

Modular S_3 flavored Pati-Salam model with a two family seesaw mechanism

A. E. Cárcamo Hernández^{*}

*Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile;
Centro Científico-Tecnológico de Valparaíso, Casilla 110-V, Valparaíso, Chile;
and Millennium Institute for Subatomic Physics at the High-Energy Frontier, SAPHIR,
Calle Fernández Concha No 700, Santiago, Chile*

Ivo de Medeiros Varzielas[†]

*CFTP, Departamento de Física, Instituto Superior Técnico, Universidade de Lisboa,
Avenida Rovisco Pais 1, 1049 Lisboa, Portugal*

S. F. King[‡]

School of Physics and Astronomy, University of Southampton, SO17 1BJ Southampton, United Kingdom

Vishnudath K. N.[§]

*Department of Physics, School of Advanced Sciences, Vellore Institute of Technology,
Vellore 632014, Tamilnadu, India*



(Received 11 August 2025; accepted 24 September 2025; published 24 November 2025; corrected 16 June 2026)

We present a unified model of quarks and leptons with modular S_3 flavor symmetry, where the two lightest family masses are naturally suppressed via a Pati-Salam version of the type I seesaw mechanism, mediated through heavier vectorlike fermions. Majorana neutrino masses are further suppressed through a double seesaw mechanism. The viable parameter space has a preferred range of the modulus field with $\text{Im}(\tau) \sim 2$, leading to successful fermion masses and mixing. The prediction for neutrinoless double beta decay is partly within the reach of the nEXO experiment. In particular, the Dirac CP violating neutrino oscillation phase is predicted to lie in the range $\delta_{CP}^\nu \sim 270^\circ - 360^\circ$.

DOI: [10.1103/PhysRevD.112.095042](https://doi.org/10.1103/PhysRevD.112.095042)

I. INTRODUCTION

Despite its remarkable agreement with experimental data, the current theory of strong and electroweak interactions—the Standard Model (SM) of particle physics—lacks an underlying mechanism to explain the strong hierarchy in the masses of elementary charged fermions. Additionally, the different mixing patterns in the quark and lepton sectors remain unexplained within the SM. The theory also fails to account for several issues, such as the tiny masses of active neutrinos and the origin of parity violation in the electroweak interaction, whose basic V-A nature is introduced by hand in the formulation of the SM. This has motivated the

development of several new physics models that aim to explain some or all of these unresolved issues.

Recently, the use of modular symmetries in extensions of the SM as a way of explaining the observed pattern of SM fermion masses and mixing angles has received a lot of attention from the theoretical particle physics community. See, for instance, [1–18]. Models based on discrete flavor symmetry, along with modular symmetry, do not include flavon fields in the particle spectrum excepting the modulus τ , thus making the scalar sector of these theories more minimal than that of models not having modular symmetries. When the complex modulus τ acquires a non-vanishing vacuum expectation value (v_{ev}), the flavor symmetry is spontaneously broken. Theories with modular flavor symmetries do not require the implementation of a mechanism responsible for the vacuum alignment; instead they need a mechanism to determine the modulus τ , which however we shall not address here. However, in modular flavor models, the Yukawa couplings depend on the modular forms, which are holomorphic functions of τ [1], which may thus be determined phenomenologically. For example, such models have been proposed with Pati-Salam unification together with A_4 modular symmetry [15].

^{*}Contact author: antonio.carcamo@usm.cl

[†]Contact author: ivo.de@udo.edu

[‡]Contact author: king@soton.ac.uk

[§]Contact author: vishnudath.kn@vit.ac.in

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

The lightness of the first two families is not fully addressed in many such approaches, and to remedy this the weighton mechanism has been proposed [19], together with other strategies [2]. Here we shall follow a different path, motivated by the type I seesaw mechanism [20–25], in order to explain the smallness of the first and second family masses.

In this paper, in order to address the SM fermion flavor puzzle and to provide the dynamical origin of the parity violation of the electroweak interactions, we propose a minimal modular model based on the smallest quark-lepton unified symmetry, the Pati-Salam $SU(4)_C \times SU(2)_L \times SU(2)_R$ gauge group [26], and the smallest modular symmetry, S_3 . The modular S_3 symmetry is perfect for impa two-family seesaw mechanism, since it admits doublet representations. The masses of the third family of SM charged fermion arise from renormalizable Yukawa interactions involving a colorless scalar bidoublet as well as a bidoublet scalar in the adjoint representation of $SU(4)_C$. The Dirac masses of the first and second families (including neutrinos) arise from a generalized version of the type I seesaw mechanism, but applied to both charged and neutral Dirac masses [27–31]. In our proposed model, the tiny active Majorana neutrino masses then arise from a double seesaw mechanism. The model is shown to describe all quark and lepton (including neutrino) masses and mixing angles, in terms of high energy mass scales, together with complex dimensionless Yukawa coefficients which are all of order unity, and a single complex modulus field τ with $\text{Im}(\tau) \sim 2$.

In Sec. II, we present the details of the model, followed by a numerical study of the input parameters leading to viable observables in Sec. III. Section IV summarizes our findings. A review of modular flavor symmetry can be found in the Appendix.

II. THE MODEL

We propose an extended Pati-Salam theory where the $SU(4)_C \times SU(2)_L \times SU(2)_R$ gauge symmetry is supplemented by an S_3 modular symmetry. The masses of the third-generation SM charged fermions arise from Yukawa interactions involving the scalar bidoublets Φ and Σ , which transform as singlet and adjoint representations of $SU(4)_C$, respectively.

The field content is enlarged by the inclusion of heavy vector-like fermions and right-handed Majorana neutrinos, required for the implementation of the tree level two family seesaw mechanism that yields the masses of the first and second generation of SM charged fermions as well as the Double Seesaw mechanism that produces the tiny masses of the light active neutrinos. Specifically, we have vectorlike fermions Ψ_n and Ψ_n^c ($n = 1, 2$) transforming as $(\mathbf{4}, \mathbf{1}, \mathbf{2})$ and $(\bar{\mathbf{4}}, \mathbf{1}, \bar{\mathbf{2}})$, respectively, under the Pati-Salam group. The vectorlike fermions Ψ_n and Ψ_n^c ($n = 1, 2$) are the seesaw messengers which mix with the SM fermionic multiplet fields F_i and F_i^c ($i = 1, 2, 3$), also transforming as

$(\mathbf{4}, \mathbf{1}, \mathbf{2})$ and $(\bar{\mathbf{4}}, \mathbf{1}, \bar{\mathbf{2}})$, respectively, under the Pati-Salam group. Such mixing between SM fermions and the seesaw messengers occurs due to the Yukawa interactions involving the singlet scalar fields σ_n ($n = 1, 2$) as well as the $SU(4)_C$ adjoint scalars Ξ_1 and Ξ_2 . In addition, we include three Majorana neutrinos, S_i ($i = 1, 2, 3$), which are singlets under the $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$ group, in order to implement the double seesaw mechanism for the generation of light active neutrino masses. The full symmetry $\mathcal{G}$ of our model features the following spontaneous breaking pattern:

$$\begin{aligned} \mathcal{G} &= SU(4)_C \otimes SU(2)_L \otimes SU(2)_R \otimes S_3 \\ &\Downarrow \Lambda_{PS} \\ &SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L} \\ &\Downarrow v_R \\ &SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \\ &\Downarrow v \\ &SU(3)_C \otimes U(1)_Q, \end{aligned} \quad (1)$$

where $v = 246$ GeV, and it is assumed that the Pati-Salam gauge symmetry is broken at the scale $\Lambda_{PS} \gtrsim 10^6$ GeV, which arises from the experimental bound on the branching ratio for the rare meson decays $K_L^0 \rightarrow \mu^\pm e^\mp$ mediated by the vector leptoquarks, as indicated in Refs. [32,33]. The $SU(4)_C \times SU(2)_L \times SU(2)_R$ Pati-Salam symmetry is spontaneously broken down to the $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ gauge group by the vev 's of the scalar multiplets Ξ_1 and Ξ_2 which transform as the adjoint representation of the Pati-Salam gauge group. The second stage of symmetry breaking is triggered by the vev of the scalar multiplet χ_R that transforms as $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ under the Pati-Salam group. The scalars χ_R , Φ , and Σ develop vev 's of the form

$$\begin{aligned} \langle \chi_R \rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & v_R \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \langle \Phi \rangle &= \frac{1}{\sqrt{2}} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \quad \langle \Sigma \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} v_{\Sigma_1} T^{15} & 0_{4 \times 4} \\ 0_{4 \times 4} & v_{\Sigma_2} T^{15} \end{pmatrix}, \end{aligned} \quad (2)$$

with $T^{15} = \frac{1}{2\sqrt{6}} \text{diag}(1, 1, 1, -3)$.

The SM fermions can be written in component form as follows:

$$\begin{aligned} F_i &= \begin{pmatrix} u_i & u_i & u_i & \nu_i \end{pmatrix}^T, \\ F_i^c &= \begin{pmatrix} u_i^c & u_i^c & u_i^c & \nu_i^c \end{pmatrix}, \quad i = 1, 2, 3. \end{aligned} \quad (3)$$

TABLE I. Transformation properties of the scalar and fermionic fields, as well as those of the Yukawa couplings, under the Pati-Salam gauge group and S_3 modular symmetry, where the modular weights of the fields are labeled with k .

	$SU(4)_C$	$SU(2)_L$	$SU(2)_R$	S_3	k
$F = (F_1, F_2)$	4	2	1	2	0
F_3	4	2	1	1'	-1
F_1^c	$\bar{\mathbf{4}}$	1	$\bar{\mathbf{2}}$	1	1
F_2^c	$\bar{\mathbf{4}}$	1	$\bar{\mathbf{2}}$	1'	-1
F_3^c	$\bar{\mathbf{4}}$	1	$\bar{\mathbf{2}}$	1'	1
Ψ_1	4	1	2	1'	-2
Ψ_2	4	1	2	1'	-4
Ψ_1^c	$\bar{\mathbf{4}}$	1	$\bar{\mathbf{2}}$	1'	2
Ψ_2^c	$\bar{\mathbf{4}}$	1	$\bar{\mathbf{2}}$	1'	4
$S^c = (S_1^c, S_2^c)$	1	1	1	2	1
S_3^c	1	1	1	1'	1
Φ	1	2	2	1	0
χ_R	$\bar{\mathbf{4}}$	1	1	1	2
Σ	15	2	2	1	0
σ_1	1	1	1	1	7
σ_2	1	1	1	1	5
Ξ_1	15	1	1	1	7
Ξ_2	15	1	1	1	5
$Y_1^{(4)}(\tau)$	1	1	1	1	4
$Y_1^{(6)}(\tau)$	1	1	1	1	6
$Y_2^{(2)}(\tau)$	1	1	1	2	2
$Y_2^{(4)}(\tau)$	1	1	1	2	4
$Y_2^{(6)}(\tau)$	1	1	1	2	6

Similarly, the heavy vectorlike fermionic multiplets Ψ_n and Ψ_n^c ($n = 1, 2$) containing the two family seesaw messengers are expressed as follows:

$$\Psi_n = \begin{pmatrix} U_n & U_n & U_n & N_n \\ D_n & D_n & D_n & E_n \end{pmatrix}^T, \quad \Psi_n^c = \begin{pmatrix} U_n^c & U_n^c & U_n^c & N_n^c \\ D_n^c & D_n^c & D_n^c & E_n^c \end{pmatrix}, \quad n = 1, 2. \quad (4)$$

The transformation properties of the scalar and fermionic fields, as well as those of the Yukawa couplings, under the Pati-Salam gauge group and S_3 modular symmetry are given in Table I. With this particle content and symmetries, the Yukawa superpotential compatible with the S_3 modular symmetry is

$$\begin{aligned}
-\mathcal{W} = & y_1 Y_2^{(2)}(\tau) F \Phi \Psi_1^c + y_2 Y_2^{(4)}(\tau) F \Phi \Psi_2^c + y_3 F_3 \Phi F_3^c \\
& + z_1 Y_1^{(6)}(\tau) \Psi_1 \sigma_1 F_1^c + z_2 Y_1^{(4)}(\tau) \Psi_1 \sigma_1 F_2^c + z_3 Y_1^{(4)}(\tau) \Psi_1 \sigma_2 F_3^c + z_4 \Psi_2 \sigma_2 F_2^c + z_5 Y_1^{(4)}(\tau) \Psi_2 \sigma_1 F_3^c \\
& + w_1 Y_1^{(6)}(\tau) \Psi_1 \Xi_1 F_1^c + w_2 Y_1^{(4)}(\tau) \Psi_1 \Xi_1 F_2^c + w_3 Y_1^{(4)}(\tau) \Psi_1 \Xi_2 F_3^c + w_4 \Psi_2 \Xi_2 F_2^c + w_5 Y_1^{(4)}(\tau) \Psi_2 \Xi_1 F_3^c \\
& + x_1 Y_2^{(2)}(\tau) F \Sigma \Psi_1^c + x_2 Y_2^{(4)}(\tau) F \Sigma \Psi_2^c + x_3 F_3 \Sigma F_3^c + m_{\Psi_1} \Psi_1 \Psi_1^c + m_{\Psi_2} \Psi_2 \Psi_2^c \\
& + \gamma_1 Y_2^{(4)}(\tau) F_1^c \chi_R S^c + \gamma_2 Y_2^{(2)}(\tau) F_2^c \chi_R S^c + \gamma_3 Y_2^{(4)}(\tau) F_3^c \chi_R S^c \\
& + \gamma_4 Y_1^{(4)}(\tau) F_3^c \chi_R S_3^c + M_1 Y_2^{(2)}(\tau) (S^c S^c)_2 + M_2 Y_2^{(2)}(\tau) (S^c S_3^c) + \text{H.c.}
\end{aligned} \quad (5)$$

Note that the above superpotential is completely renormalizable and is valid at all scales. Higher dimensional operators are strongly suppressed by the Planck scale, and constraints arising from those operators are discussed below. Although our model predicts no proton decay at the renormalizable level, it can be induced at the nonrenormalizable level by the following Yukawa operators:

$$\begin{aligned} & \frac{1}{\Lambda} Y_1^{(4)}(\tau) F F \chi_R \chi_R, & \frac{1}{\Lambda} Y_1^{(6)}(\tau) F_1^c F_1^c \chi_R \chi_R, \\ & \frac{1}{\Lambda} Y_1^{(6)}(\tau) F_1^c F_2^c \chi_R \chi_R, \end{aligned} \quad (6)$$

where we have not included the nonrenormalizable Yukawa operator,

$$\frac{1}{\Lambda} F_2^c F_2^c \chi_R \chi_R, \quad (7)$$

as it is forbidden by the S_3 modular flavor symmetry since there is no modular form of weight 2 corresponding to a trivial S_3 singlet. As discussed in detail in Ref. [34], assuming $\Lambda = 10^{19}$ GeV, the constraints arising from nucleon decay in a Pati-Salam theory set the scale of spontaneous breaking of the $SU(2)_R$ symmetry to be larger than 5×10^6 GeV. As shown in Sec. III, in our numerical analysis of fermion masses and mixings, we assume the scale of spontaneous breaking of the $SU(2)_R$ symmetry to be larger than 10^{11} GeV, which respects the constraints arising from nucleon decay, making proton decay practically unobservable in our model.

The flavor structure of the superpotential is replicated with Φ, Σ having the same S_3 assignments (being distinguished by the gauge group), and the same holds for the pairs σ_1, Ξ_1 and for σ_2, Ξ_2 . The two pairs are distinct as σ_1, Ξ_1 have modular weight $k=5$ whereas σ_2, Ξ_2 have modular weight $k=7$. This structure is also visible in

the diagrams in Fig. 1. The structure of the diagrams is similar to the diagram of the seesaw mechanism. In particular, the first two families of all the charged fermions obtain their masses via seesaw mechanism mediated by the heavy vectorlike fermions Ψ_1 and Ψ_2 , whereas the third families obtain their masses via their Yukawa couplings to Φ and Σ . The neutrinos also obtain Dirac masses in a similar way, which is further extended to double seesaw by the inclusion of the singlet fields S_i^c .

Due to the difference in modular weights, the superpotential terms are such that the effective Yukawa terms arise with the respective modular forms, leading to the following mass terms for charged fermions and neutrinos:

$$\begin{aligned} & (u_i \quad U_k) \begin{pmatrix} M_c^{(u)} & M_a^{(u)} \\ M_b^{(u)} & M_U \end{pmatrix} \begin{pmatrix} u_j^c \\ U_k^c \end{pmatrix}, \\ & (d_i \quad D_k) \begin{pmatrix} M_c^{(d)} & M_a^{(d)} \\ M_b^{(d)} & M_D \end{pmatrix} \begin{pmatrix} d_j^c \\ D_k^c \end{pmatrix}, \end{aligned} \quad (8)$$

$$\begin{aligned} & (e_i \quad E_k) \begin{pmatrix} M_c^{(e)} & M_a^{(e)} \\ M_b^{(e)} & M_E \end{pmatrix} \begin{pmatrix} e_j^c \\ E_k^c \end{pmatrix}, \\ & (\nu_i \quad N_k) \begin{pmatrix} M_c^{(\nu)} & M_a^{(\nu)} \\ M_b^{(\nu)} & M_N \end{pmatrix} \begin{pmatrix} \nu_j^c \\ N_k^c \end{pmatrix}, \end{aligned} \quad (9)$$

where the heavy vectorlike seesaw mediator mass matrices are

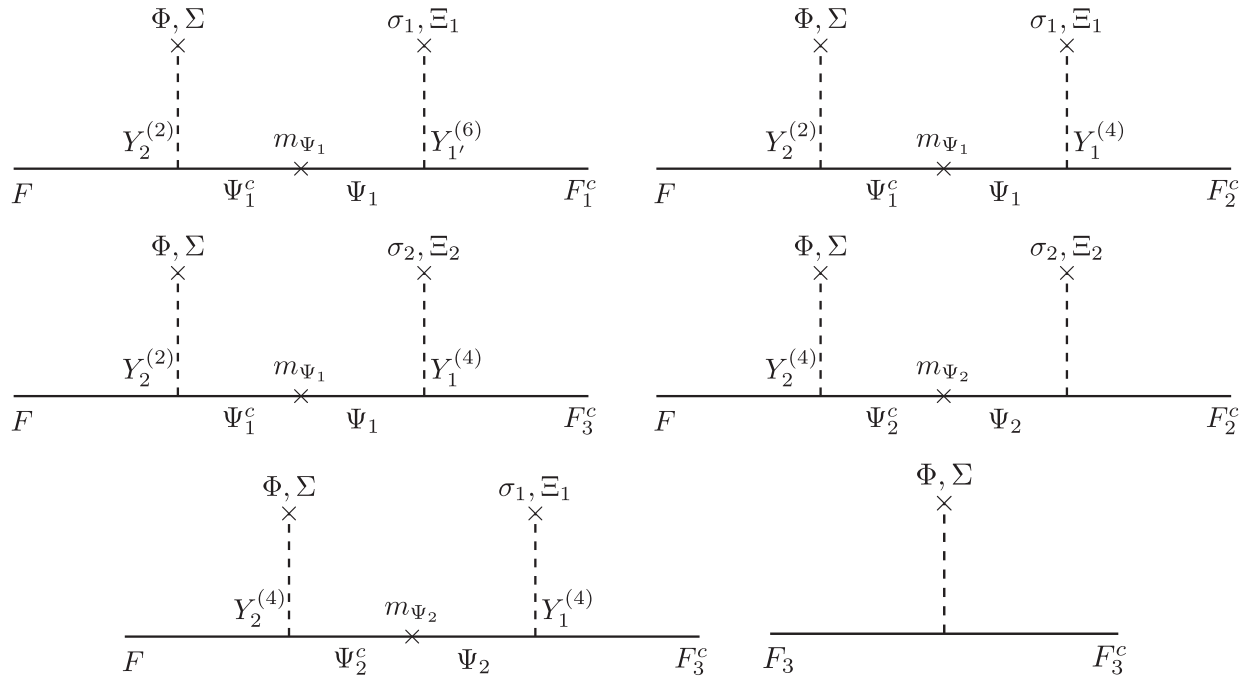


FIG. 1. Feynman diagrams corresponding to the generation of the masses of the charged fermions as well as for the Dirac neutrino submatrix. The charged fermions of the first and second generations get their masses via seesawlike diagrams mediated by ψ_1 and ψ_2 , whereas those of the third generation get their masses through their Yukawa couplings to Φ and Σ fields.

$$M_U = M_D = M_E = M_N = \begin{pmatrix} m_{\Psi_1} & 0 \\ 0 & m_{\Psi_2} \end{pmatrix}. \quad (10)$$

The 3×3 submatrices $M_c^{(u,d,e,\nu)}$ in Eqs. (8) and (9) have only the (3,3) element as nonzero and are given as

$$(M_c^{(u)})_{ij} = \frac{1}{\sqrt{2}}(y_3 v_1 + x_3 v_{\Sigma_1})\delta_{i3}\delta_{j3}, \quad (M_c^{(d)})_{ij} = \frac{1}{\sqrt{2}}(y_3 v_2 + x_3 v_{\Sigma_2})\delta_{i3}\delta_{j3}, \quad (11)$$

$$(M_c^{(e)})_{ij} = \frac{1}{\sqrt{2}}(y_3 v_2 - 3x_3 v_{\Sigma_2})\delta_{i3}\delta_{j3}, \quad (M_c^{(\nu)})_{ij} = \frac{1}{\sqrt{2}}(y_3 v_1 - 3x_3 v_{\Sigma_1})\delta_{i3}\delta_{j3}, \quad (12)$$

and these correspond to the masses of the third families of fermions generated from their Yukawa couplings to Φ and Σ . The remaining submatrices appearing in the quark sector are given as

$$M_a^{(u)} = \frac{1}{\sqrt{2}} \begin{pmatrix} (y_1 v_1 + x_1 v_{\Sigma_1})Y_{2,2}^{(2)}(\tau) & (y_2 v_1 + x_2 v_{\Sigma_1})Y_{2,2}^{(4)}(\tau) \\ -(y_1 v_1 + x_1 v_{\Sigma_1})Y_{2,1}^{(2)}(\tau) & -(y_2 v_1 + x_2 v_{\Sigma_1})Y_{2,1}^{(4)}(\tau) \\ 0 & 0 \end{pmatrix}, \quad (13)$$

$$M_b^{(u)} = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_1^{(6)}(z_1 v_{\sigma_1} + w_1 v_{\Xi_1}) & Y_1^{(4)}(\tau)(z_2 v_{\sigma_1} + w_2 v_{\Xi_1}) & Y_1^{(4)}(\tau)(z_3 v_{\sigma_2} + w_3 v_{\Xi_2}) \\ 0 & z_4 v_{\sigma_2} + w_4 v_{\Xi_2} & Y_1^{(4)}(\tau)(z_5 v_{\sigma_1} + w_5 v_{\Xi_1}) \end{pmatrix} = M_b^{(d)}, \quad (14)$$

$$M_a^{(d)} = \frac{1}{\sqrt{2}} \begin{pmatrix} (y_1 v_2 + x_1 v_{\Sigma_2})Y_{2,2}^{(2)}(\tau) & (y_2 v_2 + x_2 v_{\Sigma_2})Y_{2,2}^{(4)}(\tau) \\ -(y_1 v_2 + x_1 v_{\Sigma_2})Y_{2,1}^{(2)}(\tau) & -(y_2 v_2 + x_2 v_{\Sigma_2})Y_{2,1}^{(4)}(\tau) \\ 0 & 0 \end{pmatrix}, \quad (15)$$

whereas those in the lepton sector are given as

$$M_a^{(e)} = \frac{1}{\sqrt{2}} \begin{pmatrix} (y_1 v_2 - 3x_1 v_{\Sigma_2})Y_{2,2}^{(2)}(\tau) & (y_2 v_2 - 3x_2 v_{\Sigma_2})Y_{2,2}^{(4)}(\tau) \\ -(y_1 v_2 - 3x_1 v_{\Sigma_2})Y_{2,1}^{(2)}(\tau) & -(y_2 v_2 - 3x_2 v_{\Sigma_2})Y_{2,1}^{(4)}(\tau) \\ 0 & 0 \end{pmatrix}, \quad (16)$$

$$M_b^{(e)} = \frac{1}{\sqrt{2}} \begin{pmatrix} Y_1^{(6)}(z_1 v_{\sigma_1} - 3w_1 v_{\Xi_1}) & Y_1^{(4)}(\tau)(z_2 v_{\sigma_1} - 3w_2 v_{\Xi_1}) & Y_1^{(4)}(\tau)(z_3 v_{\sigma_2} - 3w_3 v_{\Xi_2}) \\ 0 & z_4 v_{\sigma_2} - 3w_4 v_{\Xi_2} & Y_1^{(4)}(\tau)(z_5 v_{\sigma_1} - 3w_5 v_{\Xi_1}) \end{pmatrix} = M_b^{(\nu)}, \quad (17)$$

$$M_a^{(\nu)} = \frac{1}{\sqrt{2}} \begin{pmatrix} (y_1 v_1 - 3x_1 v_{\Sigma_1})Y_{2,2}^{(2)}(\tau) & (y_2 v_1 - 3x_2 v_{\Sigma_1})Y_{2,2}^{(4)}(\tau) \\ -(y_1 v_1 - 3x_1 v_{\Sigma_1})Y_{2,1}^{(2)}(\tau) & -(y_2 v_1 - 3x_2 v_{\Sigma_1})Y_{2,1}^{(4)}(\tau) \\ 0 & 0 \end{pmatrix}. \quad (18)$$

Thus, once we integrate out the heavy vectorlike fermions Ψ_1 and Ψ_2 , the masses for the first and second generation of the SM charged fermions are obtained via the seesaw mechanism, which also yields the Dirac neutrino submatrix, $\tilde{M}_\nu$. The resulting effective low energy 3×3 mass matrices for the SM charged fermions as well as the Dirac neutrino matrix are:

$$\tilde{M}_u = M_c^{(u)} - M_a^{(u)} M_U^{-1} M_b^{(u)}, \quad \tilde{M}_d = M_c^{(d)} - M_a^{(d)} M_D^{-1} M_b^{(d)}, \quad (19)$$

$$\tilde{M}_e = M_c^{(e)} - M_a^{(e)} M_E^{-1} M_b^{(e)}, \quad \tilde{M}_\nu = M_c^{(\nu)} - M_a^{(\nu)} M_N^{-1} M_b^{(\nu)}. \quad (20)$$

As mentioned before, our model also contains extra singlet fermions S_i^c that couple to the right handed neutrinos ν^c [last two lines in Eq. (5)]. Thus, the resultant neutral fermion mass terms (after integrating out the Ψ fields) can be written as

$$\frac{1}{2} \begin{pmatrix} \nu & \nu^c & S^c \end{pmatrix} \begin{pmatrix} 0_{3 \times 3} & \tilde{M}_\nu & 0_{3 \times 3} \\ \tilde{M}_\nu^T & 0_{3 \times 3} & M_R \\ 0_{3 \times 3} & M_R^T & M_S \end{pmatrix} \begin{pmatrix} \nu \\ \nu^c \\ S^c \end{pmatrix} + \text{H.c.} \quad (21)$$

In the above equation, all the submatrices are 3×3 with the Dirac mass matrix $\tilde{M}_\nu$ determined by Eq. (20), while M_R and M_S are given as

$$M_R = \begin{pmatrix} \gamma_1 Y_{2,1}^{(4)}(\tau) & \gamma_1 Y_{2,2}^{(4)}(\tau) & 0 \\ -\gamma_2 Y_{2,2}^{(2)}(\tau) & \gamma_2 Y_{2,1}^{(2)}(\tau) & 0 \\ -\gamma_3 Y_{2,2}^{(4)}(\tau) & \gamma_3 Y_{2,1}^{(4)}(\tau) & \gamma_4 Y_1^{(4)}(\tau) \end{pmatrix} \frac{v_R}{\sqrt{2}},$$

$$M_S = \begin{pmatrix} -M_1 Y_{2,1}^{(2)}(\tau) & M_1 Y_{2,2}^{(2)}(\tau) & -M_2 Y_{2,2}^{(2)}(\tau) \\ M_1 Y_{2,2}^{(2)}(\tau) & M_1 Y_{2,1}^{(2)}(\tau) & M_2 Y_{2,1}^{(2)}(\tau) \\ -M_2 Y_{2,2}^{(2)}(\tau) & M_2 Y_{2,1}^{(2)}(\tau) & 0 \end{pmatrix}. \quad (22)$$

In the limit $M_S \gg M_R \gg \tilde{M}_\nu$, the mass matrix in Eq. (21) corresponds to the double seesaw [35],¹ according to which, once the heavy fields ν^c and S^c are integrated out, the mass matrix for the light active neutrinos reads:

$$M_\nu = \tilde{M}_\nu M_R^{-1} M_S M_R^{-1} \tilde{M}_\nu^T. \quad (23)$$

Finally, it is worth comparing our model with some other models based on Pati-Salam gauge symmetry, supplemented with modular symmetries, such as the ones given in [15,16]. Our model has vectorlike fermionic seesaw messengers that trigger a two family seesaw mechanism yielding the masses of the first and second families of SM charged fermions, whereas the models of Refs. [15,16] do not contain vectorlike fermionic seesaw mediators in their particle spectrum. In those models, all SM charged fermions arise from Yukawa interactions involving the scalar bidoublets, whereas in our model the scalar bidoublets are used to provide masses to the third generation of SM charged fermions via Yukawa interactions invariant under the symmetries. Another difference between our model and models of Refs. [15,16] corresponds to the neutrino mass generation mechanism. In our model, the tiny active neutrino masses arise from a double seesaw mechanism, whereas in the models of Refs. [15,16] these masses are generated by a type I seesaw mechanism. Moreover, our scalar spectrum differs significantly from those in Refs. [15,16]. In the other models, there is a scalar multiplet that transforms under the Pati-Salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$ as $(\mathbf{10}, \mathbf{1}, \mathbf{3})$, while our framework does not include such a multiplet.

¹For a seesaw review, see, e.g., [36].

III. NUMERICAL ANALYSIS

In this section, we present the results of the numerical analysis conducted to evaluate the viability of the model in explaining the observed fermion masses and mixing. We vary all input parameters, including Yukawa couplings, vev 's, and masses, and minimize the function χ^2 , which is defined as

$$\chi^2 = \sum_i \left(\frac{O_{i\text{calc}} - O_{i\text{exp}}}{O_{i\text{exp}}} \right)^2, \quad (24)$$

to determine the best fit parameters that reproduce the observed fermion masses and mixing.

In Eq. (24), $O_{i\text{calc}}$ represents the model prediction, while $O_{i\text{exp}}$ denotes the experimental best fit value. The summation is performed over the masses of charged fermions [37,38], the Cabibbo-Kobayashi-Masakawa (CKM) mixing angles, the CKM CP phase, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing angles, and the mass-squared differences of light neutrinos [39,40].

While fitting the charged fermion masses, the masses of the first two generations are fitted at 10^{12} GeV [37], as they arise from the seesaw mechanism, while those of the third generation are taken at the electroweak scale [38]. This is because the double seesaw mechanism that generates light neutrino masses requires the masses of Ψ_1, Ψ_2 to be around 10^{12} GeV. Further, as discussed before, the masses of the first two generations of charged fermions are generated by integrating out these heavy fermions, and hence, we fit their masses at the mass scale of Ψ_1, Ψ_2 . On the other hand, the masses of the third generation of charged fermions are generated by renormalizable couplings and hence one could work at any scale. For simplicity, we fit them at the electroweak scale.

In addition, the constraint from cosmological observations on the sum of the active light neutrino masses, $\Sigma m_i \leq 0.12$ eV [41], as well as the 3σ bound on the CP phase of the PMNS matrix [39,40], are imposed as extra conditions. In our fit, the absolute values of the Yukawa couplings are taken to be within the range $[0.2, \sqrt{4\pi}]$, whereas their phases are varied in the range $[0, 2\pi]$. An important result is that the model only fits the normal ordering of the active light neutrino masses, whereas the inverted ordering is disfavored.

In Table II, we present a set of sample best fit parameters that reproduce the correct fermion masses and mixing along with the corresponding values of the calculated fermion masses and mixing parameters. The low-energy mass matrices for the up and down type quarks, charged leptons, and active light neutrinos are also given in this table. One can see that the rows of the charged fermion mass matrices satisfy a natural hierarchy as a consequence of the modular symmetry with two family seesaw, which in turn explains the observed fermion mass hierarchy.

In Fig. 2, we show the values of the real and imaginary components of the modular field τ that give $\chi^2 \leq 5 \times 10^{-4}$

TABLE II. Sample best fit input parameters for normal hierarchy of active neutrinos along with the corresponding values of the calculated fermion masses and mixing parameters. The χ^2 value for the given point is 6.30442×10^{-15} . The low-energy mass matrices for the up and down type quarks, charged leptons, and active light neutrinos as well as the masses of the heavy neutrinos in addition to the ones coming from $\Psi_{1,2}$ are also given. For the PMNS matrix, the Majorana phase matrix is defined as $P = \text{diagonal}(1, e^{i\alpha^M/2}, e^{i\beta^M/2})$.

Input parameters	Best fit value for normal hierarchy of light neutrinos
τ	$-0.03224 + i1.86682$
m_{Ψ_1}, m_{Ψ_2} (GeV)	$2.38 \times 10^{12}, 8.71 \times 10^{12}$
v_1, v_2 (GeV)	245.62150, 3.95665
$v_{\Sigma_1}, v_{\Sigma_2}$ (GeV)	8.37732, 10.01214
$v_{\sigma_1}, v_{\sigma_2}$ (GeV)	$6.12 \times 10^{12}, 4.64 \times 10^{12}$
v_{Ξ_1}, v_{Ξ_2} (GeV)	$7.90 \times 10^{12}, 3.53 \times 10^{12}$
v_R (GeV)	1.23×10^{11}
M_1, M_2 (GeV)	$1.0 \times 10^{12}, 6.20 \times 10^8$
x_1, x_2, x_3	$3.53382e^{i2.23402}, 2.93598e^{i3.85910}, 0.20033e^{i2.15046}$
y_1, y_2, y_3	$0.27120e^{i5.93154}, 0.46397e^{i5.47007}, 0.98750e^{i2.24996}$
w_1, w_2, w_3, w_4, w_5	$0.75168e^{i0.96442}, 0.20136e^{i2.04578}, 1.37663e^{i5.91647}, 0.23368e^{i0.86247}, 2.62639e^{i5.87924}$
z_1, z_2, z_3, z_4, z_5	$2.37549e^{i0.67600}, 0.25164e^{i1.45687}, 2.46850e^{i4.02943}, 0.32353e^{i0.15040}, 3.54392e^{i6.12983}$
$\gamma_1, \gamma_2, \gamma_3, \gamma_4$	$0.20302e^{i3.09154}, 1.99746e^{i6.05522}, 3.42165e^{i0.22612}, 1.96501e^{i5.93289}$
$M_{\text{heavy neutrinos}}$ (GeV)	$6.06521 \times 10^5, 2.49460 \times 10^9, 2.64148 \times 10^9, 3.95307 \times 10^9,$ $1.25075 \times 10^{11}, 1.28998 \times 10^{11}$
Low energy mass matrices, masses, and mixing parameters	
$\tilde{M}_u$ (GeV)	$\begin{pmatrix} -0.00018e^{i(-2.57621)} & -0.01312e^{i2.50106} & 0.00174e^{i1.43459} \\ 0.00428e^{i0.66665} & -0.19406e^{i2.37489} & -0.06547e^{i(-2.04282)} \\ 0.00000e^{i0.00000} & 0.00000e^{i0.00000} & -108.38236e^{i2.24928} \end{pmatrix}$
$\tilde{M}_d$ (GeV)	$\begin{pmatrix} 0.00016e^{i(-0.36728)} & 0.00339e^{i0.98476} & -0.00459e^{i(-2.87477)} \\ -0.00409e^{i2.87558} & -0.00659e^{i1.86300} & 0.15841e^{i0.40887} \\ 0.00000e^{i0.00000} & 0.00000e^{i0.00000} & -2.51228e^{i2.21622} \end{pmatrix}$
m_u, m_c, m_t (GeV)	0.00054, 0.2670, 172.69001
m_d, m_s, m_b (GeV)	0.00120, 0.0240, 4.180
$s_{12}^q, s_{23}^q, s_{13}^q, \delta_{CP}^q$	0.2250, 0.04182, 0.00370, 65.6750°
$\tilde{M}_e$ (GeV)	$\begin{pmatrix} 0.00011e^{i0.67714} & -0.00335e^{i1.79184} & 0.02874e^{i(-0.74991)} \\ -0.00261e^{i(-2.36319)} & -0.10577e^{i(-2.70883)} & -0.64387e^{i2.36816} \\ 0.00000e^{i0.00000} & 0.00000e^{i0.00000} & 0.59512e^{i(-1.17142)} \end{pmatrix}$
M_ν (eV)	$\begin{pmatrix} 0.00013e^{i0.26925} & 0.00065e^{i(-1.25936)} & 0.00314e^{i0.96134} \\ 0.00065e^{i(-1.25936)} & -0.02407e^{i2.14357} & 0.01431e^{i(-0.64736)} \\ 0.00314e^{i0.96134} & 0.01431e^{i(-0.64736)} & 0.00000e^{i0.00000} \end{pmatrix}$
m_e, m_μ, m_τ (GeV)	0.00048, 0.10155, 1.77686
m_{ν_1} (eV), $\Delta m_{\text{sol}}^2, \Delta m_{\text{atm}}^2$ (eV ²)	0.00276, 7.49×10^{-5} , 0.00251
$s_{12}^\nu, s_{23}^\nu, s_{13}^\nu$	0.55497, 0.68557, 0.14883
$\delta_{CP}^\nu, \alpha^M, \beta^M$	323.53560°, 194.46228°, 297.46063°

in our numerical scan. In the left panel, the masses of the seesaw mediators and the vev 's are fixed according to the values provided in Table II, while the absolute values of the Yukawa couplings are varied within the range $[0.2, \sqrt{4\pi}]$, with their phases taking any value within $[0, 2\pi]$. In the right panel, both the mediator masses and the vev 's are also varied freely in addition to the Yukawa couplings. The fundamental domain of τ ($-\frac{1}{2} \leq \text{Re}[\tau] \leq$

$\frac{1}{2}, |\tau| \geq 1$) is shown by the gray-shaded region. From the figure, one can see that the imaginary part of τ (τ_I) is more restricted than the real part (τ_R). This is because the magnitudes of the entries of the fermion mass matrices are more sensitive to τ_I than to τ_R . This can be seen, for instance, by taking the (1,1) entry of $\tilde{M}_u$ to the leading order in the q -expansion of the modular forms,

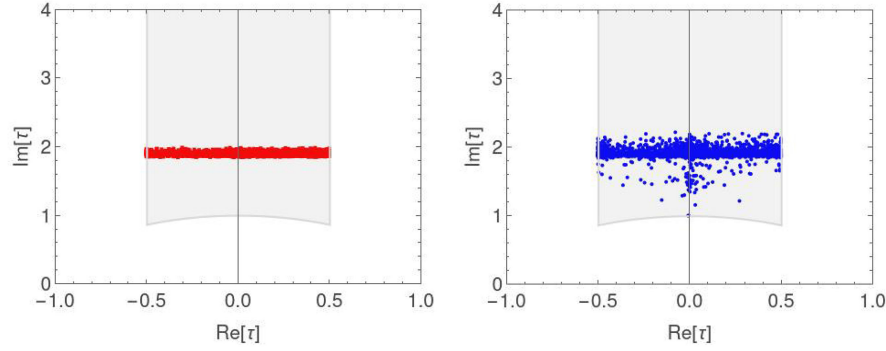


FIG. 2. Values of the real and imaginary components of the modular field τ that give $\chi^2 \leq 5 \times 10^{-4}$ in our numerical scan. The fundamental domain for τ ($-\frac{1}{2} \leq \text{Re}[\tau] \leq \frac{1}{2}, |\tau| \leq 1$) is shown by the gray-shaded region. The left panel is for fixed vev 's and mass scales, whereas they are varied in the right panel. See text for details.

$$(\tilde{M}_u)_{11} \approx -\frac{9e^{-6\pi\tau_I+2i\pi\tau_R}(e^{4\pi\tau_I}-16e^{2\pi(\tau_I+i\tau_R)}+576e^{4i\pi\tau_R})(v_{\Sigma 1}x_1+v_1y_1)(v_{\Xi 1}w_1+v_{\Sigma 1}z_1)}{128M_1}, \quad (25)$$

from which we can see that τ_R always contributes to the phases of the individual terms in the expansion. The same goes for all the mass terms. This is the reason why τ_I is restricted to be in the range $\sim [1.8, 2]$ for the case of fixed mass scales and vev 's (left panel of Fig. 2) as the fermion masses and mixing are more sensitive to τ_I . The variation in the mass scales and the vev 's can relax this bound because the variation in the absolute values of the mass terms due to τ_I can be compensated by taking different vev 's and/or mass scales. It is worth mentioning that our predicted ranges for $\text{Re}[\tau]$ and $\text{Im}[\tau]$ are different than the ones obtained in the Pati-Salam models with modular S_3 and A_4 symmetries proposed in Refs. [16] and [15], respectively. $\text{Re}[\tau]$ is predicted to be in the ranges $[0.36, 0.45]$ and $[0.39, 0.48]$ for the models of Refs. [16] and [15], respectively. In our case, we obtain $-\frac{1}{2} \leq \text{Re}[\tau] \leq \frac{1}{2}$ (across the fundamental domain, in terms of $\text{Re}[\tau]$). Furthermore, the Pati-Salam models with modular S_3 and A_4 symmetries of Refs. [16]

and [15] yield values of $\text{Im}[\tau]$ to be in the ranges $[1.77, 1.93]$ and $[1.02, 1.09]$, respectively, while our model predicts $1 \leq \text{Im}[\tau] \leq 2.2$.

Figure 3 shows the predictions for the effective Majorana mass (m_{ee}) that governs $0\nu\beta\beta$. Note that we have shown only the contributions due to the three active light neutrinos since the mixing of the heavy neutrinos with the active light neutrinos is strongly suppressed, making their contribution to $0\nu\beta\beta$ negligible. The region within the solid gray lines represents the standard predictions for m_{ee} , with the assumption that the active light neutrinos follow the normal hierarchy and no modular symmetry. The red/blue points indicate the predictions from our model. As in Fig. 2, the masses of the seesaw mediators and the vev 's are fixed according to the values provided in Table II in the left panel, while the magnitude and phase of the Yukawa couplings are varied within the ranges $[0.2, \sqrt{4\pi}]$ and $[0, 2\pi]$, respectively. In the right panel, both the mediator masses and the

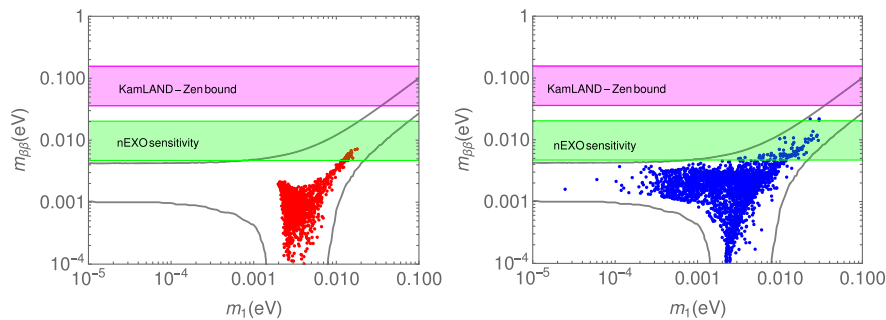


FIG. 3. Predictions for the effective Majorana mass (m_{ee}) governing $0\nu\beta\beta$. The region within the solid gray lines represents the standard predictions for m_{ee} assuming the active light neutrinos follow the normal hierarchy. The left and right panels correspond to the cases with fixed and varying mass scales, respectively, as in Fig. 2. The region above the purple band is excluded by the KamLAND-Zen experimental bound, while the green band corresponds to the projected sensitivity of the nEXO experiment.

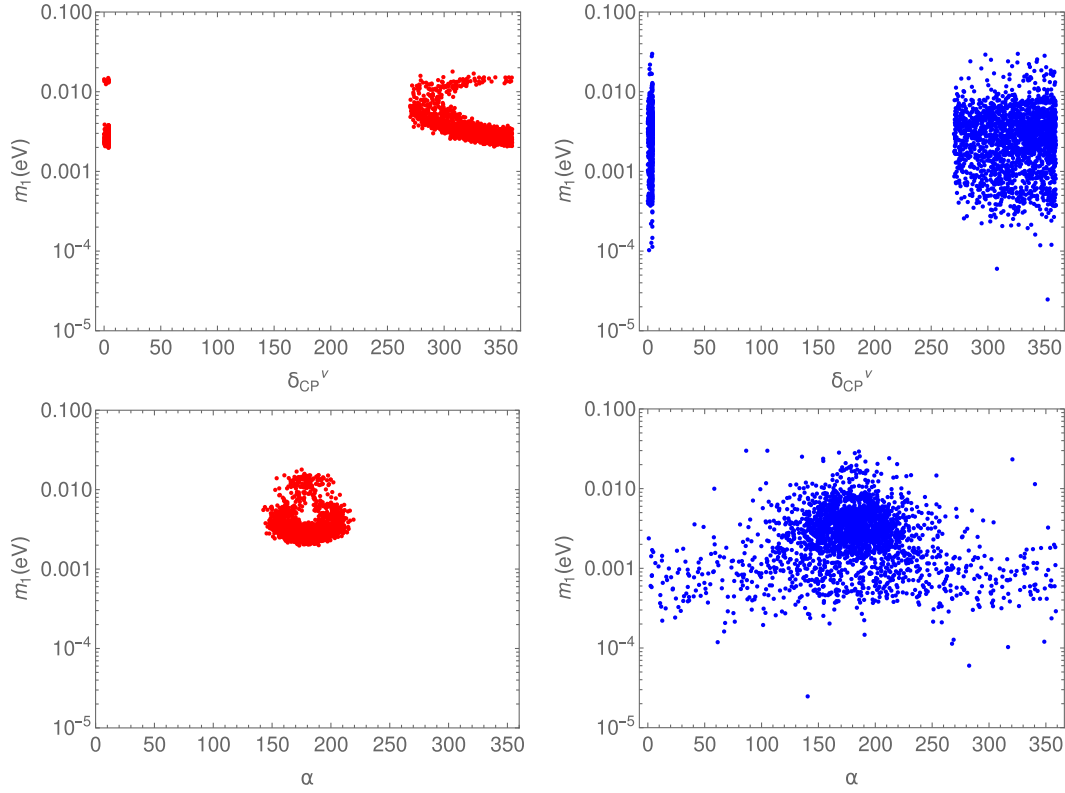


FIG. 4. Correlations of the Dirac CP phase δ_{CP}^ν and the Majorana phase α to the lightest neutrino mass m_1 . The left and right panels correspond to the cases with fixed and varying mass scales, respectively, as in the previous figures.

$\nu\nu$'s are allowed to vary freely in addition to the Yukawa couplings. The region above the purple band is excluded by the experimental bound from KamLAND-Zen [42], while the green band corresponds to the projected sensitivity of the nEXO experiment [43]. The widths of these bands are due to the uncertainty in the values of the nuclear matrix elements. One interesting feature that we can see from this figure is that the model predicts a lower bound on the mass of the lightest active neutrino. This is around 0.0025 eV for the case of fixed mass scales, while it becomes $\sim 10^{-4}$ eV in the general case. A small part of the predicted parameter space lies within the nEXO reach.

Figure 4 shows the correlations of the Dirac CP phase δ_{CP}^ν and one of the Majorana phases, α , to the lightest neutrino mass m_1 . As before, the left and right panels correspond to the cases with fixed and varying mass scales, respectively. It is interesting to see that, for the case of fixed scales, there exists a strong correlation between m_1 and δ_{CP}^ν as well as α , in particular for the case of fixed mass scales. Moreover, the Dirac CP violating neutrino oscillation phase is found to lie in the range $\delta_{CP}^\nu \sim 270^\circ - 364^\circ$, which is different from the ones $\delta_{CP}^\nu \sim 90^\circ - 342^\circ$ and $\delta_{CP}^\nu \sim 144^\circ - 360^\circ$ predicted by the models of Refs [16] and [15], respectively. Note that our bound on δ_{CP}^ν is obtained by imposing the 3σ bound on delta CP [$124^\circ - 364^\circ$ [40]]. We also tried fitting the data without imposing this bound on δ_{CP}^ν , and we found points between 4° and 124° , but the cut at 270° was still there, implying that it is a

model prediction. Furthermore, our range of values for the effective Majorana mass (m_{ee}) that governs $0\nu\beta\beta$ is larger than the ones obtained in [16] and [15], thus making our model more testable than the latter via $0\nu\beta\beta$ decay experiments.

IV. CONCLUSIONS

We have proposed a model based on the smallest quark-lepton unified symmetry, the Pati-Salam gauge group and the smallest modular symmetry, S_3 . The masses of the third family of SM charged fermion arise from renormalisable Yukawa interactions involving a colorless scalar bidoublet as well as a bidoublet scalar in the adjoint representation of $SU(4)_C$. The first and second family masses are naturally suppressed due to a Pati-Salam version of the type I seesaw mechanism for neutrinos, but here mediated through heavier vectorlike Pati-Salam fermions. Due to the Pati-Salam symmetry, the same mechanism also suppresses first and second family Dirac neutrino masses, but in the neutrino sector there are additional fields leading to tiny active Majorana neutrino masses via a Double Seesaw mechanism.

The diagrams responsible for the effective Yukawa operators are similar to those of the type I seesaw mechanism for neutrinos, with two insertions of vacuum expectation values, where one of them breaks electroweak symmetry, and one does not. The three Pati-Salam families are essentially distinguished by whether they couple to

heavier vectorlike fermions (the first two families) or not (third family), and this is controlled by their S_3 assignments. The modular S_3 is perfect for implementing such a two-family seesaw mechanism, since it admits doublet representations for the first two families, and a singlet representation for the third family.

The model is shown to describe all quark and lepton (including neutrino) masses and mixing angles, in terms of high energy mass scales, together with complex dimensionless Yukawa coefficients which are all of order unity, and a single complex modulus field τ with $\text{Im}(\tau) \sim 2$. It provides a good fit to neutrino data, assuming a normal ordering of neutrino masses, while the inverted ordering is disfavoured. The associated prediction for neutrinoless double beta decay is partly within the reach of the nEXO experiment. In particular, the Dirac CP violating neutrino oscillation phase is predicted to lie in the range $\delta_{CP}^\nu \sim 270^\circ - 360^\circ$.

ACKNOWLEDGMENTS

A. E. C. H. is supported by Agencia Nacional de Investigación y Desarrollo (ANID)-Chile FONDECYT 1210378, ANID-Chile FONDECYT 1241855, ANID Programa de Investigación Asociativa (PIA)/APOYO AFB230003, and ANID- Programa Milenio—code No. ICN2019_044. V. K. N. was supported by ANID Chile Fondecyt Postdoctoral Grant No. 3220005 during the initial stages of this work. I. d. M. V. acknowledges funding from Fundação para a Ciência e a Tecnologia (FCT) through the Projects CFTP-FCT Unit UIDB/FIS/00777/2020 and UIDP/FIS/00777/2020, CERN/FIS-PAR/0019/2021, CERN/FIS-PAR/0002/2021, 2024.02004 CERN, which are partially funded through POCTI (FEDER), COMPETE, Quadro de Referência Estratégico Nacional (QREN), and EU. A. E. C. H. thanks the Instituto Superior Técnico, Universidade de Lisboa for hospitality, where part of this work was done. S. F. K. acknowledges the STFC Consolidated Grant No. ST/X000583/1 and the European Union's Horizon 2020 Research and Innovation programme under the Marie Skłodowska-Curie Grant Agreement HIDEEN European ITN project (H2020-MSCA-ITN-2019//860881-HIDEEN).

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX: MODULAR FLAVOR SYMMETRY

In this appendix, we provide a concise review of the main features of modular flavor symmetry. The full modular group $\Gamma \cong \text{SL}(2, \mathbb{Z})$ corresponds to the group of two-dimensional matrices with integral entries and unit determinant,

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}. \quad (\text{A1})$$

The modular group is an infinite group that can be generated by two elements, conventionally denoted as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (\text{A2})$$

which fulfill the relations:

$$S^4 = (ST)^3 = 1, \quad S^2 T = T S^2. \quad (\text{A3})$$

The modular symmetry is ubiquitous in string compactifications and corresponds to the geometrical symmetry of the extra compact space. In simple toroidal compactification, the two-dimensional torus T^2 is described as the quotient $T^2 = \mathbb{C}/\Lambda_{\omega_1, \omega_2}$, where $\mathbb{C}$ stands for the whole complex plane $\mathbb{C}$ and $\Lambda_{\omega_1, \omega_2} = \{m\omega_1 + n\omega_2, m, n \in \mathbb{Z}\}$ denotes a two-dimensional lattice with the basis vectors ω_1 and ω_2 . The lattice is left invariant under a change in lattice basis vectors only and if only

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \rightarrow \begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \quad (\text{A4})$$

The torus is characterized by the complex modulus $\tau = \omega_1/\omega_2$ up to rotation and scale transformations; without loss of generality, we can limit τ to the upper half of the complex plane with $\text{Im}(\tau) > 0$. The two tori related by modular transformations would be identical, i.e.,

$$\tau \xrightarrow{\gamma} \tau' = \frac{\omega'_1}{\omega'_2} = \frac{a\tau + b}{c\tau + d} \equiv \gamma\tau, \quad \text{Im}(\tau) > 0, \quad (\text{A5})$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus, the action of the generators S and T , corresponding to modular inversion and translation, respectively, take the form:

$$\tau \xrightarrow{S} -\frac{1}{\tau}, \quad \tau \xrightarrow{T} \tau + 1. \quad (\text{A6})$$

Notice that $\gamma \in \Gamma$ and $-\gamma \in \Gamma$ define the same transformation of τ . Making use of the modular transformations, it is always possible to restrict τ to the fundamental domain defined as follows:

$$\mathcal{D} = \left\{ \tau \mid \text{Im}(\tau) > 0, |\text{Re}(\tau)| \leq \frac{1}{2}, |\tau| \geq 1 \right\}.$$

Any value of τ in the upper-half plane can be mapped into the fundamental domain $\mathcal{D}$ by performing an appropriate

modular transformation, but no two points inside the fundamental domain $\mathcal{D}$ are related under the modular group. Consequently, the fundamental domain $\mathcal{D}$ is a representative set of the physically inequivalent modulus. Notice that the left boundary of $\mathcal{D}$ with $\text{Re}(\tau) = -1/2$ is related to the right boundary of $\text{Re}(\tau) = 1/2$ by the T transformation, and the S transformation maps the left unit arc $\tau = e^{i\theta}$ ($\pi/2 \leq \theta \leq 2\pi/3$) on the boundary is related to the right unit arc $\tau = e^{i\theta}$ ($\pi/3 \leq \theta \leq \pi/2$) by the S transformation.

The modular symmetry provides an origin of the discrete flavor symmetry through the quotient

$$\Gamma_N = \Gamma / \pm \Gamma(N), \quad \Gamma'_N = \Gamma / \Gamma(N), \quad (\text{A7})$$

where Γ_N and Γ'_N are the inhomogeneous and homogeneous finite modular groups, respectively, and $\Gamma(N)$ is the principal normal subgroup of level N ,

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}, \quad (\text{A8})$$

which implies $T^N \in \Gamma(N)$. The inhomogeneous finite modular groups Γ_N for $N = 2, 3, 4, 5$ are isomorphic to the permutation groups S_3 , A_4 , S_4 , and A_5 respectively [1,44], and Γ'_N is the double cover of Γ_N [6].

In the framework of $\mathcal{N} = 1$ global supersymmetry, the modulus τ is a chiral supermultiplet, its scalar component is restricted to the upper half of the complex plane, and the action takes the form

$$S = \int d^4x d^2\theta d^2\bar{\theta} \mathcal{K}(\tau, \bar{\tau}; \Phi_I, \bar{\Phi}_I) + \left[\int d^4x d^2\theta \mathcal{W}(\tau, \Phi_I) + \text{H.c.} \right], \quad (\text{A9})$$

where the Kähler potential $\mathcal{K}(\tau, \bar{\tau}; \Phi_I, \bar{\Phi}_I)$ is a real gauge-invariant function of the chiral superfields τ , Φ_I and their conjugates, and the superpotential $\mathcal{W}(\tau, \Phi_I)$ is a holomorphic gauge invariant function of the chiral superfields τ , Φ_I . Under the action of $\gamma \in \Gamma$, the superfield Φ_I has the following nonlinear transformation:

$$\Phi_I \xrightarrow{\gamma} (c\tau + d)^{-k_I} \rho_I(\gamma) \Phi_I, \quad (\text{A10})$$

where the weight k_I is an integer and ρ_I is a unitary representation of the finite modular group Γ_N or Γ'_N . The Kähler potential is assumed to take the following minimal form:

$$\mathcal{K}(\tau, \bar{\tau}; \Phi_I, \bar{\Phi}_I) = -h \log(-i\tau + i\bar{\tau}) + \sum_I \frac{\bar{\Phi}_I \Phi_I}{(-i\tau + i\bar{\tau})^{k_I}}, \quad (\text{A11})$$

which is invariant up to Kähler transformations. It yields the kinetic terms of for the scalar components of τ and Φ_I after the modulus acquires a *v*ev.

Concerning the superpotential $\mathcal{W}(\tau, \Phi_I)$, it can be expressed as follows:

$$\mathcal{W}(\tau, \Phi_I) = \sum_n Y_{I_1 \dots I_n}(\tau) \Phi_{I_1} \dots \Phi_{I_n}. \quad (\text{A12})$$

Modular invariance of $\mathcal{W}$ requires that $Y_{I_1 \dots I_n}(\tau)$ should be a modular form of weight k_Y and level N transforming in the representation ρ_Y of Γ_N (or Γ'_N), i.e.,

$$Y_{I_1 \dots I_n}(\tau) \xrightarrow{\gamma} Y_{I_1 \dots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 \dots I_n}(\tau). \quad (\text{A13})$$

The modular weights and the representations should fulfill the following conditions:

$$k_Y = k_{I_1} + \dots + k_{I_n}, \quad \rho_Y \otimes \rho_{I_1} \otimes \dots \otimes \rho_{I_n} \supset \mathbf{1}, \quad (\text{A14})$$

where $\mathbf{1}$ denotes the trivial singlet of Γ_N (or Γ'_N).

In the present work, we are concerned with the inhomogeneous finite modular group S_3 . The group $\Gamma_2 \cong S_3$ is the permutation group of order 3 with 6 elements, which can be expressed in terms of the two S and T generators satisfying the following relations [45]:

$$S^2 = T^2 = (ST)^3 = 1. \quad (\text{A15})$$

The six elements of $\Gamma_2 \cong S_3$ can be grouped into three conjugacy classes:

$$1C_1 = \{1\}, \quad 3C_2 = \{S, T, TST\}, \quad 2C_3 = \{ST, TS\}, \quad (\text{A16})$$

where nC_k stands for the conjugacy class of k elements of order n . The irreducible representations of the finite modular S_3 group are two singlets $\mathbf{1}$ and $\mathbf{1}'$, and one doublet $\mathbf{2}$. Here we work in the basis of diagonal matrix representation for the T generator. The representation matrices for the S and T generators in the three S_3 irreducible representations take the form:

$$\begin{aligned} \mathbf{1}: \rho_1(S) &= 1, & \rho_1(T) &= 1, \\ \mathbf{1}': \rho_{1'}(S) &= -1, & \rho_{1'}(T) &= -1, \\ \mathbf{2}: \rho_2(S) &= -\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, & \rho_2(T) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (\text{A17})$$

The tensor product rules between the S_3 irreducible representations are given by

$$\mathbf{1} \otimes \mathbf{1}' = \mathbf{1}', \quad \mathbf{1}^a \otimes \mathbf{2} = \mathbf{2}, \quad \mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}, \quad (\text{A18})$$

where $a, b = 0, 1$ and we denote $\mathbf{1}^0 \equiv \mathbf{1}$ and $\mathbf{1}^1 \equiv \mathbf{1}'$. Regarding the product of the singlet $\mathbf{1}'$ with a doublet, we have

$$\mathbf{1}' \otimes \mathbf{2} = \mathbf{2} \sim \theta \begin{pmatrix} \varphi_2 \\ -\varphi_1 \end{pmatrix}, \quad (\text{A19})$$

whereas the tensor product rule of two S_3 doublets takes the form:

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}, \quad \text{with} \quad \begin{cases} \mathbf{1} = \theta_1 \varphi_1 + \theta_2 \varphi_2, \\ \mathbf{1}' = \theta_1 \varphi_2 - \theta_2 \varphi_1, \\ \mathbf{2} = \begin{pmatrix} \theta_2 \varphi_2 - \theta_1 \varphi_1 \\ \theta_1 \varphi_2 + \theta_2 \varphi_1 \end{pmatrix}. \end{cases}$$

In the finite modular S_3 group, there are two linearly independent modular forms of the lowest nontrivial weight 2, which can be accommodated into a S_3 doublet $\mathbf{2}$ of S_3 , and the doublet is given by

$$Y_2^{(2)} = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \quad (\text{A20})$$

where the modular forms $Y_1(\tau)$ and $Y_2(\tau)$ take the form [46]:

$$Y_1(\tau) = \frac{i}{4\pi} \left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - 8 \frac{\eta'(2\tau)}{\eta(2\tau)} \right],$$

$$Y_2(\tau) = \frac{\sqrt{3}i}{4\pi} \left[\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} \right]. \quad (\text{A21})$$

Furthermore, $\eta(\tau)$ is the Dedekind function which is defined as follows:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q \equiv e^{2\pi i \tau}. \quad (\text{A22})$$

Then, the modular forms $Y_{1,2}(\tau)$ can be expressed as follows [47]:

$$Y_1(\tau) = 1/8 + 3q + 3q^2 + 12q^3 + 3q^4 + 18q^5 + 12q^6 + 24q^7 + 3q^8 + 39q^9 + 18q^{10} \dots,$$

$$Y_2(\tau) = \sqrt{3}q^{1/2}(1 + 4q + 6q^2 + 8q^3 + 13q^4 + 12q^5 + 14q^6 + 24q^7 + 18q^8 + 20q^9 \dots). \quad (\text{A23})$$

The modular multiplets of level $N = 2$ up to weight 8 are given by

$$Y_1^{(4)} = (Y_2^{(2)} Y_2^{(2)})_1 = (Y_{2,1}^{(2)})^2 + (Y_{2,2}^{(2)})^2, \quad Y_2^{(4)} = (Y_2^{(2)} Y_2^{(2)})_2 = \begin{pmatrix} (Y_{2,2}^{(2)})^2 - (Y_{2,1}^{(2)})^2 \\ 2Y_{2,1}^{(2)} Y_{2,2}^{(2)} \end{pmatrix},$$

$$Y_1^{(6)} = (Y_2^{(2)} Y_2^{(4)})_1 = Y_{2,1}^{(2)} Y_{2,1}^{(4)} + Y_{2,2}^{(2)} Y_{2,2}^{(4)}, \quad Y_{1'}^{(6)} = (Y_2^{(2)} Y_2^{(4)})_{1'} = Y_{2,1}^{(2)} Y_{2,2}^{(4)} - Y_{2,2}^{(2)} Y_{2,1}^{(4)},$$

$$Y_2^{(6)} = (Y_2^{(2)} Y_1^{(4)})_2 = \begin{pmatrix} Y_{2,1}^{(2)} Y_1^{(4)} \\ Y_{2,2}^{(2)} Y_1^{(4)} \end{pmatrix}, \quad Y_1^{(8)} = (Y_1^{(4)} Y_1^{(4)})_1 = (Y_1^{(4)})^2,$$

$$Y_{2a}^{(8)} = (Y_1^{(4)} Y_2^{(4)})_2 = \begin{pmatrix} Y_1^{(4)} Y_{2,1}^{(4)} \\ Y_1^{(4)} Y_{2,2}^{(4)} \end{pmatrix}, \quad Y_{2b}^{(8)} = (Y_2^{(4)} Y_2^{(4)})_2 = \begin{pmatrix} (Y_{2,2}^{(4)})^2 - (Y_{2,1}^{(4)})^2 \\ 2Y_{2,1}^{(4)} Y_{2,2}^{(4)} \end{pmatrix}. \quad (\text{A24})$$

[1] F. Feruglio, Are neutrino masses modular forms?, *From My Vast Repertoire ...: Guido Altarelli's Legacy* (World Scientific, Singapore, 2019), pp. 227–266.

[2] G.-J. Ding and S. F. King, Neutrino mass and mixing with modular symmetry, *Rep. Prog. Phys.* **87**, 084201 (2024).

[3] P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov, Modular A_5 symmetry for flavour model building, *J. High Energy Phys.* **04** (2019) 174.

[4] S. F. King and Y.-L. Zhou, Trimaximal TM_1 mixing with two modular S_4 groups, *Phys. Rev. D* **101**, 015001 (2020).

- [5] H. Okada and Y. Orikasa, Modular S_3 symmetric radiative seesaw model, *Phys. Rev. D* **100**, 115037 (2019).
- [6] X.-G. Liu and G.-J. Ding, Neutrino masses and mixing from double covering of finite modular groups, *J. High Energy Phys.* **08** (2019) 134.
- [7] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, and T. H. Tatsuishi, Modular S_3 -invariant flavor model in SU(5) grand unified theory, *Prog. Theor. Exp. Phys.* **2020**, 053B05 (2020).
- [8] X. Du and F. Wang, SUSY breaking constraints on modular flavor S_3 invariant SU(5) GUT model, *J. High Energy Phys.* **02** (2021) 221.
- [9] M. Abbas, Fermion masses and mixing in modular A_4 symmetry, *Phys. Rev. D* **103**, 056016 (2021).
- [10] P. P. Novichkov, J. T. Penedo, and S. T. Petcov, Fermion mass hierarchies, large lepton mixing and residual modular symmetries, *J. High Energy Phys.* **04** (2021) 206.
- [11] K. Ishiguro, H. Okada, and H. Otsuka, Residual flavor symmetry breaking in the landscape of modular flavor models, *J. High Energy Phys.* **09** (2022) 072.
- [12] M.-C. Chen, S. F. King, O. Medina, and J. W. F. Valle, Quark-lepton mass relations from modular flavor symmetry, *J. High Energy Phys.* **02** (2024) 160.
- [13] D. Meloni and M. Parriciatu, A simplest modular S_3 model for leptons, *J. High Energy Phys.* **09** (2023) 043.
- [14] T. Kobayashi, T. Nomura, H. Okada, and H. Otsuka, Modular flavor models with positive modular weights: A new lepton model building, *J. High Energy Phys.* **01** (2024) 121.
- [15] G.-J. Ding, S.-Y. Jiang, S. F. King, J.-N. Lu, and B.-Y. Qu, Pati-Salam models with A_4 modular symmetry, *J. High Energy Phys.* **08** (2024) 134.
- [16] M. Belfkir, M. A. Lualidi, and S. Nasri, Fermion masses and mixing in Pati-Salam unification with S_3 modular symmetry, *Prog. Theor. Exp. Phys.* **2025**, 033B05 (2025).
- [17] S. Marciano, D. Meloni, and M. Parriciatu, Minimal seesaw and leptogenesis with the smallest modular finite group, *J. High Energy Phys.* **05** (2024) 020.
- [18] G.-J. Ding and J. W. F. Valle, The symmetry approach to quark and lepton masses and mixing, *Phys. Rep.* **1109**, 1 (2025).
- [19] S. J. D. King and S. F. King, Fermion mass hierarchies from modular symmetry, *J. High Energy Phys.* **09** (2020) 043.
- [20] P. Minkowski, $\mu \rightarrow e\gamma$ at a rate of one out of 10^9 muon decays?, *Phys. Lett.* **67B**, 421 (1977).
- [21] T. Yanagida, Horizontal gauge symmetry and masses of neutrinos, *Conf. Proc. C* **7902131**, 95 (1979).
- [22] M. Gell-Mann, P. Ramond, and R. Slansky, Complex spinors and unified theories, *Conf. Proc. C* **790927**, 315 (1979).
- [23] S. L. Glashow, The future of elementary particle physics, *NATO Sci. Ser. B* **61**, 687 (1980).
- [24] R. N. Mohapatra and G. Senjanovic, Neutrino mass and spontaneous parity nonconservation, *Phys. Rev. Lett.* **44**, 912 (1980).
- [25] J. Schechter and J. W. F. Valle, Neutrino masses in SU(2) $\times$ U(1) theories, *Phys. Rev. D* **22**, 2227 (1980).
- [26] J. C. Pati and A. Salam, Lepton number as the fourth color, *Phys. Rev. D* **10**, 275 (1974); **11**, 703(E) (1975).
- [27] C. D. Froggatt and H. B. Nielsen, Hierarchy of quark masses, Cabibbo angles and CP violation, *Nucl. Phys.* **B147**, 277 (1979).
- [28] Z. G. Berezhiani, The weak mixing angles in gauge models with horizontal symmetry: A new approach to quark and lepton masses, *Phys. Lett.* **129B**, 99 (1983).
- [29] S. Rajpoot, See-saw masses for quarks and leptons in an ambidextrous electroweak interaction model, *Mod. Phys. Lett. A* **02**, 307 (1987); **02**, 541(E) (1987).
- [30] A. Davidson and K. C. Wali, Universal Seesaw mechanism?, *Phys. Rev. Lett.* **59**, 393 (1987).
- [31] A. Davidson and K. C. Wali, SU(5)-L $\times$ SU(5)-R hybrid unification, *Phys. Rev. Lett.* **58**, 2623 (1987).
- [32] G. Valencia and S. Willenbrock, Quark-lepton unification and rare meson decays, *Phys. Rev. D* **50**, 6843 (1994).
- [33] A. D. Smirnov, Mass limits for scalar and gauge leptoquarks from $K_L^0 \rightarrow e^\mp \mu^\pm$, $B^0 \rightarrow e^\mp \tau^\pm$ decays, *Mod. Phys. Lett. A* **22**, 2353 (2007).
- [34] T. P. Dutka and J. Gargalionis, Dimension-five baryon-number violation in low-scale Pati-Salam models, *Phys. Rev. D* **107**, 035019 (2023).
- [35] R. N. Mohapatra, Mechanism for understanding small neutrino mass in superstring theories, *Phys. Rev. Lett.* **56**, 561 (1986).
- [36] S. F. King, Right-handed Neutrinos: Seesaw Models and Signatures, *arXiv:2502.07877*.
- [37] Z.-z. Xing, Flavor structures of charged fermions and massive neutrinos, *Phys. Rep.* **854**, 1 (2020).
- [38] R. Workman *et al.* (Particle Data Group), Review of particle physics, *Prog. Theor. Exp. Phys.* **2022**, 083C01 (2022).
- [39] P. F. de Salas, D. V. Forero, S. Gariazzo, P. Martínez-Miravé, O. Mena, C. A. Ternes, M. Tórtola, and J. W. F. Valle, 2020 Global reassessment of the neutrino oscillation picture, *J. High Energy Phys.* **02** (2021) 071.
- [40] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, I. Martinez-Soler, J. a. P. Pinheiro, and T. Schwetz, NuFit-6.0: Updated global analysis of three-flavor neutrino oscillations, *J. High Energy Phys.* **12** (2024) 216.
- [41] N. Aghanim *et al.* (Planck Collaboration), Planck 2018 results. VI. Cosmological parameters, *Astron. Astrophys.* **641**, A6 (2020). (2021)].
- [42] S. Abe *et al.* (KamLAND-Zen Collaboration), Search for the Majorana nature of neutrinos in the inverted mass ordering region with KamLAND-Zen, *Phys. Rev. Lett.* **130**, 051801 (2023).
- [43] G. Adhikari *et al.* (nEXO Collaboration), nEXO: Neutrino-less double beta decay search beyond 10^{28} year half-life sensitivity, *J. Phys. G* **49**, 015104 (2022).
- [44] R. de Adelhart Toorop, F. Feruglio, and C. Hagedorn, Finite modular groups and lepton mixing, *Nucl. Phys.* **B858**, 437 (2012).
- [45] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, and M. Tanimoto, Non-Abelian discrete symmetries in particle physics, *Prog. Theor. Phys. Suppl.* **183**, 1 (2010).
- [46] T. Kobayashi, K. Tanaka, and T. H. Tatsuishi, Neutrino mixing from finite modular groups, *Phys. Rev. D* **98**, 016004 (2018).
- [47] C.-C. Li and G.-J. Ding, Eclectic flavor group $\Delta(27) \rtimes S_3$ and lepton model building, *J. High Energy Phys.* **03** (2024) 054.

Correction: Reference [16] was missing source information and has been fixed.