta V_2[\sigma, S, P]}{\delta P_{ba}} \right|_{\sigma=\bar{\sigma}, S=S, P=P}, \end{aligned} \quad (\text{B6})$$

are the meson self-energies, which also depend on S and P . Notice that if we take $V_2 = 0$, in the gap equations in Eq. (B5), the propagators are replaced by the tree-level ones, thus the 2PI effective potential will be reduced to the 1PI effective action at one-loop level.

Since V_2 includes infinite 2PI diagrams, we have to make a truncation to close the stationary condition and gap equations in Eq. (B5). In the present work, we apply the double-bubble diagram truncation, as depicted in Fig. 1, for which the truncation scheme is equivalent to the so-called Hartree approximation. The thus truncated 2PI diagrams are generated from the four-meson interactions in V_0 given in Eq. (3), which reads

$$V_2[S, P] = \mathcal{F}^{abcd} \left[\int_k S_{ab}(k) \int_q S_{cd}(q) + \int_k P_{ab}(k) \int_q P_{cd}(q) \right] + 2\mathcal{H}^{abcd} \int_k S_{ab}(k) \int_q P_{cd}(q), \quad (\text{B7})$$

where

$$\begin{aligned} \mathcal{F}^{abcd} &\equiv \frac{1}{8} \frac{\partial^4 V_0}{\partial \sigma_a \partial \sigma_b \partial \sigma_c \partial \sigma_d} \Big|_{\Phi=0} = \frac{\lambda_1}{8} (d^{abn} d^{ncd} + d^{adn} d^{nbc} + d^{acn} d^{nbd}) + \frac{\lambda_2}{4} (\delta^{ab} \delta^{cd} + \delta^{ad} \delta^{bc} + \delta^{ac} \delta^{bd}), \\ \mathcal{H}^{abcd} &\equiv \frac{1}{8} \frac{\partial^4 V_0}{\partial \sigma_a \partial \sigma_b \partial \pi_c \partial \pi_d} \Big|_{\Phi=0} = \frac{\lambda_1}{8} (d^{abn} d^{ncd} + f^{acn} f^{nbd} + f^{bcn} f^{nad}) + \frac{\lambda_2}{4} \delta^{ab} \delta^{cd}, \end{aligned} \quad (\text{B8})$$

and the structure constants are defined as

$$d^{abc} = 2\text{tr}[\{T^a, T^b\}T^c], \quad f^{abc} = -2i\text{tr}[[T^a, T^b]T^c]. \quad (\text{B9})$$

2. Gap equations of meson masses

Under the Hartree approximation as in Eq. (B7), the loop correction does not generate the momentum-dependent terms. As has been done in Eq. (18) of the main text, we may therefore parametrize the full propagators as

$$\begin{aligned} [\mathcal{S}^{-1}(k)]^{ab} &= k^2 \delta^{ab} + [M_S^2]^{ab}, \\ [\mathcal{P}^{-1}(k)]^{ab} &= k^2 \delta^{ab} + [M_P^2]^{ab}, \end{aligned} \quad (\text{B10})$$

where $[M_S]^{ab}$ and $[M_P]^{ab}$ are the full masses, which include quantum and thermal corrections. Then, the gap equations in Eq. (B5) are reduced to

$$\begin{aligned} [M_S^2]^{ab} &= [m_S^2(\bar{\sigma})]^{ab} + 4\mathcal{F}^{abcd} \int_k \mathcal{S}_{cd}(k) + 4\mathcal{H}^{abcd} \int_k \mathcal{P}_{cd}(k), \\ [M_P^2]^{ab} &= [m_P^2(\bar{\sigma})]^{ab} + 4\mathcal{F}^{abcd} \int_k \mathcal{P}_{cd}(k) + 4\mathcal{H}^{abcd} \int_k \mathcal{S}_{cd}(k). \end{aligned} \quad (\text{B11})$$

At the classical level, we diagonalize the mass matrices to obtain the mass eigenstates, see Eqs. (A16) and (A21). After incorporating the quantum and thermal corrections into the meson propagators, we also introduce the following orthogonal transformations to diagonalize the full masses, in such a way that

$$[\tilde{M}_{S/P}^2]_{(i)} \delta^{ij} = (O'_{S/P})^i_a [M_{S/P}^2]^{ab} (O'_{S/P})_b^j. \quad (\text{B12})$$

The rotation angles are expressed as

$$\theta_{S/P} = \frac{1}{2} \arctan \left[\frac{2[M_{S/P}^2]^{08}}{[M_{S/P}^2]^{00} - [M_{S/P}^2]^{88}} \right]. \quad (\text{B13})$$

In general, the transformation matrices $O'_{S/P}$ are different from those at the tree level, i.e., Eqs. (A13) and (A18).

The full propagators $\mathcal{S}$ and $\mathcal{P}$ can also be diagonalized simultaneously by $O'_{S/P}$ in Eq. (B12), which goes like

$$\begin{aligned} \tilde{\mathcal{S}}_{(i)}(k) \delta_{ij} &= (O'^{-1}_S)_i^a \mathcal{S}_{ab}(k) (O'_S)^b_j, \\ \tilde{\mathcal{P}}_{(i)}(k) \delta_{ij} &= (O'^{-1}_P)_i^a \mathcal{P}_{ab}(k) (O'_P)^b_j. \end{aligned} \quad (\text{B14})$$

Then, the diagonalized meson propagators are represented fully in terms of the mass eigenvalues, i.e.,

$$\tilde{\mathcal{S}}_{(i)}^{-1}(k) = k^2 + [\tilde{M}_S^2]_{(i)}, \quad \tilde{\mathcal{P}}_{(i)}^{-1}(k) = k^2 + [\tilde{M}_P^2]_{(i)}, \quad (\text{B15})$$

where the loop corrections constructed from the meson propagators only include the form

$$\int_k \tilde{\mathcal{S}}_{(i)}(k) \quad \text{and} \quad \int_k \tilde{\mathcal{P}}_{(i)}(k). \quad (\text{B16})$$

Here, we show the expressions of the diagonalized scalar meson self-energies $\tilde{\sigma}_{(i)}$ explicitly as

$$\begin{aligned}
 \tilde{\Sigma}_{(0)} = & \int_k \tilde{\mathcal{S}}_{(0)}(k) \cdot \frac{1}{16} [27\lambda_1 + 48\lambda_2 - \lambda_1(4\cos(2\theta_S) + 7\cos(4\theta_S) + 32\sqrt{2}\cos\theta_S\sin^3\theta_S)] \\
 & + \int_k \tilde{\mathcal{S}}_{(8)}(k) \cdot \frac{1}{16} [9\lambda_1 + 16\lambda_2 + \lambda_1(7\cos(4\theta_S) - 4\sqrt{2}\sin(4\theta_S))] \\
 & + \int_k \tilde{\mathcal{S}}_{(1)}(k) \cdot \frac{3}{4} [3\lambda_1 + 4\lambda_2 + \lambda_1(\cos(2\theta_S) + 2\sqrt{2}\sin(2\theta_S))] \\
 & + \int_k \tilde{\mathcal{S}}_{(4)}(k) \cdot [3\lambda_1 + 4\lambda_2 + \lambda_1(\cos(2\theta_S) - \sqrt{2}\sin(2\theta_S))] \\
 & + \int_k \tilde{\mathcal{P}}_{(0)}(k) \cdot \frac{1}{24} [9\lambda_1 + 24\lambda_2 - \lambda_1\cos(2\theta_P) \\
 & + \lambda_1\sin(\theta_P)(-4\sqrt{2}\cos\theta_P + 4\sqrt{2}\cos(\theta_P + 2\theta_S) - 9\sin(\theta_P - 2\theta_S) + 7\sin(\theta_P + 2\theta_S))] \\
 & + \int_k \tilde{\mathcal{P}}_{(8)}(k) \cdot \frac{1}{24} [9\lambda_1 + 24\lambda_2 + \lambda_1\cos(2\theta_P) \\
 & + \lambda_1\cos(\theta_P)(-9\cos(\theta_P - 2\theta_S) + 7\cos(\theta_P + 2\theta_S) - 8\sqrt{2}\cos(\theta_P + \theta_S)\sin\theta_S)] \\
 & + \int_k \tilde{\mathcal{P}}_{(1)}(k) \cdot \frac{1}{4} [3\lambda_1 + 12\lambda_2 + \lambda_1(\cos(2\theta_S) + 2\sqrt{2}\sin(2\theta_S))] \\
 & + \int_k \tilde{\mathcal{P}}_{(4)}(k) \cdot \frac{1}{3} [9\lambda_1 + 12\lambda_2 - \lambda_1(5\cos(2\theta_S) + \sqrt{2}\sin(2\theta_S))], \tag{B17}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\Sigma}_{(8)} = & \int_k \tilde{\mathcal{S}}_{(0)}(k) \cdot \frac{1}{16} [9\lambda_1 + 16\lambda_2 + \lambda_1(7\cos(4\theta_S) - 4\sqrt{2}\sin(4\theta_S))] \\
 & + \int_k \tilde{\mathcal{S}}_{(8)}(k) \cdot \frac{1}{16} [27\lambda_1 + 48\lambda_2 + \lambda_1(4\cos(2\theta_S) - 7\cos(4\theta_S) + 32\sqrt{2}\cos^3\theta_S\sin\theta_S)] \\
 & + \int_k \tilde{\mathcal{S}}_{(1)}(k) \cdot \frac{3}{4} [3\lambda_1 + 4\lambda_2 - \lambda_1(\cos(2\theta_S) + 2\sqrt{2}\sin(2\theta_S))] \\
 & + \int_k \tilde{\mathcal{S}}_{(4)}(k) \cdot [3\lambda_1 + 4\lambda_2 - \lambda_1(\cos(2\theta_S) - \sqrt{2}\sin(2\theta_S))] \\
 & + \int_k \tilde{\mathcal{P}}_{(0)}(k) \cdot \frac{1}{24} [9\lambda_1 + 24\lambda_2 - \lambda_1\cos(2\theta_P) \\
 & + 2\lambda_1\sin(\theta_P)(\cos(2\theta_S)\sin\theta_P - 4\cos\theta_S(\sqrt{2}\cos(\theta_S + \theta_P) + 4\cos(\theta_P)\sin(\theta_S))] \\
 & + \int_k \tilde{\mathcal{P}}_{(8)}(k) \cdot \frac{1}{24} [9\lambda_1 + 24\lambda_2 + \lambda_1\cos(2\theta_P) \\
 & + \lambda_1\cos(\theta_P)(9\cos(\theta_P - 2\theta_S) - 7\cos(\theta_P + 2\theta_S) + 8\sqrt{2}\cos\theta_S\sin(\theta_P + \theta_S))] \\
 & + \int_k \tilde{\mathcal{P}}_{(1)}(k) \cdot \frac{1}{4} [3\lambda_1 + 12\lambda_2 - \lambda_1(\cos(2\theta_S) + 2\sqrt{2}\sin(2\theta_S))] \\
 & + \int_k \tilde{\mathcal{P}}_{(4)}(k) \cdot \frac{1}{3} [9\lambda_1 + 12\lambda_2 + \lambda_1(5\cos(2\theta_S) + \sqrt{2}\sin(2\theta_S))], \tag{B18}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{\Sigma}_{(1)} = & \int_k \tilde{\mathcal{S}}_{(0)}(k) \cdot \frac{1}{4} [3\lambda_1 + 4\lambda_2 + \lambda_1(\cos(2\theta_S) + 2\sqrt{2}\sin(2\theta_S))] \\
 & + \int_k \tilde{\mathcal{S}}_{(8)}(k) \cdot \frac{1}{4} [3\lambda_1 + 4\lambda_2 - \lambda_1(\cos(2\theta_S) + 2\sqrt{2}\sin(2\theta_S))] + \int_k \tilde{\mathcal{S}}_{(1)}(k) \cdot \frac{5}{2} [\lambda_1 + 2\lambda_2] \\
 & + \int_k \tilde{\mathcal{S}}_{(4)}(k) \cdot 2[\lambda_1 + 2\lambda_2] + \int_k \tilde{\mathcal{P}}_{(0)}(k) \cdot \frac{1}{12} [3\lambda_1 + 12\lambda_2 + \lambda_1(\cos(2\theta_P) + 2\sqrt{2}\sin(2\theta_P))] \\
 & + \int_k \tilde{\mathcal{P}}_{(8)}(k) \cdot \frac{1}{12} [3\lambda_1 + 12\lambda_2 - \lambda_1(\cos(2\theta_P) + 2\sqrt{2}\sin(2\theta_P))] \\
 & + \int_k \tilde{\mathcal{P}}_{(1)}(k) \cdot \frac{1}{2} [7\lambda_1 + 6\lambda_2] + \int_k \tilde{\mathcal{P}}_{(4)}(k) \cdot 2[\lambda_1 + 2\lambda_2], \tag{B19}
 \end{aligned}$$

$$\begin{aligned}
\tilde{\Sigma}_{(4)} = & \int_k \tilde{S}_{(0)}(k) \cdot \frac{1}{4} [3\lambda_1 + 4\lambda_2 + \lambda_1 (\cos(2\theta_S) - \sqrt{2} \sin(2\theta_S))] \\
& + \int_k \tilde{S}_{(8)}(k) \cdot \frac{1}{4} [3\lambda_1 + 4\lambda_2 - \lambda_1 (\cos(2\theta_S) - \sqrt{2} \sin(2\theta_S))] + \int_k \tilde{S}_{(1)}(k) \cdot \frac{3}{2} [\lambda_1 + 2\lambda_2] \\
& + \int_k \tilde{S}_{(4)}(k) \cdot 3[\lambda_1 + 2\lambda_2] + \int_k \tilde{P}_{(0)}(k) \cdot \frac{1}{12} [9\lambda_1 + 12\lambda_2 - \lambda_1 (5 \cos(2\theta_P) + \sqrt{2} \sin(2\theta_P))] \\
& + \int_k \tilde{P}_{(8)}(k) \cdot \frac{1}{12} [9\lambda_1 + 12\lambda_2 + \lambda_1 (5 \cos(2\theta_P) + \sqrt{2} \sin(2\theta_P))] \\
& + \int_k \tilde{P}_{(1)}(k) \cdot \frac{3}{2} [\lambda_1 + 2\lambda_2] + \int_k \tilde{P}_{(4)}(k) \cdot [3\lambda_1 + 4\lambda_2].
\end{aligned} \tag{B20}$$

For the diagonalized pseudoscalar meson self-energies $\tilde{\Pi}_{(i)}$, they are the same as $\tilde{\Sigma}_{(i)}$ with the replacement

$$\tilde{S}_{(0)}(k) \leftrightarrow \tilde{P}_{(0)}(k) \quad \text{and} \quad \theta_S \leftrightarrow \theta_P. \tag{B21}$$

We do not show them explicitly for readability.

To obtain the value of the rotation angles $\theta_{S/P}$ in Eq. (B13), we look at the 08 component of the gap equations in Eq. (B11) after performing the orthogonal transformation, which turns out to give the following constraints:

$$\begin{aligned}
0 &= [\tilde{m}_S^2(\bar{\sigma})]^{08} + \tilde{\Sigma}^{08}, \\
0 &= [\tilde{m}_P^2(\bar{\sigma})]^{08} + \tilde{\Pi}^{08},
\end{aligned} \tag{B22}$$

where

$$[\tilde{m}_{S/P}^2(\bar{\sigma})]^{08} = [m_{S/P}^2(\bar{\sigma})]^{08} \cos(2\theta_{S/P}) - \{[m_{S/P}^2(\bar{\sigma})]^{00} - [m_{S/P}^2(\bar{\sigma})]^{88}\} \cos(\theta_{S/P}) \sin(\theta_{S/P}), \tag{B23}$$

and

$$\begin{aligned}
\tilde{\Sigma}^{08} = & \int_k \tilde{S}_{(0)}(k) \cdot \frac{\lambda_1}{8} [\sin \theta_S (9 \cos \theta_S + 7 \cos(3\theta_S) - 4\sqrt{2} \sin(3\theta_S))] \\
& + \int_k \tilde{S}_{(8)}(k) \cdot \frac{\lambda_1}{8} [\cos \theta_S (9 \sin \theta_S - 7 \sin(3\theta_S) - 4\sqrt{2} \cos(3\theta_S))] \\
& + \int_k \tilde{S}_{(1)}(k) \cdot \frac{3\lambda_1}{4} [2\sqrt{2} \cos(2\theta_S) - \sin(2\theta_S)] - \int_k \tilde{S}_{(4)}(k) \cdot \lambda_1 [\sqrt{2} \cos(2\theta_S) + \sin(2\theta_S)] \\
& + \int_k \tilde{P}_{(0)}(k) \cdot \frac{\lambda_1}{24} \sin \theta_P [9 \cos(\theta_P - 2\theta_S) + 7 \cos(\theta_P + 2\theta_S) - 4\sqrt{2} \sin(\theta_P + 2\theta_S)] \\
& - \int_k \tilde{P}_{(8)}(k) \cdot \frac{\lambda_1}{24} \cos \theta_P [9 \sin(\theta_P - 2\theta_S) + 7 \sin(\theta_P + 2\theta_S) + 4\sqrt{2} \cos(\theta_P + 2\theta_S)] \\
& + \int_k \tilde{P}_{(1)}(k) \cdot \frac{\lambda_1}{4} [2\sqrt{2} \cos(2\theta_S) - \sin(2\theta_S)] - \int_k \tilde{P}_{(4)}(k) \cdot \frac{\lambda_1}{3} [\sqrt{2} \cos(2\theta_S) - 5 \sin(2\theta_S)].
\end{aligned} \tag{B24}$$

Again, the pseudoscalar mixing self-energy $\tilde{\Pi}^{08}$ is obtained directly through $\tilde{\Sigma}^{08}$ simply by the replacement in Eq. (B21).

3. The stationary condition for VEVs

The stationary conditions derived from the 2PI effective potential (B1) read

$$\begin{aligned}
0 &= \frac{\partial V(\bar{\sigma})}{\partial \bar{\sigma}_0} - 3\mathcal{G}^{0bc} \left[\int_k \mathcal{S}_{cb}(k) - \int_k \mathcal{P}_{cb}(k) \right] + 4\mathcal{F}^{0bcd} \bar{\sigma}_b \int_k \mathcal{S}_{dc}(k) + 4\mathcal{H}^{0bcd} \bar{\sigma}_b \int_k \mathcal{P}_{dc}(k), \\
0 &= \frac{\partial V(\bar{\sigma})}{\partial \bar{\sigma}_8} - 3\mathcal{G}^{8bc} \left[\int_k \mathcal{S}_{cb}(k) - \int_k \mathcal{P}_{cb}(k) \right] + 4\mathcal{F}^{8bcd} \bar{\sigma}_b \int_k \mathcal{S}_{dc}(k) + 4\mathcal{H}^{8bcd} \bar{\sigma}_b \int_k \mathcal{P}_{dc}(k),
\end{aligned} \tag{B25}$$

where $V(\bar{\sigma})$ is given in Eq. (A3), and

$$\begin{aligned}\mathcal{G}^{abc} &\equiv -\frac{1}{6} \frac{\partial^3 V_{\text{anom}}}{\partial \sigma_a \partial \sigma_b \partial \sigma_c} \Big|_{\Phi=0} = \frac{1}{6} \frac{\partial^3 V_{\text{anom}}}{\partial \sigma_a \partial \pi_b \partial \pi_c} \Big|_{\Phi=0} \\ &= \frac{B}{6} \left[d^{abc} - \frac{3}{2} (\delta^{a0} d^{0bc} + \delta^{b0} d^{a0c} + \delta^{c0} d^{ab0}) \right. \\ &\quad \left. + \frac{9}{2} d^{000} \delta^{a0} \delta^{b0} \delta^{c0} \right].\end{aligned}\quad (\text{B26})$$

To regularize the meson loops, we perform the vacuum-subtraction prescription, such that

$$\begin{aligned}\int_k \tilde{S}_{(i)}(k) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\epsilon_k^{S,(i)}} \left(\exp \left\{ \frac{\epsilon_k^{S,(i)}}{T} \right\} - 1 \right)^{-1}, \\ \int_k \tilde{P}_{(i)}(k) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\epsilon_k^{P,(i)}} \left(\exp \left\{ \frac{\epsilon_k^{P,(i)}}{T} \right\} - 1 \right)^{-1},\end{aligned}\quad (\text{B27})$$

where

$$\epsilon_k^{S/P,(i)} \equiv \sqrt{k^2 + [\tilde{M}_{S/P}^2]_{(i)}}. \quad (\text{B28})$$

APPENDIX C: DERIVATIONS OF WTIS AND EXTENSION TO ANOMALOUS WTIS

1. Chiral-limit WTIs in 1PI formalism and threshold property of pseudo-NG masses

Consider a continuous symmetry ($\in$ group G) associated with the infinitesimal variation (δ_ϵ) of the field ϕ (in the linear symmetry realization) and the action $S[\phi]$ as follows:

$$\delta_\epsilon S[\phi] = \frac{\delta S}{\delta \phi_a} \cdot \delta_\epsilon \phi_a = 0, \quad \delta_\epsilon \phi_a = d_a^b \phi_b \quad (\text{C1})$$

where d_a^b is an arbitrary coefficient matrix in the field space, and $A \cdot B \equiv \int_x A(x) B(x)$. The generating functional for the action $S[\phi]$ is defined together with the source J_a for ϕ_a as

$$Z[J] = \int [\mathcal{D}\phi_a] e^{-S[\phi] + J^a \cdot \phi_a} = e^{W[J]}. \quad (\text{C2})$$

Imposing this $Z[J]$ to be invariant under the symmetry transformation in Eq. (C1), we have

$$\int [\mathcal{D}\phi'_a] e^{-S[\phi'] + J^a \cdot \phi'_a} \Big|_{\phi'_a = \phi_a + i\epsilon \delta_\epsilon \phi_a} = \int [\mathcal{D}\phi_a] e^{-S[\phi] + J^a \cdot \phi_a}. \quad (\text{C3})$$

Supposing that the path integral measure $[\mathcal{D}\phi_a]$ is invariant under this symmetry, i.e., no anomaly there, we find the following identity:

$$\begin{aligned}&\int [\mathcal{D}\phi_a] [e^{-S[\phi'] + J^a \cdot \phi'_a} - e^{-S[\phi] + J^a \cdot \phi_a}]_{\phi'_a = \phi_a + i\epsilon \delta_\epsilon \phi_a} \\ &= -i\epsilon \int [\mathcal{D}\phi_a] \left[\frac{\delta S}{\delta \phi_a} \cdot \delta_\epsilon \phi_a - J^a \cdot \delta_\epsilon \phi_a \right] e^{-S[\phi] + J^a \cdot \phi_a} + \mathcal{O}(\epsilon^2) \\ &= i\epsilon [J^a \cdot \langle \delta_\epsilon \phi_a \rangle_J] Z[J] + \mathcal{O}(\epsilon^2) \\ &= 0,\end{aligned}\quad (\text{C4})$$

where

$$\langle O \rangle_J \equiv \frac{1}{Z[J, \mathcal{M}]} \int [\mathcal{D}\phi_a] O e^{-S[\phi, \mathcal{M}] + J^a \cdot \phi_a}. \quad (\text{C5})$$

Equation (C4) gives the WTI at the generating functional level,

$$J^a \cdot \langle \delta_\epsilon \phi_a \rangle_J = 0. \quad (\text{C6})$$

Defining the 1PI effective action

$$\Gamma[\phi_{\text{cl}}] = \sup_J (J \cdot \phi_{\text{cl}} - W[J]), \quad (\text{C7})$$

we have

$$\frac{\delta W}{\delta J^a} = \langle \phi_a \rangle_J \equiv (\phi_{\text{cl}})_a, \quad \frac{\delta \Gamma}{\delta (\phi_{\text{cl}})_a} = J^a. \quad (\text{C8})$$

Replacing J_a with Γ as above, the WTI in Eq. (C6) is cast into the form

$$\frac{\delta \Gamma}{\delta (\phi_{\text{cl}})_a} \cdot \delta_\epsilon (\phi_{\text{cl}})_a = 0, \quad (\text{C9})$$

where $\delta_\epsilon (\phi_{\text{cl}})_a = d_a^b (\phi_{\text{cl}})_b$.

We apply the WTI arguments above to the present LSM case. First of all, consider the chiral limit. In the chiral limit, the LSM action is invariant under the chiral $SU(3)_L \times SU(3)_R$ transformation,

$$\Phi \rightarrow g_L \cdot \Phi \cdot g_R^\dagger, \quad (\text{C10})$$

with $g_{L/R} = \exp\{i(\theta_{L/R})_\alpha T^\alpha\}$ and $\alpha = 1, \dots, 8$. Under the $SU(3)$ axial rotation with $(\theta_L)_\alpha = -(\theta_R)_\alpha = (\theta_A)_\alpha$, the σ_a and π_a thus infinitesimally transform like

$$\begin{aligned}\delta^\alpha (\sigma_a T^a) &= \{T^\alpha, \pi_a T^a\}, \\ \delta^\alpha (\pi_a T^a) &= -\{T^\alpha, \sigma_a T^a\}.\end{aligned}\quad (\text{C11})$$

Or equivalently,

$$\delta^\alpha \sigma_a = d^{ab}{}_a \pi_b, \quad \delta^\alpha \pi_a = -d^{ab}{}_a \sigma_b, \quad (\text{C12})$$

where

$$d^{ab}_a \equiv 2\text{tr}[\{T^a, T^b\}T_a], \quad d^{ab}_0 = \sqrt{\frac{2}{3}}\delta^{ab}. \quad (\text{C13})$$

When the general WTI in Eq. (C9) is applied to the 1PI effective action in the LSM, we have

$$0 = \frac{\delta\Gamma[\sigma_{\text{cl}}, \pi_{\text{cl}}]}{\delta(\sigma_{\text{cl}})_a} \cdot \delta^a(\sigma_{\text{cl}})_a + \frac{\delta\Gamma[\sigma_{\text{cl}}, \pi_{\text{cl}}]}{\delta(\pi_{\text{cl}})_a} \cdot \delta^a(\pi_{\text{cl}})_a = \frac{\delta\Gamma[\sigma_{\text{cl}}, \pi_{\text{cl}}]}{\delta(\sigma_{\text{cl}})_a} \cdot d^{ab}_a(\pi_{\text{cl}})_b + \frac{\delta\Gamma[\sigma_{\text{cl}}, \pi_{\text{cl}}]}{\delta(\pi_{\text{cl}})_a} \cdot d^{ab}_a(\sigma_{\text{cl}})_b. \quad (\text{C14})$$

Performing one more functional derivative with respect to the external (background) pseudoscalar field $(\pi_{\text{cl}}(y))_{b_1}$ on the WTIs in Eq. (C14), we get

$$\begin{aligned} & \frac{\delta}{\delta(\pi_{\text{cl}}(y))_{b_1}} \left(\frac{\delta\Gamma[\sigma_{\text{cl}}, \pi_{\text{cl}}]}{\delta(\sigma_{\text{cl}})_a} \cdot d^{ab}_a(\pi_{\text{cl}})_b + \frac{\delta\Gamma[\sigma_{\text{cl}}, \pi_{\text{cl}}]}{\delta(\pi_{\text{cl}})_a} \cdot d^{ab}_a(\sigma_{\text{cl}})_b \right) \\ &= \frac{\delta\Gamma}{\delta(\sigma_{\text{cl}})_a} \cdot \delta^{(4)}(x-y) d^{ab_1}_a - \frac{\delta^2\Gamma}{\delta(\pi_{\text{cl}}(y))_{b_1} \delta(\pi_{\text{cl}}(x))_a} \cdot d^{ab}_a(\sigma_{\text{cl}})_b. \end{aligned} \quad (\text{C15})$$

Imposing the quantum equations of motion such that $\frac{\delta\Gamma}{\delta(\sigma_{\text{cl}})_a} = 0$ at the background field values $\bar{\Phi}$, Eq. (C15) is simplified to

$$-\frac{\delta^2\Gamma}{\delta(\pi_{\text{cl}}(y))_{b_1} \delta(\pi_{\text{cl}}(x))_a} \cdot d^{ab}_a \bar{\sigma}_b = - \int_x [\mathcal{P}^{-1}(y, x)]^{b_1 a} d^{ab}_a \bar{\sigma}_b, \quad (\text{C16})$$

where $(\mathcal{P}^{-1}(y, x))^{b_1 a}$ is the full inverse propagator of the pseudoscalar mesons satisfying the gap equations in Eq. (B4). In the present system, the translation invariance holds when we turn off the external sources, thus in Euclidean momentum space, $(\mathcal{P}^{-1}(y, x))^{b_1 a}$ can be Fourier transformed like

$$[\mathcal{P}^{-1}(y, x)]^{b_1 a} = \int \frac{d^4 k}{(2\pi)^4} [\mathcal{P}^{-1}(k)]^{b_1 a} e^{i(y-x) \cdot k}. \quad (\text{C17})$$

In the Hartree approximation, as done in Eq. (B10), $\mathcal{P}^{-1}(k)$ can have momentum dependence only from the canonical kinetic part, which reads

$$[\mathcal{P}^{-1}(k)]^{b_1 a} = k^2 \delta^{b_1 a} + [M_P^2]^{b_1 a}. \quad (\text{C18})$$

Putting Eqs. (C17) and (C18) into Eq. (C16), we are able to perform the above x integral in Eq. (C16), to get

$$\begin{aligned} & - \int_x [\mathcal{P}^{-1}(y, x)]^{b_1 a} d^{ab}_a \bar{\sigma}_b \\ &= - \int d^4 x \int \frac{d^4 k}{(2\pi)^4} [\mathcal{P}^{-1}(k)]^{b_1 a} e^{i(y-x) \cdot k} d^{ab}_a \bar{\sigma}_b \\ &= - [\mathcal{P}^{-1}(0)]^{b_1 a} d^{ab}_a \bar{\sigma}_b \\ &= - [M_P^2]^{b_1 a} d^{ab}_a \bar{\sigma}_b. \end{aligned} \quad (\text{C19})$$

Taking $b_1 = 1$ in Eq. (C19), we find the right-hand side to be

$$- [M_P^2]^{1a} d^{ab}_a (\bar{\sigma}_0 \delta_{0b} + \bar{\sigma}_8 \delta_{8b}). \quad (\text{C20})$$

Since the $SU(3)$ vectorial symmetry is present in the system, the $SU(2)$ triplet pions and the $SU(2)$ doublet kaons do not mix with each other, hence we have $[M_P^2]^{1a} = [M_P^2]^{11} \delta^{1a}$. Noting also that $d^{a0}_1 = \sqrt{2/3} \delta_1^a$ and $d^{a8}_1 = 1/\sqrt{3} \delta_1^a$, we then get

$$\begin{aligned} - [M_P^2]^{11} d^{ab}_1 (\bar{\sigma}_0 \delta_{0b} + \bar{\sigma}_8 \delta_{8b}) &= -M_\pi^2 \left(\sqrt{\frac{2}{3}} \bar{\sigma}_0 + \frac{1}{\sqrt{3}} \bar{\sigma}_8 \right) \delta_1^a \\ &= -2M_\pi^2 \bar{\Phi}_1 \delta_1^a. \end{aligned} \quad (\text{C21})$$

By taking $b_1 = 4$ in Eq. (C19), the right-hand side is evaluated to be

$$- [M_P^2]^{44} d^{ab}_4 (\bar{\sigma}_0 \delta_{0b} + \bar{\sigma}_8 \delta_{8b}). \quad (\text{C22})$$

Here, we have $d^{a0}_4 = \sqrt{2/3} \delta_0^a$ and $d^{a8}_4 = -1/(2\sqrt{3}) \delta_4^a$, such that

$$\begin{aligned}
 -[M_P^2]^{44} d^{ab} (\bar{\sigma}_0 \delta_{0b} + \bar{\sigma}_8 \delta_{8b}) &= -M_\kappa^2 \left(\sqrt{\frac{2}{3}} \bar{\sigma}_0 - \frac{1}{2\sqrt{3}} \bar{\sigma}_0 \right) \delta_4^\alpha \\
 &= -M_\kappa^2 (\bar{\Phi}_1 + \bar{\Phi}_3) \delta_4^\alpha. \quad (C23)
 \end{aligned}$$

Taking $\alpha = 1$ and $\alpha = 4$ individually, we thus find the following threshold property for the pion and kaon masses

$$M_\pi^2 \bar{\Phi}_1 = 0, \quad M_\kappa^2 (\bar{\Phi}_1 + \bar{\Phi}_3) = 0, \quad (C24)$$

which precisely follows the massless NG boson when $\bar{\Phi}_1$ or $\bar{\Phi}_8$ takes a nonzero value, i.e., the spontaneous breaking of the chiral symmetry takes place.

2. Anomalous WTIs and GMOR relations

In the present LSM, actually, the chiral $SU(3)$ symmetry is explicitly broken by the current quark masses encoded in

the spurion field $\mathcal{M}$. We therefore extend the formulation described above on the chiral-limit WTIs by including $\mathcal{M}$.

As in the main text, the spurion field $\mathcal{M}$ is introduced so as to compensate for the explicit chiral breaking by quark masses in the action. The spurion field $\mathcal{M}$ therefore transforms under the chiral $SU(3)$ symmetry in the same way as ϕ does in Eq. (C10),

$$\mathcal{M} \rightarrow g_L \cdot \mathcal{M} \cdot g_R^\dagger. \quad (C25)$$

Thus, the chiral variation of the action, as generically given in Eq. (C1), gets shifted as

$$\delta_\epsilon S[\phi] = \frac{\delta S}{\delta \phi_a} \cdot \delta_\epsilon \phi_a + \frac{\delta S}{\delta \mathcal{M}} \cdot \delta_\epsilon \mathcal{M} + \frac{\delta S}{\delta \mathcal{M}^\dagger} \cdot \delta_\epsilon \mathcal{M}^\dagger = 0. \quad (C26)$$

On the level of the generating functional, Eq. (C4) is then modified as

$$\begin{aligned}
 &\int [\mathcal{D}\phi_a] [e^{-S[\phi', \mathcal{M}] + J^a \cdot \phi'_a} - e^{-S[\phi, \mathcal{M}] + J^a \cdot \phi_a}]_{\phi'_a = \phi_a + i\epsilon \delta_\epsilon \phi_a} \\
 &= -i\epsilon \int [\mathcal{D}\phi_a] \left[\frac{\delta S}{\delta \phi_a} \cdot \delta_\epsilon \phi_a - J^a \cdot \delta_\epsilon \phi_a \right] e^{-S[\phi, \mathcal{M}] + J^a \cdot \phi_a} + \mathcal{O}(\epsilon^2) \\
 &= i\epsilon \int [\mathcal{D}\phi_a] \left[\frac{\delta S}{\delta \mathcal{M}} \cdot \delta_\epsilon \mathcal{M} + \frac{\delta S}{\delta \mathcal{M}^\dagger} \cdot \delta_\epsilon \mathcal{M}^\dagger + J^a \cdot \delta_\epsilon \phi_a \right] e^{-S[\phi, \mathcal{M}] + J^a \cdot \phi_a} + \mathcal{O}(\epsilon^2) \\
 &= i\epsilon \left[\left\langle \frac{\delta S}{\delta \mathcal{M}} \right\rangle_J \cdot \delta_\epsilon \mathcal{M} + \left\langle \frac{\delta S}{\delta \mathcal{M}^\dagger} \right\rangle_J \cdot \delta_\epsilon \mathcal{M}^\dagger + J^a \cdot \langle \delta_\epsilon \phi_a \rangle_J \right] Z[J, \mathcal{M}] + \mathcal{O}(\epsilon^2) \\
 &= 0. \quad (C27)
 \end{aligned}$$

We then find the general form of the anomalous WTIs including $\mathcal{M}$,

$$\left\langle \frac{\delta S}{\delta \mathcal{M}} \right\rangle_J \cdot \delta_\epsilon \mathcal{M} + \left\langle \frac{\delta S}{\delta \mathcal{M}^\dagger} \right\rangle_J \cdot \delta_\epsilon \mathcal{M}^\dagger + J^a \cdot \langle \delta_\epsilon \phi_a \rangle_J = 0. \quad (C28)$$

Written in terms of the 1PI effective action in Eq. (C7), the anomalous WTIs take the form

$$\begin{aligned}
 &\frac{\delta S}{\delta \mathcal{M}} \Big|_{\phi_a = (\phi_{cl})_a + \Delta_{ab} \frac{\delta}{\delta (\phi_{cl})_b}} \cdot \delta_\epsilon \mathcal{M} + \frac{\delta S}{\delta \mathcal{M}^\dagger} \Big|_{\phi_a = (\phi_{cl})_a + \Delta_{ab} \frac{\delta}{\delta (\phi_{cl})_b}} \\
 &\cdot \delta_\epsilon \mathcal{M}^\dagger + \frac{\delta \Gamma}{\delta (\phi_{cl})_a} \cdot \delta_\epsilon (\phi_{cl})_a = 0, \quad (C29)
 \end{aligned}$$

where

$$\Delta_{ab}(x, y) = \frac{\delta^2 W}{\delta J(x) \delta J(y)} \quad (C30)$$

denotes the full connected two-point function.

With Eq. (C29) at hand, we are ready to derive the GMOR relations at the 1PI level. In the last subsection, we have dealt with the first term in Eq. (C29), hence, we focus on the latter two terms. Following Eq. (C25) the infinitesimal transformation of $\mathcal{M}$ goes like

$$\delta^\alpha \mathcal{M}_{mn} = \{T^\alpha, \mathcal{M}\}_{mn}. \quad (C31)$$

Decomposing $\mathcal{M}$ into scalar and pseudoscalar parts as $\mathcal{M} = \mathcal{M}^s + i\mathcal{M}^p = \mathcal{M}_a^s T^a + i\mathcal{M}_a^p T^a$, the transformation of $\mathcal{M}_a^{s/p}$ then reads as

$$\delta^\alpha \mathcal{M}_a^s = d^{ab}{}_a \mathcal{M}_b^p, \quad \delta^\alpha \mathcal{M}_a^p = -d^{ab}{}_a \mathcal{M}_b^s. \quad (C32)$$

Consider

$$\begin{aligned}
 \frac{\delta S_{\text{LSM}}}{\delta \mathcal{M}_{mn}} &= -c(\Phi^\dagger)_{nm} - kc\epsilon_{mbc}\epsilon^{nef}\Phi_e^b\Phi_f^c \\
 &= -c(\sigma_a - i\pi_a)T_{nm} - kc(\sigma_{a_1} + i\pi_{a_1}) \\
 &\quad \times (\sigma_{a_2} + i\pi_{a_2})\epsilon_{mbc}\epsilon^{nef}(T^{a_1})_e^b(T^{a_2})_f^c. \quad (C33)
 \end{aligned}$$

Replacing ϕ_a by $(\phi_{\text{cl}})_a + \Delta_{ab} \cdot \frac{\delta}{\delta(\phi_{\text{cl}})_b}$, we obtain

$$\begin{aligned} \left. \frac{\delta S_{\text{LSM}}}{\delta \mathcal{M}_{mn}} \right|_{\phi_a=(\phi_{\text{cl}})_a + \Delta_{ab} \cdot \frac{\delta}{\delta(\phi_{\text{cl}})_b}} \cdot \delta^\alpha \mathcal{M}_{mn} = & -c[(\sigma_{\text{cl}})_a - i(\pi_{\text{cl}})_a] \cdot [\mathcal{M}_{a_2}^p - i\mathcal{M}_{a_2}^s] T_{nm}^b T_{nm}^a d^{aa_2}_b \\ & - kc[\mathcal{S}_{a_1 a_2}(x, x) + i\mathcal{P}_{a_1 a_2}(x, x)] \cdot [\mathcal{M}_{a_3}^p - i\mathcal{M}_{a_3}^s] \epsilon_{mbc} \epsilon^{nef} (T^g)_n^m (T^{a_1})_e^b (T^{a_2})_f^c d^{aa_3}_g \\ & - kc[(\sigma_{\text{cl}})_{a_1}(\sigma_{\text{cl}})_{a_2} + i(\sigma_{\text{cl}})_{a_1}(\pi_{\text{cl}})_{a_2} + i(\pi_{\text{cl}})_{a_1}(\sigma_{\text{cl}})_{a_2} - (\pi_{\text{cl}})_{a_1}(\pi_{\text{cl}})_{a_2}] \cdot [\mathcal{M}_{a_3}^p - i\mathcal{M}_{a_3}^s] \\ & \times \epsilon_{mbc} \epsilon^{nef} (T^g)_n^m (T^{a_1})_e^b (T^{a_2})_f^c d^{aa_3}_g, \end{aligned} \quad (\text{C34})$$

where $\mathcal{S}_{a_1 a_2}(x, y)$ and $\mathcal{P}_{a_1 a_2}(x, y)$ are the full-dressed scalar and pseudoscalar propagators. We further get the following.

$$\begin{aligned} \frac{\delta}{\delta(\pi_{\text{cl}}(y))_{b_1}} (\text{C34}) = & c\delta^{(4)}(x-y)[\delta_a^{b_1}] \cdot [\mathcal{M}_{a_2}^s + i\mathcal{M}_{a_2}^p] T_{mn}^b T_{nm}^a d^{aa_2}_b \\ & - kc \left[\frac{\delta}{\delta(\pi_{\text{cl}}(y))_{b_1}} \mathcal{S}_{a_1 a_2}(x, x) + i \frac{\delta}{\delta(\pi_{\text{cl}}(y))_{b_1}} \mathcal{P}_{a_1 a_2}(x, x) \right] \cdot [\mathcal{M}_{a_3}^p - i\mathcal{M}_{a_3}^s] \\ & \times \epsilon_{mbc} \epsilon^{nef} (T^g)_n^m (T^{a_1})_e^b (T^{a_2})_f^c d^{aa_3}_g \\ & - kc\delta^{(4)}(x-y)[((\sigma_{\text{cl}})_{a_1} + i(\pi_{\text{cl}})_{a_1})\delta_{a_2}^{b_1} + \delta_{a_1}^{b_1}((\sigma_{\text{cl}})_{a_2} + i(\pi_{\text{cl}})_{a_2})] \cdot [\mathcal{M}_{a_3}^s + i\mathcal{M}_{a_3}^p] \\ & \times \epsilon_{mbc} \epsilon^{nef} (T^g)_n^m (T^{a_1})_e^b (T^{a_2})_f^c d^{aa_3}_g \\ = & \frac{c}{2}\delta^{(4)}(x-y)[\delta_a^{b_1}] \cdot [\mathcal{M}_{a_2}^s + i\mathcal{M}_{a_2}^p] \delta^{ab} d^{aa_2}_g - kc[H^{b_1}_{a_1 a_2}(y, x)] \cdot [\mathcal{M}_{a_3}^p - i\mathcal{M}_{a_3}^s] \mathcal{T}^{ga_1 a_2} d^{aa_3}_g \\ & - kc\delta^{(4)}(x-y)[((\sigma_{\text{cl}})_{a_1} + i(\pi_{\text{cl}})_{a_1})\delta_{a_2}^{b_1} + \delta_{a_1}^{b_1}((\sigma_{\text{cl}})_{a_2} + i(\pi_{\text{cl}})_{a_2})] \cdot [\mathcal{M}_{a_3}^s + i\mathcal{M}_{a_3}^p] \mathcal{T}^{ga_1 a_2} d^{aa_3}_g, \end{aligned} \quad (\text{C35})$$

where

$$\mathcal{T}^{ga_1 a_2} \equiv \epsilon_{mbc} \epsilon^{nef} (T^g)_n^m (T^{a_1})_e^b (T^{a_2})_f^c. \quad (\text{C36})$$

In Eq. (C35), we have defined the *meson scattering kernel* $H^{b_1}_{a_1 a_2}(y, x)$ as

$$H^{b_1}_{a_1 a_2}(y, x) \equiv \frac{\delta}{\delta(\pi_{\text{cl}}(y))_{b_1}} \mathcal{S}_{a_1 a_2}(x, x) + i \frac{\delta}{\delta(\pi_{\text{cl}}(y))_{b_1}} \mathcal{P}_{a_1 a_2}(x, x). \quad (\text{C37})$$

The corresponding diagrammatic interpretation is available in Fig. 6.

The meson scattering kernel $H^{b_1}_{a_1 a_2}(y, x)$ includes the fully dressed three-point meson vertex. In the present work, we adopt the truncation scheme such that the three-point meson vertex is replaced by the tree-level vertex, i.e.,

$$\frac{\delta^3 \Gamma}{\delta \sigma_a(x) \delta \sigma_b(y) \delta \sigma_c(z)} = -\frac{\delta^3 \Gamma}{\delta \sigma_a(x) \delta \pi_b(y) \delta \pi_c(z)} = \frac{\delta^3 S_{\text{LSM}}}{\delta \sigma_a(x) \delta \sigma_b(y) \delta \sigma_c(z)} = -6\mathcal{G}^{abc} \delta^{(4)}(x-y) \delta^{(4)}(x-z), \quad (\text{C38})$$

where $\mathcal{G}^{abc}$ is defined in Eq. (B26). We find that within the current truncation scheme, the contribution from the meson scattering kernel vanishes in the anomalous WTIs, in Eq. (C35).

Taking the background field values $\bar{\sigma}_a T^a = \text{diag}\{\bar{\Phi}_1, \bar{\Phi}_1, \bar{\Phi}_3\}$, $\mathcal{M} = \text{diag}\{m_l, m_l, m_s\}$, and $\bar{\pi}_a = 0$, we see that for $b_1 = \alpha = 1$,

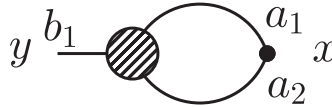


FIG. 6. Feynman diagram of the meson scattering kernel defined in Eq. (C37). The solid line connected to the point y denotes the pseudoscalar meson propagator, and the lines involving the point x represent the scalar or pseudoscalar meson propagators. The hatched blob stands for the dressed three-point meson vertex in the 1PI formulation.

$$(C35) = cm_l + 2kcm_l\bar{\Phi}_3, \quad (C39)$$

and for $b_1 = \alpha = 4$,

$$(C35) = \frac{1}{2}(cm_l + cm_s) + k(cm_l + cm_s)\bar{\Phi}_1, \quad (C40)$$

where we have integrated spacetime over x . The Hermitian conjugate term $\propto \mathcal{M}^\dagger$ gives the same result as above.

Combining Eqs. (C21), (C23), (C39), and (C40), we get the GMOR relations expressed in terms of the LSM parameters, as in Eq. (27),

$$\begin{aligned} M_\pi^2 \bar{\Phi}_1 &= cm_l(1 + 2k\bar{\Phi}_3), \\ M_\kappa^2(\bar{\Phi}_1 + \bar{\Phi}_3) &= (cm_l + cm_s)(1 + 2k\bar{\Phi}_1). \end{aligned} \quad (C41)$$

These two relations coincide with those obtained from the tree-level action, as we expected, meaning that the contributions to the explicit chiral breaking terms generated from the quantum fluctuations are dropped in the current truncation scheme. In the 1PI formulation, the above GMOR relations are actually equivalent to the stationary conditions for $\bar{\Phi}_1$ and $\bar{\Phi}_3$, which substantially deviate from

each other in the 2PI formalism, as noted in the main text and will be further clarified in the next section.

3. (Anomalous) WTIs in 2PI formalism

Considering the generating functional with a bilocal source $K^{ab}(x, y)$,

$$Z[J, K] = \int [\mathcal{D}\phi_a] e^{-S[\phi] + J^a \cdot \phi_a + \frac{1}{2} \phi_a \cdot K^{ab} \cdot \phi_b} = e^{W[J, K]}, \quad (C42)$$

we have

$$\begin{aligned} \frac{\delta W[J, K]}{\delta J^a} &= \langle \phi_a \rangle_{J, K} \equiv (\phi_{cl})_a, \\ \frac{\delta W}{\delta J^a \delta J^b} &= \langle \phi_a \phi_b \rangle_{J, K} - \langle \phi_a \rangle_{J, K} \langle \phi_b \rangle_{J, K} \equiv \Delta_{ab}, \end{aligned} \quad (C43)$$

and

$$\frac{\delta W[J, K]}{\delta K^{ab}} = \frac{1}{2}(\Delta_{ab} + (\phi_{cl})_a (\phi_{cl})_b). \quad (C44)$$

The 2PI effective action is obtained by performing the following double-Legendre transformation,

$$\begin{aligned} \Gamma[\phi_{cl}, \Delta] &= \sup_K \left[K^{ab} \cdot \frac{\delta \Gamma[\phi_{cl}, K]}{\delta K^{ab}} - \sup_J (J^a \cdot (\phi_{cl})_a - W[J, K]) \right] \\ &= \sup_K \inf_J \left[W[J, K] - J^a \cdot (\phi_{cl})_a - \frac{1}{2} K^{ab} \cdot (\Delta_{ab} + (\phi_{cl})_a (\phi_{cl})_b) \right], \end{aligned} \quad (C45)$$

where we have used

$$\frac{\delta \Gamma[\phi_{cl}, K]}{\delta K^{ab}} = -\frac{\delta W[J, K]}{\delta K^{ab}}. \quad (C46)$$

We then have

$$\frac{\delta \Gamma[\phi_{cl}, \Delta]}{\delta (\phi_{cl})_a} = -J^a - K^{ab} \cdot (\phi_{cl})_b, \quad \frac{\delta \Gamma[\phi_{cl}, \Delta]}{\delta \Delta_{ab}} = -\frac{1}{2} K^{ab}. \quad (C47)$$

In a manner similar to what has been done in Eq. (C4), the symmetry variation as in Eq. (C1) and its invariance of the generating functional $Z[J, K]$ in Eq. (C42) leads to the WTIs in terms of $Z[J, K]$,

$$\begin{aligned} &\int [\mathcal{D}\phi_a] [e^{-S[\phi'] + J^a \cdot \phi'_a + \frac{1}{2} \phi'_a \cdot K^{ab} \cdot \phi'_b} - e^{-S[\phi] + J^a \cdot \phi_a + \frac{1}{2} \phi_a \cdot K^{ab} \cdot \phi_b}] \\ &= -i\epsilon \int [\mathcal{D}\phi_a] \left[\frac{\delta S}{\delta \phi_a} \cdot \delta_\epsilon \phi_a - J^a \cdot \delta_\epsilon \phi_a - \frac{1}{2} \phi_a \cdot K^{ab} \cdot \delta_\epsilon \phi_b - \frac{1}{2} \delta_\epsilon \phi_a \cdot K^{ab} \cdot \phi_b \right] e^{-S[\phi] + J^a \cdot \phi_a + \frac{1}{2} \phi_a \cdot K^{ab} \cdot \phi_b} + \mathcal{O}(\epsilon^2) \\ &= i\epsilon \left[J^a \cdot \langle \delta_\epsilon \phi_a \rangle + \frac{1}{2} K^{ab} \cdot \langle \phi_a \cdot \delta_\epsilon \phi_b \rangle + \frac{1}{2} K^{ab} \cdot \langle \delta_\epsilon \phi_a \cdot \phi_b \rangle \right] Z[J, K] + \mathcal{O}(\epsilon^2) \\ &= 0. \end{aligned} \quad (C48)$$

Taking into account Eqs. (C43) and (C47), we rewrite Eq. (C48) into the form

$$\frac{\delta\Gamma[\phi_{\text{cl}}, \Delta]}{\delta(\phi_{\text{cl}})_a} \cdot d_a^b(\phi_{\text{cl}})_b + \frac{\delta\Gamma[\phi_{\text{cl}}, \Delta]}{\delta\Delta_{ab}} \cdot (d_b^c \Delta_{ac} + d_a^c \Delta_{cb}) = 0. \quad (\text{C49})$$

This is the exact form of WTIs in the 2PI formalism. Taking the functional derivative of both sides in Eq. (C49) with respect to $(\phi_{\text{cl}})_c$, we reach the form of Eq. (32) in the main text,

APPENDIX D: THERMAL EVOLUTIONS OF CHIRAL ORDER PARAMETERS AND MESON MASSES AT CRITICAL POINTS

In this appendix, we present the results based on the SICJT formalism on the T dependences of the chiral order parameters $\bar{\Phi}_{1,3}$ and scalar and pseudoscalar meson masses

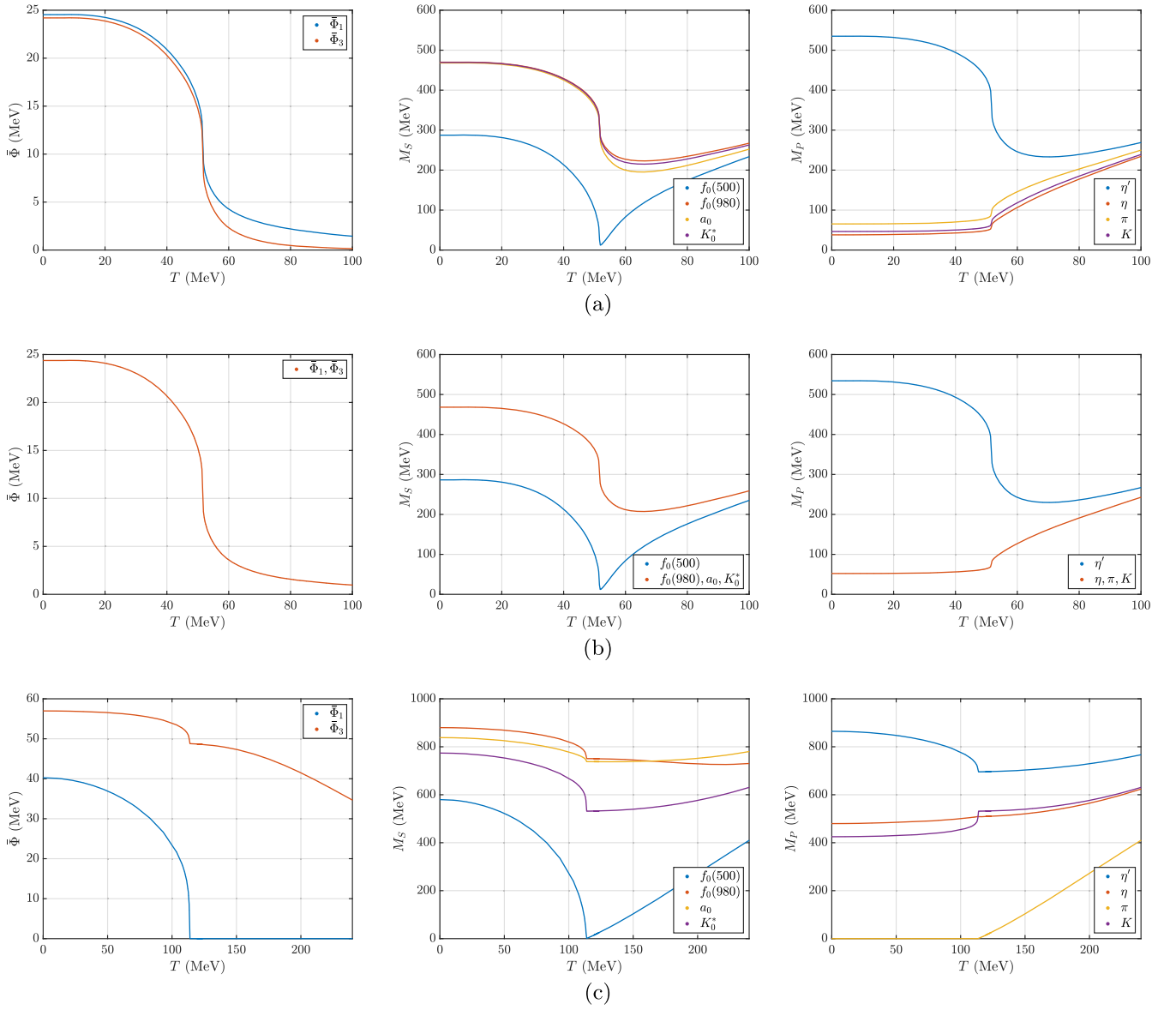


FIG. 7. Benchmarked chiral phase transitions at critical points (a), (b), and (c) defined in the text.

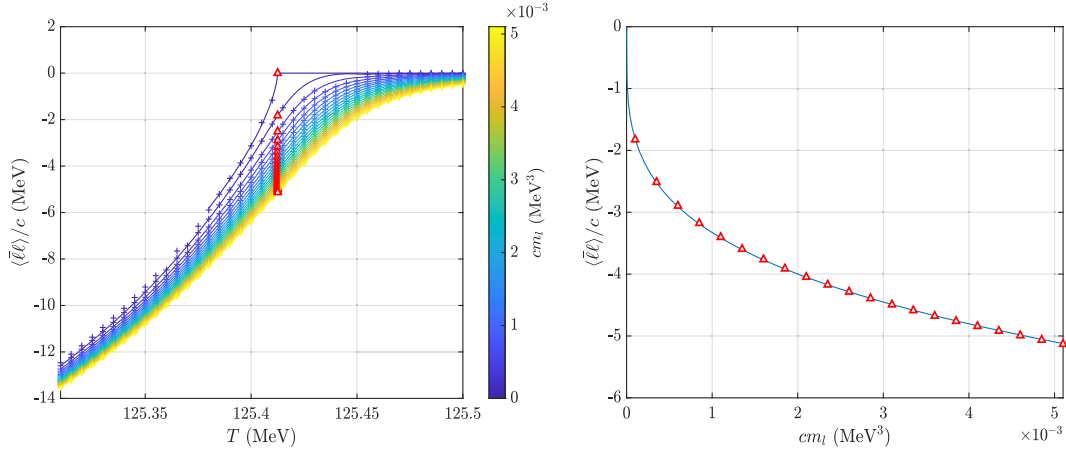


FIG. 8. Plots of the light quark condensate $\langle \bar{\ell}\ell \rangle$ around the critical points where $m_l^c = 0$ and $m_s^c = m_s^{\text{phys}}$ as a function of the temperature T (left panel) and the light current quark mass cm_l (right panel). The plus markers in the left panel denote the benchmark points evaluated from the numerical approach, the solid line in the top denotes the fit result of the scaling function, and the rest of the solid lines represent the spline-interpolation functions. The correspondence between the benchmarks and the light current quark mass is shown in the colorbar. The red-triangle markers denote the points obtained by the spline interpolation at the critical temperature T_c . In the right panel, the red-triangle markers denote the same points obtained from the left panel, and the blue solid line shows the scaling fit function.

at several critical points in the Columbia plot, as seen in Fig. 5. We take the parameter set I in Sec. IV B 1 to fix the $f_0(500)$ mass: $m[f_0(500)] = 672.4$ MeV.

The chosen critical points are:

- (a) $(m_l, m_s)/m_l^{\text{phys}} \simeq (0.1474, 0)$: the second-order critical line on the m_l -axis,
- (b) $(m_l, m_s)/m_l^{\text{phys}} \simeq (0.0931, 0.0939)$: the second-order critical line on the three-flavor symmetric curve, and
- (c) $(m_l, m_s)/m_s^{\text{phys}} \simeq (0, 0.696)$: the second-order critical line on the m_s -axis, i.e., vicinity of the tricritical point.

In Fig. 7, we plot the T dependences of the chiral order parameters $\Phi_{1,3}$ and scalar and pseudoscalar meson masses at those critical points.

APPENDIX E: SYSTEMATICS ON EVALUATION OF THE CRITICAL EXPONENTS

We read off the critical exponents of the light quark condensate $\langle \bar{\ell}\ell \rangle$ in the vicinity of the critical points based on the formulas in Eq. (41). To this end, we first try to observe the order of the phase transition by varying the temperature for selected points in the Columbia plot on the (cm_l, cm_s) plane. Once the critical point is found out, we next shrink the range of the external parameters (T, cm_l) with the step sizes $\Delta T = 0.3$ MeV and $\Delta(cm_l) = 5 \times 10^{-3}$ (MeV³). For the current strange quark mass cm_s , we fix it to the critical values listed in Table I, as also reflected in Fig. 7. We then read off the cm_l dependence of the light quark condensate to fit the scaling functions with the spline interpolation.

For the case of $cm_l^c = 0$, we use the following three-parameter ansatz to fit the phase transition curve across the critical point with the variation of T ,

$$\langle \bar{\ell}\ell \rangle(T, 0) = \theta(T_c - T)A \left(\frac{T - T_c}{T_c} \right)^\beta, \quad (\text{E1})$$

where A , B , and T_c are parameters to be determined. When going along the cm_l direction, we apply the following scaling function:

$$\langle \bar{\ell}\ell \rangle(T_c, cm_l) = C(cm_l)^\delta, \quad (\text{E2})$$

to extract the exponent δ , where C and δ are the parameters to be determined.

For the case of $cm_l^c \neq 0$, to fit the scaling condensate with T , we introduce the following fit ansatz:

$$\langle \bar{\ell}\ell \rangle(T, cm_l^c) = A \left(\frac{T - T_c}{T_c} \right)^\beta + B, \quad (\text{E3})$$

with a constant B part. In moving in the cm_l direction, we follow the following scaling function:

$$\langle \bar{\ell}\ell \rangle(T_c, cm_l) = C \left(\frac{cm_l - cm_l^c}{cm_l^c} \right)^\delta + B. \quad (\text{E4})$$

When the B part becomes relevant, the size of the scaling steps for (T, cm_l) would be crucial to determine the critical exponent. In that case, we have taken the step sizes as $\Delta T = 1 \times 10^{-3}$ MeV and $\Delta(cm_l)/cm_l^c = 1 \times 10^{-5}$.

We use the least-squares fitting to determine the parameters in the scaling functions versus our numerical data. One example at the critical point is shown in Fig. 8, where we have taken $m_l^c = 0$ and $m_s^c = m_s^{\text{phys}}$ together with the parameter set I in Sec. IV B 1.

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