Generalized Mandelstam-Leibbrandt regularization

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Algebraic noncovariant gauges are used often in string theory, Chern-Simons theory, and gravitation and gauge theories. Loop integrals, however, have spurious singularities that need to be regularized. The most popular and consistent regularization is the Mandelstam-Leibbrandt (ML) prescription. This paper extends the ML prescription outside the light cone. It shares all the properties of light cone ML regularization: it preserves naive power counting and gauge invariance. Moreover, using dimensional regularization, we get a closed form for the basic integrals, including divergent and finite pieces. These results simplify calculations in gauge theories and open new avenues for applications in nonlocal models.

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I. INTRODUCTION

The modern description of particles and interactions in the subatomic world has been based on gauge theories. In a gauge theory we have redundant degrees of freedom that need to be removed by a gauge choice, which is a constraint imposed upon the gauge fields. The election of a gauge can significantly simplify the computation of Green functions.

For several years, various groups of researchers have been working on the quantization of gauge theories in algebraic noncovariant gauges [1,2]. These gauges became popular because they simplify the analysis of gauge theories due to (i) the decoupling of Fadeev-Popov ghost contributions,[1] (ii) starting at the classical level, only physical degrees of freedom are present, and (iii) absence of Gribov ambiguities. Noncovariant gauges have been used to discuss Yang-Mills (YM) theories [3], supersymmetric YM theories [4], supergravity, and superstrings [5].

The main difficulty of algebraic noncovariant gauges is that Feynman integrals have singularities in the k_0 complex plane, with k_μ being the loop variable being integrated over. Several regularizations have been tested to deal with this situation but they present problems and inconsistencies.

One possible way to regularize the singularity is the use of the principal value prescription (PV). But in this case, naive power counting is lost and double poles in dimensional regularization (DR) are present already at the one-loop level, introducing logarithmic dependence on the external momenta in the divergent part of the integral. The loss of naive power counting is related to the fact that using PV regularization, Wick's rotation is not allowed.

The α prescription [6] is a deformation of the gluon propagator in the temporal gauge to regularize the integrals. But, it does not preserve the Ward identities of the gauge theory [1].

Mandelstam-Leibbrandt (ML) prescription [4,7] was introduced to deal with these problems in the light cone gauge (lcg): it preserves naive power counting and gauge invariance, and Wick's rotation from Minkowski to Euclidean space is justified.

Also, renormalization in the lcg can be implemented using ML regularization [2].

Moving away from the lcg presents problems though. Although a generalization of ML prescription in the lcg was implemented for other axial-type gauges, a canonical derivation of the prescription is lacking. Besides, the classification of counterterms and the renormalization in these gauges is problematic [2].

To understand some problems of the Standard Model, such as the origin of neutrino masses and oscillations, very special relativity (VSR) has been proposed [8]. In this approach, the four parameters $Sim(2)$ subgroup of the Lorentz group is assumed to be the symmetry of nature. $Sim(2)$ is characterized by changing a fixed null vector n_μ by a scale factor, so ratios $\frac{n\cdot p_1}{n\cdot p_2}$, where p_i are particle momenta, are $Sim(2)$ invariant but break Lorentz invariance. Such nonlocal terms permit the introduction of chiral neutrino masses [9] and gauge invariant masses for photon [10] and graviton [11].

In VSR, the propagators and vertexes of fermions and bosons are allowed by $Sim(2)$ to have the aforementioned nonlocal terms. In loop computations, these nonlocal terms share the same singularities as algebraic noncovariant gauges.

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[1]There are some subtleties related to this point. Please see [1], chapter 4.4.

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A few years ago, we derived complete formulas for ML regularized integrals in the lcg [12]. A scale symmetry and a regularity condition are enough to determine the integrals in closed form, using DR. These formulas have been fundamental to define a gauge $Sim(2)$ invariant regularization for VSR models, with interesting phenomenological predictions [13].

In this paper we implement the same program for algebraic noncovariant gauges, outside the light cone. We derived the integrals by leveraging a scale symmetry and imposing a regularity condition. They preserve naive power counting and the divergences are polynomials in the external momenta. They preserve gauge invariance. Moreover, they have a smooth light cone limit.

It turns out that the integrals agree with the ones computed using the generalized ML prescription.

In Sec. II, we review the computation of integrals in the lcg. In Sec. III, we compute the single spurious pole integrals. Section IV contains the computation of integrals with higher-order spurious poles. It remains to solve a recurrence relation. This is done in Sec. V. In Sec. VI, we draw some conclusions. Appendix A lists some basic loop integrals. In Appendix B, we mention the existence of a symmetry of single spurious pole integrals. In Appendix C, a computer routine to solve the recurrence relation introduced in Sec. IV is presented. Appendix D deals with the calculation of finite integrals in two-dimensional spacetime that serve as a test of our results. In Appendix E, we compute some integrals to compare with [1,2].

II. ML PRESCRIPTION IN THE LIGHT CONE GAUGE

We are assuming a Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

The light cone gauge is

$$n^\mu A_\mu = 0, \quad n\cdot n = 0,$$

where A_μ is a gauge field.

In loop calculations, spurious singularities appear. A typical loop integral, such as[2]

$$\int dp \frac{1}{[p^2+2p\cdot q - m^2]^a}\frac{1}{(n\cdot p)^b},$$

has a singularity when $n\cdot p = 0$.

These singularities have been treated using various prescriptions: PV and α prescription [1]. All of these prescriptions have some problems: α prescription does not preserve the Ward identities of the gauge symmetry, while PV does not permit the rotation to Euclidean space. As a consequence, naive power counting is lost.

To illustrate the problem of PV prescription, we recall the integral defined in Eq. (6.1) of [2],

$$I = \int \frac{dp}{p^2(p-k)^2 p^+}, \qquad p^+ = \frac{p^0+p^3}{\sqrt{2}}. \tag{1}$$

PV is defined by

$$\frac{1}{p^+} = \lim_{\varepsilon\to 0}\frac{1}{2}\left(\frac{1}{p^+ + i\varepsilon} + \frac{1}{p^+ - i\varepsilon}\right). \tag{2}$$

Naive power counting would imply that I is finite. Instead, the result for the divergent part of I, using PV, is

$$\mathbb{P}I_{PV} = i(-\pi)^\omega \frac{2\omega-3}{\omega-2}\frac{(k^2)^{\omega-2}}{k^+}\frac{\Gamma(2-\omega)[\Gamma(\omega-1)]^2}{\Gamma(2\omega-2)}. \tag{3}$$

Here, $\omega = \frac{d}{2}$, where d is the spacetime dimension in DR.

We see that $\mathbb{P}I_{PV}$ contains a double pole at $\omega = 2$, which is unusual for a one-loop integral. Worse yet, the single pole of $\mathbb{P}I_{PV}$ exhibits a log dependence on the external momentum k_μ. This will create problems in the renormalization of the theory.

The source of the problem mentioned above is that PV puts the poles in the second and third quadrants of the p_0 complex plane (for $p^3 > 0$). So the Wick's rotation to Euclidean space picks up an extra factor from the residue of the encircled pole.

To solve the inconsistencies of previous prescriptions, the ML regularization introduces a second null vector $\bar{n}_\mu$ and defines

$$\frac{1}{n\cdot p} = \lim_{\varepsilon\to 0}\frac{\bar{n}\cdot p}{n\cdot p\bar{n}\cdot p + i\varepsilon}, \bar{n}\cdot\bar{n} = 0. \tag{4}$$

In a given Lorentz frame, we can choose $n_\mu = (n_0,\vec{n}), \bar{n}_\mu = (n_0,-\vec{n})$ with $n_0^2 - \vec{n}\cdot\vec{n} = 0$. The poles in the p_0 complex plane are situated following the same pattern as covariant poles (the poles are in the II and IV quadrants) such that the Wick's rotation from Minkowski to Euclidean space does not find any encircled pole. It follows that ML prescription preserves naive power counting of loop integrals.[3] Moreover, in gauge theories, it maintains the Ward identities of the gauge symmetry.

Without loss of generality, we can impose the condition $\bar{n}\cdot n = 1$.

A disadvantage of ML regularization is that calculations are very long compared with usual DR in covariant gauges. Moreover, explicit formulas for the ML integrals, including finite and divergent parts, were not readily available.

In [12] we developed a technique to calculate such integrals based on a symmetry and a regularity condition.

[2]Here as well as the rest of the paper m^2 actually means $m^2 - i\varepsilon$ with $\varepsilon > 0$.

[3]In Appendix E we compute integral (1) using ML regularization. It is finite.

We got a closed form for the integrals, using DR. This is a great simplification compared with previous methods that computed mainly the pole part because the calculation of the finite part was complicated.

To illustrate the techniques of [12], let us compute the following simple integral:

$$A_\mu = \int dp \frac{f(p^2)p_\mu}{n\cdot p}, \tag{5}$$

where f is an arbitrary function, dp is the integration measure in d dimensional space, and n_μ is a fixed null vector ($n\cdot n=0$). The integrand is singular when $n\cdot p=0$.

To compute A_μ, using ML prescription, we have to know the specific form of f, provide a form of n_μ, $\bar{n}_\mu$, and evaluate the residues of all poles of $\frac{f(p^2)}{n.p}$ in the p_0 complex plane, a rather formidable task for an arbitrary f.

Instead, we notice the following symmetry:

$$n_\mu \to \lambda n_\mu, \quad \bar{n}_\mu \to \lambda^{-1}\bar{n}_\mu, \quad \lambda\neq 0, \quad \lambda\varepsilon R. \tag{6}$$

The properties of n_μ and $\bar{n}_\mu$ are preserved,

$$0 = n\cdot n \to \lambda^2 n\cdot n = 0$$
$$0 = \bar{n}\cdot\bar{n} \to \lambda^{-2}\bar{n}.\cdot\bar{n} = 0$$
$$1 = n\cdot\bar{n} \to n\cdot\bar{n} = 1.$$

We see from (4) that

$$\frac{1}{n\cdot p} \to \frac{1}{n\cdot p}\lambda^{-1}. \tag{7}$$

Now we compute A_μ, using its symmetries. It is a Lorentz vector which scales under (6) as λ^{-1}. The only Lorentz vectors that are available in this case are n_μ and $\bar{n}_\mu$. But (6) forbids n_μ. That is,

$$A_\mu = a\bar{n}_\mu.$$

Multiply by n^μ to find $A\cdot n = a$. Thus, $a=\int dp f(p^2)$. Finally, we get

$$\int dp \frac{f(p^2)p_\mu}{n\cdot p} = \bar{n}_\mu \int dp f(p^2). \tag{8}$$

We consider now a more general integral. The regularity of the answer will determine it uniquely,

$$A = \int dp \frac{H(p^2, p\cdot q)}{n\cdot p} = \bar{n}\cdot q f(q^2, n\cdot q\bar{n}\cdot q), \tag{9}$$

where q_μ is an external momentum, a Lorentz vector. H is an arbitrary function. The last relation follows from (6), for a certain f we will find next.

Derive with respect to q^μ and multiply by n^μ,

$$\frac{\partial A}{\partial q^\mu}n^\mu = \int dp H_{,u} = g(x)$$
$$= f(x,y) + 2y\frac{\partial}{\partial x}f(x,y) + y\frac{\partial}{\partial y}f(x,y). \tag{10}$$

We defined $u = p\cdot q$, $x=q^2$, $y = n\cdot q\bar{n}\cdot q$, and $()_{,u}$ means derivative with respect to u.

Assuming that the solution and its partial derivatives are finite in the neighborhood of $y=0$, it follows from the equation that $f(x,0)=g(x)$. That is, the partial differential equation has a unique regular solution. In the next section, we will explain how to solve this type of equation, using the method of characteristics [14]. In the same way, we solved Eq. (10) in [12].

Now we apply Eq. (10) to compute integrals that appear in gauge theory loops,

$$\int dp \frac{1}{[p^2+2p\cdot q - m^2]^a}\frac{1}{n\cdot p} = \bar{n}\cdot q f(x,y). \tag{11}$$

In this case we have

$$g(x) = -2a\int dp \frac{1}{[p^2 - x - m^2]^{a+1}}. \tag{12}$$

The unique regular solution of (10) is

$$f(x,y) = -\frac{1}{y}\left\{\int dp[p^2 - x - m^2]^{-a} - \int dp[p^2 - x + 2y - m^2]^{-a}\right\}.$$

We can check that $f(x,0) = -2a\int dp[p^2-x-m^2]^{-a-1} = g(x)$. The remaining integral can be computed using DR. The result is

$$\int dp \frac{1}{[p^2+2p\cdot q - m^2]^a}\frac{1}{n\cdot p} = (-1)^{a+1}i(\pi)^\omega(-2)\frac{\Gamma(a+1-\omega)}{\Gamma(a)}\bar{n}\cdot q\int_0^1 dt \frac{1}{(m^2+q\cdot q - 2n\cdot q\bar{n}\cdot q t)^{a+1-\omega}}, \quad \omega = d/2. \tag{13}$$

The technique can be extended to obtain the general integral

$$\int dp \frac{1}{[p^2+2p\cdot q-m^2]^a}\frac{1}{(n\cdot p)^b}=(-1)^{a+b}i(\pi)^{\omega}(-2)^b\frac{\Gamma(a+b-\omega)}{\Gamma(a)\Gamma(b)}(\bar{n}\cdot q)^b\int_0^1 dt t^{b-1}\frac{1}{(m^2+q\cdot q-2n\cdot q\bar{n}\cdot qt)^{a+b-\omega}},$$
$$\omega=d/2. \tag{14}$$

Here, d is the dimension of spacetime in DR.

These results followed from the scale symmetry (6) plus the condition that (14) must be regular at $n\cdot q\bar{n}\cdot q=0$.

In the next section we will apply the same method to the same type of integrals for arbitrary $n\cdot n\neq 0, \bar{n}\cdot\bar{n}\neq 0$.

III. TREATMENT OF SINGULARITIES OUTSIDE THE LIGHT CONE: SINGLE SPURIOUS POLE

Numerous applications of algebraic noncovariant gauges to Yang-Mills theories, supergravity, and superstrings motivate the need to compute loop integrals with singularities of the type we described in the last section.

In this and the following section, we will calculate such integrals outside the lcg. We will find the whole answer, including divergent and finite parts.

In this section, integrals with a single spurious pole will be treated. They are somehow special in ML prescription and are the basis for computation of integrals with higher-order poles.

A central role will be played by a vector F_μ, which is a function of $n_\mu, \bar{n}_\mu$. F_μ plays here the same role that $\bar{n}_\mu$ played in the lcg. Due to a consistency condition that we will derive below or a hidden symmetry that we present in Appendix B, F_μ must be a null vector, $F\cdot F=0$. This property determines the form of F_μ.

The algebraic noncovariant gauge we are considering here is

$$n^\mu A_\mu=0,\quad n\cdot n\neq 0,$$

where A_μ is a gauge field. This includes the temporal gauge ($n\cdot n>0$) and axial gauge ($n\cdot n<0$).

In loop calculations spurious singularities appear, which can be embedded in the following integral:

$$A=\int dp\frac{H(p^2,p\cdot q)}{n\cdot p}, \tag{15}$$

dp is the integration measure in d dimensional space and H is an arbitrary function.

The generalized ML prescription is

$$\frac{1}{n\cdot p}=\lim_{\varepsilon\to 0}\frac{\bar{n}\cdot p}{n\cdot p\,\bar{n}\cdot p+i\varepsilon},\quad \bar{n}\cdot\bar{n}\neq 0,\quad n\cdot n\neq 0. \tag{16}$$

In $d=4$, we can choose a Lorentz frame where $n_\mu=(n_0,0,0,n_3), \bar{n}_\mu=(n_0,0,0,-n_3)$, and $n_0\neq 0$, $n_3\neq 0$.

Under the scale transformation,

$$n_\mu\to\lambda n_\mu,\qquad \bar{n}_\mu\to\lambda^{-1}\bar{n}_\mu,\qquad A\to\lambda^{-1}A. \tag{17}$$

Let us assume the existence of a vector $F_\mu(n,\bar{n})$ that transforms under (17) as

$$F_\mu\to\lambda^{-1}F_\mu. \tag{18}$$

We normalize it so that $n\cdot F=1$.

We will determine the exact form of F_μ later.

To simplify the notation, we define

$$x=\bar{n}\cdot n,\qquad y=\bar{n}\cdot\bar{n},\qquad z=n\cdot n. \tag{19}$$

We introduce the following definitions:

$$u=p\cdot q,\quad v=q\cdot q,\quad w=\frac{n\cdot q}{\sqrt{z}},\quad s=\sqrt{z}F\cdot q. \tag{20}$$

This set provides a complete list of functions of q_μ, which are Lorentz scalar and scale invariant under (17).

Therefore, we can write in all generality

$$A=\int dp\frac{H(p^2,p\cdot q)}{n\cdot p}=F\cdot q f(v,w,s). \tag{21}$$

That is,

$$\frac{\partial A}{\partial q^\mu}=\int dp\frac{H_{,u}p_\mu}{n\cdot p}$$
$$=F_\mu(f+sf_s)+2z^{-1/2}sf_v q_\mu+z^{-1}sf_w n_\mu, \tag{22}$$

where $()_{,u}$ means derivative with respect to u.

Multiplying by n^μ, we get a partial differential equation for $f(v,w,s)$,

$$\frac{\partial A}{\partial q^\mu}n^\mu=\int dpH_{,u}=B(v),$$
$$(f+f_s s)+2s\omega f_v+sf_w=B(v),$$
$$f_s=\frac{\partial f}{\partial s},\qquad f_v=\frac{\partial f}{\partial v},\qquad f_w=\frac{\partial f}{\partial w}. \tag{23}$$

As in the lcg, we impose a regularity condition: f_s, f_v, and f_w must be finite at $s=0$. From Eq. (23) we get $f(v,w,0)=B(v)$. Thus, Eq. (23) has a unique solution, which is regular at $s=0$.

To find the solution we use the method of characteristics [14],

$$\dot{s} = s, \qquad \dot{v} = 2sw, \qquad \dot{w} = s$$
$$f + \dot{f} = B(v), \tag{24}$$

$$s = s_0 e^t, \qquad \dot{w} = s_0 e^t, \qquad w = s_0 e^t + w_0,$$
$$\dot{v} = 2s_0^2 e^{2t} + 2s_0 w_0 e^t, \qquad v = s_0^2 e^{2t} + 2s_0 w_0 e^t + v_0$$
$$w - s = w_0, \qquad v + s^2 - 2sw = v_0.$$

The homogeneous equation

$$\dot{f} = -f, \qquad f = f_0 e^{-t} = f_0 s_0 \frac{1}{s}.$$

Thus, the general solution of the homogeneous ($B = 0$) equation is

$$f_h = \frac{1}{s}\Pi(v + s^2 - 2sw, w - s).$$

Where Π is an arbitrary function.

A particular solution of the inhomogeneous equation is

$$f_p = a(t)e^{-t}, \qquad \dot{a}e^{-t} = B(v(t)),$$
$$f_p = e^{-t}\int_{-\infty}^{t} dt' e^{t'} B(v(t')). \tag{25}$$

Then, the most general solution of Eq. (23) is

$$f = f_p + \frac{1}{s}\Pi(v + s^2 - 2sw, w - s). \tag{26}$$

Regularity at $s = 0$ implies $\Pi(v, w) = 0$, all v, w.

Moreover, we see that

$$f_p(v, w, 0) = B(v).$$

Therefore, f_p defined in Eq. (25) is the unique regular solution to Eq. (23).

A. Loop integrals

Let us apply Eq. (25) to

$$L_1 = \int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a}\frac{1}{n\cdot p}. \tag{27}$$

We get

$$B(v) = -2a\int dp \frac{1}{[p^2 - v - m^2]^{a+1}}. \tag{28}$$

Then, using Eq. (25), we obtain

$$f = -2a\int dp \int_0^1 dt \frac{1}{[p^2 - (v - 2swt + s^2t^2) - m^2]^{a+1}}. \tag{29}$$

Using dimensional regularization, we get

$$\int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a}\frac{1}{n\cdot p} = 2(-1)^a i\pi^\omega \frac{\Gamma(a + 1 - \omega)}{\Gamma(a)} F\cdot q \int_0^1 dt \frac{1}{(q\cdot q - 2F\cdot qn\cdot qt + z(F\cdot q)^2 t^2 + m^2)^{a+1-\omega}}. \tag{30}$$

We can see that naive power counting is preserved. Moreover, we recover the light cone integral (14), when $n\cdot n = \bar{n}\cdot\bar{n} = 0$. To get this we require that $F_\mu \to \frac{\bar{n}_\mu}{x}$ in the lcg limit. We should keep this condition in mind when we will derive the form of F_μ in the next subsection.

Other integrals can be obtained deriving with respect to q^μ. For instance,

$$\int dp \frac{1}{[p^2 + 2p.q - m^2]^a}\frac{p_\mu}{n.p} = (-1)^a i\pi^\omega \frac{\Gamma(a - \omega)}{\Gamma(a)} F_\mu \int_0^1 dt \frac{1}{(q.q - 2F.qn.qt + n.n(F.q)^2 t^2 + m^2)^{a-\omega}}$$
$$- 2(-1)^a i\pi^\omega \frac{\Gamma(a + 1 - \omega)}{\Gamma(a)} F.q \int_0^1 dt \frac{q_\mu - F_\mu n.qt - F.qn_\mu t + n.nF.qF_\mu t^2}{(q.q - 2F.qn.qt + n.n(F.q)^2 t^2 + m^2)^{a+1-\omega}}. \tag{31}$$

We integrate by part the second integral to obtain

$$\begin{aligned}
&2(-1)^a i\pi^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a)} \int_0^1 dt \frac{F.qn.qt - n.n(F.q)^2 t^2}{(q.q - 2F.qn.qt + n.n(F.q)^2 t^2 + m^2)^{a+1-\omega}} \\
&= 2(-1)^a i\pi^\omega \frac{\Gamma(a-\omega)}{\Gamma(a)} \frac{1}{2} \int_0^1 dtt \frac{d}{dt} \frac{1}{(q.q - 2F.qn.qt + n.n(F.q)^2 t^2 + m^2)^{a-\omega}} \\
&= (-1)^a i\pi^\omega \frac{\Gamma(a-\omega)}{\Gamma(a)} \frac{1}{(q.q - 2F.qn.q + n.n(F.q)^2 + m^2)^{a-\omega}} \\
&\quad - (-1)^a i\pi^\omega \frac{\Gamma(a-\omega)}{\Gamma(a)} \int_0^1 dt \frac{1}{(q.q - 2F.qn.qt + n.n(F.q)^2 t^2 + m^2)^{a-\omega}}.
\end{aligned} \tag{32}$$

So, finally, the integral is

$$\begin{aligned}
\int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a} \frac{p_\mu}{n\cdot p} &= (-1)^a i\pi^\omega \frac{\Gamma(a-\omega)}{\Gamma(a)} \frac{F_\mu}{(q\cdot q - 2F\cdot qn\cdot q + z(F\cdot q)^2 + m^2)^{a-\omega}} \\
&\quad - 2(-1)^a i\pi^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a)} F\cdot q \int_0^1 dt \frac{q_\mu - F\cdot qn_\mu t}{(q\cdot q - 2F\cdot qn\cdot qt + z(F\cdot q)^2 t^2 + m^2)^{a+1-\omega}}.
\end{aligned} \tag{33}$$

Integration by parts in a t integral will be used several times in this work.

But still we do not know what F_μ is. We will determine it in the next subsection.

B. Consistency condition

Consider the third derivative of Eq. (30) with respect to q^μ, and evaluate it at $q^\mu = 0$. We get

$$\begin{aligned}
\int dp \frac{1}{[p^2 - m^2]^{a+3}} \frac{p_\mu p_\nu p_\lambda}{n\cdot p} &= \frac{1}{2} \frac{(-1)^a i\pi^\omega}{\Gamma(a+3)} \frac{\Gamma(a+2-\omega)}{(m^2)^{a+2-\omega}} \\
&\quad \times \Big\{ F_\mu \int_0^1 dt(\eta_{\nu\lambda} - t(n_\lambda F_\nu + F_\lambda n_\nu) + t^2 z F_\lambda F_\nu) \\
&\quad + F_\nu \int_0^1 dt(\eta_{\mu\lambda} - t(n_\lambda F_\mu + F_\lambda n_\mu) + t^2 z F_\lambda F_\mu) \\
&\quad + F_\lambda \int_0^1 dt(\eta_{\mu\nu} - t(n_\nu F_\mu + F_\nu n_\mu) + t^2 z F_\nu F_\mu) \Big\}.
\end{aligned} \tag{34}$$

Let d_{ph} be the physical dimension. Choose $a + 2 = \frac{d_{ph}}{2}$. Define $\varepsilon = d - d_{ph}$. Then the pole part ($\mathbb{P}$) is given by

$$\begin{aligned}
&\mathbb{P} \int dp \frac{1}{[p^2 - m^2]^{\frac{d_{ph}}{2}+1}} \frac{p_\mu p_\nu p_\lambda}{n\cdot p} \\
&= i(-1)^{\frac{d_{ph}}{2}} \pi^{\frac{d_{ph}}{2}} \frac{1}{\varepsilon} \frac{1}{\Gamma\left(\frac{d_{ph}}{2}+1\right)} \{F_\mu F_\nu F_\lambda z - F_\mu F_\nu n_\lambda - F_\mu F_\lambda n_\nu - F_\lambda F_\nu n_\mu + F_\mu \eta_{\nu\lambda} + F_\nu \eta_{\mu\lambda} + F_\lambda \eta_{\mu\nu}\}.
\end{aligned} \tag{35}$$

Contracting $\nu = \lambda$ we get

$$\mathbb{P} \int dp \frac{1}{[p^2 - m^2]^{\frac{d_{ph}}{2}+1}} \frac{p_\mu p^2}{n\cdot p} = i(-1)^{\frac{d_{ph}}{2}} \pi^{\frac{d_{ph}}{2}} \frac{1}{\varepsilon} \frac{1}{\Gamma\left(\frac{d_{ph}}{2}+1\right)} [F_\mu(d_{ph} + zF\cdot F) - F\cdot Fn_\mu].$$

Since naive power counting is preserved, $\mathbb{P}$ can be computed using (33),

$$\mathbb{P}\int dp\frac{1}{[p^2-m^2]^{\frac{d_{ph}}{2}+1}}\frac{p_\mu p^2}{n\cdot p}=\mathbb{P}\int dp\frac{1}{[p^2-m^2]^{\frac{d_{ph}}{2}}}\frac{p_\mu}{n\cdot p}$$
$$=F_\mu i(-1)^{\frac{d_{ph}}{2}}\pi^{\frac{d_{ph}}{2}}\frac{1}{\Gamma\left(\frac{d_{ph}}{2}\right)}\frac{2}{\varepsilon}.$$

To have consistency between the two ways of getting the pole part of the previous integral, we must have that

$$\frac{1}{d_{ph}}[F_\mu(d_{ph}+zF\cdot F)-F\cdot Fn_\mu]=F_\mu,\quad \text{or}$$
$$(F_\mu z-n_\mu)F\cdot F=0.$$

$F_\mu=\frac{n_\mu}{z}$ does not go to $\frac{\bar n_\mu}{\bar n.n}$ when $z\to 0$. So the consistency condition implies $F\cdot F=0$, independently of the physical dimension d_{ph}.

To find F_μ, we use Lorentz symmetry, plus the conditions $n\cdot F=1$ and $F\cdot F=0$.

We can write, for certain Lorentz scalars a, b,

$$F_\mu=a\bar n_\mu+b\frac{n_\mu}{n\cdot n},$$
$$n\cdot F=1\to b=1-a\bar n\cdot n,$$
$$F\cdot F=0\to a^2\bar n\cdot\bar n+2ab\frac{\bar n\cdot n}{n\cdot n}+b^2\frac{1}{n\cdot n}=0,$$
$$a=\pm\frac{1}{\sqrt{(\bar n\cdot n)^2-\bar n\cdot\bar n n\cdot n}}.$$

To fix the sign in a we used the boundary condition $\lim_{z\to 0}b=0$, to recover the lcg result $F_\mu\to\frac{\bar n_\mu}{x}$.

That is,

$$F_\mu=\frac{\bar n_\mu}{D}-\frac{n_\mu}{z}\left(\frac{x}{D}-1\right),\qquad D=\frac{1}{\sqrt{x^2-yz}},\tag{36}$$

where F_μ scales as Eq. (18) required.

We see that F_μ agrees with Eq. (A6.42) of [2].

As a further check that Eq. (33) does provide the complete integral, we compute a finite integral which is not listed in [1,2]. Choose $d_{ph}=2$, $n_\mu=(n_0,n_1)$, $\bar n_\mu=(n_0,-n_1)$, and calculate

$$A_\mu=\int dp\frac{1}{[p^2-m^2]^2}\frac{p_\mu}{n\cdot p}.\tag{37}$$

We get, using the generalized ML prescription (16),

$$\int dp\frac{1}{[p^2-m^2]^2}\frac{p_1}{n\cdot p}=-i\pi\frac{n_1}{m^2}\left(\frac{1}{n_1(n_0+n_1)}\right)=i\pi\frac{1}{m^2}F_1,\tag{38}$$

$$\int dp\frac{1}{[p^2-m^2]^2}\frac{p_0}{n\cdot p}=\frac{i\pi n_0}{m^2}\frac{1}{n_0(n_0+n_1)}=i\pi\frac{1}{m^2}F_0,\tag{39}$$

which coincide with Eq. (33) at $q_\mu=0,a=2$. Integrals (38) and (39) are calculated in Appendix D.

It is easy to check that (30) includes all integrals with a single power of $\frac{1}{n\cdot p}$ in Appendix 6 of [2] or Appendix D of [1]. In the aforementioned books, the pole part of the integrals is listed, but Eq. (30) provides the whole integral for arbitrary values of a, d.

IV. HIGHER SINGULARITIES, $(\frac{1}{n\cdot p})^b$, $b\geq 2$

In the lcg, the scaling symmetry and the regularity condition at $\bar n\cdot qn\cdot q=0$ were enough to determine the integral for $b\geq 2$. Outside the lcg, this is not so.

But outside the lcg, both n_μ and $\bar n_\mu$ are unconstrained, so it makes sense to compute derivatives with respect to n_μ.

To proceed further, we notice that

$$\frac{\partial}{\partial n^\nu}\left(\frac{\bar n.p}{\bar n\cdot pn\cdot p+i\varepsilon}\right)=-\frac{(\bar n.p)^2p_\nu}{(\bar n\cdot pn\cdot p+i\varepsilon)^2}=-\frac{p_\nu}{(n\cdot p)^2}.\tag{40}$$

Using this identity in Eq. (30), we get

$$\int dp\frac{1}{[p^2+2p\cdot q-m^2]^a}\frac{p_\mu}{(n\cdot p)^2}$$
$$=2(-1)^a i\pi^\omega\frac{\Gamma(a+1-\omega)}{\Gamma(a)}E_{\nu\mu}q^\nu\int_0^1 dt\frac{1}{(q\cdot q-2F\cdot qn\cdot qt+n\cdot n(F\cdot q)^2t^2+m^2)^{a+1-\omega}}$$
$$+2^2(-1)^a i\pi^\omega\frac{\Gamma(a+2-\omega)}{\Gamma(a)}F\cdot q\int_0^1 dt\frac{E_{\nu\mu}q^\nu n\cdot qt-F\cdot qq_\mu t+n_\mu(F\cdot q)^2t^2-n\cdot nF\cdot qE_{\nu\mu}q^\nu t^2}{(q\cdot q-2F\cdot qn\cdot qt+n\cdot n(F\cdot q)^2t^2+m^2)^{a+2-\omega}},$$

where

$$E_{\mu\nu} = \frac{x}{D^3}\bar{n}_\mu\bar{n}_\nu + \eta_{\mu\nu}\frac{y}{D(x+D)} + n_\mu n_\nu \frac{y^2}{D^2(x+D)}\left(\frac{1}{D}+\frac{1}{x+D}\right) - (n_\mu\bar{n}_\nu + n_\nu\bar{n}_\mu)\frac{y}{D^3}$$
$$\frac{\partial F_\mu}{\partial n^\nu} = -E_{\mu\nu}, E_{\mu\nu} = E_{\nu\mu}, \tag{41}$$

where $E_{\mu\nu}$ coincides with Eq. (A6.44) of [2].

Consider the coefficient of $E_{\nu\mu}q^\nu$ in the second integral. Integrating by parts in t, we get

$$\int dp \frac{1}{[p^2+2p\cdot q-m^2]^a}\frac{p_\mu}{(n\cdot p)^2} = 2(-1)^a i\pi^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a)}\frac{E_{\nu\mu}q^\nu}{(q\cdot q - 2F\cdot qn\cdot q + n\cdot n(F\cdot q)^2+m^2)^{a+1-\omega}}$$
$$+ 2^2(-1)^a i\pi^\omega \frac{\Gamma(a+2-\omega)}{\Gamma(a)}\int_0^1 dt \frac{-(F\cdot q)^2 q_\mu t + n_\mu (F\cdot q)^3 t^2}{(q\cdot q - 2F\cdot qn\cdot qt + n\cdot n(F\cdot q)^2 t^2 + m^2)^{a+2-\omega}}. \tag{42}$$

Using Eq. (36) we get

$$E_{\mu\nu} = F_\mu F_\nu(\rho z+1) + \rho(\eta_{\mu\nu} - n_\mu F_\nu - n_\nu F_\mu), \qquad \rho = \frac{y}{D(x+D)} \tag{43}$$

and

$$E_{\mu\nu}q^\nu = \frac{x}{D}(F\cdot q)F_\mu + \rho(q_\mu - n_\mu F\cdot q - n\cdot q F_\mu). \tag{44}$$

Since the $b=2$ integral is well defined, we can write

$$\frac{\partial}{\partial q^\mu}\int dp \frac{1}{[p^2+2p\cdot q-m^2]^{a-1}}\frac{1}{(n\cdot p)^2} = -2(a-1)\int dp \frac{1}{[p^2+2p\cdot q-m^2]^a}\frac{p_\mu}{(n\cdot p)^2}. \tag{45}$$

Equation (45) defines the $b=2$ integral up to an additive constant, which is independent of q_μ.

For a Lorentz invariant g, we can write

$$\int dp \frac{1}{[p^2+2p\cdot q-m^2]^{a-1}}\frac{1}{(n\cdot p)^2} = g(v,w,s),$$
$$\frac{\partial}{\partial q^\mu}g(v,w,s) = 2g_v q_\mu + g_w\frac{n_\mu}{\sqrt{z}} + g_s F_\mu\sqrt{z}. \tag{46}$$

We use the notation

$$g_s = \frac{\partial g}{\partial s}, \qquad g_v = \frac{\partial g}{\partial v}, \qquad g_w = \frac{\partial g}{\partial w}.$$

From Eqs. (42), (44), and (46) we get

$$g_v = 2\rho(-1)^{a+1}i\pi^\omega\frac{\Gamma(a+1-\omega)}{\Gamma(a-1)}\frac{1}{(q\cdot q - 2F\cdot qn\cdot q + n\cdot n(F\cdot q)^2+m^2)^{a+1-\omega}}$$
$$+ (2^2)i\pi^\omega(-1)^a\frac{\Gamma(a+2-\omega)}{\Gamma(a-1)}\int_0^1 dt\frac{(F\cdot q)^2 t}{(q\cdot q - 2F\cdot qn\cdot qt + n\cdot n(F\cdot q)^2t^2+m^2)^{a+2-\omega}} \tag{47}$$

$$g_w = (-1)^a 2^2 i\pi^\omega\frac{\Gamma(a+1-\omega)}{\Gamma(a-1)}\frac{\rho(F.q)\sqrt{z}}{(q\cdot q - 2F\cdot qn\cdot q + n\cdot n(F\cdot q)^2+m^2)^{a+1-\omega}}$$
$$+ 2^3(-1)^{a+1}i\pi^\omega\frac{\Gamma(a+2-\omega)}{\Gamma(a-1)}\int_0^1 dt\frac{t^2\sqrt{z}(F\cdot q)^3}{(q\cdot q - 2F\cdot qn\cdot qt + n\cdot n(F\cdot q)^2t^2+m^2)^{a+2-\omega}} \tag{48}$$

$$g_s = 2^2(-1)^{a+1} i\pi^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a-1)} \frac{1}{\sqrt{z}} \frac{\frac{x}{D}(F\cdot q) - \rho n\cdot q}{(q\cdot q - 2F\cdot qn\cdot q + n\cdot n(F\cdot q)^2 + m^2)^{a+1-\omega}}. \tag{49}$$

Integrating g_v over $v = q\cdot q$, we obtain

$$\begin{aligned} g = \bar{g}(w,s) &+ 2\rho(-1)^a i\pi^\omega \frac{\Gamma(a-\omega)}{\Gamma(a-1)} \frac{1}{(q\cdot q - 2F\cdot qn\cdot q + n\cdot n(F\cdot q)^2 + m^2)^{a-\omega}} \\ &+ (2^2)(-1)^{a+1} i\pi^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a-1)} \int_0^1 dt \frac{(F\cdot q)^2 t}{(q\cdot q - 2F\cdot qn\cdot qt + n\cdot n(F\cdot q)^2 t^2 + m^2)^{a+1-\omega}}. \end{aligned} \tag{50}$$

Since the integral exists there must be a $\bar{g}(w,s)$ that produces agreement between Eqs. (50), (48), and (49). Imposing the boundary condition that the integral vanishes when $v \to -\infty$, we get $\bar{g}(w,s) = 0$.[4] As a further check, we used Eq. (50) to obtain Eqs. (48) and (49). We got that $\bar{g}(w,s)$ is a constant. We must have $a > \omega$. But the integral is analytic almost everywhere, so the result can be extended by analytic continuation.

We have proved that

$$\begin{aligned} &\int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a} \frac{1}{(n\cdot p)^2} \\ &= 2\rho(-1)^{a+1} i\pi^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a)} \frac{1}{(q\cdot q - 2F\cdot qn\cdot q + n\cdot n(F\cdot q)^2 + m^2)^{a+1-\omega}} \\ &\quad + (2^2)(-1)^a i\pi^\omega \frac{\Gamma(a+2-\omega)}{\Gamma(a)} (F\cdot q)^2 \int_0^1 dt \frac{t}{(q\cdot q - 2F\cdot qn\cdot qt + n\cdot n(F\cdot q)^2 t^2 + m^2)^{a+2-\omega}}. \end{aligned} \tag{51}$$

We see that in the light cone limit, we recover Eq. (14). Moreover, we preserve the scaling property

$$\int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a} \frac{1}{(n\cdot p)^2} = m^{d-2a-2} \int dp \frac{1}{[p^2 + 2p.\frac{q}{m} - 1]^a} \frac{1}{(n\cdot p)^2}.$$

Equation (51) contains all double pole integrals. We have checked that the result coincides with Appendix 6 of [2] and Appendix D of [1].

Using the same approach we proceed to find the value of the integral for b arbitrary.

Inspired by the lcg result, Eq. (14), we write the ansatz

$$\begin{aligned} &\int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a} \frac{1}{(n\cdot p)^b} \\ &= T(a,b) + (-1)^{a+b} i(\pi)^\omega (-2)^b \frac{\Gamma(a+b-\omega)}{\Gamma(a)\Gamma(b)} (F\cdot q)^b \int_0^1 dt \frac{t^{b-1}}{(q\cdot q - 2F\cdot qn\cdot qt + z(F\cdot q)^2 t^2 + m^2)^{a+b-\omega}}, \end{aligned} \tag{52}$$

where $T(a,b)$ is a function of v, w, s, x, y, z.

Using the identity (40) in Eq. (52), we get

$$\begin{aligned} &-b \int dp \frac{1}{[p^2 + 2p\cdot q - m^2]^a} \frac{p_\mu}{(n\cdot p)^{b+1}} \\ &= \frac{\partial}{\partial n^\mu} T(a,b) + (-1)^{a+1} i(\pi)^\omega 2^b \frac{\Gamma(a+b-\omega)}{\Gamma(a)\Gamma(b)} (F\cdot q)^{b-1} b E_{\mu\alpha} q^\alpha \int_0^1 dt \frac{t^{b-1}}{(q\cdot q - 2F\cdot qn\cdot qt + z(F\cdot q)^2 t^2 + m^2)^{a+b-\omega}} \\ &\quad + (-1)^{a+1} i(\pi)^\omega 2^{b+1} \frac{\Gamma(a+b+1-\omega)}{\Gamma(a)\Gamma(b)} (F\cdot q)^b \int_0^1 dt \frac{t^{b-1}(E_{\mu\alpha} q^\alpha n\cdot qt - F\cdot q q_\mu t + n_\mu (F\cdot q)^2 t^2 - z t^2 F\cdot q E_{\mu\alpha} q^\alpha)}{(q\cdot q - 2F\cdot qn\cdot qt + z(F\cdot q)^2 t^2 + m^2)^{a+b+1-\omega}}. \end{aligned} \tag{53}$$

[4] g_v must be $\mathcal{O}(v^{-(1+\delta)}), \delta > 0$; and g_s, g_w must vanish, at $v \to -\infty$.

Integrating by parts the coefficient of $E_{\mu\alpha}q^\alpha$ in the second integral, we get

$$F\cdot q\int_0^1 dt\frac{t^{b-1}2(n\cdot qt-zt^2F\cdot q)(a+b-\omega)(-)}{(q\cdot q-2F\cdot qn\cdot qt+z(F\cdot q)^2t^2+m^2)^{a+b+1-\omega}}$$
$$=-\int_0^1 dtt^b\frac{d}{dt}\frac{1}{(q\cdot q-2F\cdot qn\cdot qt+z(F\cdot q)^2t^2+m^2)^{a+b-\omega}}$$
$$=-\frac{1}{(q\cdot q-2F\cdot qn\cdot q+z(F\cdot q)^2+m^2)^{a+b-\omega}}+b\int_0^1 dtt^{b-1}\frac{1}{(q\cdot q-2F\cdot qn\cdot qt+z(F\cdot q)^2t^2+m^2)^{a+b-\omega}}. \quad (54)$$

We see that the first integral in Eq. (53) is canceled by the second term of Eq. (54). Finally, we obtain

$$-b\int dp\frac{1}{[p^2+2p.q-m^2]^a}\frac{p_\mu}{(n.p)^{b+1}}=\frac{\partial}{\partial n^\mu}T(a,b)+(-1)^{a+b+1}i(\pi)^\omega(-2)^b\frac{\Gamma(a+b-\omega)}{\Gamma(a)\Gamma(b)}$$
$$\times\frac{(F\cdot q)^{b-1}(\frac{x}{D}(F\cdot q)F_\mu+\rho(q_\mu-n_\mu F\cdot q-n\cdot qF_\mu)}{(q\cdot q-2F\cdot qn\cdot q+z(F\cdot q)^2+m^2)^{a+b-\omega}}+(-1)^{a+1}i(\pi)^\omega 2^{b+1}\frac{\Gamma(a+b+1-\omega)}{\Gamma(a)\Gamma(b)}$$
$$\times(F\cdot q)^{b+1}\int_0^1 dt\frac{t^b(q_\mu-n_\mu F\cdot qt)}{(q\cdot q-2F\cdot qn\cdot qt+z(F\cdot q)^2t^2+m^2)^{a+b+1-\omega}}. \quad (55)$$

We have used Eq. (44) in the second term of (55).
We also know that

$$\frac{\partial}{\partial n^\mu}T(a,b)=T_w(a,b)\left(q_\mu\frac{1}{\sqrt{z}}-w\frac{n_\mu}{z}\right)+T_s(a,b)\left(-\sqrt{z}E_{\mu\alpha}q_\alpha+s\frac{n_\mu}{z}\right)+T_x(a,b)\bar{n}_\mu+2n_\mu T_z(a,b). \quad (56)$$

We defined

$$T_s=\frac{\partial T}{\partial s},\qquad T_v=\frac{\partial T}{\partial v},\qquad T_w=\frac{\partial T}{\partial w},\qquad T_x=\frac{\partial T}{\partial x},\qquad T_z=\frac{\partial T}{\partial z}.$$

Write, for a Lorentz scalar h,

$$\int dp\frac{1}{[p^2+2p\cdot q-m^2]^{a-1}}\frac{1}{(n\cdot p)^{b+1}}=h(v,w,s)$$
$$\frac{\partial}{\partial q^\mu}h(v,w,s)=2h_vq_\mu+h_w\frac{n_\mu}{\sqrt{z}}+h_sF_\mu\sqrt{z}. \quad (57)$$

We use the notation

$$h_s=\frac{\partial h}{\partial s},\qquad h_v=\frac{\partial h}{\partial v},\qquad h_w=\frac{\partial h}{\partial w}.$$

Picking up the coefficient of q_μ in Eqs. (55) and (57) we get

$$h_v=\rho(-1)^{a+b+1}i(\pi)^\omega(-2)^b\frac{\Gamma(a+b-\omega)}{\Gamma(a-1)\Gamma(b+1)}(F\cdot q)^{b-1}\frac{1}{(q\cdot q-2F\cdot qn\cdot q+n\cdot n(F\cdot q)^2+m^2)^{a+b-\omega}}$$
$$+(-1)^{a+b+1}i(\pi)^\omega(-2)^{b+1}\frac{\Gamma(a+b+1-\omega)}{\Gamma(a-1)\Gamma(b+1)}(F\cdot q)^{b+1}\int_0^1 dt\frac{t^b}{(q\cdot q-2F\cdot qn\cdot qt+n\cdot n(F\cdot q)^2t^2+m^2)^{a+b+1-\omega}}$$
$$+\frac{a-1}{b}\left(-\rho\sqrt{z}T_s(a,b)+T_w(a,b)\frac{1}{\sqrt{z}}\right). \quad (58)$$

Integrating h_v over $v=q\cdot q$, we get

$$h = \bar{h}(w,s) + \rho(-1)^{a+b} i(\pi)^{\omega}(-2)^{b} \frac{\Gamma(a+b-1-\omega)}{\Gamma(a-1)\Gamma(b+1)} (F\cdot q)^{b-1} \frac{1}{(q\cdot q - 2F\cdot qn\cdot q + n\cdot n(F\cdot q)^2 + m^2)^{a+b-1-\omega}}$$
$$+ (-1)^{a+b} i(\pi)^{\omega}(-2)^{b+1} \frac{\Gamma(a+b-\omega)}{\Gamma(a-1)\Gamma(b+1)} (F\cdot q)^{b+1} \int_0^1 dt \frac{t^b}{(q\cdot q - 2F\cdot qn\cdot qt + n\cdot n(F\cdot q)^2 t^2 + m^2)^{a+b-\omega}}$$
$$+ \frac{a-1}{b} \int dv \left(-\rho\sqrt{z} T_s(a,b) + T_w(a,b) \frac{1}{\sqrt{z}} \right). \tag{59}$$

From this expression we can derive h_w, h_s and compare it with Eq. (55). Since the integral exists, the integrability conditions for h are satisfied, as was the case for $b = 2$.

To fix $\bar{h}(w,s)$, we impose that the integral vanishes at $v \to -\infty$ (Please see footnote 4). We get $\bar{h}(w,s) = 0$.

Reintroducing the scale invariant variables v, w, s and using Eq. (52), we can write

$$T(a-1,b+1) = \frac{\rho(-1)^{a+b} i(\pi)^{\omega}(-2)^{b} \frac{\Gamma(a+b-1-\omega)}{\Gamma(a-1)\Gamma(b+1)} \frac{1}{z^{\frac{b-1}{2}}} s^{b-1}}{(v - 2sw + s^2 + m^2)^{a+b-1-\omega}} + \frac{a-1}{b} \int_{-\infty}^{v} dv' \left(-\rho\sqrt{z} T_s(a,b) + T_w(a,b) \frac{1}{\sqrt{z}} \right). \tag{60}$$

Notice that the term proportional to $(F\cdot q)^b$ in Eq. (52) is absent from (60). This says that the ansatz was right.

To simplify the recursion relation, we introduce the function $S(a,b)$ as follows:

$$T(a,b) = \rho(-1)^{a+b} i(\pi)^{\omega}(-2)^{b-1} \frac{1}{\Gamma(a)\Gamma(b)} \frac{1}{z^{\frac{b-2}{2}}} S(a,b). \tag{61}$$

We get

$$S(a-1,b+1) = \frac{s^{b-1}\Gamma(a+b-1-\omega)}{(v - 2sw + s^2 + m^2)^{a+b-1-\omega}} - \frac{1}{2} \int_{-\infty}^{v} dv' (-\rho z S_s(a,b) + S_w(a,b)), \tag{62}$$

subject to the condition $S(a-1,1) = 0$.

This recurrence relation is easy to solve because the v integral is trivial. Below we write the first terms of the recurrence relation:

$$S(a-1,2) = \frac{\Gamma(a-\omega)}{(v - 2sw + s^2 + m^2)^{a-\omega}}, \tag{63}$$

$$S(a-1,3) = \frac{(s(\rho z + 2) - w\rho z)\Gamma(a+1-\omega)}{(v - 2sw + s^2 + m^2)^{a+1-\omega}}, \tag{64}$$

$$S(a-1,4) = \frac{(s^2(3 + 3\rho z + \rho^2 z^2) - ws(3\rho z + 2\rho^2 z^2) + w^2\rho^2 z^2)\Gamma(a+2-\omega)}{(v - 2sw + s^2 + m^2)^{a+2-\omega}}$$
$$- \frac{1}{2} \frac{(3\rho z + \rho^2 z^2)\Gamma(a+1-\omega)}{(v - 2sw + s^2 + m^2)^{a+1-\omega}}. \tag{65}$$

It is straightforward to write a routine to solve the recurrence relation. We have solved it using the computer program FORM [15]. The code is written in Appendix C.

As a further check of Eq. (51), we have calculated the following finite integral that is not listed neither in [2] nor [1]. Using Eq. (16) we get the two-dimensional integral,

$$\int \frac{dp}{(2\pi)^2} \frac{1}{[p^2 - m^2]} \frac{1}{(n\cdot p)^2} = \frac{i}{4\pi m^2} \frac{n_0 - n_1}{n_0 n_1 (n_0 + n_1)} = 2\pi\rho \frac{i}{m^2}. \tag{66}$$

We have chosen $n_\mu = (n_0, n_1)$, $\bar{n}_\mu = (n_0, -n_1)$. We see that the result coincides with Eq. (51). Integral (66) is calculated in Appendix D.

In the next section we will find the solution of recurrence relation Eq. (62).

V. SOLVING THE RECURRENCE RELATION FOR $S(a,b)$

Define the function $G(t,b)$ by the following expression:

$$S(a,b)=\int_0^\infty dtG(t,b)t^{a+b-2-\omega}e^{-t(v-2sw+s^2+m^2)}. \tag{67}$$

Introducing it into Eq. (62), we see that $G(t,b)$ satisfies the recurrence relation

$$G(t,b+1)=s^{b-1}+\lambda G(t,b)-\frac{1}{t}\hat{O}G(t,b). \tag{68}$$

We have defined

$$\hat{O}=\frac{1}{2}(\rho z\partial_s-\partial_w),\qquad \lambda=s(1+\rho z)-\rho zw, \tag{69}$$

where $\partial_{s(w)}$ denotes the partial derivative with respect to $s(w)$.

Notice that $G(t,b)$ does not depend on a.

Equation (68) has the solution

$$G(t,b)=\frac{s^{b-1}-\lambda^{b-1}}{s-\lambda}-\sum_{i=2}^{b-2}\left(\lambda-\frac{1}{t}\hat{O}\right)^{i-2}\frac{\hat{O}}{t}\frac{s^{b-i}-\lambda^{b-i}}{s-\lambda},b\geq 1. \tag{70}$$

Equation (70) can be proved using induction on b.

Notice that λ depends on s, w, so $\hat{O}$ does not commute with λ.

Using Eq. (70), we get

$$S(a,2)=\frac{\Gamma(a+1-\omega)}{(v-2sw+s^2+m^2)^{a+1-\omega}}, \tag{71}$$

$$S(a,3)=(s+\lambda)\frac{\Gamma(a+2-\omega)}{(v-2sw+s^2+m^2)^{a+2-\omega}}, \tag{72}$$

$$S(a,4)=\frac{s^3-\lambda^3}{s-\lambda}\frac{\Gamma(a+3-\omega)}{(v-2sw+s^2+m^2)^{a+3-\omega}}-\hat{O}(s+\lambda)\frac{\Gamma(a+2-\omega)}{(v-2sw+s^2+m^2)^{a+2-\omega}}, \tag{73}$$

$$S(a,5)=\frac{s^4-\lambda^4}{s-\lambda}\frac{\Gamma(a+4-\omega)}{(v-2sw+s^2+m^2)^{a+2-\omega}}-\left(\hat{O}\frac{s^3-\lambda^3}{s-\lambda}+\lambda\hat{O}\frac{s^2-\lambda^2}{s-\lambda}\right)\frac{\Gamma(a+3-\omega)}{(v-2sw+s^2+m^2)^{a+1-\omega}}, \tag{74}$$

$$\begin{aligned}S(a,6)=&\frac{s^5-\lambda^5}{s-\lambda}\frac{\Gamma(a+5-\omega)}{(v-2sw+s^2+m^2)^{a+3-\omega}}-\frac{\Gamma(a+4-\omega)}{(v-2sw+s^2+m^2)^{a+2-\omega}}\left(\hat{O}\frac{s^4-\lambda^4}{s-\lambda}+\lambda\hat{O}\frac{s^3-\lambda^3}{s-\lambda}+\lambda^2\hat{O}(s+\lambda)\right)\\&+\frac{\Gamma(a+3-\omega)}{(v-2sw+s^2+m^2)^{a+1-\omega}}\left(\hat{O}^2\frac{s^3-\lambda^3}{s-\lambda}+\hat{O}\lambda\hat{O}(s+\lambda)\right).\end{aligned} \tag{75}$$

We have verified that Eq. (70) gives the same answers provided by a direct solution of Eq. (62).

We finally write the main result of this work,

$$\int dp \frac{1}{[p^2+2p\cdot q-m^2]^a}\frac{1}{(n\cdot p)^b}$$
$$=\rho(-1)^{a+b}i(\pi)^\omega(-2)^{b-1}\frac{1}{\Gamma(a)\Gamma(b)}\frac{1}{z^{\frac{b-2}{2}}}\int_0^\infty dt t^{a+b-2-\omega}e^{-t(v-2sw+s^2+m^2)}\left(\frac{s^{b-1}-\lambda^{b-1}}{s-\lambda}-\sum_{i=2}^{b-2}\left(\lambda-\frac{1}{t}\hat{O}\right)^{i-2}\frac{\hat{O}}{t}\frac{s^{b-i}-\lambda^{b-i}}{s-\lambda}\right)$$
$$+(-1)^{a+b}i(\pi)^\omega(-2)^b\frac{\Gamma(a+b-\omega)}{\Gamma(a)\Gamma(b)}(F\cdot q)^b\int_0^1 dt\frac{t^{b-1}}{(q\cdot q-2F\cdot qn\cdot qt+z(F\cdot q)^2t^2+m^2)^{a+b-\omega}}. \tag{76}$$

We can see that Eq. (76) preserves naive power counting and goes to the lcg result when $y\to 0, z\to 0$.

VI. CONCLUSIONS

In this paper we have obtained a closed form for the integrals that appear in loop calculations of gauge models in algebraic noncovariant gauges, outside the lcg. We employed the same method we used to derive the light cone integrals in [12].

The procedure is very simple, being based on a scale symmetry plus regularity conditions. For the single spurious pole, the integral satisfies a partial differential equation whose unique regular solution determines the value of the integral. Integrals containing higher-order spurious poles are obtained by derivation with respect to n_μ using (40) and then integrating a simple partial differential equation.

The procedure provides pole and finite parts of the integrals using DR.

We have verified that the integrals obtained in this way coincide with the ones computed using the generalized ML prescription, Eq. (16), which requires long calculations, having to fix from the beginning a form of $n_\mu, \bar{n}_\mu$.

We have also clarified the role of F_μ. It must be a null vector, $F\cdot F=0$, in any spacetime dimension. We derived this property from a consistency condition. Lorentz invariance plus $F\cdot F=0$ determines the form of F_μ in terms of $n_\mu, \bar{n}_\mu$. We have also exhibited a symmetry of the single spurious pole integral, which provides an independent way to determine F_μ.

Having obtained a closed form for the loop integrals in algebraic noncovariant gauges will greatly facilitate the calculation of Green functions and the discussion of renormalization in these gauges.

Our results provide a robust framework for regularizing integrals in noncovariant gauges. Future work could envisage applications in VSR and other nonlocal gauge theories.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: INTEGRALS EVALUATED AT $q_\mu=0$

In this appendix we write a list of several integrals that facilitate the comparison with [1,2].

First, we recall some definitions,

$$\int dp\frac{1}{[p^2-m^2]^a}\frac{1}{(n\cdot p)^b}=\rho(-1)^{a+1}i(\pi)^\omega(2)^{b-1}\frac{1}{\Gamma(a)\Gamma(b)}\frac{1}{z^{\frac{b-2}{2}}}S(a,b)\Big|_{q_\mu=0},$$

$$\int dp\frac{1}{[p^2-m^2]^a}\frac{p_\mu}{(n\cdot p)^b}=\rho(-1)^{a+1}i(\pi)^\omega 2^{b-2}\frac{1}{\Gamma(a)\Gamma(b)}\frac{1}{z^{\frac{b-2}{2}}}\frac{\partial S}{\partial q^\mu}(a-1,b)\Big|_{q_\mu=0},\quad b=2,3\ldots \tag{A1}$$

Below we list some integrals for $b = 1, \ldots, 8$. For $b \geq 2$, we rely either on the FORM routine presented in Appendix C or in Eq. (76).

$$\int dp \frac{1}{[p^2 - m^2]^a} \frac{p_\mu}{(n \cdot p)} = (-1)^a i\pi^\omega \frac{\Gamma(a-\omega)}{\Gamma(a)} \frac{1}{(m^2)^{a-\omega}} F_\mu \tag{A2}$$

$$\int dp \frac{p_\mu p_\nu p_\lambda}{[p^2 - m^2]^a} \frac{1}{n \cdot p} = (-1)^{a+1} \frac{i\pi^\omega}{2} \frac{\Gamma(a-1-\omega)}{\Gamma(a)(m^2)^{a-1-\omega}},$$
$$(F_\mu F_\nu F_\lambda z - F_\mu F_\nu n_\lambda - F_\mu F_\lambda n_\nu - F_\lambda F_\nu n_\mu + F_\mu \eta_{\nu\lambda} + F_\nu \eta_{\mu\lambda} + F_\lambda \eta_{\mu\nu}), \tag{A3}$$

$$\int dp \frac{1}{[p^2 - m^2]^a} \frac{1}{(n \cdot p)^2} = 2\rho(-1)^{a+1} i(\pi)^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a)(m^2)^{a+1-\omega}}, \tag{A4}$$

$$\int dp \frac{1}{[p^2 - m^2]^a} \frac{p_\mu}{(n \cdot p)^3} = \rho(-1)^{a+1} i(\pi)^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a)} \frac{(F_\mu(\rho z + 2) - n_\mu \rho)}{(m^2)^{a+1-\omega}}, \tag{A5}$$

$$\int dp \frac{1}{[p^2 - m^2]^a} \frac{1}{(n \cdot p)^4} = \frac{2}{3}\rho^2(-1)^a i(\pi)^\omega \frac{\Gamma(a+2-\omega)}{\Gamma(a)(m^2)^{a+2-\omega}} (3 + \rho z), \tag{A6}$$

$$\int dp \frac{1}{[p^2 - m^2]^a} \frac{p_\mu}{(n \cdot p)^5} = \rho^2(-1)^a i(\pi)^\omega \frac{\Gamma(a+2-\omega)}{6\Gamma(a)(m^2)^{a+2-\omega}} (F_\mu(12 + 12\rho z + 3\rho^2 z^2) - n_\mu \rho(8 + 3\rho z)), \tag{A7}$$

$$\int dp \frac{1}{[p^2 - m^2]^a} \frac{1}{(n \cdot p)^6} = \rho^3(-1)^{a+1} i(\pi)^\omega \frac{1}{15\Gamma(a)} \frac{\Gamma(a+3-\omega)}{(m^2)^{a+3-\omega}} (20 + 15\rho z + 3\rho^2 z^2), \tag{A8}$$

$$\int dp \frac{1}{[p^2 - m^2]^a} \frac{p_\mu}{(n \cdot p)^7} = \rho^3(-1)^{a+1} i(\pi)^\omega \frac{1}{90\Gamma(a)} \frac{\Gamma(a+3-\omega)}{(m^2)^{a+3-\omega}}$$
$$\times (F_\mu(120 + 180\rho z + 90\rho^2 z^2 + 15\rho^3 z^3) - n_\mu \rho(90 + 72\rho z + 15\rho^2 z^2)), \tag{A9}$$

$$\int dp \frac{1}{[p^2 - m^2]^a} \frac{1}{(n \cdot p)^8} = \rho^4(-1)^a i(\pi)^\omega \frac{2^7}{\Gamma(a)7!} \frac{\Gamma(a+4-\omega)}{(m^2)^{a+4-\omega}} \left(\frac{105}{4} + \frac{63}{2}\rho z + \frac{105}{8}\rho^2 z^2 + \frac{15}{8}\rho^3 z^3\right). \tag{A10}$$

We want to mention that there are relations among these integrals that serve as additional checks of the results of this paper.

Let us recall that

$$\rho = \frac{y}{D(x+D)}, \qquad \frac{\partial}{\partial n^\mu}\rho = \rho^2 n_\mu - F_\mu \rho(\rho z + 2). \tag{A11}$$

Then,

$$-\frac{1}{2}\frac{\partial}{\partial n^\mu} \int dp \frac{1}{[p^2 - m^2]^a} \frac{1}{(n \cdot p)^2} = \int dp \frac{1}{[p^2 - m^2]^a} \frac{p_\mu}{(n \cdot p)^3}.$$

Using Eq. (A11) we can easily get Eq. (A5) from Eq. (A4).

Let us take a second derivative of Eq. (A4), to get

$$\int dp \frac{1}{[p^2 - m^2]^a} \frac{p^2}{(n \cdot p)^4} = \frac{1}{3}(-1)^{a+1} i(\pi)^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a)(m^2)^{a+1-\omega}} \frac{\partial}{\partial n_\mu} \frac{\partial}{\partial n^\mu} \rho. \tag{A12}$$

But, we have that

$$\int dp \frac{1}{[p^2-m^2]^a}\frac{p^2}{(n\cdot p)^4} = \int dp \frac{1}{[p^2-m^2]^{a-1}}\frac{1}{(n\cdot p)^4} + \frac{m^2}{a-1}\partial_{m^2}\int dp \frac{1}{[p^2-m^2]^{a-1}}\frac{1}{(n\cdot p)^4}. \tag{A13}$$

The solution of the differential equation for $\int dp \frac{1}{[p^2-m^2]^{a-1}}\frac{1}{(n\cdot p)^4}$ that preserves the scaling in m^2 is

$$\int dp \frac{1}{[p^2-m^2]^{a-1}}\frac{1}{(n\cdot p)^4} = \frac{1}{3}(-1)^{a+1} i(\pi)^\omega \frac{\Gamma(a+1-\omega)}{\Gamma(a-1)(\omega-2)}(m^2)^{\omega-2+1-a}\frac{\partial}{\partial n_\mu}\frac{\partial}{\partial n^\mu}\rho. \tag{A14}$$

Using that

$$\frac{\partial}{\partial n_\mu}\frac{\partial}{\partial n^\mu}\rho = (d-4)\rho^2(\rho z+3), \tag{A15}$$

we get agreement with Eq. (A6).

In this way we can generate all integrals listed above, starting from Eq. (A2).

In addition to the previous relations, a symmetric $E_{\mu\nu}$ implies that F_μ is the gradient of a potential, $F_\mu = \phi_{,\mu}$. Here, $\phi_{,\mu} = \frac{\partial\phi}{\partial n^\mu}$. We get

$$\phi = \log\left(\frac{x+D}{\sqrt{y}}\right). \tag{A16}$$

Therefore, we can write

$$\int dp \frac{1}{(p^2-m^2+i\varepsilon)^a}\frac{p_\mu}{n\cdot p} = A_a F_\mu, \tag{A17}$$

$$A_a = (-1)^a i\pi^\omega \frac{\Gamma(a-\omega)}{\Gamma(a)}\frac{1}{(m^2)^{a-\omega}}. \tag{A18}$$

This permits us to prove that

$$\int dp \frac{1}{(p^2-m^2+i\varepsilon)^a}\frac{p_\mu p_\nu}{(n\cdot p)^2} = A_a E_{\mu\nu}, \qquad E_{\mu\nu} = -\phi_{,\mu\nu}, \tag{A19}$$

$$\int dp \frac{1}{(p^2-m^2+i\varepsilon)^a}\frac{p_\mu p_\nu p_\lambda}{(n\cdot p)^3} = \frac{1}{2}\phi_{,\mu\nu\lambda}A_a, \tag{A20}$$

$$\int dp \frac{1}{(p^2-m^2+i\varepsilon)^a}\frac{p_{\mu_1}\cdots p_{\mu_n}}{(n\cdot p)^b} = \frac{(-1)^{b+1}}{\Gamma(b)}\phi_{,\mu_1\ldots\mu_b}A_a. \tag{A21}$$

APPENDIX B: A SYMMETRY OF SINGLE POLE INTEGRALS

We want to mention that F_μ is invariant under the following infinitesimal transformations:

$$\delta n_\mu = 0, \qquad \delta\bar{n}_\mu = c\frac{n_\mu}{n\cdot n} + d\bar{n}_\mu. \tag{B1}$$

Here, c and d are arbitrary parameters.

This means that Eq. (30) is invariant under (B1).

This symmetry determines F_μ uniquely, without imposing $F\cdot F = 0$.

Higher pole integrals do not respect the symmetry corresponding to the infinitesimal parameter c. In fact, $\delta\rho = \frac{c}{zD}$. D is defined in Eq. (36).

From Eq. (16) it is easy to see that the scaling defined by the parameter d is always a symmetry.

APPENDIX C: SOLVING EQ. (62) USING FORM

```
**********************rec.frm*********************************
#-;
F Ts,Tw,s,w,int,q;
S rho,a,x,b,z,om;
*q includes the gamma factor in the numerator
Off statistics;
.global
GL t2=q(a-om);
p;
.store
* Change next line to get other b's
#do i=2,10
GL t'i'2=(-rho*z*Ts+Tw)*t'i';
id q(a?)=-int*q(a+1)/2;
repeat;
id int*q(a?)=-q(a-1);
id Ts*q(a?)=(2*w(1)-2*s(1))*q(a+1);
id Tw*q(a?)=(2*s(1))*q(a+1);
id Ts*s(a?)=a*s(a-1)+s(a)*Ts;
id Ts*w(a?)=w(a)*Ts;
id Tw*s(a?)=s(a)*Tw;
id Tw*w(a?)=a*w(a-1)+w(a)*Tw;
id s(a?)*w(b?)=w(b)*s(a);
id s(a?)*s(b?)=s(a+b);
id w(a?)*w(b?)=w(a+b);
id w(0)=1;
id s(0)=1;
endrepeat;
.store
GL t{'i'+1}=s('i'-1)*q(a+'i'-1-om)+t'i'2;
p;
.store
#enddo
.end
```

We have defined

$$q(a-om)=\frac{\Gamma(a-\omega)}{(v-2sw+s^2+m^2)^{a-\omega}},\qquad t'i'=S(a-1,i)\qquad s(n)=s^n,\qquad w(n)=w^n,\qquad rho=\rho.$$

APPENDIX D: SOME FINITE INTEGRALS

In this appendix, we calculate the integrals defined in Eqs. (38), (39), and (66) using Eq. (16).

To calculate the integrals, we see that the even functions of p_0 and p_1 survive. Then we introduce the Feynman parameter x to write

$$
\begin{aligned}
&\int \frac{dp}{(2\pi)^2}\frac{1}{[p_0^2-p_1^2-m^2+i\varepsilon]^2}p_0\frac{n_0p_0+n_1p_1}{(n_0p_0)^2-(n_1p_1)^2+i\varepsilon}\\
&=2n_0\int_0^1 dxx\int\frac{dp}{(2\pi)^2}\frac{p_0^2}{[p_0^2x-p_1^2x-m^2x+(1-x)(n_0^2p_0^2-n_1^2p_1^2)+i\varepsilon]^3}\\
&=2n_0\int dx\frac{x}{(x+(1-x)n_0^2)^3}\int\frac{dp}{(2\pi)^2}\frac{p_0^2}{[p_0^2-p_1^2\frac{x}{A(x)}-m^2\frac{x}{A(x)}-\frac{(1-x)}{A(x)}n_1^2p_1^2+i\varepsilon]^3}\\
&=n_0\frac{i}{\sqrt{4\pi}}\int_0^1 dx\frac{x}{(x+(1-x)n_0^2)^3}\int\frac{dp_1}{(2\pi)}\frac{\Gamma(3/2)}{2}\frac{1}{(p_1^2\frac{x}{A(x)}+m^2\frac{x}{A(x)}+\frac{(1-x)}{A(x)}n_1^2p_1^2-i\varepsilon)^{3/2}}\\
&=n_0\frac{i}{2^3}\int_0^1 dx\frac{x}{(x+(1-x)n_0^2)^3}\frac{1}{B(x)^{3/2}}\int\frac{dp_1}{(2\pi)}\frac{1}{(p_1^2+m^2\frac{x}{B(x)A(x)})^{3/2}}\\
&=n_0\frac{i}{2^3}\int dx\frac{x}{(x+(1-x)n_0^2)^3}\frac{1}{B(x)^{3/2}}\left(m\sqrt{\frac{x}{B(x)A(x)}}\right)^{-2}\frac{1}{\pi}\\
&=\frac{in_0}{8\pi m^2}\int_0^1 dx\frac{1}{\sqrt{(x+(1-x)n_0^2)^3}}\frac{1}{\sqrt{x+(1-x)n_1^2}}\\
&=\frac{in_0}{8\pi m^2}\left(-\frac{2}{n_0(n_0^2-n_1^2)}(n_1-n_0)\right)=\frac{in_0}{4\pi m^2}\frac{1}{n_0(n_0+n_1)}.
\end{aligned}
\tag{D1}
$$

We have defined

$$A(x)=x+(1-x)n_0^2, \tag{D2}$$

$$B(x)=\frac{x}{A(x)}+\frac{(1-x)}{A(x)}n_1^2. \tag{D3}$$

That is, we get

$$\int dp\frac{1}{[p^2-m^2]^2}\frac{p_0}{n.p}=\frac{in_0\pi}{m^2}\frac{1}{n_0(n_0+n_1)}. \tag{D4}$$

Similarly, we get

$$
\begin{aligned}
\int\frac{dp}{(2\pi)^2}\frac{1}{[p^2-m^2]^2}\frac{p_1}{n.p}&=\int\frac{dp}{(2\pi)^2}\frac{1}{[p_0^2-p_1^2-m^2+i\varepsilon]^2}p_1\frac{n_0p_0+n_1p_1}{(n_0p_0)^2-(n_1p_1)^2+i\varepsilon}\\
&=\int\frac{dp}{(2\pi)^2}\frac{1}{[p_0^2-p_1^2-m^2+i\varepsilon]^2}\frac{n_1p_1^2}{(n_0p_0)^2-(n_1p_1)^2+i\varepsilon}\\
&=2n_1\int_0^1\frac{dx}{A^3}x\int\frac{dp}{(2\pi)^2}\frac{p_1^2}{[p_0^2-p_1^2\frac{x}{A(x)}-m^2\frac{x}{A(x)}-\frac{(1-x)}{A(x)}n_1^2p_1^2+i\varepsilon]^3}\\
&=2n_1\frac{1}{(2\pi)}\frac{-i}{\sqrt{4\pi}}\frac{\Gamma(5/2)}{2}\int_0^1\frac{dx}{A^3B^{5/2}}x\int dp_1\frac{p_1^2}{(p_1^2+m^2\frac{x}{A(x)B})^{5/2}}\\
&=2n_1\frac{1}{(2\pi)}\frac{-i}{\sqrt{4\pi}}\frac{\Gamma(5/2)}{2}\frac{2}{3}\int_0^1\frac{dx}{A^3B^{5/2}}\frac{x}{m^2\frac{x}{A(x)B}}\\
&=-i\frac{n_1}{8\pi m^2}\int_0^1 dx\frac{1}{\sqrt{(x+(1-x)n_1^2)^3}}\frac{1}{\sqrt{x+(1-x)n_0^2}}\\
&=-i\frac{n_1}{8\pi m^2}\left(-\frac{2}{n_1(n_0^2-n_1^2)}(n_1-n_0)\right)=-i\frac{n_1}{4\pi m^2}\left(\frac{1}{n_1(n_0+n_1)}\right).
\end{aligned}
\tag{D5}
$$

Therefore, we get

$$\int dp \frac{1}{[p^2-m^2]^2}\frac{p_1}{n.p} = -i\frac{n_1\pi}{m^2}\left(\frac{1}{n_1(n_0+n_1)}\right). \tag{D6}$$

Now we wish to calculate the following integral:

$$\begin{aligned}
\int \frac{dp}{(2\pi)^2}\frac{1}{[p^2-m^2]}\frac{1}{(n.p)^2} &= \int \frac{dp}{(2\pi)^2}\frac{1}{[p_0^2-p_1^2-m^2+i\varepsilon]}\frac{(n_0p_0+n_1p_1)^2}{(n_0^2p_0^2-n_1^2p_1^2+i\varepsilon)^2}\\
&= 2\int_0^1 dx(1-x)\int \frac{dp}{(2\pi)^2}\frac{(n_0p_0+n_1p_1)^2}{[(p_0^2-p_1^2-m^2)x+(1-x)(n_0^2p_0^2-n_1^2p_1^2)+i\varepsilon]^3}\\
&= 2\int_0^1 dx(1-x)\int \frac{dp}{(2\pi)^2}\frac{(n_0^2p_0^2+n_1^2p_1^2)}{[(p_0^2-p_1^2-m^2)x+(1-x)(n_0^2p_0^2-n_1^2p_1^2)+i\varepsilon]^3}\\
&= 2\int_0^1 dx(1-x)\int \frac{dp_1}{(2\pi)}\frac{n_0^2}{A^3}\frac{i}{2^4}\frac{1}{[(p_1^2+m^2)\frac{x}{A}+n_1^2p_1^2\frac{1-x}{A}]^{3/2}}\\
&\quad +2\int_0^1 dx(1-x)\int \frac{dp_1}{(2\pi)}\frac{n_1^2}{A^3}\frac{-i}{\sqrt{4\pi}}\frac{\Gamma(5/2)}{2}\frac{p_1^2}{[(p_1^2+m^2)\frac{x}{A}+n_1^2p_1^2\frac{1-x}{A}]^{5/2}}\\
&= \frac{i}{2^3m^2\pi}\int_0^1 dx\frac{(1-x)}{x}\left\{\frac{n_0^2}{A^2B^{1/2}}-\frac{n_1^2}{A^2B^{3/2}}\right\}\\
&= \frac{i}{2^3m^2\pi}\int_0^1 dx\frac{x}{1-x}\left(\frac{n_0^2}{(x+(1-x)n_0^2)^{3/2}(x+(1-x)n_1^2)^{1/2}}-\frac{n_1^2}{(x+(1-x)n_0^2)^{1/2}(x+(1-x)n_1^2)^{3/2}}\right)\\
&= \frac{i}{2^3m^2\pi}2\frac{n_0^2+n_1^2-2n_0n_1}{n_0n_1(n_0^2-n_1^2)} = \frac{i}{4\pi m^2}\frac{n_0-n_1}{n_0n_1(n_0+n_1)}.
\end{aligned} \tag{D7}$$

Therefore, we get

$$\int dp \frac{1}{[p^2-m^2]}\frac{1}{(n.p)^2} = \frac{i\pi}{m^2}\frac{n_0-n_1}{n_0n_1(n_0+n_1)}. \tag{D8}$$

On the other hand, we have that

$$x = n_0^2+n_1^2, \qquad y=z=n_0^2-n_1^2, \qquad D=2n_0n_1, \qquad x+D=(n_0+n_1)^2, \qquad \rho=\frac{n_0-n_1}{2n_0n_1(n_0+n_1)}, \tag{D9}$$

and

$$F_0=\frac{1}{n_0+n_1}, \qquad F_1=-\frac{1}{n_0+n_1}. \tag{D10}$$

We can verify that Eqs. (38), (39), and (66) are correct.

APPENDIX E: COMPARISON WITH [1,2]

In this section, we want to compute some of the integrals listed in [1,2] to provide a test of our results,

$$\mathbb{P}\int \frac{dp}{(n\cdot p)^2}\frac{1}{((p-q)^2-m^2)} = 2\rho\frac{i\pi^2}{2-\omega}. \tag{E1}$$

This is (A6.36) of [2]. It is obtained from Eq. (51) evaluated at $a=1$.

To compute the pole part in the next integral, we expand

$$\frac{1}{(p-q)^2-m^2} = \frac{1}{p^2-m^2}\left(1+2\frac{p\cdot q}{p^2-m^2}+o(q^2)\right), \tag{E2}$$

$$\mathbb{P}\int \frac{dp}{n\cdot p}\frac{p_\mu p_\nu}{((p-q)^2-m^2)p^2}=2q^\lambda\mathbb{P}\int\frac{dp}{n\cdot p}\frac{p_\mu p_\nu p_\lambda}{(p^2-m^2)^3}$$
$$=\frac{i\pi^2}{2-\omega}\frac{1}{2}\{F_\mu F_\nu F\cdot qz-F_\mu F_\nu n\cdot q-F\cdot q(F_\mu n_\nu+F_\nu n_\mu)+F_\mu q_\nu+F_\nu q_\mu+F\cdot q\eta_{\mu\nu}\}. \quad \text{(E3)}$$

The contribution to the integral of the first term in the expansion (E2) vanishes because the integrand is odd in p_μ. To obtain integral (E3) we used Eq. (35). We agree with Appendix D of [1]. Notice that $F_\mu=\frac{F^L_\mu}{n\cdot F^L}$, where F^L_μ is Leibbrandt's F_μ.

From Eq. (51) we get

$$\int dp\frac{1}{[p^2-m^2]^a}\frac{(p\cdot q)^4}{(n\cdot p)^2}=\frac{3}{2}(-1)^{a+1}i\pi^\omega\frac{\Gamma(a-1-\omega)}{\Gamma(a)(m^2)^{a-1-w}}\left(\rho U^2+(F\cdot q)^2\left(2q\cdot q-\frac{8}{3}F\cdot qn\cdot q+z(F\cdot q)^2\right)\right)$$
$$U=q\cdot q-2F\cdot qn\cdot q+z(F\cdot q)^2. \quad \text{(E4)}$$

For $a=3$ this equation provides the pole part of the integral written in (A6.45) of [2]. Additionally, it gives the value of the whole integral for arbitrary a.

Finally, we compute integral (1) using ML prescription,

$$I=\int\frac{dp}{p^2(p-k)^2n\cdot p}=\int_0^1dx\int\frac{dp}{n\cdot p[p^2-2k\cdot px+xk^2]^2}$$
$$=\int_0^1dx2i\pi^2\bar n\cdot(-kx)\int_0^1dt\frac{1}{(-xk^2+k^2x^2-2n\cdot(-kx)(\bar n\cdot(-kx))t)}$$
$$=i\pi^2\frac{1}{n\cdot k}\int_0^1dx\frac{1}{x}\log\left(1+\frac{2n\cdot k\bar n\cdot kx}{(1-x)k^2}\right)$$
$$=i\pi^2\frac{1}{n\cdot k}\left(\frac{\pi^2}{6}+\int_0^{1-\frac{2n\cdot k\bar n\cdot k}{k^2}}du\frac{\log(1-u)}{u}\right). \quad \text{(E5)}$$

We have used Eq. (13) with $q_\mu=-k_\mu x, m^2=-xk^2$. The integral is finite and agrees with Eq. (6.5) of [2].

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