

Genuine multientropy and holography

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Is bipartite entanglement sufficient for holography? Through the analysis of the Markov gap, it is known that the answer is no. In this paper, we give a new perspective on this issue from a different angle using a multientropy. We define a genuine α -partite multientropy from a α -partite multientropy by subtracting appropriate linear combinations of $\tilde{\alpha}$ -partite multientropies for $\tilde{\alpha} < \alpha$, in such a way that the genuine α -partite multientropy vanishes for all $\tilde{\alpha}$ -partite entangled states. After studying several aspects, we apply it to black holes and holography. For the application to black holes, we see that such a genuine α -partite multientropy is important only after the Page time. For the application to holography, we prove that nonbipartite multientropies are always positive and $\mathcal{O}(1/G_N)$, as long as boundary subregions are connected. This indicates that for holography, genuine multipartite entanglement is not small and plays an important role.

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I. INTRODUCTION

Since the Ryu-Takayanagi (RT) formula [1], which relates quantum entanglement entropy between boundary regions into minimal geometrical surfaces (RT-surface) in the bulk, was proposed, significant progress has been made in our understanding of holography from a quantum information viewpoint. The random tensor network (RTN) [2] is one such interesting development, which shows a very analogous formula to RT for the entanglement entropy and this analogy gives the view that RTN is a toy model of quantum gravity. Another interesting picture is given by a bit thread picture [3], which gives a new perspective to the entanglement entropy as the maximum number of “bit threads.” Given that entanglement entropy is a bipartite measure, it is natural to ask if bipartite entanglement is enough for holography and it is conjectured that the entanglement of holographic conformal field theory (CFT) states is mostly bipartite (mostly bipartite conjecture) in [4].

In general quantum systems, one can divide the total system into three or more subsystems, and many states are involved in entanglement. In general, there is more than

bipartite entanglement involved. For example, in spin systems, generic states contain more than bipartite entanglement, i.e., they contain genuine multipartite entanglement such as tripartite and quadripartite entanglement. Famous examples of tripartite entanglement are the Greenberger-Horne-Zeilinger (GHZ) state and the W state [5], which are distinguished from bipartite entangled states.

Even though the mostly bipartite conjecture is quite interesting, it was disproved through the study of the Markov gap¹ [6,7]. The Markov gap is defined as the difference between reflected entropy [8] and mutual information. The Markov gap is a useful quantity to detect genuine tripartite entanglement contribution in holography since it vanishes for all states that contain only bipartite entanglement. Recently, another new multipartite quantum entanglement measure has been proposed, which is called the multientropy [9–11]. In this paper, we would like to obtain a new perspective on the above-mentioned point by defining a *genuine* α -partite multientropy using the multientropy. This is because the multientropy by itself can be nonzero even by a bipartite contribution only. On the other hand, the genuine α -partite multientropy we define can become nonzero *only if* the state contains genuine α -partite entanglement. By genuine α -partite entanglement, we mean the entanglement that cannot be decomposed into tensor products of lower-partite (i.e., less than α -partite) entanglement. We emphasize that the genuine multientropy can be

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¹We review the argument of [6] in footnote 11 at the end of Sec. IV, and compare it with our holographic proof.

defined for any integer $q \geq 3$. Note that how to construct such a quantity is quite nontrivial, and in this paper, we give a prescription for how to construct the genuine multipartropy with a new $q = 4$ concrete example. We also point out the analogy between the Markov gap and genuine multipartropy for $q = 3$. The Markov gap is defined using reflected entropy. On the other hand, genuine multipartropy is defined using multipartropy. We will see more of their analogy in detail.²

The outline of this paper is as follows. In Sec. II, after reviewing the definition of a multipartropy and Rényi multipartropy, we define the genuine q -partite multipartropy for the $q = 3$ case as a linear combination of the $q = 3$ and $q = 2$ multipartropies. We also study the genuine multipartropy for $q = 3$ of two qubit states, namely the GHZ and W states. In Sec. III, we use this genuine $q = 3$ multipartropy to study genuine tripartite entanglement in an evaporating black hole. The crucial approximation here is that we approximate an evaporating black hole and its radiation with a Haar-random state. Then we divide the total system into three subsystems, two for Hawking radiation subsystems and one for an evaporating black hole subsystem. By computing the genuine $q = 3$ multipartropy in random states, we observe that the time scale when the effect of genuine tripartite entanglement becomes large is always after the Page time. In Sec. IV, we also apply the genuine multipartropy in holography. We give a simple proof that non-bipartite, generic q entanglement is always positive and of order $\mathcal{O}(N^2) = \mathcal{O}(1/G_N)$ in holographic settings as far as the boundary subregions are connected. This explains why bipartite entanglement is not enough in holography, and this is similar to the nonnegative Markov gap [6,7]. In this paper, we start a genuine q -partite multipartropy from the $q = 3$ tripartite genuine case and also define genuine multipartropy for $q = 4$. However, we would like to stress that one can define genuine multipartropy for any q as well. In Sec. V, we discuss the genuine multipartropy for $q \geq 4$. We also show the black hole genuine multipartropy curves for $q = 4$ and see that just as $q = 3$, the time scale when the effect of genuine quadripartite entanglement becomes large is always after the Page time. After briefly giving the prescription on how to construct on genuine multipartropy, we end in Sec. VI with open questions. Appendix A is for the maximum value of the genuine tripartite multipartropy in W class, and Appendix B is for other measures constructed as products of bipartite measures. Appendix C is for the evaluation of the q -partite information I_q curves for black hole evaporation.

²Other than the Markov gap, in [12,13] aspects of holographic multipartite entanglement are studied. In [14], the absence of distillable bipartite entanglement in some holographic regions is suggested. See [15] as well.

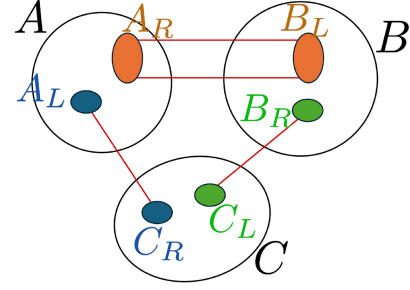


FIG. 1. Triangle state is composed of bipartite entangled states only. Red lines represent bipartite entanglements.

II. DEFINING A GENUINE MULTIPARTROPY IN TRIPARTITE SYSTEMS

In this section, we introduce the definition of multipartropy and *genuine* tripartite multipartropy by the linear combination of multipartropies. Before we proceed, we first give our criterion for “genuineness.”

For that purpose, let us first consider a state that contains *only* bipartite entanglement. A typical example of such states is a triangle state $|\Delta\rangle_{ABC}$, which is defined in [16] as

$$|\Delta\rangle_{ABC} = |\psi\rangle_{A_R B_L} |\psi\rangle_{B_R C_L} |\psi\rangle_{C_R A_L}, \quad (2.1)$$

for given three subsystems A, B, C . See Fig. 1. *Genuine* tripartite entanglement measures must be zero for all of the triangle states. Later in Sec. II B, we define the genuine tripartite multipartropy, which we denote $\text{GM}^{(3)}(A:B:C)$, to satisfy this genuineness. In other words, our genuine tripartite multipartropy, $\text{GM}^{(3)}(A:B:C)$, excludes all bipartite entanglement contributions. Similarly we define a genuine q -partite multipartropy, which we denote $\text{GM}^{(q)}(A_1:A_2:\dots:A_q)$, as a quantity that vanishes for all states, which can be written as tensor products of entangled subsystems involving fewer than q parties.

The Markov gap, defined as $S_R - I$ in [6,7], also satisfies this criterion [16]; that is, it vanishes for all states that exhibit only bipartite entanglement, and therefore serves as an indicator for the presence of tripartite entanglement. Here S_R is the reflected entropy³ and I is the mutual information.⁴ Although both our genuine multipartropy $\text{GM}^{(3)}$ and the Markov gap vanish for states with only bipartite entanglement, they differ in their construction: the

³Suppose a pure state ρ_{ABC} is defined on a tripartite system consisting of subsystems A, B, C . The reflected entropy S_R for the pair A and B is defined as follows [8]: First, trace out subsystem C to obtain a reduced density matrix $\rho_{AB} = \text{Tr}_C \rho_{ABC}$. Then canonically purify it to $\rho_{AA^* BB^*}$, where A^* and B^* are auxiliary purifying systems for A and B , respectively. The reflected entropy is given by the entanglement entropy between AA^* and BB^* , namely the von Neumann entropy of the $\rho_{AA^*} = \text{Tr}_{BB^*} \rho_{AA^* BB^*}$.

⁴The mutual information is simply $I(A, B) = S(A) + S(B) - S(AB) = S(A) + S(B) - S(C)$, if the total system ABC is pure.

Markov gap is defined via the reflected entropy, whereas the genuine tripartite multientropy is defined in terms of the multientropy. We will comment further on their similarities and differences later, but before doing so, we briefly review the definitions of multientropy and Rényi multientropy.

A. Multientropy in tripartite systems

The Rényi multi-entropy [9–11] was defined as a symmetric multipartite entanglement measure in $\mathfrak{q}$ -partite systems. For instance, let us consider a tripartite pure state $|\psi\rangle$ on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. For such $\mathfrak{q} = 3$ tripartite systems, Rényi multientropy $S_n^{(3)}(A:B:C)$ of $|\psi\rangle$ is defined by⁵

$$S_n^{(3)}(A:B:C) := \frac{1}{1-n} \frac{1}{n} \log \frac{Z_n^{(3)}}{(Z_1^{(3)})^{n^2}}, \quad (2.2)$$

$$Z_n^{(3)} := \langle \psi |^{\otimes n^2} \Sigma_A(g_A) \Sigma_B(g_B) \Sigma_C(g_C) | \psi \rangle^{\otimes n^2}, \quad (2.3)$$

where $\Sigma_{A,B,C}$ are the twist operators for the permutation action of $g_{A,B,C}$ on indices of density matrices. Their explicit representation of the permutation group is

$$g_A = (1, 2, \dots, n)(n+1, n+2, \dots, 2n) \cdots \\ \times (n^2 - n + 1, n^2 - n + 2, \dots, n^2), \quad (2.4)$$

$$g_B = (1, n+1, \dots, n^2 - n + 1) \\ \times (2, n+2, \dots, n^2 - n + 2) \cdots (n, 2n, \dots, n^2), \quad (2.5)$$

$$g_C = (1)(2) \cdots (n^2). \quad (2.6)$$

See Fig. 2. The multientropy $S^{(3)}(A:B:C)$ is defined by

$$S^{(3)}(A:B:C) := \lim_{n \rightarrow 1} S_n^{(3)}(A:B:C). \quad (2.7)$$

One can understand this multientropy as a generalization of the purity $\text{Tr}(\rho^2)$, as follows. Consider the case where subsystem B is one-dimensional. Then, in Fig. 2, the indices i_B and j_B (represented by vertical lines) become trivial since $i_B = j_B = 1$. In this limit, $Z_2^{(3)}$ reduces to two copies of the purity of the reduced density matrix for subsystem A , i.e., $Z_2^{(3)} \rightarrow (\text{Tr}(\rho_A^2))^2$. Similarly, if instead subsystem A is one-dimensional, the horizontal indices i_A and j_A lose their meaning, and $Z_2^{(3)}$ reduces to two copies of the purity of ρ_B : $Z_2^{(3)} \rightarrow (\text{Tr}(\rho_B^2))^2$.

⁵This definition of multientropy follows [10], which is different from the definition [9] by a factor $1/n$. Thus, for generic $\mathfrak{q}$, there are factor $1/n^{\mathfrak{q}-2}$ difference [17]. In our previous paper [18], we define the multientropy without this $1/n^{\mathfrak{q}-2}$ factor.

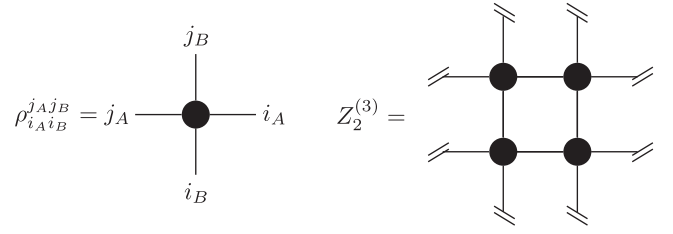


FIG. 2. Contraction pattern of a reduced density matrix $\rho = \text{Tr}_C |\psi\rangle\langle\psi|$ for $Z_2^{(3)}$.

Note that the purity-type contractions for subsystems A and B are taken along the horizontal and vertical directions, respectively. More generally, for a tripartite density matrix ρ_{ABC} , after tracing out subsystem C , we compute purity-type contractions for A along horizontal lines and for B along vertical lines. This procedure yields the $\mathfrak{q} = 3$ and $n = 2$ Rényi multientropy represented in Fig. 2.

This construction can be generalized to arbitrary $\mathfrak{q}$ and n : by performing n -fold purity-type contractions $\text{Tr}(\rho^n)$ along $\mathfrak{q} - 1$ mutually orthogonal directions, thus we have $n^{\mathfrak{q}-1}$ replicas and in this way, one obtains the Rényi multientropy $S_n^{(\mathfrak{q})}$.

The Rényi multientropy has the following properties [10,17].

- (i) $S_n^{(3)}(A:B:C)$ is invariant under local unitary transformations of $|\psi\rangle$.
- (ii) $S_n^{(3)}(A:B:C)$ is symmetric in all the parties of tripartite systems:

$$S_n^{(3)}(A:B:C) = S_n^{(3)}(A:C:B) = S_n^{(3)}(B:A:C) \\ = S_n^{(3)}(B:C:A) = S_n^{(3)}(C:A:B) = S_n^{(3)}(C:B:A). \quad (2.8)$$

- (iii) $S_n^{(3)}(A:B:C)$ is additive under tensor products of pure states. If $|\psi\rangle_{A_1 B_1 C_1}$ and $|\phi\rangle_{A_2 B_2 C_2}$ are pure states, $S_n^{(3)}(A_1 A_2 : B_1 B_2 : C_1 C_2)_{|\psi\rangle \otimes |\phi\rangle}$ of $|\psi\rangle_{A_1 B_1 C_1} \otimes |\phi\rangle_{A_2 B_2 C_2}$ is given by

$$S_n^{(3)}(A_1 A_2 : B_1 B_2 : C_1 C_2)_{|\psi\rangle \otimes |\phi\rangle} \\ = S_n^{(3)}(A_1 : B_1 : C_1)_{|\psi\rangle} + S_n^{(3)}(A_2 : B_2 : C_2)_{|\phi\rangle}. \quad (2.9)$$

These properties also hold for $(\mathfrak{q}, n)$ -Rényi multientropy in $\mathfrak{q}$ -partite systems. As we will see soon, the additive property given by (2.9) plays an important role in defining genuine multientropy.

As concrete examples, we consider $S_n^{(3)}$ of the W state and GHZ state in three-qubit systems, where the W state and GHZ state [5] are given by

$$|\text{GHZ}\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \quad (2.10)$$

$$|\text{W}\rangle := \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle). \quad (2.11)$$

For small $n = 2, 3, 4$, $S_n^{(3)}$ of the W state is evaluated in [9] but a generic n formula for Rényi multientropy is not known. They behave as follows:

$$S_{2|\text{W}}^{(3)} = \frac{1}{2} \log 9 = \log 3, \quad (2.12)$$

$$S_{3|\text{W}}^{(3)} = \frac{1}{6} \log \left(\frac{6561}{14} \right) \approx \frac{1}{6} \log \frac{3^8}{3^{2.4}} \approx 0.93 \log 3, \quad (2.13)$$

$$S_{4|\text{W}}^{(3)} = \frac{1}{12} \log \left(\frac{14348907}{139} \right) \approx \frac{1}{12} \log \frac{3^{15}}{3^{4.5}} \approx 0.88 \log 3, \quad (2.14)$$

$$S_{5|\text{W}}^{(3)} = \frac{1}{20} \log \left(\frac{94143178827}{847} \right) \approx \frac{1}{20} \log \frac{3^{23}}{3^{6.1}} \approx 0.84 \log 3, \quad (2.15)$$

$$S_{6|\text{W}}^{(3)} = \frac{1}{30} \log \left(\frac{50031545098999707}{87880} \right) \approx \frac{1}{30} \log \frac{3^{35}}{3^{10.4}} \approx 0.82 \log 3. \quad (2.16)$$

From these, we conjecture that

$$S_{1|\text{W}}^{(3)} = c_{\text{W}} \log 3, \quad \text{where } c_{\text{W}} > 1. \quad (2.17)$$

On the other hand, for the GHZ state, generic n dependence for Rényi multientropy is known

$$S_{2|\text{GHZ}}^{(3)} = \frac{3}{2} \log 2, \quad S_{3|\text{GHZ}}^{(3)} = \frac{4}{3} \log 2, \quad S_{4|\text{GHZ}}^{(3)} = \frac{5}{4} \log 2, \quad (2.18)$$

$$S_{n|\text{GHZ}}^{(3)} = \frac{1+n}{n} \log 2, \quad S_{|\text{GHZ}}^{(3)} = 2 \log 2. \quad (2.19)$$

As another example, consider a triangle state $|\Delta\rangle$ given by (2.1), which has no genuine tripartite entanglement. For such a triangle state, its Rényi multientropy $S_n^{(3)}(A:B:C)_{|\Delta\rangle}$ satisfies the following relationship [10]

$$S_n^{(3)}(A:B:C)_{|\Delta\rangle} = \frac{1}{2} (S_n^{(2)}(A)_{|\Delta\rangle} + S_n^{(2)}(B)_{|\Delta\rangle} + S_n^{(2)}(C)_{|\Delta\rangle}), \quad (2.20)$$

where

$$S_n^{(2)}(A)_{|\Delta\rangle} := \frac{1}{1-n} \log \text{Tr}_A \rho_A^n \quad (2.21)$$

is a Rényi entanglement entropy for reduced density matrix $\rho_A := \text{Tr}_{BC} |\Delta\rangle\langle\Delta|$. Thus, we also write $S_n^{(2)}(A:BC) = S_n^{(2)}(A)$, $S_n^{(2)}(B:CA) = S_n^{(2)}(B)$, $S_n^{(2)}(C:AB) = S_n^{(2)}(C)$. In particular, for three-qubit systems A, B, C , we obtain

$$S_n^{(3)}(A:B:C)_{|\Delta\rangle} \leq \log 2, \quad (2.22)$$

which is smaller than $S_{n|\text{GHZ}}^{(3)}$ if $n > 0$.

By comparing (2.16) and (2.18), one can see that $S_{n|\text{W}}^{(3)}$ is larger than $S_{n|\text{GHZ}}^{(3)}$ in the calculation examples above for $n \geq 2$. In addition, our numerical computations imply that $S_{2|\text{W}}^{(3)}$ is maximum among $S_2^{(3)}$ in three qubit systems. As a partial proof, in Appendix A, we show that $S_{2|\text{W}}^{(3)}$ is maximum among $S_2^{(3)}$ in the W class. Therefore, we conjecture that $S_{n|\text{W}}^{(3)}$ is maximum among $S_n^{(3)}$ in three qubit systems.

B. Genuine multientropy in tripartite systems

Generally, $S_n^{(3)}$ is nonzero for pure states with bipartite entanglement as seen in (2.20). In tripartite systems, there are two types of entanglement: bipartite entanglement between two subsystems and genuine tripartite entanglement between three subsystems [5].

For generic $\mathfrak{q}$, we define a genuine $\mathfrak{q}$ -partite multientropy from a $\mathfrak{q}$ -partite multientropy by subtracting appropriate linear combinations of $\tilde{\mathfrak{q}}$ -partite multientropies for $\tilde{\mathfrak{q}} < \mathfrak{q}$, in such a way that the genuine $\mathfrak{q}$ -partite multientropy vanishes for all $\tilde{\mathfrak{q}}$ -partite entangled states. Thus, for the $\mathfrak{q} = 3$ case, we define the genuine multientropy in such a way that it gives zero for pure states with only bipartite entanglement. Then the following linear combination works,

$$\begin{aligned} \text{GM}_n^{(3)}(A:B:C) &:= S_n^{(3)}(A:B:C) - \frac{1}{2} (S_n^{(2)}(A:BC) \\ &\quad + S_n^{(2)}(B:CA) + S_n^{(2)}(C:AB)). \end{aligned} \quad (2.23)$$

From (2.20), it immediately follows that $\text{GM}_n^{(3)}(A:B:C)$ of the triangle state $|\Delta\rangle$, which has only bipartite entanglement, is zero. We call $\text{GM}_n^{(3)}(A:B:C)$ genuine n -th Rényi multientropy ($\mathfrak{q} = 3$) in tripartite systems. As an independent work from us, in [19] the same quantity (2.23) is defined as κ_n and values on 2D CFTs are studied.

We define the genuine $\mathfrak{q} = 3$ multientropy $\text{GM}^{(3)}(A:B:C)$ by taking the $n \rightarrow 1$ limit as

$$\text{GM}^{(3)}(A:B:C) := \lim_{n \rightarrow 1} \text{GM}_n^{(3)}(A:B:C). \quad (2.24)$$

Later in Sec. IV, we prove that

$$\text{GM}^{(3)}(A:B:C) \geq 0 \quad (2.25)$$

using holographic duals.

The Rényi $n = 2$ case is interesting as well since in this case, the multientropy $S_{n=2}^{(3)}(A:B:C)$ essentially reduces to the reflected entropy. See Fig. 13 of [18] for the four replica contraction diagram. In this case, one can show that the genuine $n = 2$ Rényi multientropy ($\mathfrak{q} = 3$) is related to the Rényi Markov gap, which is the difference between the Rényi reflected entropy and the Rényi mutual entropy, as [19]

$$\text{GM}_{n=2}^{(3)}(A:B:C) \propto ((m = 2, n = 2) \text{ Rényi Markov gap}). \quad (2.26)$$

We will comment on this relation more in detail in the next section.

Note that both the Markov gap and the genuine multi-entropy $\mathfrak{q} = 3$ vanish for the states containing only bipartite entanglement. Thus for $\mathfrak{q} = 3$ tripartite case, their properties are expected to be similar in general. Next, we will comment on properties of the genuine $\mathfrak{q} = 3$ multi-entropy through explicit examples.

For the W state, one can obtain $\text{GM}_n^{(3)}$ for small n as follows,

$$\text{GM}_{2|W}^{(3)} = \log\left(\frac{5\sqrt{5}}{9}\right), \quad \text{GM}_{3|W}^{(3)} = \frac{1}{12} \log\left(\frac{2187}{196}\right), \quad (2.27)$$

$$\text{GM}_{4|W}^{(3)} = \frac{1}{12} \log\left(\frac{24137569}{2735937}\right),$$

$$\text{GM}_{5|W}^{(3)} = \frac{1}{40} \log\left(\frac{285311670611}{234365481}\right), \quad (2.28)$$

$$\text{GM}_{6|W}^{(3)} = \frac{1}{120} \log\left(\frac{72890483685103052142902866787761839379440139451}{725120831374733747546618320576719360000}\right), \quad (2.29)$$

using

$$S_{n|W}^{(2)} = \frac{\log((2^n + 1)/3^n)}{1 - n}, \quad \text{with} \quad S_{|W}^{(2)} = \log 3 - \frac{2}{3} \log 2. \quad (2.30)$$

All of $\text{GM}_n^{(3)}$ for $n = 2, 3, 4, 5, 6$ are positive. Finally, using (2.17) and (2.30), for the $n \rightarrow 1$ case we have

$$\text{GM}_{|W}^{(3)} = \log(23^{c_W - \frac{3}{2}}) > \log(23^{-\frac{1}{2}}) \approx 0.14 > 0 \quad \text{for } c_W > 1. \quad (2.31)$$

This is the new result as far as we are aware.

In contrast, $\text{GM}_n^{(3)}$ of the GHZ state is [19]

$$\text{GM}_{n|\text{GHZ}}^{(3)} = \left(\frac{1+n}{n} - \frac{3}{2}\right) \log 2. \quad (2.32)$$

Especially this means for $n \rightarrow 1$ and $n \rightarrow 2$ cases,

$$\text{GM}_{|\text{GHZ}}^{(3)} = \frac{1}{2} \log 2 > 0 (n \rightarrow 1), \quad (2.33)$$

$$\text{GM}_{2|\text{GHZ}}^{(3)} = 0 (n \rightarrow 2), \quad (2.34)$$

where genuine multientropy $\text{GM}_{|\text{GHZ}}^{(3)}$ is positive, but $\text{GM}_{n|\text{GHZ}}^{(3)}$ is negative if $n > 2$. Thus, even though $\text{GM}^{(3)}$

is expected to behave similarly to the Markov gap [6], there are important differences. One difference between the Markov gap and genuine multientropy $\text{GM}^{(3)}$ is that Markov gap of GHZ state is zero, but $\text{GM}^{(3)}$ of GHZ state is nonzero as (2.33). This means that the Markov gap cannot capture genuine tripartite entanglement in the GHZ state, on the other hand, $\text{GM}^{(3)}$ can capture it. Another note is that even though we have not yet proved it, it is likely that $c_W > 1$ in (2.17) from (2.12)–(2.16). This implies that for both the GHZ state and W state, $\text{GM}^{(3)} > 0$ as (2.31) and (2.33). This inequality is the same as (2.25); however the origin is totally different since there is no guarantee that the GHZ state or W state can be holographic.

So far, we focus on the $\mathfrak{q} = 3$ tripartite case. However similarly it is possible to define genuine multientropy for higher $\mathfrak{q}$ cases as well, using the same principles. We will show this more explicitly for the $\mathfrak{q} = 4$ case in Sec. V.

Other than the simple spin systems, several papers studied 2D CFT cases as in [10,11,20]. In particular, $\text{GM}_n^{(3)}(A:B:C)$ in 2D CFTs and a free fermion model on a 2D square lattice was studied in [19]. See also [21,22] for 2D CFT computations for some generalizations of the multientropy.

Before we proceed, we comment on other proposal for the genuine multipartite entanglement. One such proposal is the “L entropy” defined by [23]. The bipartite L-entropy l_{AB} was defined as

$$l_{AB} := \text{Min}\{2S^{(2)}(A), 2S^{(2)}(B)\} - S_R^{(1,1)}(A:B), \quad (2.35)$$

which is motivated by the inequality $S_R^{(1,1)}(A:B) \leq \text{Min}\{2S^{(2)}(A), 2S^{(2)}(B)\}$, just as the Markov gap was motivated by the inequality $I^{(1)}(A:B) \leq S_R^{(1,1)}(A:B)$. The tripartite L entropy is a product of three bipartite L entropies, namely l_{AB}, l_{BC}, l_{CA} . However, the L entropy does not satisfy the additive property (2.9) due to

$$\begin{aligned} & \text{Min}\{S^{(2)}(A_1), S^{(2)}(B_1)\} + \text{Min}\{S^{(2)}(A_2), S^{(2)}(B_2)\} \\ & \neq \text{Min}\{S^{(2)}(A_1) + S^{(2)}(A_2), S^{(2)}(B_1) + S^{(2)}(B_2)\}. \end{aligned} \quad (2.36)$$

The lack of additive property is crucial since this leads to the property that L entropy does not vanish for the triangle state (2.1). Thus L entropy does not measure the genuine tripartite entanglement in our criteria.

III. BLACK HOLE GENUINE MULTIENTROPY CURVE

As an application of the multientropy to a black hole evaporation, the black hole multientropy curve was introduced and studied by [18]. This is a natural generalization of the Page curve of entanglement entropy. The crucial point is that we approximate an evaporating black hole and its radiation with a Haar-random state for this purpose. This is based on the fact that through the gauge/gravity correspondence, asymptotic AdS quantum gravity is equivalent to ordinary quantum mechanical systems living on the boundary, and in general, such quantum systems are generally highly chaotic [24,25] at the deconfinement

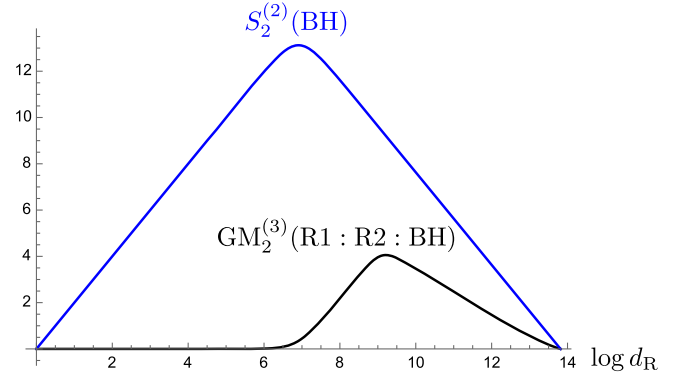


FIG. 3. Black curve: black hole Rényi multientropy curve of $\text{GM}_2^{(3)}(\text{R1}:\text{R2}:\text{BH})$. Blue curve: Page curve of $S_2^{(2)}(\text{BH})$. We fix the total dimension $d_{\text{Total}} = d_{\text{BH}}d_{\text{R}}^2 = 10^{12}$.

phase dual to a black hole dominant phase [26,27]. Since chaotic boundary states can be very well approximated by Haar-random states, it is reasonable to model the state of an evaporating black hole and its radiation using a boundary Haar-random state. In this section, similarly to the previous work, we study the genuine multientropy $\text{GM}^{(3)}(A:B:C)$ curve for the following tripartite systems: two Hawking radiation subsystems $A = \text{R1}$ and $B = \text{R2}$, and an evaporating black hole subsystem $C = \text{BH}$.

More concretely, we first compute the black hole genuine $n = 2$ Rényi tripartite entropy curve of $\text{GM}_2^{(3)}(\text{R1}:\text{R2}:\text{BH})$ by using a single random tensor model as in the original work of the Page curve [28,29]. The explicit expressions of Rényi multientropy are given by,⁶ [18]

$$S_2^{(3)}(\text{R1}:\text{R2}:\text{BH}) = -\frac{1}{2} \log \left[\frac{d_{\text{R}}^2 d_{\text{BH}} (9 + 2d_{\text{R}}^2 + d_{\text{BH}}^2) + 2(2d_{\text{R}}^2 + d_{\text{BH}}^2 + d_{\text{R}}^4 + 2d_{\text{R}}^2 d_{\text{BH}}^2)}{(1 + d_{\text{BH}} d_{\text{R}}^2)(2 + d_{\text{BH}} d_{\text{R}}^2)(3 + d_{\text{BH}} d_{\text{R}}^2)} \right], \quad (3.1)$$

$$S_2^{(2)}(\text{R1}) = S_2^{(2)}(\text{R2}) = -\log \left[\frac{d_{\text{R}} + d_{\text{R}} d_{\text{BH}}}{1 + d_{\text{BH}} d_{\text{R}}^2} \right], \quad S_2^{(2)}(\text{BH}) = -\log \left[\frac{d_{\text{BH}} + d_{\text{R}}^2}{1 + d_{\text{BH}} d_{\text{R}}^2} \right], \quad (3.2)$$

where we set $\dim \mathcal{H}_{\text{R1}} = \dim \mathcal{H}_{\text{R2}} = d_{\text{R}}$ and $\dim \mathcal{H}_{\text{BH}} = d_{\text{BH}}$. We also fix the dimension of total system

$$d_{\text{Total}} = d_{\text{BH}} d_{\text{R}}^2 = \text{fixed}. \quad (3.3)$$

By using $d_{\text{BH}} = d_{\text{Total}}/d_{\text{R}}^2$, we can plot $\text{GM}_2^{(3)}(\text{R1}:\text{R2}:\text{BH})$ as a function of d_{R} .

Figure 3 shows the black hole genuine $\mathfrak{q} = 3, n = 2$ Rényi multientropy curve of $\text{GM}_2^{(3)}(\text{R1}:\text{R2}:\text{BH})$. The black curve shows $\text{GM}_2^{(3)}(\text{R1}:\text{R2}:\text{BH})$ with $d_{\text{Total}} = 10^{12}$, where the horizontal axis is $\log d_{\text{R}}$. In the black hole $\mathfrak{q} = 3$ Rényi

multientropy curve, there is only one time scale, which is the multientropy time

$$d_{\text{R}} = d_{\text{BH}} = d_{\text{Total}}/d_{\text{R}}^2 \Leftrightarrow d_{\text{R}} = (d_{\text{Total}})^{1/3} (\text{multientropy time}). \quad (3.4)$$

For bipartite entanglement entropy, there is only one time scale, which is the Page time

⁶As explained in footnote 5 our definition of $S_2^{(3)}$ differs from the definition in [18] by a factor $1/2$.

$$d_R^2 = d_{\text{BH}} = d_{\text{Total}}/d_R^2 \Leftrightarrow d_R = (d_{\text{Total}})^{1/4}(\text{Page time}). \quad (3.5)$$

Since $\alpha = 3$ genuine multientropy is a linear combination of $\alpha = 3$ and $\alpha = 2$ multientropies, there are only two time scales for that, and one can see this in Fig. 3.

The black hole genuine $n = 2$ Rényi multientropy curve of $\text{GM}_2^{(3)}(\text{R1}:\text{R2}:\text{BH})$ is initially zero and starts to increase after the Page time. Then, $\text{GM}_2^{(3)}(\text{R1}:\text{R2}:\text{BH})$ starts to decrease from the multientropy time, and finally vanishes. In particular, we can see that the genuine $n = 2$ Rényi multientropy $\text{GM}_2^{(3)}(\text{R1}:\text{R2}:\text{BH})$ becomes nonzero and large *only after* the Page time. Thus, the Page time determines the time scale when it deviates from zero, and the multientropy time determines the time scale when it reaches maximum.

For the comparison of the magnitudes, we also plot the Page curve of $S_2^{(2)}(\text{BH})$ by the blue curve, which initially increases and starts to decrease from the Page time. One can see that even though the genuine tripartite multientropy contribution is smaller than the bipartite entanglement entropy, their magnitudes are similar in order. This implies that the genuine multipartite entanglement is not small.

Next, we study the $n \rightarrow 1$ limit. The asymptotic behaviors of $S_n^{(3)}(\text{R1}:\text{R2}:\text{BH})$ at $d_R \ll d_{\text{BH}}$ and $d_R \gg d_{\text{BH}}$ were studied by [18]. By using these expressions, we can plot the asymptotic behaviors in the black hole genuine multientropy curve of $\text{GM}^{(3)}(\text{R1}:\text{R2}:\text{BH})$ at $n \rightarrow 1$. Figure 4 shows the asymptotic behaviors of $\text{GM}^{(3)}(\text{R1}:\text{R2}:\text{BH})$ and $S_1^{(2)}(\text{BH})$. Since these are only the asymptotic behaviors, these plots are not smooth at the Page time and the multientropy time. Comparing Figs. 3 and 4, one can see that the qualitative behavior of $\text{GM}_n^{(3)}(\text{R1}:\text{R2}:\text{BH})$ is independent of n . This is due to the fact that the asymptotic behaviors (5.23) in [18] are not sensitive to n if we divide them by a factor $n^{\alpha-2}$ as explained in footnote 5. In particular, the two time scales, the Page time and the multientropy time, do not depend on n .

We also plot a similar curve of the Markov gap by using the study of reflected entropy in a single random tensor model by [30]. Rényi generalization of the Markov gap

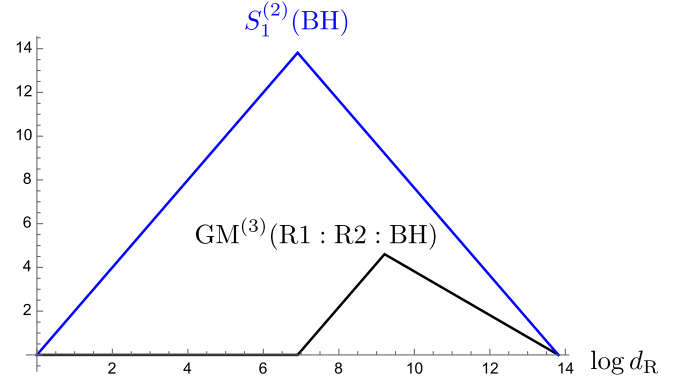


FIG. 4. Asymptotic behaviors of $\text{GM}^{(3)}(\text{R1}:\text{R2}:\text{BH})$ and $S_{n \rightarrow 1}^{(2)}(\text{BH})$.

$h^{m,n}(A:B)$ is defined by [19,31,32]

$$h^{m,n}(A:B) := S_R^{(m,n)}(A:B) - I^{(n)}(A:B), \quad (3.6)$$

where $I^{(n)}(A:B) := S_n^{(2)}(A) + S_n^{(2)}(B) - S_n^{(2)}(AB)$ is Rényi mutual information, and $S_R^{(m,n)}(A:B)$ is Rényi reflected entropy defined in [8] as

$$S_R^{(m,n)}(A:B) := \frac{1}{1-n} \log \frac{Z_{m,n}}{(Z_{m,1})^n}, \quad (3.7)$$

$$Z_{m,n} := \text{Tr}_{AA^*}(\text{Tr}_{BB^*} |\rho_{AB}^{m/2}\rangle \langle \rho_{AB}^{m/2}|)^n, \quad (3.8)$$

where $|\rho_{AB}^{m/2}\rangle$ is the canonical purified state of ρ_{AB}^m . Markov gap h is defined by

$$h := \lim_{m \rightarrow 1} \lim_{n \rightarrow 1} h^{m,n}(A:B). \quad (3.9)$$

Here, the reduced density matrix ρ_{AB} is defined by

$$\rho_{AB} := \text{Tr}_C |\psi\rangle \langle \psi|, \quad (3.10)$$

for a given pure state $|\psi\rangle$ on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. The explicit expression of $S_R^{(2,2)}(\text{R1}:\text{R2})$ for a random state $|\psi\rangle$ on $\mathcal{H}_{\text{R1}} \otimes \mathcal{H}_{\text{R2}} \otimes \mathcal{H}_{\text{BH}}$ is [18]

$$S_R^{(2,2)}(\text{R1}:\text{R2}) = -\log \frac{d_R^2 d_{\text{BH}} (9 + 2d_R^2 + d_{\text{BH}}^2) + 2(2d_R^2 + d_{\text{BH}}^2 + d_R^4 + 2d_R^2 d_{\text{BH}}^2)}{d_{\text{BH}}^3 d_R^2 + d_{\text{BH}}^2 (2d_R^4 + 4) + d_{\text{BH}} (d_R^6 + 10d_R^2) + 4d_R^4 + 2}, \quad (3.11)$$

where we set $\dim \mathcal{H}_{\text{R1}} = \dim \mathcal{H}_{\text{R2}} = d_R$ and $\dim \mathcal{H}_{\text{BH}} = d_{\text{BH}}$.

Figure 5 shows a curve of $h^{2,2}(\text{R1}:\text{R2})$ with the fixed total dimension $d_{\text{Total}} = d_{\text{BH}} d_R^2 = 10^{12}$. This plot has a similar behavior to $\text{GM}_2^{(3)}(\text{R1}:\text{R2}:\text{BH})$ in Fig. 3. In fact, there is a relation⁷ between $\text{GM}_2^{(3)}(A:B:C)$ and $h^{2,2}(A:B)$ [19]

⁷To derive the results in a single random tensor model such as Eqs. (3.1) and (3.11), an approximation that is valid when $d_{\text{Total}} = d_{\text{BH}} d_R^2 \gg 1$ was used.

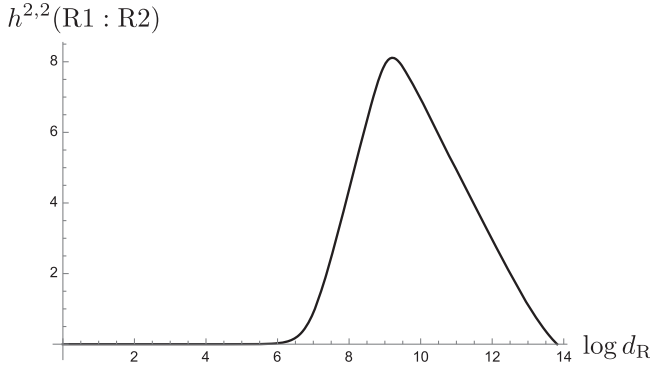


FIG. 5. Curve of $h^{2,2}(R1:R2)$ with the total dimension $d_{\text{Total}} = d_{\text{BH}}d_{\text{R}}^2 = 10^{12}$.

$$\text{GM}_2^{(3)}(A:B:C) = \frac{1}{2}h^{2,2}(A:B). \quad (3.12)$$

One difference between $\text{GM}_n^{(3)}(R1:R2:BH)$ and $h^{m,n}(R1:R2)$ is that the time scale of $h^{m,n}(R1:R2)$ to start decreasing depends on n . From the result (3.96) in [30], this time scale of $h^{1,n}(R1:R2)$ is given by

$$d_{\text{R}}^{2/n} = d_{\text{BH}} = d_{\text{Total}}/d_{\text{R}}^2 \Leftrightarrow d_{\text{R}} = (d_{\text{Total}})^{\frac{n}{2(1+n)}}, \quad (3.13)$$

which coincides the Page time when $n \rightarrow 1$. Therefore, there is the transition in the asymptotic behavior of the Markov gap $h(R1:R2)$ at the Page time as shown in Fig. 6.

Before we end, we summarize the characteristic natures of these curves.

- (1) It is always only after the Page time that this genuine multientropy becomes large. This means before the Page time, essentially there is no contribution by genuine tripartite multientropy, compared with the bipartite entanglement.
- (2) As Figs. 3 and 4 show, after the Page time, the amount of genuine tripartite multientropy is not parametrically small compared with the bipartite entanglement.
- (3) The genuine $\mathfrak{q} = 3$ multientropy curve Fig. 4 shows the peak at the multientropy time. On the other hand, the Markov gap Fig. 6 shows the peak at the Page time. Even though both capture the genuine tripartite entanglement, the peak is at different time scales. This is because they are different measures. Note that a multientropy treats all subsystems on equal footing, however, the Markov gap does not. However, both quantities show nonzero value only after the Page time.

Quite interestingly, the L entropy [23] also shows a very similar curve. See their Fig. 17. However, as we commented before, since L entropy does not vanish for the triangle state, their contributions are not necessary for

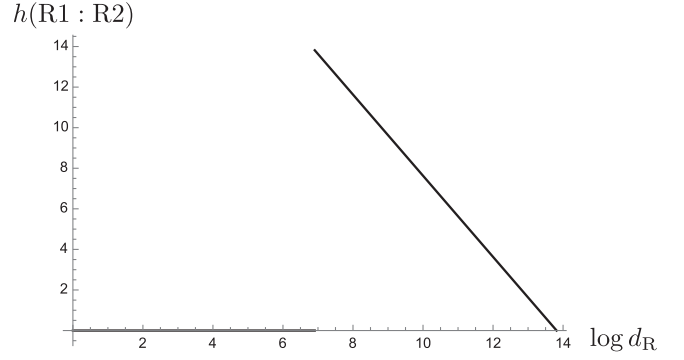


FIG. 6. Asymptotic behavior of Markov gap $h(R1:R2)$. There is the transition of asymptotic behavior at the Page time.

genuine tripartite entanglement. In fact, quite a similar curve can be obtained from the products of logarithmic negativity, see Appendix B for more detail. It is interesting to investigate the difference between L-entropy and genuine multipartite quantities in detail and see why Fig. 17 of [23] and Fig. 3 are similar.

IV. WHY BIPARTITE ENTANGLEMENT IS NOT ENOUGH FOR HOLOGRAPHY

In this section, we show the inequality (2.25) for $\text{GM}^{(3)}(A:B:C)$ holds in holographic settings.⁸ The proof is very simple. To show this, let's divide the boundary space into three regions/subsystems and let us call them A , B , C respectively. The holographic multientropy $S^{(3)}(A:B:C)$ is given by the sum of the three surfaces, γ_{xo} , γ_{yo} , and γ_{zo} [9,11], as

$$S^{(3)}(A:B:C) = \frac{1}{4G_N}(\gamma_{xo} + \gamma_{yo} + \gamma_{zo}) \quad (4.1)$$

where O is their meeting point. See the left figure of Fig. 7. In this section, we *assume* that holographic multientropy is given by the formula as (4.1).

On the other hand, the entanglement entropies S_A , S_B , S_C are given by the RT surfaces γ_A , γ_B , γ_C [1]. Thus,

$$S_A + S_B + S_C = \frac{1}{4G_N}(\gamma_A + \gamma_B + \gamma_C). \quad (4.2)$$

See the right figure of Fig. 7, where RT surfaces γ_A , γ_B , γ_C represent the minimal surface connecting x - y , y - z , and z - x through the bulk, respectively.

⁸After writing the draft, we learned that the similar proof is also shown in [20] for the $\mathfrak{q} = 3$ case.

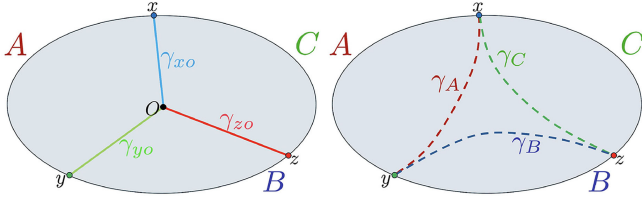


FIG. 7. Left: Holographic multientropy is given by the sum of the three surfaces, γ_{xo} , γ_{yo} , and γ_{zo} . Right: RT surfaces γ_A , γ_B , γ_C for the entanglement entropy S_A , S_B , S_C .

Thus, the genuine multientropy is always non-negative

$$\begin{aligned} S^{(3)}(A:B:C) - \frac{1}{2}(S_A + S_B + S_C) \\ = \frac{1}{8G_N}(\gamma_{xo} + \gamma_{yo} - \gamma_A) + \frac{1}{8G_N}(\gamma_{yo} + \gamma_{zo} - \gamma_B) \\ + \frac{1}{8G_N}(\gamma_{zo} + \gamma_{xo} - \gamma_C) > 0 \end{aligned} \quad (4.3)$$

since each parenthesis is always positive because RT-surfaces γ_A , γ_B , γ_C are *minimal* surfaces. See Fig. 8. The similar proof is also shown in [20].

This holographic proof is analogous to the one given for the Markov gap given by [6]. See their Fig. 3. In this way, even holographic-wise, genuine multientropy and the Markov gap share some similarities. However, there are also several differences. We summarize their analogies and differences here.

- (1) Both the Markov gap and genuine multientropy are measures for genuine multipartite entanglement in the sense that both quantities vanish for bipartite entangled states.
- (2) However for the GHZ state, which is one of the genuine multipartite states, there is a difference. The Markov gap vanishes for the GHZ state, on the other hand, genuine multientropy gives a positive value for the GHZ state.
- (3) The Markov gap is defined for tripartite states in the sense that the reflected entropy is defined by tracing C and leaving only subsystems A and B . There is no way to further divide the subsystem C into finer subsystems. On the other hand, since the multientropy is defined in a generic $\mathfrak{q}$ -partite system, it is possible to define a genuine multientropy even for

the $\mathfrak{q} \geq 4$ case as well. We will comment on this in more detail later in the next section.

In this way, both the genuine multientropy and the Markov gap give evidence that in holography genuine tripartite entanglement is not small.

So far, we have focused on the $\mathfrak{q} = 3$ case. However, this proof can be easily generalized for higher $\mathfrak{q}$ as well. The n -th Rényi multientropy of a pure state $|\psi\rangle$ on $\mathfrak{q}$ -partite subsystems $A_1, A_2, \dots, A_{\mathfrak{q}}$ is defined by [9,11,17]

$$S_n^{(\mathfrak{q})}(A_1:A_2:\dots:A_{\mathfrak{q}}) := \frac{1}{1-n} \frac{1}{n^{\mathfrak{q}-2}} \log \frac{Z_n^{(\mathfrak{q})}}{(Z_1^{(\mathfrak{q})})^{n^{\mathfrak{q}-1}}}, \quad (4.4)$$

$$Z_n^{(\mathfrak{q})} := \langle \psi |^{\otimes n^{\mathfrak{q}-1}} \Sigma_1(g_1) \Sigma_2(g_2) \cdots \Sigma_{\mathfrak{q}}(g_{\mathfrak{q}}) | \psi \rangle^{\otimes n^{\mathfrak{q}-1}}, \quad (4.5)$$

where $\Sigma_k(g_k)$ are twist operators for the permutation action of g_k on indices of density matrices for A_k . The action of g_k can be expressed as

$$\begin{aligned} g_k \cdot (x_1, \dots, x_k, \dots, x_{\mathfrak{q}-1}) &= (x_1, \dots, x_k + 1, \dots, x_{\mathfrak{q}-1}), \\ 1 \leq k &\leq \mathfrak{q} - 1, \end{aligned} \quad (4.6)$$

$$g_{\mathfrak{q}} = e, \quad (4.7)$$

where $(x_1, x_2, \dots, x_{\mathfrak{q}-1})$ represents an integer lattice point on a $(\mathfrak{q} - 1)$ -dimensional hypercube of length n , and we identify $x_k = n + 1$ with $x_k = 1$. The $\mathfrak{q}$ -partite multientropy $S^{(\mathfrak{q})}(A_1:A_2:\dots:A_{\mathfrak{q}})$ is defined by taking the $n \rightarrow 1$ limit as

$$S^{(\mathfrak{q})}(A_1:A_2:\dots:A_{\mathfrak{q}}) := \lim_{n \rightarrow 1} S_n^{(\mathfrak{q})}(A_1:A_2:\dots:A_{\mathfrak{q}}). \quad (4.8)$$

As a $\mathfrak{q} = 4$ example, let us consider the following combination of the $n \rightarrow 1$ limit,

$$\begin{aligned} S^{(4)}(A:B:C:D) - \frac{1}{2}(S^{(2)}(A:BCD) + S^{(2)}(B:CDA) \\ + S^{(2)}(C:DAB) + S^{(2)}(D:ABC)). \end{aligned} \quad (4.9)$$

Note that even though this combination excludes all of the bipartite contributions as $\text{GM}_n^{(3)}(A:B:C)$ in (2.23), it contains *both* genuine tripartite and quadripartite

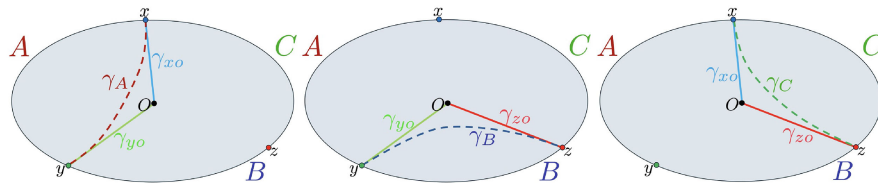


FIG. 8. Genuine multientropy $\mathfrak{q} = 3$ reduces to the subtraction of three minimal surfaces. However, due to the fact that γ_A , γ_B , γ_C are minimal surfaces, each parenthesis in (4.3) is always positive quantities. Thus genuine multientropy is positive.

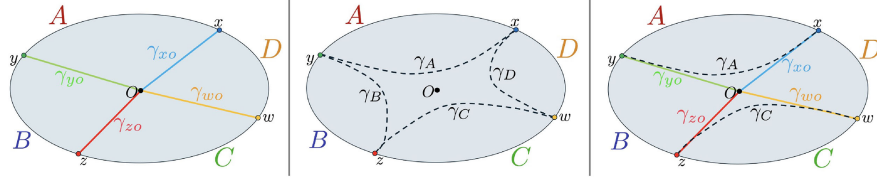


FIG. 9. Left: For the case where holographic multientropy is given by the sum of the four surfaces, γ_{xo} , γ_{yo} , γ_{zo} , and γ_{wo} . Middle: RT surfaces γ_A , γ_B , γ_C , γ_D for the entanglement entropy S_A , S_B , S_C , S_D . Right: (4.9) reduces to the summing over positive surfaces.

entanglement. Since it includes genuine tripartite entanglement, we will not call this combination a genuine quadripartite one.

It is straightforward to show this combination is always positive in holography

$$S^{(4)}(A:B:C:D) - \frac{1}{2}(S^{(2)}(A:BCD) + S^{(2)}(B:CDA) + S^{(2)}(C:DAB) + S^{(2)}(D:ABC)) > 0. \quad (4.10)$$

The proof is the same as the previous section. See Figs. 9 and 10. Again we *assume* that holographic multientropy is given by the formula as the left figures of Figs. 9 and 10 depending on the subregions A , B , C , D . A crucial assumption here is that holographic dual admits “intersection surfaces,” such as O in Fig. 9 and O_1 , O_2 in Fig. 10.

- (1) For the case where holographic multientropy is given by the sum of the four surfaces, γ_{xo} , γ_{yo} , γ_{zo} , and γ_{wo} as Fig. 9,

$$\begin{aligned} S^{(4)}(A:B:C:D) - \frac{1}{2}(S^{(2)}(A:BCD) + S^{(2)}(B:CDA) + S^{(2)}(C:DAB) + S^{(2)}(D:ABC)) \\ = \frac{1}{8G_N}(\gamma_{xo} + \gamma_{yo} - \gamma_A) + \frac{1}{8G_N}(\gamma_{yo} + \gamma_{zo} - \gamma_B) \\ + \frac{1}{8G_N}(\gamma_{zo} + \gamma_{wo} - \gamma_C) + \frac{1}{8G_N}(\gamma_{wo} + \gamma_{xo} - \gamma_D) > 0. \end{aligned} \quad (4.11)$$

- (2) For the case where holographic multientropy is given by the sum of the five surfaces, γ_{xo_2} , γ_{yo_1} , γ_{zo_1} , γ_{wo_2} , and $\gamma_{o_1o_2}$ as Fig. 10,

$$\begin{aligned} S^{(4)}(A:B:C:D) - \frac{1}{2}(S^{(2)}(A:BCD) + S^{(2)}(B:CDA) + S^{(2)}(C:DAB) + S^{(2)}(D:ABC)) \\ = \frac{1}{8G_N}(\gamma_{xo_2} + \gamma_{yo_1} - \gamma_A) + \frac{1}{8G_N}(\gamma_{yo_1} + \gamma_{zo_1} - \gamma_B) \\ + \frac{1}{8G_N}(\gamma_{zo_1} + \gamma_{wo_2} - \gamma_C) + \frac{1}{8G_N}(\gamma_{wo_2} + \gamma_{xo_2} - \gamma_D) > 0. \end{aligned} \quad (4.12)$$

$$\begin{aligned} &= \frac{1}{8G_N}(\gamma_{xo_2} + \gamma_{yo_1} + \gamma_{zo_1} - \gamma_A) \\ &+ \frac{1}{8G_N}(\gamma_{yo_1} + \gamma_{zo_1} - \gamma_B) \\ &+ \frac{1}{8G_N}(\gamma_{zo_1} + \gamma_{wo_2} + \gamma_{xo_2} - \gamma_C) \\ &+ \frac{1}{8G_N}(\gamma_{wo_2} + \gamma_{xo_2} - \gamma_D) > 0. \end{aligned} \quad (4.14)$$

Therefore, in any case,⁹ we obtain the inequality (4.10).

It is pretty straightforward to generalize this to general $\mathfrak{q}$, where we obtain the following inequality

$$S^{(\mathfrak{q})}(A_1:A_2:\dots:A_{\mathfrak{q}}) - \frac{1}{2}\left(\sum_{k=1}^{\mathfrak{q}} S^{(2)}(A_k)\right) > 0 \quad (4.15)$$

from holography, where $S^{(2)}(A_k)$ is bipartite entanglement entropy between subregion A_k and the rest. Here, we assume the bulk geometry is the dual of a pure state on the subsystems $A_1, A_2, \dots, A_{\mathfrak{q}}$, and the boundary subregions are all connected. Note that the left-hand side of (4.15) excludes all bipartite entanglement contributions. Thus, this generic $\mathfrak{q}$ inequality implies that in holography, bipartite entanglement is not enough.

So far we have assumed that the boundary subregions $A_1, A_2, \dots, A_{\mathfrak{q}}$ are connected subregions. It is straightforward to generalize these proofs to the cases where the boundary subregions are disconnected.¹⁰ However, for such cases, there is a possibility that in the bulk, the holographic dual does not admit “intersection surfaces,” such as O in Fig. 9 and O_1 , O_2 in Fig. 10. For example, see Fig. 11 in [20]. In such cases, there is a case that an inequality (4.15) can be saturated. For example, if the disconnected subregion A is much larger than subregions B and C as shown in the right figure of Fig. 11, the holographic multientropy and the holographic entanglement entropy are given by

⁹In fact, the configuration in Fig. 9 never becomes dominant.
¹⁰We thank Simon Lin for making this point.

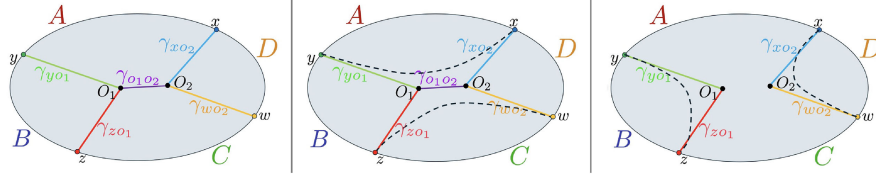


FIG. 10. Left: For the case where holographic multientropy is given by the sum of the five surfaces, $\gamma_{x o_2}$, $\gamma_{y o_1}$, $\gamma_{z o_1}$, $\gamma_{w o_2}$, and $\gamma_{o_1 o_2}$. Middle and Right: Subtracting the RT surfaces gives positive in both cases.

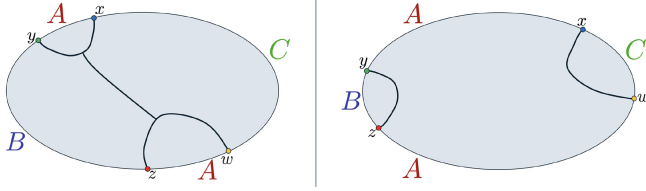


FIG. 11. For the case where holographic multientropy is given by surfaces admitting “intersection surfaces” as in the left figure, (4.17) is positive, but if it is not, as shown in the right figure, (4.17) is saturated.

Thus, if there are “intersection surfaces” in the bulk for a holographic dual of multientropy, inequality (4.15) holds, but if not, the inequality (4.15) can be saturated. Thus, we conclude

$$S^{(\mathfrak{q})}(A_1 : A_2 : \dots : A_{\mathfrak{q}}) - \frac{1}{2} \left(\sum_{k=1}^{\mathfrak{q}} S^{(2)}(A_k) \right) \geq 0. \quad (4.17)$$

$$\begin{aligned} S^{(3)}(A : B : C) &= S_A = \frac{1}{4G_N} (\gamma_B + \gamma_C), \\ S_B &= \frac{1}{4G_N} \gamma_B, \quad S_C = \frac{1}{4G_N} \gamma_C. \end{aligned} \quad (4.16)$$

The equality can hold only if there are no “intersection surfaces” in the bulk.

This argument is parallel to the case of the holographic Markov gap.¹¹ In holography, there are cases where the Markov gap becomes zero as well.

V. GENERIC $\mathfrak{q}$ GENUINE MULTIENTROPY

A. $\mathfrak{q}=4$ genuine multientropy

Similarly to $\mathfrak{q} = 3$, by taking appropriate linear combinations of multientropy, one can define genuine multientropy for the $\mathfrak{q} \geq 4$ case as well. In fact, for the $\mathfrak{q} = 4$ case, we can define the following linear combination:

$$\begin{aligned} \text{GM}_n^{(4)}(A : B : C : D) &:= S_n^{(4)}(A : B : C : D) - \frac{1}{3} (S_n^{(3)}(AB : C : D) + S_n^{(3)}(AC : B : D) \\ &\quad + S_n^{(3)}(AD : B : C) + S_n^{(3)}(BC : A : D) + S_n^{(3)}(BD : A : C) + S_n^{(3)}(CD : A : B)) \\ &\quad + a(S_n^{(2)}(AB : CD) + S_n^{(2)}(AC : BD) + S_n^{(2)}(AD : BC)) \\ &\quad + b(S_n^{(2)}(ABC : D) + S_n^{(2)}(ABD : C) + S_n^{(2)}(ACD : B) + S_n^{(2)}(BCD : A)), \end{aligned} \quad (5.1)$$

where $S_n^{(\mathfrak{q})}$ is $\mathfrak{q}$ -partite Rényi multientropy, and a and b are parameters satisfying

$$a + b = \frac{1}{3}. \quad (5.2)$$

This can also be expressed in terms of the genuine tripartite multientropies such as $\text{GM}_n^{(3)}(AB : C : D)$ for three subsystems AB , C , D defined in (2.23) as

¹¹See Fig. 3 in [6]. The holographic Markov gap is proportional to $(\text{KRT}(A) - \text{RT}(A)) + (\text{KRT}(B) - \text{RT}(B))$. The non-negativity of the geometrical quantities (as indicated in each parenthesis) is parallel to our argument.

$$\begin{aligned}
\text{GM}_n^{(4)}(A:B:C:D) &:= S_n^{(4)}(A:B:C:D) - \frac{1}{3}(\text{GM}_n^{(3)}(AB:C:D) + \text{GM}_n^{(3)}(AC:B:D) \\
&\quad + \text{GM}_n^{(3)}(AD:B:C) + \text{GM}_n^{(3)}(BC:A:D) + \text{GM}_n^{(3)}(BD:A:C) + \text{GM}_n^{(3)}(CD:A:B)) \\
&\quad + \tilde{a}(S_n^{(2)}(AB:CD) + S_n^{(2)}(AC:BD) + S_n^{(2)}(AD:BC)) \\
&\quad + \tilde{b}(S_n^{(2)}(ABC:D) + S_n^{(2)}(ABD:C) + S_n^{(2)}(ACD:B) + S_n^{(2)}(BCD:A)),
\end{aligned} \tag{5.3}$$

where

$$\tilde{a} = a - \frac{1}{3}, \quad \tilde{b} = b - \frac{1}{2}, \quad \tilde{a} + \tilde{b} = -\frac{1}{2}. \tag{5.4}$$

The claim is that this combination excludes the tripartite and bipartite entanglement and captures only the quadripartite entanglement. This means, $\text{GM}_n^{(4)}(A:B:C:D)$ is zero for pure states with tripartite or bipartite entanglement such as

$$|\psi_1\rangle_{ABC} \otimes |\psi_2\rangle_D, \quad |\psi_1\rangle_{AB} \otimes |\psi_2\rangle_{CD}. \tag{5.5}$$

This can be seen as follows.

(A) Consider a tripartite state, for example, $|\psi_1\rangle_{ABC} \otimes |\psi_2\rangle_D$. In this case, we do not need to contract the index of subsystem D , and the contraction of n^3 reduced density matrices for $S_n^{(4)}$ is factorized to n product of the contraction of n^2 reduced density matrices due to the definition of the multientropy. Then, the following identities hold

$$S_n^{(4)}(A:B:C:D) = S_n^{(3)}(AD:B:C) = S_n^{(3)}(BD:A:C) = S_n^{(3)}(CD:A:B), \tag{5.6}$$

$$S_n^{(3)}(AB:C:D) = S_n^{(2)}(AB:CD) = S_n^{(2)}(ABD:C), \tag{5.7}$$

$$S_n^{(3)}(AC:B:D) = S_n^{(2)}(AC:BD) = S_n^{(2)}(ACD:B), \tag{5.8}$$

$$S_n^{(3)}(BC:A:D) = S_n^{(2)}(AD:BC) = S_n^{(2)}(BCD:A), \tag{5.9}$$

$$S_n^{(2)}(ABC:D) = 0. \tag{5.10}$$

(5.6)–(5.9) give

$$S_n^{(4)}(A:B:C:D) - \frac{1}{3}(S_n^{(3)}(AD:B:C) + S_n^{(3)}(BD:A:C) + S_n^{(3)}(CD:A:B)) = 0, \tag{5.11}$$

$$-\frac{1}{3}S_n^{(3)}(AB:C:D) + aS_n^{(2)}(AB:CD) + bS_n^{(2)}(ABD:C) = 0, \tag{5.12}$$

$$-\frac{1}{3}S_n^{(3)}(AC:B:D) + aS_n^{(2)}(AC:BD) + bS_n^{(2)}(ACD:B) = 0, \tag{5.13}$$

$$-\frac{1}{3}S_n^{(3)}(BC:A:D) + aS_n^{(2)}(AD:BC) + bS_n^{(2)}(BCD:A) = 0. \tag{5.14}$$

Combining these with (5.10), one can see that $\text{GM}_n^{(4)}(A:B:C:D)$ given by (5.1) vanishes.

(B) Similarly, consider a bipartite state, for example, $|\psi_1\rangle_{AB} \otimes |\psi_2\rangle_{CD}$. Its contribution is decomposed into the sum of contributions of $|\psi_1\rangle_{AB}$ and $|\psi_2\rangle_{CD}$ due to the additive property (2.9). Thus, we consider only for a state $|\psi_1\rangle_{AB}$, where subsystems C and D are not entangled. Note that in this case, since genuine tripartite multientropies such as $\text{GM}_n^{(3)}(AC:B:D)$ do not contain any bipartite contributions, they vanish. In this case, the following identities hold

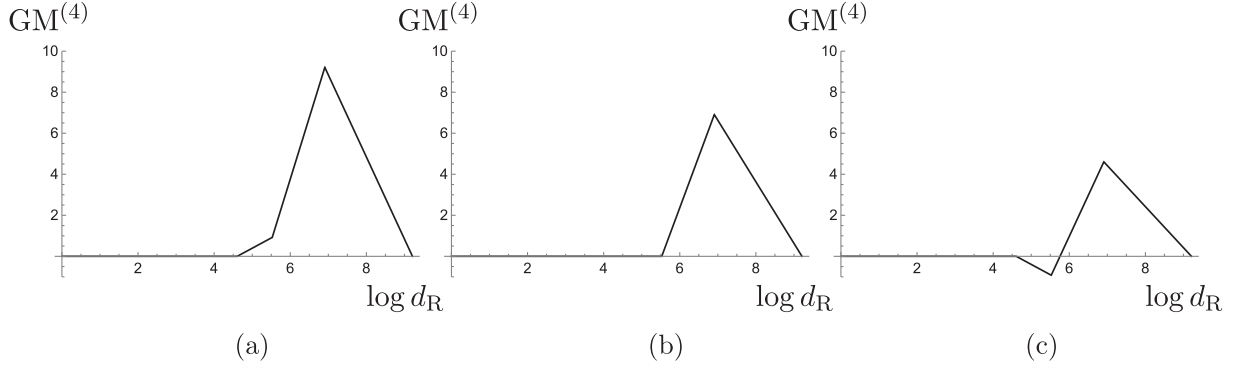


FIG. 12. Asymptotic behaviors of $\text{GM}^{(4)}(\text{R1}:\text{R2}:\text{R3}:\text{BH})$ for a Haar-random state with different values of a and b , where $\dim \mathcal{H}_{\text{R1}} = \dim \mathcal{H}_{\text{R2}} = \dim \mathcal{H}_{\text{R3}} = d_{\text{R}}$, $\dim \mathcal{H}_{\text{BH}} = d_{\text{BH}}$, and we fix $d_{\text{Total}} = d_{\text{BH}} d_{\text{R}}^3 = 10^{12}$. There are three time scales: (1) $d_{\text{R}}^3 = d_{\text{BH}}$, (2) $d_{\text{R}}^2 = d_{\text{BH}}$, and (3) $d_{\text{R}} = d_{\text{BH}}$. (a) $a = 1/2$, $b = -1/6$. (b) $a = 1/3$, $b = 0$. (c) $a = 1/6$, $b = 1/6$.

$$\begin{aligned} S_n^{(4)}(A:B:C:D) &= S_n^{(2)}(AC:BD) = S_n^{(2)}(AD:BC) \\ &= S_n^{(2)}(ACD:B) = S_n^{(2)}(BCD:A), \end{aligned} \quad (5.15)$$

$$S_n^{(2)}(AB:CD) = S_n^{(2)}(ABC:D) = S_n^{(2)}(ABD:C) = 0. \quad (5.16)$$

(5.15) can be written as

$$\begin{aligned} S_n^{(4)}(A:B:C:D) &+ \tilde{a}(S_n^{(2)}(AC:BD) + S_n^{(2)}(AD:BC)) \\ &+ \tilde{b}(S_n^{(2)}(ACD:B) + S_n^{(2)}(BCD:A)) = 0 \end{aligned} \quad (5.17)$$

Using this with (5.16), one can see that $\text{GM}_n^{(4)}(A:B:C:D)$ given by (5.3) vanishes.

Thus, $\text{GM}_n^{(4)}(A:B:C:D)$ for tensor products of states like (5.5) are zero.

B. $\mathfrak{q} = 4$ genuine multientropy curve

As a generalization of Fig. 4, we plot the black hole genuine multientropy curve of $\text{GM}^{(4)}(\text{R1}:\text{R2}:\text{R3}:\text{BH})$ in Fig. 12, where $\dim \mathcal{H}_{\text{R1}} = \dim \mathcal{H}_{\text{R2}} = \dim \mathcal{H}_{\text{R3}} = d_{\text{R}}$, $\dim \mathcal{H}_{\text{BH}} = d_{\text{BH}}$, and we fix $d_{\text{Total}} = d_{\text{BH}} d_{\text{R}}^3 = 10^{12}$. One

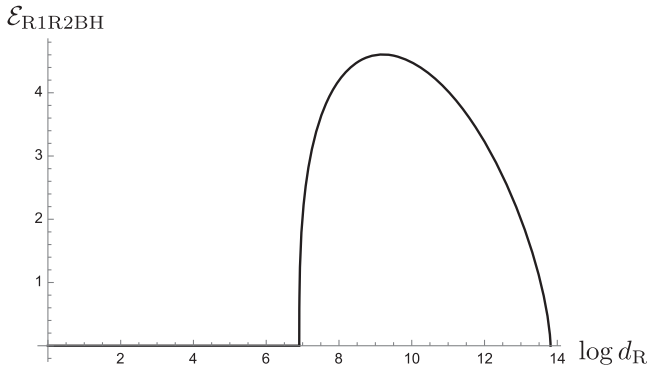


FIG. 13. Asymptotic behavior of tripartite logarithmic negativity $\mathcal{E}_{\text{R1R2BH}}$.

can see that $\text{GM}^{(4)}$ can be negative if b is positive. Thus, if we impose that $\text{GM}^{(4)}$ is non-negative, b should be non-positive. The important point we emphasize is that, as we have seen for the $\mathfrak{q} = 3$ case, $\mathfrak{q} = 4$ genuine multientropy is almost zero until the Page time, where $d_{\text{R}}^3 = d_{\text{BH}}$. They become important only after the Page time for whatever value of the parameter.

For comparison, in Appendix C, we present a detailed evaluation of the $\mathfrak{q}$ -partite information $I_{\mathfrak{q}}$ during black hole evaporation. For $\mathfrak{q} = 3$, I_3 fails to reflect the emergence of multipartite entanglement. In contrast, for $\mathfrak{q} = 4$, the behavior of $I_{\mathfrak{q}}$ —up to an overall sign—exhibits a marked increase only after the Page time, as illustrated in Fig. 14.

C. How to construct generic $\mathfrak{q}$ genuine multientropy

It is straightforward to generalize this construction for $\mathfrak{q} \geq 5$. The general prescription to construct the genuine $\mathfrak{q}$ -partite multientropy is the following.

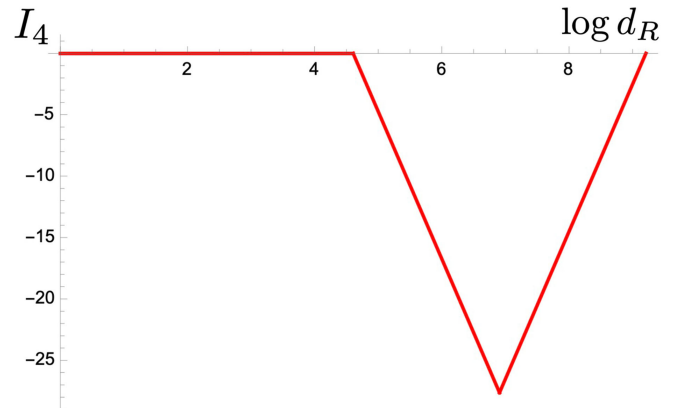


FIG. 14. Asymptotic behaviors of $I_{\mathfrak{q}=4}$ for a Haar-random state where $\dim \mathcal{H}_{\text{R1}} = \dim \mathcal{H}_{\text{R2}} = \dim \mathcal{H}_{\text{R3}} = d_{\text{R}}$, $\dim \mathcal{H}_{\text{BH}} = d_{\text{BH}}$, and we fix $d_{\text{Total}} = d_{\text{BH}} d_{\text{R}}^3 = 10^{12}$. There are two time scales: (1) The Page time: $d_{\text{R}}^3 = d_{\text{BH}}$, and (2) the multi-entropy time: $d_{\text{R}} = d_{\text{BH}}$.

- (1) First, consider a general expression for the linear combination of $S_n^{(\mathfrak{q})}$, $S_n^{(\mathfrak{q}-1)}$, ..., $S_n^{(2)}$ with symmetry in all the parties.
- (2) Then, determine each coefficient of the linear combination so that the genuine $\mathfrak{q}$ -partite multi-entropy vanishes for states with $(\mathfrak{q}-1)$ -partite entanglement such as

$$|\psi\rangle = |\psi_1\rangle_{A_1 A_2 \dots A_{\mathfrak{q}-1}} \otimes |\psi_2\rangle_{A_{\mathfrak{q}}}. \quad (5.18)$$

We emphasize that the symmetric property (2.8) and the additive property (2.9) are crucial in this construction. It is interesting to investigate this higher $\mathfrak{q}$ genuine multipartite entanglement in more detail from both CFTs and also holography. It is quite interesting to investigate more details for a higher $\mathfrak{q}$ case.

VI. DISCUSSIONS AND OPEN QUESTIONS

The central message of this paper is that one can define a genuine multientropy for any $\mathfrak{q} \geq 3$. Even though a multi-entropy $S^{(\mathfrak{q})}$ proposed in [9–11] is an interesting quantity that captures multipartite entanglement, it is not particularly useful on its own, since it also receives contributions from lower-partite $\tilde{\mathfrak{q}} < \mathfrak{q}$ entanglement. In other words, $S^{(\mathfrak{q})}$ can be nonzero even in the presence of only $\tilde{\mathfrak{q}}$ -partite entanglement.

To isolate the contribution from genuine $\mathfrak{q}$ -partite entanglement, we define the genuine multientropy, $\text{GM}^{(\mathfrak{q})}$, which vanishes unless genuine $\mathfrak{q}$ -partite entanglement is present. Intuitively, one can think of $\text{GM}^{(\mathfrak{q})}$ as an “irreducible representation decomposition,” separating out the distinct contributions from different levels of multipartite entanglement contained within $S^{(\mathfrak{q})}$.

The remainder of this paper can be summarized as follows:

- (1) Explicit construction of genuine multientropies: We construct explicitly $\mathfrak{q} = 3$ and $\mathfrak{q} = 4$ examples and give a concise yet general prescription on how to construct generic $\mathfrak{q}$ genuine multientropy for $\mathfrak{q} \geq 5$ cases.
- (2) Application to evaporating black holes: Applying our framework to the case of black hole evaporation, we compute the evolution of $\text{GM}^{(\mathfrak{q})}$ as a function of time for both $\mathfrak{q} = 3$ and $\mathfrak{q} = 4$ cases, extending the analysis of [18]. In this setup, the Hawking radiation is partitioned into $\mathfrak{q} - 1$ subsystems. We find that genuine multipartite entanglement becomes significant only after the Page time; before that, bipartite entanglement between the black hole and the radiation suffices to characterize the system.
- (3) Application to holography: We prove that the non-bipartite components of the multientropy are always

positive and of order $\mathcal{O}(1/G_N)$, as long as the boundary subregions are connected. This implies that genuine multipartite entanglement is non-negligible in holographic systems and plays an important role in their entanglement structure.

We conclude this paper with a discussion of open questions and future directions.

- (1) We conjectured that $c_W > 1$ in (2.17) by evaluating $S_n^{(3)}$ of the W-state for small n . Can we rigorously prove $c_W > 1$? And, in three qubit systems, what state gives the maximum value of multientropy and genuine tripartite multientropy?
- (2) By assuming the holographic prescription of multi-entropy [9,11], we argued the inequality (4.17). Does this inequality hold even for nonholographic CFTs or general spin systems? Or does it only work in special cases, like the monogamy of holographic mutual information [33]?
- (3) We assumed the holographic prescription of multi-entropy at $n \rightarrow 1$ [9,11]. However, as demonstrated by [10], in holographic CFTs, $S_n^{(3)}$ with $n \geq 3$ is not dual to the minimal triway cuts in the bulk, and thus the validity of holographic assumption at $n \rightarrow 1$ is unclear. At least, for pure states in RTN, the holographic assumption is plausible [10], and our geometrical proof could be applied.
- (4) Further detailed analysis on the genuine multientropy for $\mathfrak{q} \geq 4$ is needed for both holography and generic quantum states. We would like to report it in the near future.

Note added. After writing the draft, we learned that the $\mathfrak{q} = 3$ part of the results in Sec. IV overlaps with [20].

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: THE MAXIMUM VALUE OF $S_2^{(3)}$ IN THE W CLASS

Let us consider a pure state in the W class [5]

$$|\psi_W\rangle = \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle + \sqrt{d}|000\rangle, \quad (\text{A1})$$

where $a, b, c > 0$, and $d = 1 - a - b - c \geq 0$. We want to find the value of a, b, c, d to maximize $S_{2|\psi_W}^{(3)}$. To maximize $S_2^{(3)}$, the replica partition function $Z_2^{(3)}$ should be minimum. By using $c = 1 - a - b - d$, one can express $Z_2^{(3)}$ of $|\psi_W\rangle$ as

$$\begin{aligned} Z_2^{(3)} &= (2a^2 + 2a(b-1) + 2b^2 - 2b + 1)^2 \\ &\quad + d(4(2a^3 + 2a^2(-1+b) + a(1-2b+2b^2) + b(1-2b+2b^2))) + (4a^2 + 4b^2)d^2 \\ &=: f(a, b, d). \end{aligned} \quad (\text{A2})$$

Our goal is to find the minimum value of $f(a, b, d)$.

First, we show that $f(a, b, d)$ is minimum at $d = 0$ by analyzing

$$\frac{\partial f(a, b, d)}{\partial d} = -8a^2 + 8a^3 + 4b + 8a^2b - 8b^2 + 8b^3 + 4a(1-2b+2b^2) + 2(4a^2 + 4b^2)d. \quad (\text{A3})$$

One can check that $\frac{\partial f(a, b, d)}{\partial d} \geq 0$ for $a, b > 0$, and $d \geq 0$. Thus, $f(a, b, d)$ is minimum at $d = 0$. Note that $\frac{\partial f(a, b, d)}{\partial d} = 0$ at $a = b = 0$. In this case, $f(0, 0, d) = 1$.

Next, we consider

$$f(a, b, 0) = (2a^2 + 2a(b-1) + 2b^2 - 2b + 1)^2, \quad (\text{A4})$$

$$\frac{\partial f(a, b, 0)}{\partial b} = 4(-1 + a + 2b)(1 + 2a^2 + 2a(-1 + b) - 2b + 2b^2). \quad (\text{A5})$$

$\frac{\partial f(a, b, 0)}{\partial b}$ becomes zero at $b = (1-a)/2$, where $a + (1-a)/2 = (1+a)/2 \leq 1$ for $0 < a \leq 1$. There is no real solution of b for $1 + 2a^2 + 2a(-1 + b) - 2b + 2b^2 = 0$. We also obtain

$$\left. \frac{\partial^2 f(a, b, 0)}{\partial b^2} \right|_{b=(1-a)/2} = 4 - 8a + 12a^2 > 0. \quad (\text{A6})$$

Thus, $f(a, b, 0)$ is minimum at $b = (1-a)/2$.

Finally, we consider

$$f(a, (1-a)/2, 0) = 1/4(1 - 2a + 3a^2)^2. \quad (\text{A7})$$

Its minimum value is $1/9$ at $a = 1/3$. Therefore, the minimum value of $Z_2^{(3)}$ is $1/9$ at $a = 1/3, b = (1-a)/2 = 1/3, c = 1 - a - b - d = 1/3, d = 0$. This state is the W state

$$|W\rangle = \frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle, \quad (\text{A8})$$

and the maximum value of $S_2^{(3)}$ in the W class is $S_{2|W}^{(3)} = \frac{1}{2} \log 9$.

APPENDIX B: L ENTROPY AND TRIPARTITE LOGARITHMIC NEGATIVITY

A curve of L entropy, which has been proposed by [23], also has a similar behavior to genuine multi-entropy as in Fig. 3. L-entropy l_{ABC} in tri-partite systems is defined by

$$l_{ABC} := [l_{AB}l_{BC}l_{CA}]^{1/3}, \quad (\text{B1})$$

$$l_{AB} := \text{Min}\{2S^{(2)}(A), 2S^{(2)}(B)\} - S_R^{(1,1)}(A:B). \quad (\text{B2})$$

Assuming the holographic correspondence, a curve of L entropy was computed by the bulk area in a three-boundary wormhole as shown in Fig. 17 of [23], which is similar to our black hole multi-entropy curve of genuine multi-entropy.

Note that l_{AB} does not have the additive property (2.9), and thus l_{ABC} of a triangle state $|\Delta\rangle_{ABC} = |\psi_1\rangle_{A_R B_L} |\psi_2\rangle_{B_R C_L} |\psi_3\rangle_{C_R A_L}$ can be nonzero. To see it, let us compute l_{AB} of $|\Delta\rangle_{ABC}$. By using the additive property of $S^{(2)}$ and $S_R^{(1,1)}$, and

$$S_R^{(1,1)}(A:B)_{|\psi_1\rangle_{A_R B_L}} = 2S^{(2)}(A)_{|\psi_1\rangle_{A_R B_L}} = 2S^{(2)}(B)_{|\psi_1\rangle_{A_R B_L}}, \quad (\text{B3})$$

$$S_R^{(1,1)}(A:B)_{|\psi_2\rangle_{B_R C_L}} = S_R^{(1,1)}(A:B)_{|\psi_3\rangle_{C_R A_L}} = 0, \quad (\text{B4})$$

we obtain

$$\begin{aligned} I_{AB|\Delta} &= \text{Min} \left\{ 2S^{(2)}(A)_{|\psi_1\rangle_{A_R B_L}} + 2S^{(2)}(A)_{|\psi_3\rangle_{C_R A_L}}, 2S^{(2)}(B)_{|\psi_1\rangle_{A_R B_L}} + 2S^{(2)}(B)_{|\psi_2\rangle_{B_R C_L}} \right\} - S_R^{(1,1)}(A:B)_{|\psi_1\rangle_{A_R B_L}} \\ &= \text{Min} \left\{ 2S^{(2)}(A)_{|\psi_3\rangle_{C_R A_L}}, 2S^{(2)}(B)_{|\psi_2\rangle_{B_R C_L}} \right\}, \end{aligned} \quad (\text{B5})$$

which can be nonzero. Similarly,

$$I_{BC|\Delta} = \text{Min} \left\{ 2S^{(2)}(B)_{|\psi_1\rangle_{A_R B_L}}, 2S^{(2)}(C)_{|\psi_3\rangle_{C_R A_L}} \right\}, \quad (\text{B6})$$

$$I_{CA|\Delta} = \text{Min} \left\{ 2S^{(2)}(C)_{|\psi_2\rangle_{B_R C_L}}, 2S^{(2)}(A)_{|\psi_1\rangle_{A_R B_L}} \right\}, \quad (\text{B7})$$

can be also nonzero. Therefore, the tri-partite L-entropy I_{ABC} (B1) of the triangle state $|\Delta\rangle_{ABC} = |\psi_1\rangle_{A_R B_L} |\psi_2\rangle_{B_R C_L} |\psi_3\rangle_{C_R A_L}$ can be nonzero.

By using the negativity [34], the tripartite negativity was defined by [35]. In a similar way, we define tripartite logarithmic negativity $\mathcal{E}_{ABC}$ as follows.

$$\mathcal{E}_{ABC} := [\mathcal{E}_{AB} \mathcal{E}_{BC} \mathcal{E}_{CA}]^{1/3}, \quad (\text{B8})$$

where $\mathcal{E}_{AB}$ is the logarithmic negativity defined by

$$\mathcal{E}_{AB} := \log \|\rho_{AB}^{T_A}\|_1, \quad (\text{B9})$$

where T_A means the partial transpose, and $\|\cdots\|_1$ is the trace norm. The asymptotic behavior of $\mathcal{E}_{AB}$ in a single random tensor model was derived by [36–38]. By using their results, we plot a curve of $\mathcal{E}_{R_1 R_2 \text{BH}}$ with the total dimension $d_{\text{Total}} = d_{\text{BH}} d_R^2 = 10^{12}$ in Fig. 13, where the behavior is qualitatively similar to Fig. 17 of [23] for L-entropy $I_{R_1 R_2 B}$.

APPENDIX C: $\mathfrak{q}$ -PARTITE INFORMATION $I_{\mathfrak{q}}$ CURVES FOR BLACK HOLE EVAPORATION

1. Setup for multipartite systems

In this appendix, we study the $\mathfrak{q}$ -partite information $I_{\mathfrak{q}}$, which is defined for $\mathfrak{q}$ subsystems as follows:

$$I_{\mathfrak{q}}(R_1, \dots, R_{\mathfrak{q}-1}, \text{BH}) := - \sum_{\sigma \subseteq \{R_1, \dots, R_{\mathfrak{q}-1}, \text{BH}\}} (-1)^{|\sigma|} S(\sigma), \quad (\text{C1})$$

where the sum is taken over all subsets σ of $\{R_1, \dots, R_{\mathfrak{q}-1}, \text{BH}\}$, and $|\sigma|$ denotes the number of elements in the subset σ . This particular linear combination of entanglement entropies is known to cancel all ultraviolet (UV) divergences [33]. Note that despite being termed $\mathfrak{q}$ -partite information [39,40], it is constructed as a linear combination of bipartite entanglement entropies. The total Hilbert space is given by

$$\mathcal{H}_{\text{total}} = \mathcal{H}_{R_1} \otimes \mathcal{H}_{R_2} \otimes \cdots \otimes \mathcal{H}_{R_{\mathfrak{q}-1}} \otimes \mathcal{H}_{\text{BH}}. \quad (\text{C2})$$

This setup is motivated by scenarios in black hole evaporation where Hawking radiation is approximately isotropic. In such cases, it is natural to divide the radiation into $\mathfrak{q} - 1$ symmetric subsystems, as studied for instance in [18]. Due to the symmetry, we assume the following:

$$\dim(\mathcal{H}_{R_1}) = \cdots = \dim(\mathcal{H}_{R_{\mathfrak{q}-1}}) = d_R, \quad \dim(\mathcal{H}_{\text{BH}}) = d_{\text{BH}}, \quad (\text{C3})$$

so that the total dimension of the Hilbert space is

$$d_{\text{total}} = (d_R)^{\mathfrak{q}-1} d_{\text{BH}}. \quad (\text{C4})$$

In our analysis, we keep d_{total} fixed while varying d_R and d_{BH} accordingly.

We also assume that the entropy S_X of a subsystem X is determined only by its dimension d_X , and follows the typical form for reduced subsystems in random pure states,

$$S_X = \min [\log d_X, \log d_{\text{total}} - \log d_X]. \quad (\text{C5})$$

This form captures the behavior of the entanglement entropy in typical pure states and reflects the Page curvelike symmetry between a subsystem and its complement [28].

Thus, we have the following properties:

(i) The total system is pure, thus

$$S_{R_1 \dots R_{\mathfrak{q}-1} \text{BH}} = 0. \quad (\text{C6})$$

(ii) By symmetry between radiation subsystems,

$$S_{R_1} = \cdots = S_{R_{\mathfrak{q}-1}}, \quad S_{R_1 \text{BH}} = \cdots = S_{R_{\mathfrak{q}-1} \text{BH}}, \text{ etc.} \quad (\text{C7})$$

2. $\mathfrak{q}=3$ case: Tripartite information I_3

First, let us consider the $\mathfrak{q}=3$ case. The tripartite information is

$$I_3(R_1:R_2:BH) = S_{R_1} + S_{R_2} + S_{BH} - S_{R_1R_2} - S_{R_1BH} - S_{R_2BH} + S_{R_1R_2BH}. \quad (C8)$$

Using (C6) and (C7), the expression simplifies to:

$$I_3 = 2S_{R_1} + S_{BH} - S_{R_1R_2} - 2S_{R_1BH}. \quad (C9)$$

We now evaluate each term using (C4), i.e., $d_{\text{total}} = d_R^2 d_{BH}$ and (C5).

(1) Entropy of subsystem R_1 ,

$$S_{R_1} = \min[\log d_R, \log d_{\text{total}} - \log d_R] = \min[\log d_R, \log d_R + \log d_{BH}] = \log d_R \quad (C10)$$

since $\log d_{BH} \geq 0$.

(2) Entropy of subsystem BH ,

$$S_{BH} = \min[\log d_{BH}, \log d_{\text{total}} - \log d_{BH}] = \min[\log d_{BH}, 2 \log d_R] \quad (C11)$$

(3) Entropy of subsystem R_1R_2 ,

$$S_{R_1R_2} = \min[2 \log d_R, \log d_{\text{total}} - 2 \log d_R] = \min[2 \log d_R, \log d_{BH}] = S_{BH} \quad (C12)$$

due to (C11), or equivalently, since the total system is pure.

(4) Entropy of subsystem R_1BH ,

$$\begin{aligned} S_{R_1BH} &= \min[\log d_R + \log d_{BH}, \log d_{\text{total}} - \log d_R - \log d_{BH}] \\ &= \min[\log d_R + \log d_{BH}, \log d_R] = \log d_R \end{aligned} \quad (C13)$$

since $\log d_{BH} \geq 0$. Then from (C10), we have $S_{R_1BH} = S_{R_1} (= S_{R_2})$, which is consistent since the total system is pure. Thus, substituting all into (C9), we find,

$$I_3 = 0 \quad (C14)$$

Therefore, the tripartite information I_3 always vanishes in this symmetric setup.

3. $q=4$ case: Quadripartite information I_4

Similarly, let us consider the $q=4$ case. Using the symmetry (C7), the quadripartite information reduces to

$$I_4(R_1:R_2:R_3:BH) = \sum_{i=1}^3 S_{R_i} + S_{BH} - \sum_{1 \leq i < j \leq 3} S_{R_iR_j} - \sum_{i=1}^3 S_{R_iBH} + \sum_{1 \leq i < j \leq 3} S_{R_iR_jBH} + S_{R_1R_2R_3} - S_{R_1R_2R_3BH} \quad (C15)$$

$$= 3S_{R_1} + S_{BH} - 3S_{R_1R_2} - 3S_{R_1BH} + 3S_{R_1R_2BH} + S_{R_1R_2R_3}. \quad (C16)$$

Here we use (C6) and (C7). Furthermore, the fact that the total system is pure combined with the symmetry (C6) implies

$$S_{R_1} = S_{R_3} = S_{R_1R_2BH}, \quad S_{BH} = S_{R_1R_2R_3}, \quad S_{R_1BH} = S_{R_3BH} = S_{R_1R_2}, \quad \text{etc.} \quad (C17)$$

These furthermore simplify (C16) as

$$I_4(R_1:R_2:R_3:BH) = 6S_{R_1} + 2S_{BH} - 6S_{R_1BH}. \quad (C18)$$

We now evaluate each term using (C4), i.e., $d_{\text{total}} = d_R^3 d_{BH}$ and (C5).

(1) Entropy of subsystem R_1 ,

$$S_{R_1} = \min[\log d_R, \log d_{\text{total}} - \log d_R] = \min[\log d_R, 2 \log d_R + \log d_{\text{BH}}] = \log d_R \quad (\text{C19})$$

(since $\log d_R \geq 0, \log d_{\text{BH}} \geq 0$).

(2) Entropy of subsystem BH,

$$\begin{aligned} S_{\text{BH}} &= \min[\log d_{\text{BH}}, \log d_{\text{total}} - \log d_{\text{BH}}] = \min[\log d_{\text{BH}}, 3 \log d_R] \\ &= \begin{cases} 3 \log d_R & \text{Before the Page time: when } d_{\text{BH}} \geq d_R^3 \\ \log d_{\text{BH}} & \text{After the Page time: when } d_{\text{BH}} \leq d_R^3. \end{cases} \end{aligned} \quad (\text{C20})$$

The behavior changes at the Page time, defined by the condition $d_{\text{BH}} = d_R^3$.

(3) Entropy of subsystem $R_1\text{BH}$,

$$S_{R_1\text{BH}} = \min[\log d_R + \log d_{\text{BH}}, 2 \log d_R] \quad (\text{C21})$$

$$= \begin{cases} 2 \log d_R & \text{Before the multientropy time: when } d_{\text{BH}} \geq d_R \\ \log d_R + \log d_{\text{BH}} & \text{After the multientropy time: when } d_{\text{BH}} \leq d_R. \end{cases} \quad (\text{C22})$$

The behavior changes at the *multientropy time*, as defined in [18],¹² which occurs when $d_R = d_{\text{BH}}$, i.e., when all subsystems have the same Hilbert space dimension.

Thus, we find: (C18) reduces to

$$I_4 = \begin{cases} 0 & \text{Before the Page time: when } d_{\text{BH}} \geq d_R^3 > d_R \\ 2(\log d_{\text{BH}} - 3 \log d_R) & \text{After the Page time but before the multientropy time:} \\ & \text{when } d_R < d_{\text{BH}} \leq d_R^3 \\ -4 \log d_{\text{BH}} & \text{After the multientropy time (near the end of evaporation):} \\ & \text{when } d_{\text{BH}} \leq d_R < d_R^3. \end{cases}$$

¹²The multientropy time is independent of α .

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