

Low-energy DD scattering in lattice QCD

Pan-Pan Shi^{1,2,3,*} Feng-Kun Guo^{2,4,†} Chuan Liu^{5,6,7} Liuming Liu^{8,4,‡} Peng Sun,^{8,4}
Jia-Jun Wu^{4,3,§} and Hanyang Xing^{8,4}

¹*Instituto de Física Corpuscular (IFIC), CSIC-UV, 46980 Paterna, Valencia, Spain*

²*CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

³*Southern Center for Nuclear-Science Theory (SCNT), Institute of Modern Physics, Chinese Academy of Sciences, Huizhou 516000, China*

⁴*School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*

⁵*School of Physics, Peking University, Beijing 100871, China*

⁶*Center for High Energy Physics, Peking University, Beijing 100871, China*

⁷*Collaborative Innovation Center of Quantum Matter, Beijing 100871, China*

⁸*Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China*



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We present the first lattice QCD calculation of single-channel DD scattering with quantum numbers $I(J^P) = 1(0^+)$ and $0(1^-)$. The calculation is performed on the $2 + 1$ flavor Wilson-Clover ensembles with a lattice spacing $a \simeq 0.077$ fm and two different pion masses, $m_\pi \simeq 207$ and 305 MeV. The scattering parameters are determined using the Lüscher finite-volume method. Our results indicate a weak repulsive interaction in the $1(0^+)$ channel and a slightly attractive interaction in the $0(1^-)$ channel. The S -wave isovector DD scattering length and effective range, extrapolated to the physical pion mass, are $(-0.25 \pm 0.08 \pm 0.12)$ fm and $(-5.7 \pm 4.5 \pm 1.7)$ fm, respectively.

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I. INTRODUCTION

Quantum chromodynamics (QCD) is the fundamental theory that describes the strong interaction in terms of the quark and gluon degrees of freedom. However, in the low-energy region (at the energy scale close to Λ_{QCD}), a perturbative expansion in powers of the strong coupling constant fails, and thus, alternative methods are required to study the strong interaction at this scale. One such method is lattice QCD (LQCD), which allows for investigations of the strong interaction in the nonperturbative regime from first principles. In LQCD, hadron scattering information can be extracted using Lüscher's finite-volume method [1–3] and its generalizations (for a review, see Ref. [4]). The method relates the energy spectrum of a two-particle system in the finite volume and the scattering information of the two particles in the continuum. Following this path,

tremendous progress has been made with LQCD in understanding hadron spectra and interactions; see recent reviews in Refs. [4–7]. As for scatterings of identical spinless mesons, $\pi\pi$ scattering has been extensively investigated by several LQCD groups for various isospin channels $I = 0, 1$, and 2 [8–16], and KK scattering in the $I = 1$ channel has also been explored in Refs. [15–18]. Here, we will focus on DD scattering in the low-energy regime.

DD scattering is related to the study of exotic states in the doubly charmed sector. In particular, the doubly charmed exotic meson $T_{cc}(3875)^+$, which couples strongly to the isoscalar DD^* in the S wave, was observed in the $D^0 D^0 \pi^+$ invariant mass distribution by LHCb [19,20]. Information on DD scattering can provide insights into the DD final state interactions that influence predictions on the T_{cc}^+ width [21–23].¹ Actually, the phenomenological calculation in Ref. [21] displayed a sizable impact of the DD final state interaction on the T_{cc} width based on the one-boson-exchange model [26]. In addition, the DDK three-body bound state has been predicted in both continuum [27–30] and finite-volume formalisms [31,32]. So far, in the inclusive decay processes of $\Upsilon(1S)$ and $\Upsilon(2S)$ [33], the Belle

*Contact author: Panpan.Shi@ific.uv.es

†Contact author: fkguo@itp.ac.cn

‡Contact author: liuming@impcas.ac.cn

§Contact author: wujiajun@ucas.ac.cn

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¹For explorations on the influence of the analogous $D\bar{D}$ final state interaction on the open-charm decay width for $X(3872) \rightarrow D^0 \bar{D}^0 \pi$, see Refs. [24,25].

Collaboration has not found significant signals associated with this state in the $D^+D_s^{*+}$ invariant mass distribution. To further investigate the possible DDK bound state, model-independent input on DD two-body scattering is essential.

LQCD has been employed to study the interaction of the DD^* system. The DD^* scattering [34–36] and the $DD^* - D^*D^*$ coupled-channel scattering [37] have been explored using the interpolating operators of DD^* and both DD^* and D^*D^* , respectively. Additionally, the quark-mass dependence of T_{cc}^+ has been analyzed with DD^* and D^*D^* interpolating operators [38] (see also Ref. [39]). While analyzing the LQCD data for the DD^* system, it was noted that the one-pion exchange long-range force (the associated left-hand cut and a zero of the scattering amplitude) could significantly affect the extraction of scattering information [40]. Several approaches have been proposed to address this issue [41–45]. One natural solution is to include the $DD\pi$ three-body operator in the DD^* scattering process. The formalism for the $DD\pi \rightarrow DD\pi$ three-body scattering can be utilized to analyze the lattice data, allowing the extraction of the $D^*D \rightarrow D^*D$ amplitude at energies where D^* is treated as a $D\pi$ bound state or resonance. This method remains valid below the $DD\pi$ left-hand cut. However, the lack of isovector DD and isoscalar $DD\pi$ scattering information hinders further precise studies of the T_{cc}^+ [44].

This work presents the first LQCD calculation of DD scattering based on the $2+1$ flavor full QCD ensembles. The Lüscher finite-volume method is utilized to connect the finite-volume energy levels to infinite-volume scattering parameters. We compute the energy levels with the DD interpolating operators for both $I=1$ and $I=0$. The scattering length and effective range for both S - and P -wave channels are extracted. The calculations are performed at two different pion masses, $m_\pi \simeq 305$ MeV and $m_\pi \simeq 207$ MeV. The pion-mass dependence of the S -wave scattering length and effective range is explored.

This paper is organized as follows. In Sec. II, details of the lattice setup and methodology are introduced. In Sec. III, we discuss the single- and two-meson operators as well as the method for the extraction of the energy levels from correlation functions. The formula, involving the Lüscher equation modified by the dispersion relation and the parametrization of phase shift, is discussed in Sec. IV. The S - and P -wave scattering information for the isovector and isoscalar DD channels, including the scattering lengths, effective ranges, and the phase shifts, are presented in Sec. V. We explore the pion-mass dependence of the S -wave scattering length and effective range in Sec. VI. A conclusion is given in Sec. VII. Finally, the Appendix contains details of the derivation of the Lüscher equation with the modified dispersion relation.

II. LATTICE SETUP

The results presented in this paper are based on the gauge configurations generated by the CLQCD Collaboration

with $2+1$ dynamical quark flavors using the tadpole improved tree-level Symanzik gauge action and Clover fermion action [46]. Numerous studies have been performed on these configurations; see, e.g., Refs. [47–53]. In this work, we use four ensembles with the same lattice spacing, $a = 0.07746$ fm, and two different pion masses, $m_\pi \simeq 305$ MeV and $m_\pi \simeq 207$ MeV. Details of the ensembles are listed in Table I. Among these ensembles, F32P21/F48P21 and F32P30/F48P30 are two couples that share the same pion mass but have different volumes to obtain more kinematic points in the finite-volume spectra, rendering a more stable and precise determination of the scattering parameters. The valence charm quark action is the same as the light/strange quark action used in the ensembles, and the charm quark mass is tuned to reproduce the spin-averaged $1S$ -charmonium mass, $m_{av} = (m_{\eta_c} + 3m_{J/\psi})/4$.

The quark propagators are computed with the distillation quark smearing method [54], which improves the precision and allows for the inclusion of many interpolating operators in the calculation of the correlation functions. The smearing operator is composed of a small number (N_{ev}) of the eigenvectors associated with the N_{ev} lowest eigenvalues of the three-dimensional Laplacian defined in terms of the HYP (hypercubic)-smeared gauge field. The number of eigenvectors N_{ev} is 100 for all four ensembles. We also tried $N_{ev} = 200$ for the ensembles F48P21 and F48P30, and the resulting energies were consistent with those obtained using $N_{ev} = 100$. The statistical uncertainty is estimated by the bootstrap method with 4000 bootstrap samples.

III. CALCULATION OF THE ENERGY LEVELS

In LQCD, the finite-volume spectra are extracted from the two-point correlation functions

$$C_{ij}(t) = \sum_{t_s} \langle \mathcal{O}_i(t+t_s) \mathcal{O}_j^\dagger(t_s) \rangle, \quad (1)$$

where $\mathcal{O}_i$ and $\mathcal{O}_j$ are the interpolating operators carrying the quantum numbers of the system of interest. The source time t_s spans all time slices to increase statistics.

The single D -meson operators are $\mathcal{O}_{D^+}(\mathbf{x}, t) = \bar{d}\gamma_5 c(\mathbf{x}, t)$ and $\mathcal{O}_{D^0}(\mathbf{x}, t) = \bar{u}\gamma_5 c(\mathbf{x}, t)$ for the charged and neutral D , respectively. The momentum-projected operator

TABLE I. Parameters of the ensembles, including the name, the lattice spacing a , the pion mass m_π , the volume size $L^3 \times T$, the number of configurations N_{conf} , and the value of $m_\pi L$.

Ensembles	a [fm]	m_π [MeV]	$(L/a)^3 \times T/a$	N_{conf}	$m_\pi L$
F32P30	0.07746(18)	305.2(13)	$32^3 \times 96$	371	3.83(2)
F48P30	0.07746(18)	305.4(9)	$48^3 \times 96$	201	5.75(2)
F32P21	0.07746(18)	207.9(22)	$32^3 \times 64$	201	2.61(3)
F48P21	0.07746(18)	207.2(11)	$48^3 \times 96$	223	3.90(2)

TABLE II. Parameters for the dispersion relation of the D meson.

	F32P30	F48P30	F32P21	F48P21
m_D [MeV]	1966.6(8)	1966.0(6)	1903.9(16)	1900.3(5)
Z	0.926(7)	0.946(9)	0.941(1)	0.976(6)

is defined as $\mathcal{O}(\mathbf{p}, t) = \sum_{\mathbf{x}} e^{-i\mathbf{p}\cdot\mathbf{x}} \mathcal{O}(\mathbf{x}, t)$. The dispersion relation of the D meson $E^2 = m_D^2 + Zp^2$, with $p \equiv \sqrt{\mathbf{p}^2}$, is investigated by calculating the single-particle energy at the five lowest momenta on lattice: $\mathbf{p} = (0,0,0)$, $(0,0,1)$, $(0,1,1)$, $(1,1,1)$, and $(0,0,2)$ (in units of $2\pi/L$). The parameters m_D and Z , listed in Table II, are determined by fitting the five energies to the dispersion relation. As an example, the five energies and the dispersion relation for the ensemble F48P21 are shown in Fig. 1. In the continuum limit, Z should be unity. However, due to lattice artifacts, it may take a different value. The original Lüscher's formula assumes the continuum dispersion relation. In order to account for the effects of the deviation from the continuum dispersion relation, we modify the Lüscher's formula as elaborated in the Appendix.

The interacting energy levels of two particles are extracted from the correlation functions of the two-particle operators. The interpolating operators for the isovector and isoscalar DD systems are given by

$$\mathcal{O}_{DD,I=1}^{\mathbf{P},\Lambda,\mu} = \frac{1}{\sqrt{2}} \sum_{\mathbf{p}_1, \mathbf{p}_2} \mathcal{C}(\mathbf{P}, \Lambda, \mu, \mathbf{p}_1, \mathbf{p}_2) (\mathcal{O}_{D^+}(\mathbf{p}_1) \mathcal{O}_{D^0}(\mathbf{p}_2) + \mathcal{O}_{D^0}(\mathbf{p}_1) \mathcal{O}_{D^+}(\mathbf{p}_2)), \quad (2)$$

$$\mathcal{O}_{DD,I=0}^{\mathbf{P},\Lambda,\mu} = \frac{1}{\sqrt{2}} \sum_{\mathbf{p}_1, \mathbf{p}_2} \mathcal{C}(\mathbf{P}, \Lambda, \mu, \mathbf{p}_1, \mathbf{p}_2) (-\mathcal{O}_{D^+}(\mathbf{p}_1) \mathcal{O}_{D^0}(\mathbf{p}_2) + \mathcal{O}_{D^0}(\mathbf{p}_1) \mathcal{O}_{D^+}(\mathbf{p}_2)), \quad (3)$$

where $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ is the total momentum; Λ denotes the irreducible representation (irrep) of the octahedral group and its subgroups, which are the spatial rotational symmetry groups on lattice; μ denotes the irrep row; and

$\mathcal{C}(\mathbf{P}, \Lambda, \mathbf{p}_1, \mathbf{p}_2)$'s are the Clebsch-Gordan coefficients. In this work, we compute the energies only in the rest frame ($\mathbf{P} = \mathbf{0}$) and the momenta of the single particles up to $|\mathbf{p}_1| = |\mathbf{p}_2| = 2$. For $I = 1$, the operators reside in the A_1^+ , E^+ , and T_2^+ irreps, while for $I = 0$, the operators are in the T_1^- , T_2^- , and A_2^- irreps. The relevant Clebsch-Gordan coefficients are provided in Appendix A of Ref. [11].

We do not include compact tetraquarklike interpolating operators in the present calculation. Local color-neutral four-quark operators can be Fierz rearranged into products of quark bilinears in general [55]. For lattice diquark-antidiquark interpolators, this relation and its limitations for smeared and momentum-projected operators are discussed explicitly in Ref. [56]. Thus, such operators do not introduce an additional two-hadron channel below the DD threshold, although they may have different overlap with a compact component of the finite-volume eigenstates. In the near-threshold region analyzed here, the lowest relevant two-meson channel is the DD channel, while other meson-meson components with the same valence content correspond to higher thresholds or excited states. We therefore expect the low-lying levels entering the DD scattering analysis to be dominated by the DD interpolators included in our basis. This expectation is supported by the lattice study of Ref. [57], where in doubly charmed channels, including the DD_s sector, adding compact tetraquarklike operators to a meson-meson basis did not significantly change the extracted low-lying finite-volume spectra. A dedicated calculation including such operators in the present DD setup would be useful to quantify this possible source of systematic uncertainty.

After computing the correlation functions, we employ the generalized eigenvalue problem (GEVP) method [58] to determine the energy levels in each irrep. The GEVP is solved as follows:

$$\mathcal{C}(t)v_n(t, t_0) = \lambda_n(t, t_0)\mathcal{C}(t_0)v_n(t, t_0). \quad (4)$$

Here, t_0 is set to 2, and we have checked that different choices of t_0 do not lead to a noticeable impact on the determination of the energy levels. The energy levels

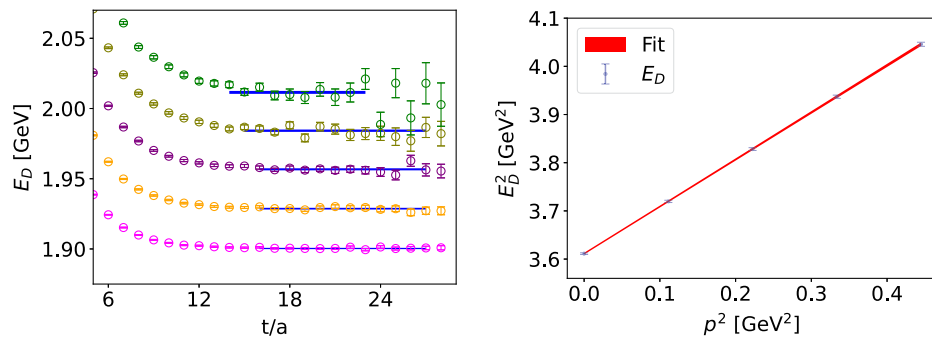


FIG. 1. Effective energies and the dispersion relation of the D meson for the ensemble F48P21. The left panel presents five energies of the D meson at the five lowest momenta, and the right panel presents a fit of the dispersion relation to these energies.

are extracted from the ratio of the eigenvalues $\lambda_n(t)$ to the single-particle correlation functions

$$R_n(t) = \frac{\lambda_n(t)}{\mathcal{C}_D^{\mathbf{p}_n}(t)\mathcal{C}_D^{\mathbf{p}_n}(t)} = A'e^{-\Delta E_n t}, \quad (5)$$

where $\mathcal{C}_D^{\mathbf{p}_n}(t)$ is the single D correlation function with momentum $\mathbf{p}_n$, and ΔE_n is the splitting between the energies of the interacting DD system and two free D mesons. In the DD system, where the interacting energy levels are quite close to the free energy levels, this method can partially cancel correlated statistical fluctuations and reduce the contributions of excited states by fitting the ratio $R_n(t)$, and therefore, distinguish energy levels that nearly degenerate with the free ones. The interacting energies are then obtained by $E_n = \Delta E_n + E_n^{\text{free}}$, with E_n^{free} the free energy levels.

To evaluate systematic errors associated with the choice of the fit window, we perform fits to each GEVP eigenvalue over all time intervals $[t_0, t_1]$ within a broad range $[t_{\min}, t_{\max}]$, subject to the constraint $t_1 - t_0 \geq 6/a$. For each fit result, we assign a weight based on the Akaike information criterion (AIC) [59,60]:

$$\omega^l = \frac{1}{Z} e^{-\frac{1}{2}\text{AIC}^l}, \quad \text{AIC}^l = \chi_l^2 + 2n^{\text{para}} - n_{\text{data}}^l, \quad (6)$$

where l labels the fit window; n^{para} is the number of parameters in the fit formula, which is 2 in our analysis; and n_{data}^l is the number of data points in window l . The normalization constant Z is the sum of all ω^l , which makes ω^l interpretable as a probability distribution over the set of fits. This procedure is carried out for each bootstrap sample. For every bootstrap sample, we compute the weighted mean $\Delta \tilde{E}_n = \sum_l \omega^l \Delta E_{n,l}$. The central value and statistical uncertainty of $\Delta \tilde{E}_n$ are then given by the mean and standard deviation of $\Delta \tilde{E}_n$ across all bootstrap samples.

The systematic uncertainty is estimated from the weighted standard deviation over all fit windows, evaluated on one specific bootstrap sample (the one corresponding to the mean of all configurations):

$$\sigma_{n,\text{sys}} = \sqrt{\sum_l \omega_n^l (\Delta E_{n,l} - \Delta \tilde{E}_n)^2}. \quad (7)$$

The total uncertainty is obtained by adding the statistical and systematic uncertainties in quadrature.

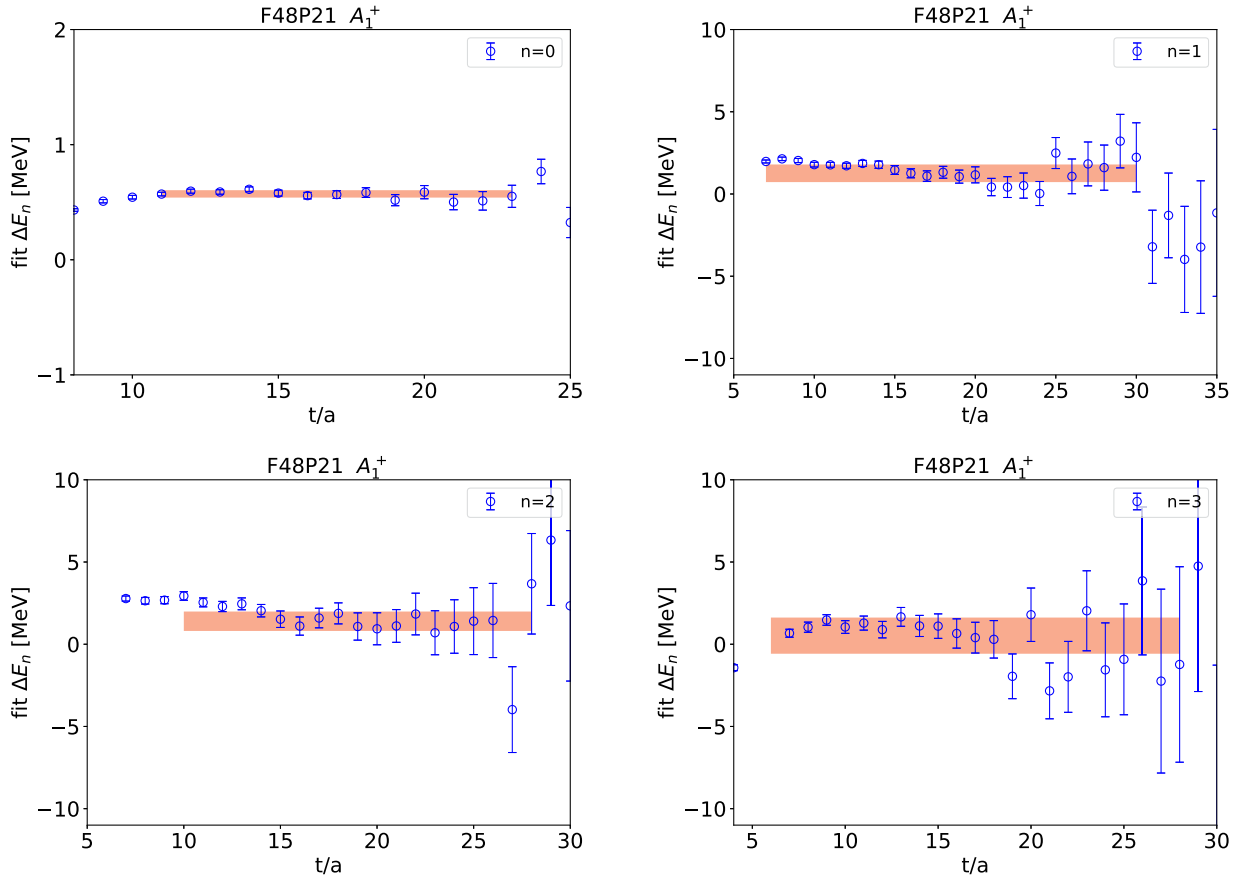
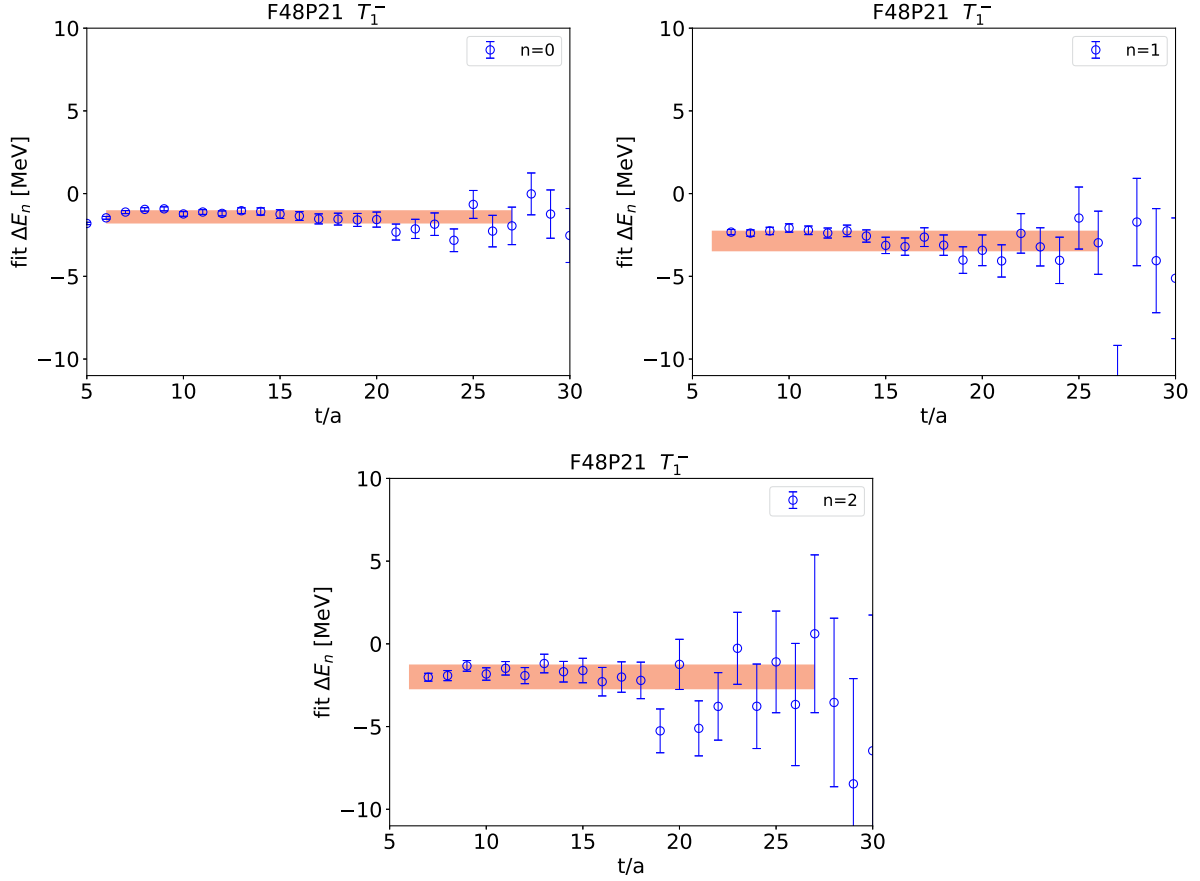


FIG. 2. Effective energies of ΔE_n for the A_1^+ irrep on the ensemble F48P21. The red horizontal bars represent the fitted values of ΔE_n with errors that include both statistical and systematic errors. The time range expanded by the bars represents the range within which various fit windows are chosen to do the fit.

FIG. 3. Same as Fig. 2 but for the T_1^- irrep.

The effective energies of the lowest four and three energy splittings ΔE_n for A_1^+ and T_1^- irreps on the F48P21 ensemble are presented in Figs. 2 and 3, respectively. The horizontal bars indicate the central value and total uncertainty of ΔE_n . The time ranges expanded by the horizontal bars represent $[t_{\min}, t_{\max}]$.

To estimate the systematic uncertainty in the subsequent scattering analysis (see Sec. IV), we randomly select 4000 sets of energy levels from different fit windows. Taking the A_1^+ irrep of F48P21 as an example, four energy levels are used to determine the scattering parameters. For these four levels, the numbers of fit windows are 28, 171, 91, and 153, respectively. Thus, the total number of combinations of energy levels taken from different windows is $28 \times 171 \times 91 \times 153$. We randomly select 4000 such combinations as inputs to the scattering analysis and take the variance among the 4000 results as the systematic error arising from the choice of fit windows.

The obtained energy levels for all irreps are presented in Figs. 4 and 5. The results show that, in the A_1^+ irreps, the lowest energy level for each pion mass is slightly above the DD threshold (see also Fig. 2), indicating a weak repulsive interaction in the S -wave $I = 1$ DD system. The energy levels in the E^+ and T_2^+ irreps almost coincide with

the free ones within the current statistical precision, indicating negligible interactions in the D wave and higher partial waves (the angular momenta of different irreps are listed in Table III). In the following, we ignore the mixing from higher partial waves and utilize the energy levels below the D^*D^* threshold to avoid possible contamination from D^*D^* , in the A_1^+ irrep, to determine the S -wave DD scattering parameters in the $I = 1$ channel. Based on similar observations in the T_1^- , T_2^- , and A_2^- irreps, we use the energy levels in the T_1^- irrep to determine the P -wave DD scattering parameters in the $I = 0$ channel.

IV. FORMALISM AND FITTING STRATEGY

For two-meson scattering, the partial-wave phase shift δ_l is obtained from the determinant of the Lüscher matrix,

$$\det(M_{lm,l'm'}(\tilde{p}) - \delta_{ll'}\delta_{mm'} \cot \delta_l) = 0, \quad (8)$$

where $\tilde{p} = pL/(2\pi)$ is related to the on-shell momentum p . The energy levels are calculated within the specific little group. By selecting a particular basis for the irrep Λ , the determinant, according to Schur's lemma, reduces to [3,14,61]

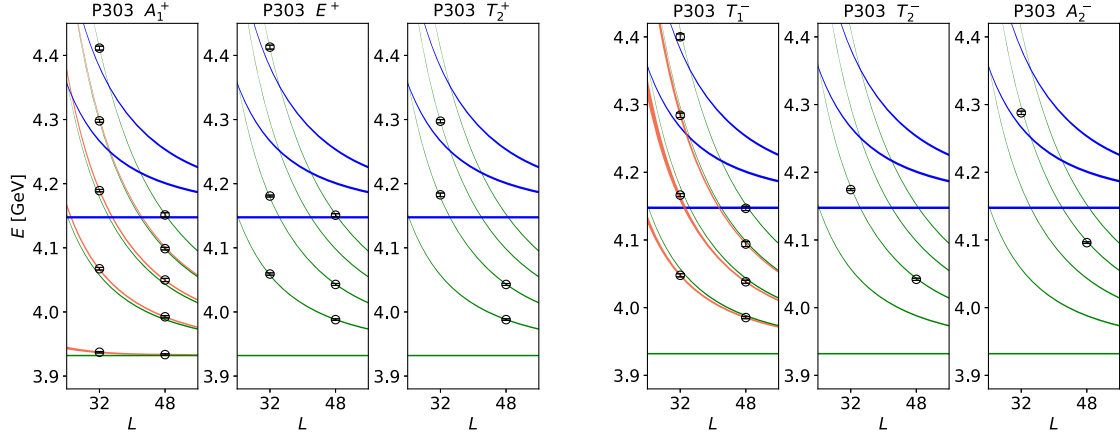


FIG. 4. Energy levels for the DD system with a pion mass $m_\pi \simeq 305$ MeV. The green and blue bands represent the free energy levels for DD and D^*D^* systems, respectively. These free energy levels are calculated using the continuum dispersion relation (i.e., $Z = 1$) with the parameters from the ensembles with lattice size $L = 48$. For the $L = 32$ lattice, the interacting energy levels are shifted based on the gap between the free energy levels in different finite volumes. The red bands denote the interacting energy levels fitted by the ERE parametrization. The width of the bands reflects both the statistical and systematic uncertainties.

$$\det(M_{ln,l'n'}^\Lambda(\tilde{p}) - \delta_{ll'}\delta_{nn'} \cot \delta_l) = 0, \quad (9)$$

where n and n' are indices indicating the occurrence of the lattice irrep Λ in the irrep of the $SO(3)$ group labeled by the angular momentum l . In the cases considered in this work, the number of occurrences is 1. To further simplify the Lüscher matrix in a specific irrep, we define the function

$$\omega_{js}(\tilde{p}) = \frac{1}{\pi^{3/2} \sqrt{2j+1}} \frac{\omega_1(p) + \omega_2(p)}{Z_1 \omega_2(p) + Z_2 \omega_1(p)} \frac{\mathcal{Z}_{js}(\tilde{p})}{\tilde{p}^{j+1}}, \quad (10)$$

where $\omega_{1,2}(p) \equiv \sqrt{m_{1,2}^2 + Z_{1,2} p^2}$, and $\mathcal{Z}_{js}$ is the zeta function. Compared to the original Lüscher formula, there is an extra factor $[\omega_1(p) + \omega_2(p)]/[Z_1 \omega_2(p) + Z_2 \omega_1(p)]$, accounting for the deviation of the dispersion relation from the continuum one. We refer to the Appendix for detailed discussion.

In this work, we focus on extracting the partial-wave phase shifts for $L = 0$ and $L = 1$, while the high partial-wave contributions are neglected. For the A_1^+ and T_1^- irreps, the determinants of Eq. (9) are reduced to

$$-\cot \delta_0 + \omega_{00}(\tilde{p}_A) = 0, \quad -\cot \delta_1 + \omega_{00}(\tilde{p}_T) = 0, \quad (11)$$

where the coefficients of $\omega_{js}(\tilde{p})$ are taken from Ref. [3], and the subscripts A and T denote the momenta for the A_1^+ and T_1^- irreps, respectively.

Instead of translating the phase shift from interacting energy levels via Eq. (11), we parametrize the phase shift using the effective range expansion (ERE) and constrain the relevant parameters by fitting the energy levels in different irreps. The ERE reads

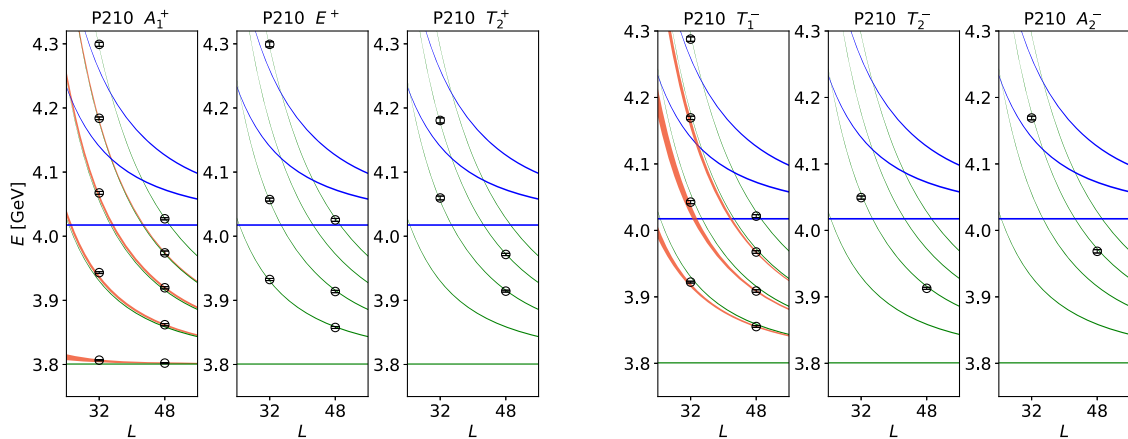


FIG. 5. Energy levels for the DD system with pion mass $m_\pi \simeq 207$ MeV for the case $Z = 1$. The color bands are the same as those in Fig. 4.

TABLE III. Irreps and the relative quantum numbers (orbital angular momentum L and parity P) for the DD system with $L \leq 4$ partial waves in the rest frame. For DD , the positive- and negative-parity irreps correspond to isospin $I = 1$ and $I = 0$, respectively.

Λ^P	A_1^+	A_2^-	E^+	T_1^-	T_2^+	T_2^-
L	0, 4	3	2, 4	1, 3	2, 4	3

$$\cot \delta_l = p^{-2l-1} \left(\frac{1}{a_l} + \frac{1}{2} r_l p^2 + \mathcal{O}(p^4) \right), \quad (12)$$

where a_l and r_l are the scattering length and effective range for the l wave, respectively. The parameters a_l and r_l are determined by minimizing the χ^2 , defined as

$$\chi^2 = \sum_{L,n,n'} [E_n(L) - E_n^{\text{sol}}(L, a_l, r_l)] C_{nn'}^{-1} [E_{n'}(L) - E_{n'}^{\text{sol}}(L, a_l, r_l)], \quad (13)$$

where $E_n(L)$ is the n th energy level obtained on the lattice with size L , $E_n^{\text{sol}}(L, a_l, r_l)$ is the n th solution of Eq. (11) with parameters a_l and r_l , and C is the covariance matrix of the energies $E_n(L)$. Two scenarios are considered to determine a_l and r_l : $Z \neq 1$, where Z is fixed to the values given in Table II, and $Z = 1$. In each case, the interacting energy levels are calculated using the formula $E_n = \Delta E_n + 2\sqrt{m_D^2 + Zq_n^2}$. To minimize possible coupled-channel effects from D^*D^* , we use only the first two energy levels and four energy levels for the ensembles F32P30 and F48P30 to fit the S -wave scattering length a_0 and effective range r_0 in the A_1^+ irrep. For the T_1^- irrep, the first one and three energy levels for the ensemble F32P30 and F48P30 are employed to get the corresponding P -wave ERE parameters, respectively. In addition, for the ensemble F48P21, the first four and three energy levels in the A_1^+ and T_1^- irreps are used to determine the S - and P -wave ERE parameters, respectively. The energy levels for the ensemble F32P21 are neglected due to the small volume.

V. SCATTERING ANALYSIS RESULTS

The S - and P -wave ERE parameters are shown in Table IV. In the case of S -wave scattering, both the scattering length a_0 and effective range r_0 are negative. In contrast, a_1 and r_1 for the P wave are positive. Near the free energy levels, the Lüscher's zeta function becomes highly sensitive to small variations in energy levels. Compared to the lowest interacting energy levels in A_1^+ irrep, the statistical uncertainties of other energy levels are enlarged by this sensitivity. Consequently, the parameters r_0 , a_1 , and r_1 have large statistical uncertainties.

As listed in Table IV, differences in the ERE parameters between the $Z \neq 1$ and $Z = 1$ cases are within 1σ . The $p \cot \delta_l$ and phase shifts as a function of p^2 for the $I(J^P) = 1(0^+)$ and $0(1^-)$ DD scattering calculated using $Z = 1$ are shown in Figs. 6 and 7, respectively. In our convention, as the energy E increases, the S -wave phase shift δ_0 decreases, while the P -wave phase shift δ_1 increases. The opposite signs for δ_0 and δ_1 indicate repulsive and attractive interactions for the S - and P -wave DD systems, respectively. Considering the heavy quark flavor symmetry, the weakly attractive interaction in the P -wave DD system is consistent with that in the BB system, as calculated using a coupled-channel Born-Oppenheimer approach [62].

VI. PION-MASS DEPENDENCE

Our results are obtained at two unphysically large pion masses. In order to determine the scattering length and effective range at the physical pion mass, we parametrize the a_0 and r_0 as functions of the pion mass and investigate their pion-mass dependence for the continuum dispersion relation (the $Z = 1$ case). We do not perform the extrapolation for the P -wave ERE parameters due to the large statistical and systematic uncertainties associated with them.

There are only two different pion masses, 207 and 305 MeV, in our calculations. Since the interaction is rather weak, it is reasonable to assume that the light-quark-mass dependence of both scattering length and effective

TABLE IV. ERE parameters for the S - and P -wave DD scattering. The χ^2 value per degree of freedom (d.o.f.) for the fits is also given. The cases for $Z \neq 1$ and $Z = 1$ denote that the discrete and continuum dispersion relations are used to determine a_l and r_l , respectively. The first and second uncertainties for the values of a_l and r_l denote the statistical and systematic errors, respectively.

	F32P30/F48P30 ($m_\pi = 305$ MeV)		F48P21 ($m_\pi = 207$ MeV)	
	$Z \neq 1$	$Z = 1$	$Z \neq 1$	$Z = 1$
a_0 [fm]	-0.25(6)(1)	-0.25(5)(1)	-0.25(5)(6)	-0.25(5)(8)
r_0 [fm]	-1.8(13)(2)	-2.0(13)(2)	-4.6(29)(11)	-4.5(30)(12)
$\chi^2/\text{d.o.f.}$	1.28	1.66	1.02	1.04
a_1 [fm ³]	11(9)(10)	12(9)(7)	6(21)(14)	6(10)(8)
r_1 [fm ⁻¹]	20.2(39)(95)	20.1(38)(69)	12.3(59)(65)	11.9(60)(36)
$\chi^2/\text{d.o.f.}$	1.19	1.23	1.08	1.03

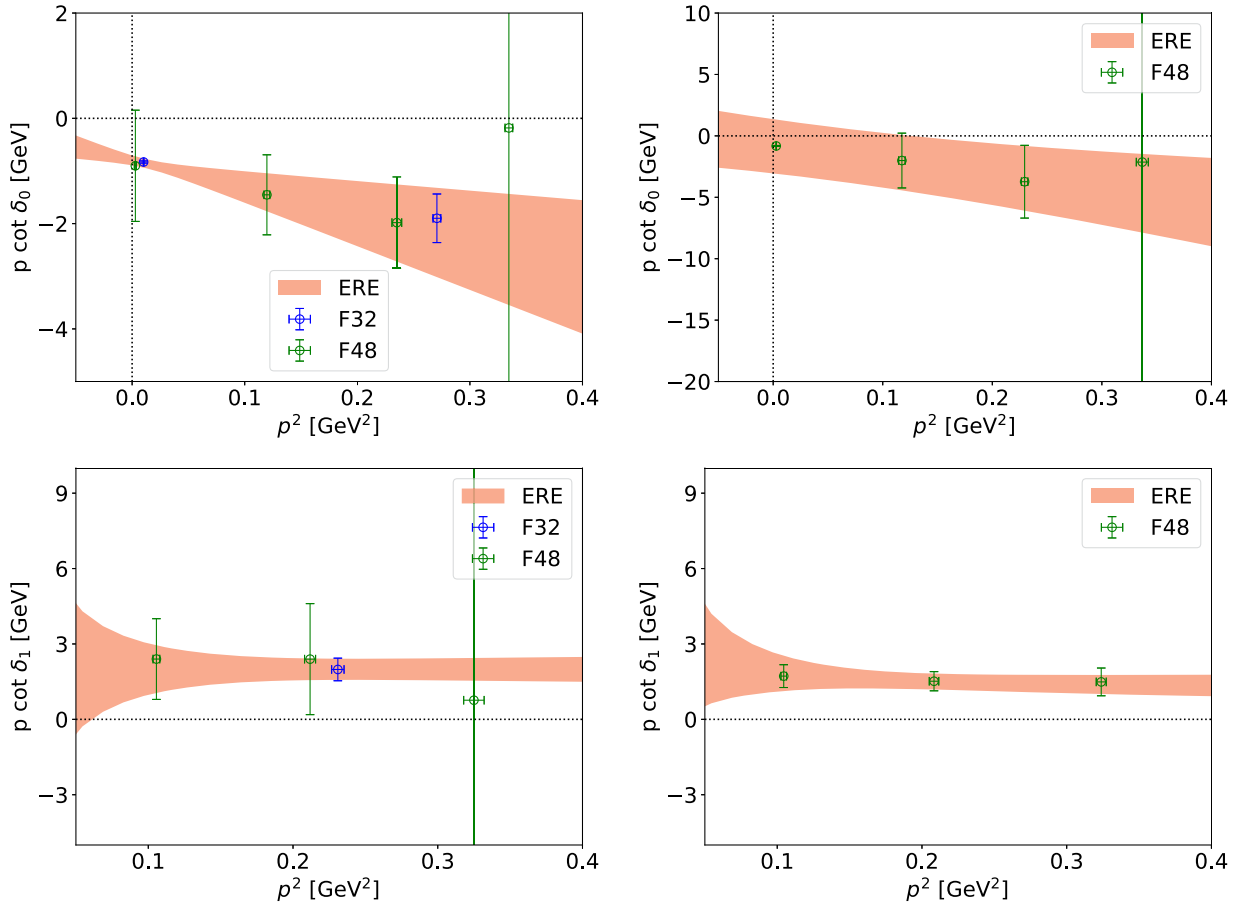


FIG. 6. The variation of $p \cot \delta_l$ as the function of p^2 for $Z = 1$ case. The left and right panels are the results for $m_\pi \simeq 305$ MeV and 207 MeV, respectively. The red bands denote the statistical and systematic uncertainties calculated from the ERE parametrization.

range is perturbative. Thus, we parametrize the scattering length and effective range as simple polynomial functions²:

$$a_0(m_\pi) = c_0^a + c_1^a m_\pi^2, \quad r_0(m_\pi) = c_0^r + c_1^r m_\pi^2, \quad (14)$$

where the parameters c_0^a , c_1^a , c_0^r , and c_1^r can be solved for each sample of the fitted values of a_0 and r_0 , corresponding to pion masses $m_\pi \approx 207$ and 305 MeV for the ensembles F32P21/F48P21 and F32P30/F48P30, respectively. The resulting parameters, including statistical and systematic errors, are found to be $c_0^a = -0.25(10)(15)$ fm, $c_1^a = -0.00(6)(6)$ fm³, $c_0^r = -6.6(56)(22)$ fm, and $c_1^r = 1.9(25)(9)$ fm³. Using the extrapolation formulae in Eq. (14), the S -wave isovector DD scattering length and effective range at the physical pion mass are predicted to be

$$a_0^{\text{phy}} = -0.25(8)(12) \text{ fm}, \quad r_0^{\text{phy}} = -5.7(45)(17) \text{ fm}, \quad (15)$$

²The formula for a_0 is equivalent to the relative parametrization in Ref. [36] upon neglecting the higher-order contributions, such as the $\mathcal{O}(m_\pi^4)$ term.

and their values in the chiral limit are

$$a_0^\chi = -0.25(10)(15) \text{ fm}, \quad r_0^\chi = -6.6(56)(22) \text{ fm}. \quad (16)$$

Here, the first and second uncertainties account for the statistical and systematic errors, respectively.

The pion-mass dependence of scattering length and effective range is displayed in Fig. 8. The large uncertainties of the effective range prevent a solid comparison of its pion-mass dependence with that of the scattering length. Quantitatively, the fitted slopes $c_1^a = -0.00(6)(6)$ fm³ and $c_1^r = 1.9(25)(9)$ fm³ are both consistent with zero (the latter at the 0.7σ level), and the effective ranges $r_0 = -4.5(30)(12)$ fm at $m_\pi \simeq 207$ MeV and $r_0 = -2.0(13)(2)$ fm at $m_\pi \simeq 305$ MeV agree within 1σ ; hence, neither a_0 nor r_0 shows a statistically significant pion-mass dependence over the range studied. A more robust chiral extrapolation to the physical pion mass requires results at more pion masses in a broader range. Since we have only one lattice spacing, we are unable to extrapolate our results to the continuum limit.

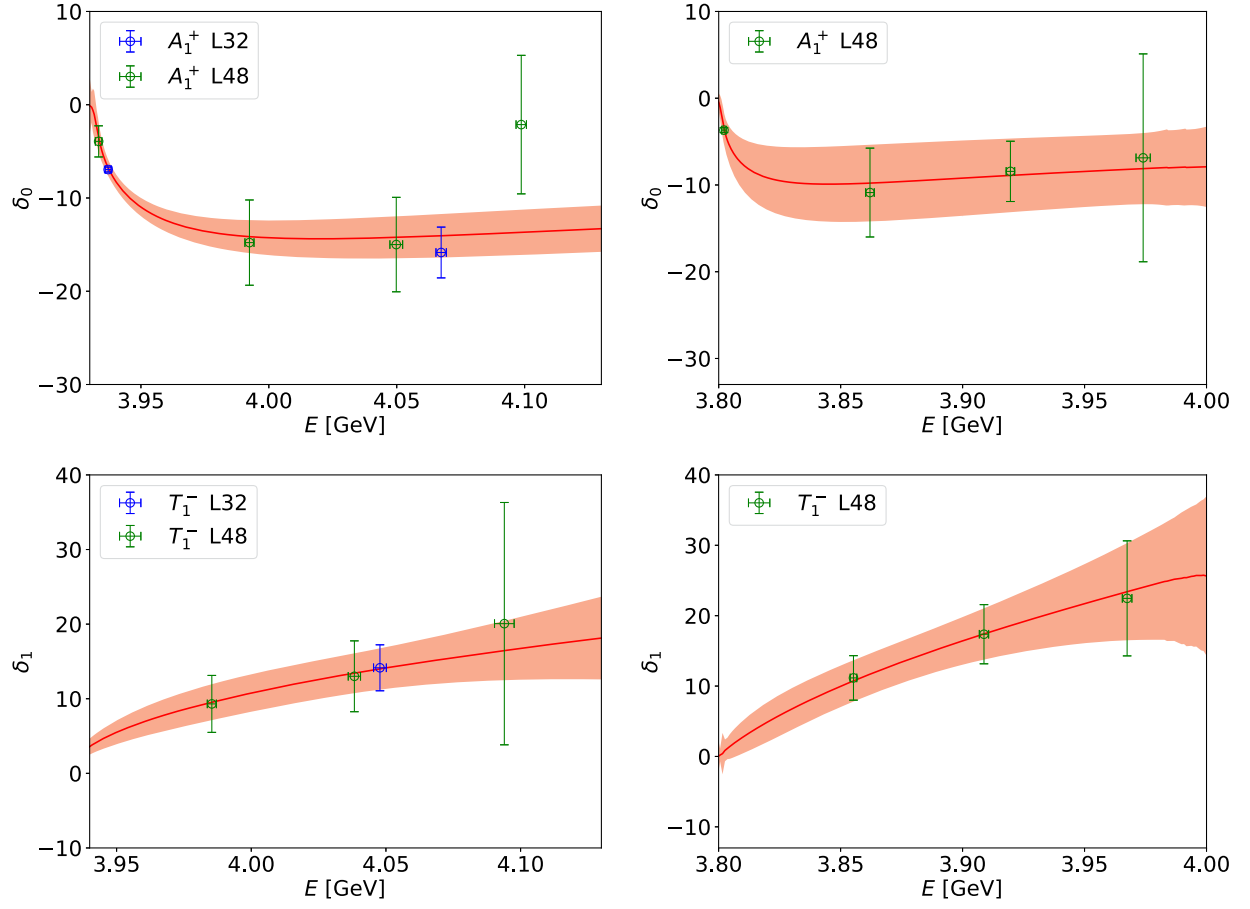


FIG. 7. S - (top row) and P -wave (bottom row) DD scattering phase shifts for the $Z = 1$ case. The left and right panels are the results for $m_\pi \approx 305$ MeV and 207 MeV, respectively. The red lines denote the phase shifts calculated from the fits using ERE parametrization, with the bands representing the combined statistical and systematic uncertainties.

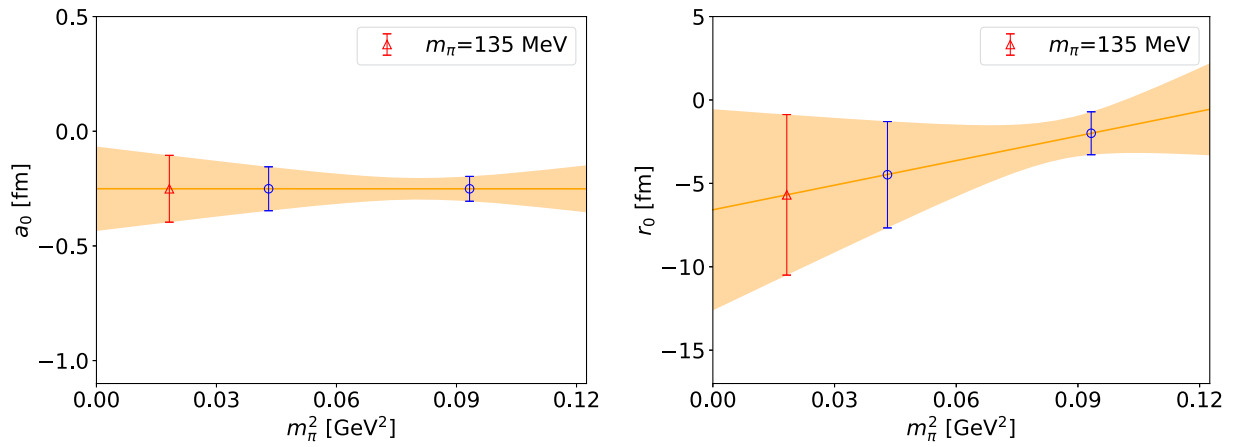


FIG. 8. Extrapolation of S -wave scattering length a_0 and effective range r_0 for the $Z = 1$ case. The blue bars are extracted from energy levels in A_1^+ irrep, and the red bars indicate the physical results, where the uncertainties originate from both the statistical and systematic errors.

VII. CONCLUSIONS

Utilizing the $2+1$ flavor full QCD Wilson-Clover configurations, we calculated the DD scattering in the $I(J^P) = 1(0^+)$ and $0(1^-)$ channels at two different pion masses, 207 and 305 MeV. Due to the lattice artifacts, the dispersion relation of the D meson on the lattice deviated from the continuum dispersion relation. We modified the original Lüscher formula to incorporate this deviation. The S - and P -wave ERE parameters were extracted from the interacting energy levels, and the chiral extrapolation to the physical pion mass was performed. Our results indicate a weak repulsive interaction in the $I(J^P) = 1(0^+)$ DD scattering channel and a slight attractive interaction in the $I(J^P) = 0(1^-)$ channel. From a phenomenological point of view, the S -wave isovector $D^{(*)}D^{(*)}$ interactions from exchanging the light vector mesons ρ and ω are repulsive while the isoscalar ones are attractive [63], and recent lattice studies indeed indicate the importance of the t -channel light vector-meson exchange in the DD^* scattering [35,64].

The physical S -wave isovector DD scattering length, $a_0^{\text{phy}} = -0.25(8)(12)$ fm, indicates that the near-threshold DD interaction is weakly repulsive. This provides useful input for phenomenological studies of three-body systems containing a DD pair. In the DDK system, where binding is driven primarily by the strongly attractive DK interaction and no DD bound-state threshold is expected [31], our lattice result supports treating the DD interaction as a weak repulsive correction. For T_{cc} and $DD\pi$ analyses, the small DD scattering length gives quantitative support to approaches in which the dominant dynamics is generated by the DD^* coupled channels, one-pion exchange, and the finite D^* width, without introducing a separate low-energy DD rescattering amplitude [65,66]. At the same time, relativistic $DD\pi$ three-body frameworks require the two-body subchannel amplitudes, including the $I = 1$ DD amplitude, as input [67]. The present calculation supplies a first-principle constraint on this input and a benchmark for testing the weak- DD approximation.

In this initial investigation, we focus on single-channel DD scattering in the energy range below the D^*D^* threshold. In future studies, the DD - D^*D^* coupled-channel scattering will be considered. Furthermore, lattice data with a broader range of pion masses and various lattice spacings are required to control the systematic uncertainties in a more robust way.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: EFFECT OF THE DISPERSION RELATION TO THE LÜSCHER EQUATION

The Lüscher finite-volume method [1–3] enables us to extract the scattering information from the finite-volume spectra, which can be calculated in lattice QCD. The original Lüscher's formula is derived by assuming the continuum dispersion relation. However, on the lattice, the continuum dispersion relation is usually broken due to lattice artifacts, especially for the heavy-flavor hadrons. To incorporate the effect of the discrete dispersion relation into the Lüscher quantization condition, we start from the partial-wave Lippmann-Schwinger equation in the finite volume. The energy levels are given by poles of the partial-wave T matrix in the finite volume with a specific lattice size L [2,72–75]. The finite-volume partial-wave T matrix with the angular momentum l (l') and its third component m (m') for initial (final) states is

$$\begin{aligned} \tilde{T}_{lm,l'm'}(p) &= V_l(p)\delta_{ll'}\delta_{mm'} \\ &+ \sum_{l''m''} V_l(p)\tilde{G}_{lm,l''m''}(p)\tilde{T}_{l''m'',l'm'}(p), \end{aligned} \quad (\text{A1})$$

where $V_l(p)$ is the short-distance potential, and p is the on-shell momentum. In the finite volume, the Green's function $\tilde{G}_{lm,l'm'}$ is

$$\tilde{G}_{lm,l'm'}(p) = \frac{4\pi}{L^3} \sum_{\mathbf{q}_n}^{|q_n| < \Lambda} \left(\frac{|\mathbf{q}_n|}{p} \right)^{l+l'} Y_{lm}^*(\hat{\mathbf{q}}_n) I(\mathbf{q}_n) Y_{l'm'}(\hat{\mathbf{q}}_n), \quad (\text{A2})$$

with a cutoff Λ and $\hat{\mathbf{q}}_n = \mathbf{q}_n/|\mathbf{q}_n|$. Considering the dispersion relation $\omega_i(\mathbf{q}) = \sqrt{m_i^2 + Z_i \mathbf{q}^2}$ and $\sqrt{s} = \omega_1(p) + \omega_2(p)$ in the center-of-mass frame, the function $I(\mathbf{q}_n)$ is expressed as

$$I(\mathbf{q}) = \frac{1}{4\omega_1(\mathbf{q})\omega_2(\mathbf{q})} \frac{1}{\sqrt{s} - (\omega_1(\mathbf{q}) + \omega_2(\mathbf{q}))} \approx \frac{1}{2[Z_1\omega_2(p) + Z_2\omega_1(p)]} \frac{1}{p^2 - \mathbf{q}^2}, \quad (\text{A3})$$

by just keeping the positive-energy pole contribution using

$$\lim_{|\mathbf{q}| \rightarrow p} \frac{1}{4\omega_1(\mathbf{q})\omega_2(\mathbf{q})} \frac{p^2 - \mathbf{q}^2}{\sqrt{s} - \omega_1(\mathbf{q}) - \omega_2(\mathbf{q})} = \lim_{|\mathbf{q}| \rightarrow p} \frac{1}{4\omega_1(\mathbf{q})\omega_2(\mathbf{q})} \frac{2p}{\frac{Z_1 p}{\omega_1(p)} + \frac{Z_2 p}{\omega_2(p)}} = \lim_{|\mathbf{q}| \rightarrow p} \frac{1}{2[Z_1\omega_2(p) + Z_2\omega_1(p)]}. \quad (\text{A4})$$

Using the identity for standard spherical harmonics $Y_{lm}(\hat{\mathbf{q}})$,

$$Y_{lm}^*(\hat{\mathbf{q}})Y_{l'm'}(\hat{\mathbf{q}}) = \sum_{js} (-1)^{m'} \sqrt{\frac{(2l+1)(2j+1)(2l'+1)}{4\pi}} Y_{js}(\hat{\mathbf{q}}) \begin{pmatrix} l & j & l' \\ m & s & m' \end{pmatrix} \begin{pmatrix} l & j & l' \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A5})$$

the finite-volume Green's function in Eq. (A2) can be reduced to

$$\tilde{G}_{lm,l'm'}(p) = \frac{4\pi}{L^3} \sum_{\mathbf{q}_n} \sum_{j,s} \left(\frac{|\mathbf{q}_n|}{p} \right)^{l+l'-j} \frac{1}{\sqrt{4\pi}} C_{lm,js,l'm'} \left(\frac{|\mathbf{q}_n|}{p} \right)^j Y_{js}(\hat{\mathbf{q}}_n) I(\mathbf{q}_n), \quad (\text{A6})$$

where $C_{lm,js,l'm'}$ are the Clebsch-Gordan coefficients given by

$$C_{lm,js,l'm'} = (-1)^{m'} \sqrt{(2l+1)(2j+1)(2l'+1)} \begin{pmatrix} l & j & l' \\ m & s & m' \end{pmatrix} \begin{pmatrix} l & j & l' \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A7})$$

Here, the factor $(|\mathbf{q}_n|/p)^{l+l'-j}$ can be replaced by unity, while $(|\mathbf{q}_n|/p)^j$ is necessary so that the subtraction does not introduce a singularity at $\mathbf{q} = 0$ for $Y_{js}(\hat{\mathbf{q}})$ [74,75].

For the single channel, the l -wave T matrix in the infinite volume is parametrized as

$$T_l(p) = \frac{8\pi\sqrt{s}}{p \cot \delta_l - ip}. \quad (\text{A8})$$

The finite-volume T -matrix element then is

$$\tilde{T}_{lm,l'm'}^{-1}(p) = T_{lm,l'm'}^{-1} - \left[\tilde{G}_{lm,l'm'}(p) - \int^{|\mathbf{q}| < \Lambda} \frac{d^3\mathbf{q}}{(2\pi)^3} I(\mathbf{q}) 4\pi Y_{lm}^*(\hat{\mathbf{q}}) Y_{l'm'}(\hat{\mathbf{q}}) \right] = \frac{p}{8\pi\sqrt{s}} [\delta_{ll'} \delta_{mm'} \cot \delta_l - \mathcal{M}_{lm,l'm'}(p)], \quad (\text{A9})$$

where the matrix $\mathcal{M}_{lm,l'm'}$ can be expressed as the Lüscher zeta function as follows:

$$\mathcal{M}_{lm,l'm'}(p) \equiv \frac{1}{\pi^{3/2}} \frac{\omega_1(p) + \omega_2(p)}{Z_1\omega_2(p) + Z_2\omega_1(p)} \sum_{j,s} \left(\frac{2\pi}{Lp} \right)^{j+1} \mathcal{Z}_{js} \left(\left(\frac{Lp}{2\pi} \right)^2 \right) C_{lm,js,l'm'}. \quad (\text{A10})$$

Here, $\mathcal{Z}_{js}(q^2)$ is the Lüscher's zeta function in the rest frame,

$$\mathcal{Z}_{js}(q^2) = \sum_{\mathbf{n} \in \mathbb{Z}^3} \frac{\mathbf{n}^j Y_{js}(\hat{\mathbf{n}})}{\mathbf{n}^2 - q^2}. \quad (\text{A11})$$

The energy levels are finally associated with the infinite-volume phase shift

$$\det [\cot \delta - \mathcal{M}(p)] = 0. \quad (\text{A12})$$

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