

Revisiting the sphaleron and axion production rates in QCD at high temperatures

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We report our new lattice results for the sphaleron rate calculated within a thermal effective field theory of soft SU(N) gluons whose momenta are below the magnetic scale, where $N = 2, 3$, for a wide range of temperatures spanning from 0.6 to 10^{15} GeV at sufficiently large volumes. Comparing these results with sphaleron rates in a nonthermal SU(N) plasma where the infrared gluons are overoccupied, we estimate the typical thermalization time for these ultrasoft gluons during the early stages of reheating after inflation. We also calculate the thermal production rate of relativistic axions due to these nonperturbatively interacting soft gluons which shows a significant deviation from its perturbative estimate even at the electroweak scale.

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I. MOTIVATION AND OUTLINE

In strongly interacting nuclear matter described by quantum chromodynamics (QCD), the axial current $j_{5,f}^\mu$ of the quarks is not conserved due to quantum fluctuations. For each flavor f of quarks in the fundamental representation of the gauge group with N colors, this violation is described by the anomaly relation,

$$\partial_\mu j_{5,f}^\mu = 2m\bar{q}_f i\gamma_5 q_f - \frac{g^2}{16\pi^2} G_{\mu\nu}^c \tilde{G}_{\mu\nu}^c, \quad (1)$$

where $G_{\mu\nu}^c$ represents the field strength tensor, $\tilde{G}_{\mu\nu}^c = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G_{\alpha\beta}^c$ is its dual, q_f are the quark fields, and g is the gauge coupling. The color index runs from $c = 1 - N^2 - 1$. For massless quarks, i.e., $m = 0$, the right-hand side of Eq. (1) can be written in terms of the divergence of the Chern-Simons current K_μ , where

$$K^\mu(x) = \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \left(A_\nu^c F_{\rho\sigma}^c(x) - \frac{g}{3} f_{cde} A_\nu^c A_\rho^d A_\sigma^e(x) \right), \quad (2)$$

such that the Chern-Simons number N_{CS} is the corresponding charge

$$N_{CS}(t) = \int d^3\mathbf{x} K^0(t, \mathbf{x}). \quad (3)$$

A unique property of this N_{CS} for non-Abelian gauge theories is that it is an integer and labels each degenerate vacuum state. Hence, each vacuum state of QCD is distinct and topologically inequivalent. The tunneling solutions between two vacua are known as instantons [1,2], which are characterized by an integer topological charge corresponding to the difference in the N_{CS} labeling these vacua. When the typical energy fluctuations are larger in magnitude than the height $\sim \Lambda_{QCD}$ of the barrier that separates two topologically distinct vacua, certain gauge field solutions will exist which represent a rollover from the top of the barrier to a vacuum state. Such a solution is known as a sphaleron [3]. The rate of change of N_{CS} with time due to topological transitions can be denoted as,

$$\frac{dN_{CS}}{dt} = \frac{g^2}{8\pi^2} \sum_{c=1}^{N^2-1} \int d^3\mathbf{x} E_i^c(\mathbf{x}) B_i^c(\mathbf{x}). \quad (4)$$

The autocorrelation of the Chern-Simons number at two different times t and $t + \Delta t$ given by

$$\Gamma_{sph} = \lim_{V \rightarrow \infty, t \rightarrow \infty} \frac{\langle [N_{CS}(t + \Delta t) - N_{CS}(t)]^2 \rangle}{V \Delta t} \quad (5)$$

thus denotes the sphaleron transition rate Γ_{sph} assuming that a change in the Chern-Simons number is diffusive at long enough time evolution $t \rightarrow \infty$. Accurately determining the sphaleron rate in QCD is important for many important physical phenomena. Sphaleron rates contribute to the damping of coherent oscillations of axions [4], to exotic

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transport phenomena, e.g., the chiral magnetic effect [5], and are a source of strong CP violation which is believed to have played a role during baryogenesis in the early Universe [6].

It was anticipated early on that the sphaleron rate can be measured controllably in classical $SU(N)$ theory [7] at finite temperatures T . However, it was soon realized that classical gauge theories in $3 + 1$ dimensions suffer from ultraviolet divergences, and the hard gluons with momenta $|\mathbf{p}| \gtrsim \pi T$ play an important role in the estimation of Γ_{sph} . These hard gluons, though not included in the classical theory, influence the sphaleron dynamics since the hard scale sets the momentum cutoff and thus the relaxation rates in the effective theory [8]. Incorporating this fact properly leads to a $\Gamma_{\text{sph}} \propto g^{10} T^4$ rather than the naive expectation $\sim g^8 T^4$ that comes just from dimensional analysis. A more careful analysis of the sphaleron rate leads to a dependence of the form $\sim \log(1/g) g^{10} T^4$ [9] arising due to the logarithmic dependence of the Debye mass on g . In the limit when $\log(1/g) \gg 1$, an effective field theory of the soft modes of the non-Abelian gauge theory at leading logarithmic order can be constructed [9], which remains valid even at next-to-leading-log order if the color conductivity which goes as an input parameter is also calculated at the same order [8]. Furthermore, this effective theory is not plagued by ultraviolet divergences that are inherent in the classical Hamiltonian for non-Abelian gauge theory [10]. The Γ_{sph} measured on the lattice [11] within this effective theory of soft gluons provides the leading $\log(1/g)$ dependence at sufficiently weak couplings, $g \lesssim 0.6$.

Sphaleron transitions are also ubiquitous in a gluonic plasma under far-from-thermal-equilibrium conditions, where the relevant scale Q_s , which denotes the gluon-saturation scale, is larger than the height of the barrier between two vacua, i.e., $Q_s \gg \Lambda_{\text{QCD}}$. A typical realization of a nonthermal state in $SU(N)$ gauge theory consists of infrared gluons whose occupation numbers in a particular momentum state are large, i.e., $1/\alpha_s(Q^2) \gg 1$, which allow for a classical-statistical description of such modes [12]. Starting from such an initial state, the classical Hamiltonian evolution leads to a self-similar scaling regime, where the momentum distribution function of gluons reaches a stationary steady state [13–15]. Magnetic, electric, and hard scales can be defined in such a system as well, and a clear scale hierarchy exists in this self-similar regime analogous to a thermal non-Abelian plasma at high enough temperatures [16]. Sphaleron transitions in such a self-similar nonthermal regime can be described as a random walk of N_{CS} values with time, whose rate has a $(Q_s t)^{-\frac{4}{3}} Q_s^4$ dependence on the gluon-saturation scale Q_s for $SU(2)$ gauge theory [12]. This rate is significantly higher than in a thermal plasma of gluons with a similar energy density.

In this work, we revisit the calculation of the Γ_{sph} in $SU(N)$ gauge theories using lattice techniques both in- and out-of-thermal-equilibrium conditions where the $N = 3$

results for the nonthermal case are new and not discussed earlier in the literature. We work in an effective theory of the soft gluons whose momenta are of the order of the magnetic scale and lower [9], which allows us to scan a wide range of temperatures not studied earlier, keeping the lattice volumes sufficiently large. This is very difficult to achieve in standard thermal lattice gauge theory simulations with a fixed number of sites along the Euclidean time direction, with new strategies being currently investigated [17]. Our emphasis is to extract some physically relevant quantities from the QCD sphaleron rates. First, we provide an estimate of the thermalization time for the ultrasoft momentum modes of $SU(2)$ and $SU(3)$ gluons during the early stages of the reheating phase by matching the sphaleron rates between a thermal and nonthermal plasma with equal energy densities.

By accurately calculating Γ_{sph} , we also estimate its contribution in the production rate of thermal (QCD) axions with momenta $|\mathbf{k}| = 0$, $\omega \rightarrow 0$ at a wide range of temperatures. We also calculate for the first time, the production rate of axions with dispersion $\omega \simeq |\mathbf{k}|$ due to nonperturbatively interacting soft gluons, which for temperatures less than the electroweak scale, is significantly larger than the perturbative predictions in the hard-thermal loop effective theory. The plan of this paper is as follows: After briefly reviewing the key aspects of the effective theory of magnetic gluons at high temperatures, we describe our algorithm to generate statistically independent gauge configurations by discretizing the effective Hamiltonian on a 3D spatial lattice. In the subsequent section, we discuss our numerical procedure to extract the sphaleron rate in an $SU(N)$ plasma in thermal equilibrium at high temperatures. Comparing these results with that of a nonthermal plasma of overoccupied gluons, we extract typical timescales relevant for the thermalization of such ultrasoft gluons in the early stages of the reheating epoch. We conclude by discussing two important physical implications of our results in the context of preheating in the early Universe and for the relic axion yields.

II. LATTICE IMPLEMENTATION

A. Algorithm for generating gauge field configurations: Thermal case

We revisit the basic features of a finite temperature effective theory of the soft magnetic gluons [9]. In a temperature regime where the soft, semihard, and the hard scales are well separated, i.e., $g^2 T / \pi \ll gT \ll \pi T$, gluons with momenta $\gtrsim gT$ can be integrated out resulting in an effective Hamiltonian description [9] of the dynamics of the soft gluons whose momenta are of the order of the magnetic scale and lower. The equation of motion of these soft gluons of an $SU(N)$ gauge theory is denoted by

$$-\partial_t E_{\mathbf{x}}^{ic} + [D_j, F^{ji}(\mathbf{x})]^c = \sigma E_{\mathbf{x}}^{ic} + \zeta_{\mathbf{x}}^{ic}(t),$$

$$c = 1, \dots, N^2 - 1. \quad (6)$$

Here, σ is the color conductivity, which is known up to next-to-leading-log order [8]

$$\sigma^{-1} = \frac{3Ng^2T}{4\pi m_D^2} \left[\ln \frac{m_D}{\gamma} + 3.041 \right] \quad (7)$$

in terms of the Debye mass in pure gauge theory, $m_D = \sqrt{\frac{N}{3}}gT + \mathcal{O}(g^3)$. The mean rate of color randomization γ is estimated self-referentially through the relation [8]

$$\gamma = \frac{Ng^2T}{4\pi} \left[\ln \frac{m_D}{\gamma} + 3.041 \right]. \quad (8)$$

The $\zeta_{\mathbf{x}}^{ic}(t)$ are the stochastic color-force fields with color index c , which satisfies the fluctuation-dissipation relation $\langle \zeta_{\mathbf{x}_1}^{ic}(t_1) \zeta_{\mathbf{x}_2}^{jd}(t_2) \rangle = 2T\sigma\delta^{ij}\delta^{cd}\delta^3(\mathbf{x}_1 - \mathbf{x}_2)\delta(t_1 - t_2)$. This effective Hamiltonian for the magnetic gluons describes their interactions in terms of a stochastic noise which mimics random kicks on them due to the hard gluons and a term proportional to the color conductivity which acts to dampen these large random forces. These two contrasting contributions from the hard modes ensure that the soft gluons attain a thermal distribution under a sufficiently long enough time evolution.

Discretizing Eq. (6), the electric fields, noise fields, and color conductivity can be written in dimensionless units $gE_{\mathbf{x}}^i a^2$, $ga^2\delta t\zeta_{\mathbf{x}}^i(t)$, and $\sigma\delta t = \frac{\sigma}{T}\delta t$ respectively, where a is the lattice spacing. We set σ/T to its perturbative estimate given in Eq. (7) and numerically implement Eq. (6) on a three-dimensional lattice with a spatial size N_s^3 by recasting it in dimensionless units. The discretized version of Eq. (6) is solved using the leapfrog method with a time step $\delta t/a = 0.01$ and evolved until $t \sim 2000\text{--}4000\delta t$ for the algorithm to produce thermal gauge configurations. We next save ~ 640 statistically independent gauge configurations at each temperature, which are separated by $\delta t/a = 50$ for performing thermal averages in order to extract physical quantities. Note that the two parameters that enter into Eq. (6), the coupling g and temperature T , are not independent but are related to each other through the two-loop beta function. The strong coupling g is calculated at the scale πT . Gauss's law constraint was implemented with a precision of 10^{-15} at $t = 0$, and it was checked to remain so at later times.

At high temperatures, the occupation numbers for the gluons with momenta $\lesssim g^2T/\pi$ are much larger than unity; hence, these interact classically. Such a classical system can be realized as consisting of $2(N^2 - 1)N_s^3$ oscillators of energy aT which has an energy density on the lattice [18] given by $\frac{2(N^2-1)}{N_s^3} \sum_{\mathbf{k}} |\mathbf{k}| \frac{aT}{|\mathbf{k}|}$, where the sum is over all allowed lattice momenta $\mathbf{k}$. We have verified that the energy density that we compute on the lattice agrees well

with this estimate. We then set the lattice spacing in the effective theory in physical units from the criterion so that the measured energy density matches its Stefan-Boltzmann value at each $T \gtrsim 600$ MeV. This results in a condition $Ta = (30/\pi^2)^{1/3}$. This criterion for setting the scale in the effective theory is motivated from lattice calculations of energy density in four-dimensional SU(2) [19] and SU(3) gauge theories [20,21], which quantify the deviation from the ideal gas limit to be $\lesssim 18\%$ at $2T_c$ where $T_c \sim 300$ MeV to $\sim 1\%$ at the electroweak scale. Nevertheless, the magnitude of these deviations sets an error on our lattice spacing estimate at each temperature which ranges from 6% to 0.6% at $2T_c$ and $20T_c$, respectively. We have checked that the sphaleron rates are insensitive to the variation of the lattice spacing within $1\text{-}\sigma$.

Evolving the color electric fields in a thermal ensemble as a function of the time, we perform cooling of the gauge links and electric fields at time steps $1 - 10\delta t$ depending on whether the coupling is $g \gtrsim 1$ or $g < 1$, in order to remove the ultraviolet fluctuations in them. Details of the cooling procedure and the implementation of Chern-Simons current is explained in the subsequent subsections. Performing time evolution of the color electric fields and gauge links from an initial thermal configuration, we measure the Chern-Simons number change as a function of the time interval of observation $\Delta t = t - t_0$, where $t_0/a = 100$.

B. Algorithm for generating gauge field configurations: Nonthermal case

In order to calculate the sphaleron rates in a nonthermal plasma, we have to first choose a suitable nonequilibrium initial state. We start from an initial condition where the gluons labeled by momentum $\mathbf{p}$ are occupied according to a phase-space distribution given by $\tilde{f}(\mathbf{p}) = g^2 f(\mathbf{p}) = n_0 \frac{Q_s}{|\mathbf{p}|} e^{\frac{-|\mathbf{p}|^2}{2Q_s^2}}$, where Q_s is the gluon-saturation scale, which is typically between 1 and 2 GeV [22], and g is the gauge coupling. This is to ensure that the occupation numbers of soft gluons is nonperturbatively large; hence, their dynamics is classical. Sampling the gauge links and the electric fields from this initial distribution at $t = 0$, the color electric fields are evolved according to Hamilton's equation which is similar to Eq. (6) but with the right-hand side of it set to zero. The classical Hamiltonian evolution of the gauge links and color electric fields on a spatial lattice of size N_s^3 and spacing a are performed using the leapfrog integrator with a time step $\delta t = 0.01a$. At sufficiently late times $Q_s \cdot t_0 \sim 50$, the scales that characterize the soft, semihard, and hard gluons in this nonthermal plasma exhibit a characteristic time dependence [16], thus separating out from each other. The late-time momentum distribution function of gluons exhibits a specific time-dependence characteristic of a nonthermal fixed point [15], during which we measure the Chern-Simons number change as a function of the observation time $\Delta t = t - t_0$

TABLE I. Parameters and statistics for nonequilibrium classical-statistical simulations of SU(2) and SU(3) gauge theories.

	$Q_s \cdot a$	N_s	N_{confs}		$Q_s \cdot a$	N_s	N_{confs}
SU(2)	1.0	64	320	SU(3)	0.5	64	320
	1.0	128	320		1.0	64	320
	0.5	64	320		1.0	48	320
	0.5	128	320		1.0	96	192

by performing a cooling of the gauge fields at every interval $10\delta t$, the details of which are described in the next subsection. The n_0 values that determine the initial gluon distributions are chosen such that the magnitude of the initial energy densities is similar to the Stefan-Boltzmann values in a thermal plasma at chosen values of the temperature, to enable a comparison of the sphaleron rates among them. The details about the different parameters and the number of nonthermal configurations generated are mentioned in Table I. We will henceforth denote all dimensional quantities in units of Q_s .

C. Measuring the Chern-Simons number change using cooling

We will next outline the details of our procedure, which we primarily follow from Ref. [12], to calculate the change in the Chern-Simons number due to sphaleron transitions. We first start with the gauge links and the electric fields obtained using the classical Hamilton's equation at a time t_1 , and successively remove the ultraviolet fluctuations of these fields through a procedure known as calibrated cooling [23,24] in order to reach the nearest vacuum configuration. The spatial gauge links are updated along the fictitious cooling time direction τ according to the following equation:

$$U_i(\mathbf{x}, t; \tau + d\tau) = e^{-i g a E_{\text{cool}}^i(\mathbf{x}, t; \tau) d\tau} U_i(\mathbf{x}, t; \tau), \quad (9)$$

where the color components of the cooled electric fields labeled by c are defined as

$$E_{\text{cool}}^{i,c}(t, \mathbf{x}) = -\frac{\delta H}{\delta A_i^c(\mathbf{x}, t)}. \quad (10)$$

In terms of the elementary plaquette variables $U_{i,j}^\square$, the above equation can be rewritten as

$$E_{\text{cool}}^{i,c}(\mathbf{x}, t; \tau) = -\frac{2}{g a^3} \sum_{j \neq i} \text{ReTr}[t^c [U_{i,j}^\square - U_{i,-j}^\square](\mathbf{x}, t; \tau)], \quad (11)$$

where t^c are the generators of SU(N). We performed successive cooling updates up to some optimal cooling time τ_c such that the short-distance fluctuations were

removed sufficiently enough to be close to a vacuum configuration. We then repeated the same procedure on a gauge field configuration at a different instant of time t_2 , which took it to another nearest vacuum state. In order to now calculate the difference in the Chern-Simons numbers between these two vacua, we first need to reconstruct the connection between the cooled configurations $U_\mu(\mathbf{x}, t_1, \tau_c)$ and $U_\mu(\mathbf{x}, t_2, \tau_c)$. We implement this through a smooth interpolation between the electric fields defined at t_1 and t_2 and reconstructing the gauge links at time t where $t_1 \leq t \leq t_2$, according to

$$E_i^{t_1 \rightarrow t_2}(\mathbf{x}, \tau_c) = \frac{i}{g a(t_2 - t_1)} \ln [U_i(\mathbf{x}, t_2, \tau_c) U_i^\dagger(\mathbf{x}, t_1, \tau_c)],$$

$$U_i(\mathbf{x}, t; \tau_c) = e^{-i g a E_i^{t_1 \rightarrow t_2}(\mathbf{x}; \tau_c)(t - t_1)} U_i(\mathbf{x}, t_1; \tau_c). \quad (12)$$

This is a well-justified procedure since the topology does not rely on the exact path connecting the two vacuum configurations in the gauge space. We next calculate the change in Chern-Simons number between these two vacuum configurations by numerically performing the time integral of the lattice discretized version of Eq. (4), using Simpson's method and summing over all sites within the lattice volume, giving us

$$N_{\text{CS}}^{\tau_c}(t_2) - N_{\text{CS}}^{\tau_c}(t_1) = \frac{g^2 a^3 (t_2 - t_1)}{8\pi^2} \sum_{x,i} E_{i,\text{imp}}^{t_1 \rightarrow t_2}(\mathbf{x}; \tau_c) \times \frac{B_{i,\text{imp}}(t_1; \tau_c) + 4B_{i,\text{imp}}(t_{\text{mid}}; \tau_c) + B_{i,\text{imp}}(t_2; \tau_c)}{6}. \quad (13)$$

Here, $t_{\text{mid}} = \frac{t_1 + t_2}{2}$ and we use an improved definition for the Chern-Simons current in terms of the $\mathcal{O}(a^2)$ -improved electric and magnetic fields. The improved electric fields are defined at each site on the lattice as

$$E_{i,\text{imp}}^c(\mathbf{x}) = -\frac{1}{12} U_i^{cd}(\mathbf{x}) E_i^d(\mathbf{x} + \hat{i}) + \frac{7}{12} U_i^{\dagger cd}(\mathbf{x} - \hat{i}) E_i^d(\mathbf{x} - \hat{i}) + \frac{7}{12} E_i^c(\mathbf{x}) - \frac{1}{12} U_i^{\dagger cd}(\mathbf{x} - \hat{i}) U_i^{\dagger de}(\mathbf{x} - 2\hat{i}) E_i^e(\mathbf{x} - 2\hat{i}), \quad (14)$$

where $U^{cd} = 2\text{Tr}[t^c U t^d U^\dagger]$. The $\mathcal{O}(a^2)$ -improved magnetic fields are constructed from a combination of the four elementary (1×1) plaquettes and the eight adjacent rectangular (2×1) plaquettes according to

$$B_{i,\text{imp}}^c(\mathbf{x}) = \frac{e^{ijk}}{g a^2} \text{ReTr} \left[i t^c \left(\frac{5}{3} \sum_{\square} U_{\pm j, \pm k}^\square(\mathbf{x}) - \frac{1}{3} \sum_{\text{rect}} U_{\pm j, \pm k}^{\text{rect}}(\mathbf{x}) \right) \right]. \quad (15)$$

III. RESULTS

A. Sphaleron rate in a thermal non-Abelian plasma: SU(2) vs SU(3)

Performing the cooling procedure described in the preceding section, we first calculate the distribution of the Chern-Simons number for different choices of the optimal cooling time τ_c for both SU(2) and SU(3) thermal plasma. In Fig. 1, the probability distribution of Chern-Simons number change ΔN_{CS} in SU(3) gauge theory is shown for different cooling times at a very high temperature denoted by $g = 0.58$. It is evident that at $(g^2 T)^2 \tau_c = 250$, the distribution of ΔN_{CS} peaks close to integer values. The peaks are more sharply concentrated near the integer values as the system evolves in time Δt , starting from $\Delta N_{\text{CS}} = 0$ at $t = 0$. Furthermore, the Chern-Simons number diffuses to larger values with time, which is evident from the lower panel of Fig. 1. Under sufficiently long time evolution, the clustering of ΔN_{CS} values around integers is already visible at $(g^2 T)^2 \tau_c \gtrsim 32$; henceforth, we will perform cooling up to this optimal depth for $g < 1$. We observe a similar trend at a lower temperature denoted by $g = 1.12$ shown in Fig. 2, but

at a comparatively larger cooling depth τ_c , which ensures that the UV fluctuations are optimally removed, and the values of ΔN_{CS} are peaked sufficiently close to integer values. Thus, for gauge configurations at low temperatures denoted by $g > 1$, we will usually perform cooling up to optimal values $(g^2 T)^2 \tau_c = 3 - 9 \times 10^3$.

We next show the autocorrelation of the Chern-Simons number change as a function of the time (interval of observation) Δt , in an SU(3) thermal plasma at $g = 0.58$, for three different physical volumes in Fig. 3. It is evident that the autocorrelation of the Chern-Simons number change is not sensitive to the physical volume when $Lg^2 T \gtrsim 8$ and varies linearly with time indicating its diffusive nature. Extracting the sphaleron rate Γ_{sph} from its slope, we next compare its volume dependence for SU(3) as well as an SU(2) plasma at $g = 0.58$ in Fig. 4. The sphaleron rate attains a plateau for spatial lengths of the lattice $Lg^2 T \geq 8$, irrespective of the gauge group, and henceforth we will calculate sphaleron rates on lattice of size $Lg^2 T > 8$. We have also verified that the volume dependence of the sphaleron rate is under control at comparatively lower temperatures denoted by $g > 1$ for lattice sizes $Lg^2 T > 8$.

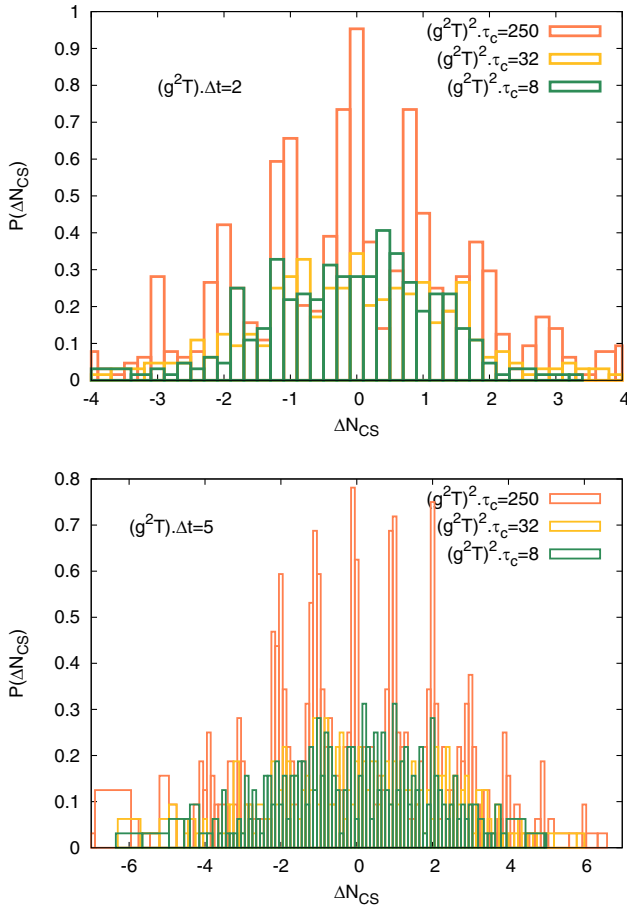


FIG. 1. Probability distribution of ΔN_{CS} at different optimal cooling depths τ_c in a thermal SU(3) gauge plasma at $g = 0.58$.

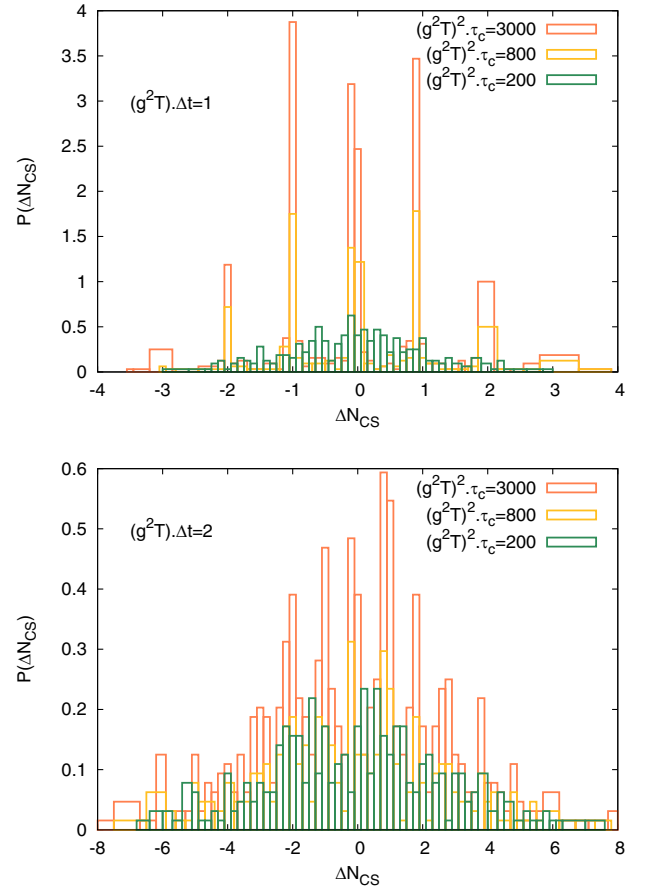


FIG. 2. Probability distribution of ΔN_{CS} at different optimal cooling depths τ_c in a thermal SU(3) gauge plasma at $g = 1.12$.

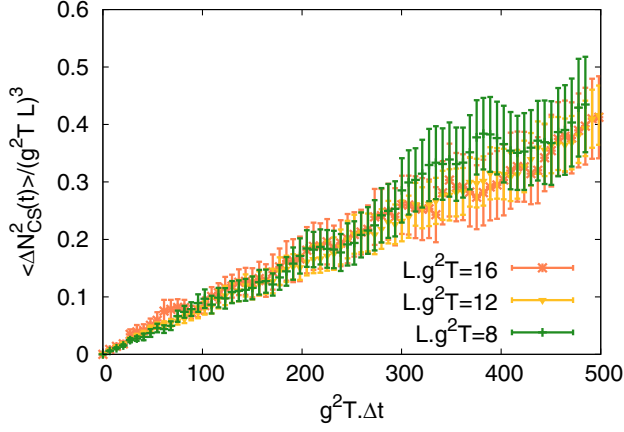


FIG. 3. Autocorrelation of the Chern-Simons number change as a function of the observation time Δt in a thermal SU(3) plasma at $g = 0.58$, estimated for three different lattice volumes.

In Figs. 5 and 6, we have shown our results for the extracted sphaleron rates as a function of the inverse gauge coupling or equivalently, the temperature, which typically varies between 0.6 and 10^{15} GeV. The corresponding spatial lattice size varies between $16 \lesssim L g^2 T \lesssim 108(72)$ for $N = 2(3)$, respectively. Our lattice results are compared with the parametric dependence of Γ_{sph} obtained by performing a fit to the earlier lattice results of the sphaleron rate at perturbatively weak couplings $\alpha_s = g^2/(4\pi) \ll 1$ [11], which for the SU(N) gauge group is

$$\Gamma_{\text{sph}} = 0.21(1) \frac{g^2 T^2}{m_D^2} \left(\ln \frac{m_D}{\gamma} + 3.041 \right) \frac{N^2 - 1}{N} (N \alpha_s)^5 T^4. \quad (16)$$

Here, γ is the damping rate, which to the leading order in g^2 is denoted in terms of the Debye mass m_D as

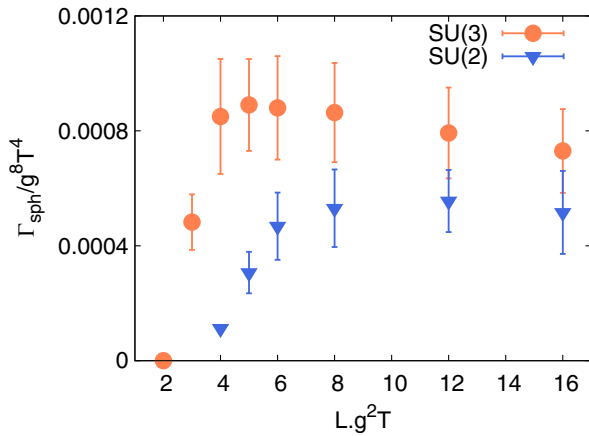


FIG. 4. Volume dependence of the sphaleron rates in a thermal SU(2) and SU(3) plasma at a particular temperature corresponding to $g = 0.58$. The rates for SU(2) are scaled by a factor ~ 9 in order to show them in the same plot.

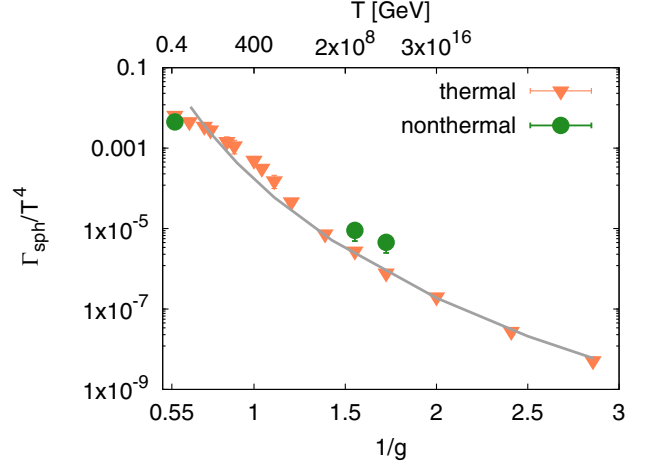


FIG. 5. Sphaleron rates for a thermal SU(2) plasma as a function of increasing temperatures, which is equivalently represented in terms of the inverse gauge coupling $1/g$. The solid line is the parametric dependence obtained by performing a fit to the lattice results of the sphaleron rate at small values of couplings in Ref. [11]. Circular data points represent sphaleron rates in nonthermal plasma, which has similar energy densities to the thermal case at three different temperatures.

$\gamma = \frac{N g^2 T}{4\pi} (\ln \frac{m_D}{\gamma} + 3.041)$. Our lattice results for the sphaleron rate start to agree with the above parametric estimate at sufficiently weak couplings $1/g \gtrsim 1.5(2.0)$ for $N = 2(3)$, which corresponds to temperatures $T > 10^8(10^{10})$ GeV. For comparison, we also show the results for sphaleron

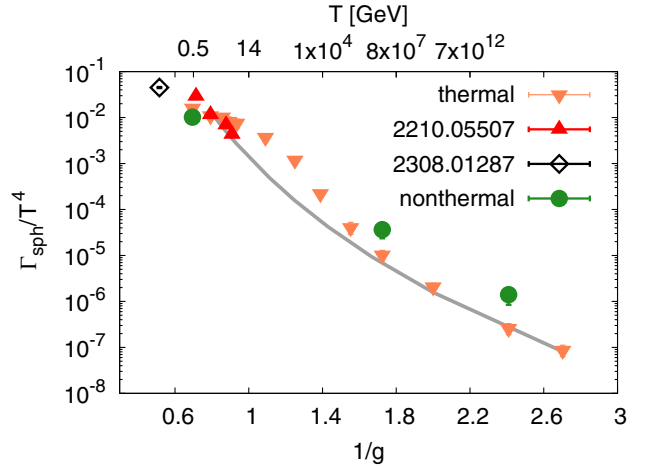


FIG. 6. Sphaleron rates for a thermal SU(3) plasma as a function of increasing temperatures, which is equivalently represented in terms of the inverse gauge coupling $1/g$. The solid line is the parametric dependence obtained by performing a fit to the lattice results of the sphaleron rate at small values of couplings in Ref. [11]. Circular data points represent sphaleron rates in nonthermal plasma, which has similar energy densities to the thermal case at three different temperatures. The triangles are data from Ref. [25] in an SU(3) gauge theory, and the black diamond is the data for 2 + 1 flavor QCD from Ref. [26].

rates in a nonthermal plasma with similar energy densities to the thermal case as circles in the same figure. We will provide a comparative discussion regarding these non-thermal data in the next section. We also perform a comparison of the existing results for the sphaleron rates in thermal SU(3) plasma with and without dynamical fermions from Refs. [25] and [26], respectively. Our results are lower in magnitude compared to the results obtained in pure SU(3) at $T \simeq 600$ MeV [25] but increase with the temperature and eventually agree with the perturbative estimates. This is due to the fact that the damping due to the hard gluons is only included at $\mathcal{O}(\ln g)$ in our case compared to the full theory; hence, our estimates for the rates are higher (lower) at $g > 1 (< 1)$. The presence of dynamical fermions does not cause a significant enhancement of the sphaleron rate at a lower temperature [26] $T \sim 350$ MeV], where the Γ_{sph} is close to our results obtained within an effective theory. However, note that our results rely on the efficient separation of scales in a thermal plasma, which might not be the case at this temperature.

Before proceeding with our calculations for the sphaleron rate in the nonthermal plasma, we also discuss an important systematic effect arising due to the choice of interval at which the cooling of gauge fields is performed. In Fig. 7, we show our results for the sphaleron rates for an SU(2) plasma at $g = 1$ as a function of how frequently we have performed cooling of the gauge fields during their Hamiltonian evolution. Evidently, the sphaleron rate changes by only 1% when the cooling procedure is performed at an interval $0.1a = 10\delta t$ as compared to

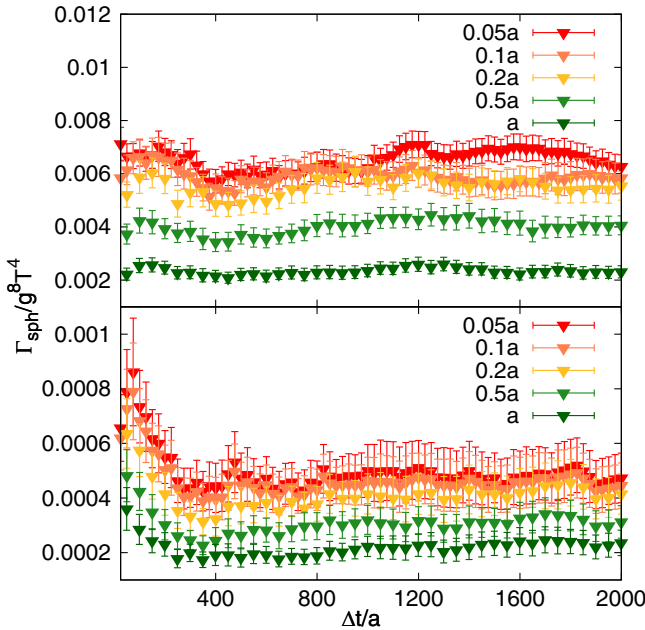


FIG. 7. Sphaleron rates as a function of the $\Delta t/a$ obtained by performing calibrated cooling of the gauge fields at time intervals ranging between 0.05 and $1.0a$ for a thermal plasma consisting of (top) SU(3) and (bottom) SU(2) gluons at $g = 1$.

$0.05a = 5\delta t$. We will henceforth calculate sphaleron rates after performing cooling at every time step $10\delta t$, for all temperatures corresponding to $g \leq 1$. However, at comparatively lower temperatures where $g > 1$, we observe a noticeable dependence on the cooling frequency. Hence, to minimize systematic errors at lower temperatures, we have chosen to perform cooling at each time step δt in order to extract ΔN_{CS} . Another source of systematic error is due to finite lattice spacing artifacts. At the lowest temperature for which the lattice spacing a is coarsest, the quantity $g^2 Ta \sim 3(4)$ for $N = 3(2)$, respectively. The size of the sphaleron, on the other hand, is typically $\sim \frac{4\pi}{g^2 T}$ [4]. Thus, even for our coarsest lattice, its spacing is much smaller compared to the typical size of a sphaleron, irrespective of the number of colors. This ensures that discretization errors on Γ_{sph} are sufficiently well controlled for the range of temperatures we have studied. Comparing our estimated sphaleron rate $\Gamma_{\text{sph}}/T^4 = 0.0109 \pm 0.002$ at $T = 4T_c$, with the continuum extrapolated rate $\Gamma_{\text{sph}}/T^4 = 0.0115 \pm 0.0006$ [25] in 4D SU(3) gauge theory at the same temperature, we observe a perfect agreement. This reinforces that the discretization errors in our results are under control.

B. Extracting sphaleron rates in a nonthermal plasma: SU(2) versus SU(3)

Starting from glasma-like initial conditions and evolving the gauge links using the classical-statistical algorithm described in Sec. II B, we calculate the autocorrelation of the Chern-Simons number for SU(3) gauge theory, which is shown in Fig. 8. The change in $\langle N_{\text{CS}}^2(t) \rangle$ as a function of $Q_s \Delta t$ is calculated for three different volumes $Q_s L = 48, 64, 98$, and $n_0 = 2$ starting from an initial time $Q_s t_0 = 50$. In contrast to the thermal case where the autocorrelation increases linearly as a function of the time, we observe a linear rise in this case until $Q_s \Delta t \lesssim 10$,

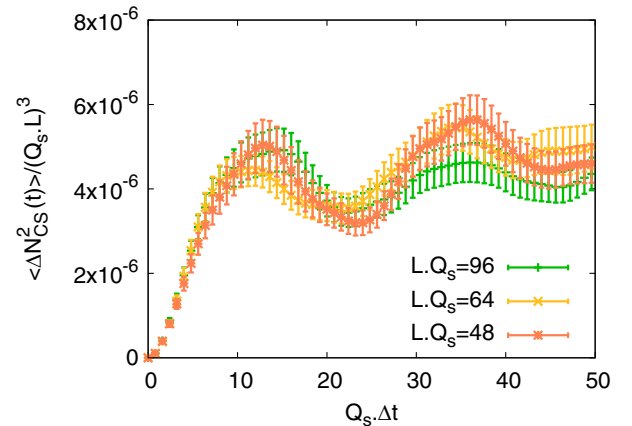


FIG. 8. Autocorrelation of the Chern-Simons number change in a nonthermal SU(3) plasma for three different volumes measured as a function of the time interval Δt for which transitions are monitored, starting from an initial time $Q_s t_0 = 50$.

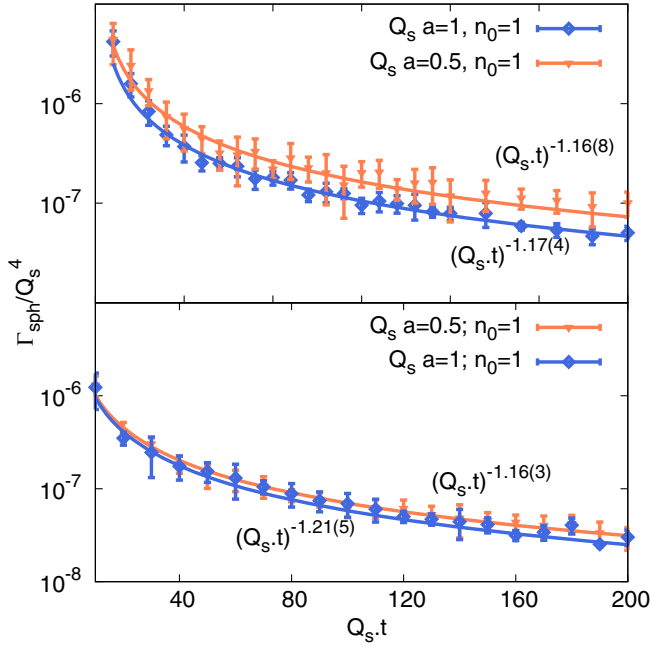


FIG. 9. Parametric dependence of sphaleron rates as a function of the time (at sufficiently late times) for a nonthermal (top) SU(3) and (bottom) SU(2) plasma where the infrared gluons are overoccupied, shown for two different lattices with $Q_s.a = 1$ (blue) and $Q_s.a = 0.5$ (orange), respectively.

beyond which an oscillatory pattern is observed, which is quite robust to different choices of lattice volumes and cooling frequencies. This observation is consistent with an earlier lattice study [12], where it was argued that such oscillations arise due to the strong collective behavior of the gluons in this overoccupied regime. We have extracted sphaleron rates from the slope of the initial linear growth of $\langle N_{CS}^2(t) \rangle$ as a function of $Q_s.t$ for two different lattice spacings $Q_s.a = 1$ (blue) and $Q_s.a = 0.5$ (orange), which are shown in Fig. 9 for the SU(3) and SU(2) plasmas, respectively. A fit to the data as a function of $Q_s.t$ reveals that the sphaleron rate has a parametric dependence $\sim Q_s^4(Q_s.t)^{-1.2}$, which is quite robust, independent of the choice of lattice spacing or the gauge group. Incidentally, the magnetic scale extracted from the spatial string tension also varies as $\sqrt{\sigma_s(t)} \sim Q_s(Q_s.t)^{-3/10}$ under sufficiently long time evolution, which ensure the onset of a nonthermal scaling regime [15]. The sphaleron rate thus parametrically behaves as $\Gamma_{\text{sph}} \simeq \sigma_s^2$ due to a significant contribution of gluons whose momenta are less than the magnetic scale. This observation is universal irrespective of the number of colors of the gauge group. Moreover, interactions among these so-called low-momentum magnetic gluons are nonperturbatively large due to their large occupation numbers; hence, the sphaleron rate is intrinsically a nonperturbative quantity.

Revisiting our discussion relating to the comparison of sphaleron rates between a thermal versus nonthermal

plasma shown in Figs. 5 and 6, we recall that such a comparison is done by keeping the energy density the same in both cases. Within the thermal effective theory of QCD, interactions with hard gluons whose momenta are larger than the Debye mass scale provide damping as well as random kicks to ultrasoft gluons leading to an additional suppression of the sphaleron rate, otherwise determined solely by the magnetic scale, by a factor $\sim 1/g^2$. In the overoccupied nonthermal plasma that we study here, the hard scale increases as a function of the time and does not influence the classical evolution of these ultrasoft modes at long enough times. Hence, the sphaleron rates in this case are determined solely by the magnetic scale. Thus, at asymptotically high temperatures, where couplings $g \ll 1$, thermal sphaleron rates get suppressed compared to the rates in a nonthermal plasma. At temperatures denoted by $g \gtrsim 1$, however, the opposite trend is visible in the data, as expected.

IV. PHYSICAL IMPLICATIONS OF THE QCD SPHALERON RATE

A. Estimating the thermalization time for gauge fields during the early reheating era

Inflation is one of the most attractive paradigms of early Universe cosmology that can explain key observations [27–29]. At the end of inflationary epoch, the Universe gets reheated due to the decay of the inflaton into radiation, the mechanism of which is not very well understood. If the coupling of the inflaton to radiation is nonperturbatively large, its decay can occur via an exponentially large production of soft radiation through a process called preheating [30]. Such a nonperturbative process can also occur at weak couplings g_Φ if the amplitude of the expectation value of the inflaton field Φ is large. In this case, the relevant parameter controlling resonant production of particles is $\sim \frac{g_\Phi^2 \Phi^2}{m_\Phi^2}$, where m_Φ is the mass of the inflaton. This ratio can be nonperturbatively large leading to efficient preheating of the Universe [31]. A possible scenario extensively discussed in the literature involves inflaton coupling weakly to a thermal bath of radiation and decaying slowly through perturbative interactions [32–34]. Since the typical mass of the inflaton is large, it will decay into very high energy Standard Model particles, say, e.g., gluons. Based on the Boltzmann kinetic theory approach, the typical timescale in which these high energy gluons will lose energy to the thermal bath and acquire a thermal distribution could be significantly longer than the Hubble time during the early stages of reheating, which can put strong restrictions on the maximum temperature achieved in the Universe [32,33,35].

We would like to stress here that the conventional Boltzmann approach can only describe thermalization of quantum fields which allow for a quasiparticle description. The hard gluons in a non-Abelian plasma whose momenta

are $\sim T$ can be, e.g., described within kinetic theory. However, it is known from classical-statistical simulations of overoccupied non-Abelian gauge theories that soft gluons interact nonperturbatively among themselves eventually thermalizing in a timescale which is faster [36] compared to the typical thermalization time required for the perturbative hard modes [37–39]. It is thus important to understand the implications of any nonperturbative collective phenomena that might occur during the very early stages of reheating [40]. We discuss one such possible scenario where a sphaleron-dominated nonthermal SU(N) plasma comprised of nonperturbatively interacting overoccupied gluons is created due to decay of the inflaton in the initial stages of reheating. We then estimate the typical time required for the occupation numbers of these ultrasoft gluons to acquire a thermal distribution starting from a nonthermal one. As discussed in the previous section, the sphaleron rate in an overoccupied nonthermal SU(N) plasma can be typically parametrized as $\Gamma \sim Q_s^4(Q_s t)^{-1.2}$ when the momentum distribution function of gluons exhibits a nonthermal scaling behavior. Comparing this with the sphaleron rate in a thermal plasma at a temperature T , one obtains a typical thermalization time t_{th} in units of $g^4 T$. Results for t_{th} as a function of the temperature or equivalently, g , are shown in Fig. 10. The data points in Fig. 10 can be described by a fit ansatz $g^{7.24(4)} T \cdot t_{\text{th}} \simeq 0.90(9)$ for SU(2) gauge theory, while for SU(3) the analogous fit ansatz that describes our data is $g^{7.27(9)} T \cdot t_{\text{th}} \simeq 0.17(6)$.

The typical temperature range that signals the end of the reheating era and is compatible with the Planck data can vary between 10^9 and 10^{13} GeV [29]. The gauge coupling in SU(2) typically varies between $g \sim 0.55$ and 0.65 in this temperature range. Using now our fit function, we can obtain typical timescales corresponding to these couplings

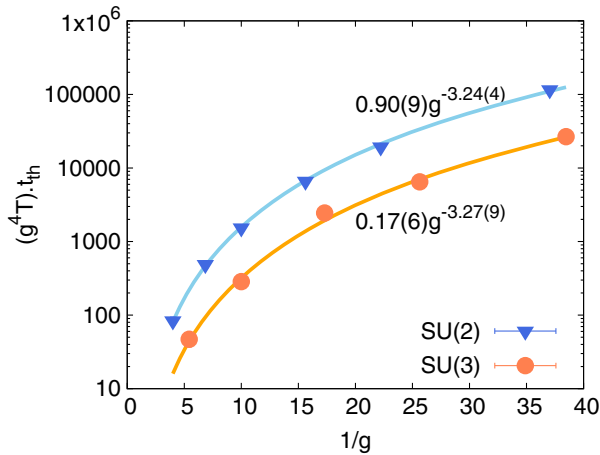


FIG. 10. Parametric dependence of the time t_{th} required for sphaleron rates to attain their values in a thermal plasma starting from nonthermal initial conditions, conserving the energy density, shown as a function of the inverse gauge coupling.

which denote the thermalization time for the ultrasoft gluons $\frac{1}{t_{\text{th}}} \sim 4.9 \times 10^7 - 1.5 \times 10^{11}$ GeV. For the SU(3) case, coupling typically varies between $g \sim 0.45$ and 0.53 and thus the thermalization time of its ultrasoft modes $\frac{1}{t_{\text{th}}} \sim 5.8 \times 10^7 - 1.8 \times 10^{11}$ GeV. These thermalization time estimates are thus insensitive to the number of colors in the gauge group. Moreover we observe that $t_{\text{th}} \ll H^{-1}$, where H denotes the Hubble parameter governing the expansion rate of the Universe during this epoch. Hence, these ultrasoft gluons undergo fast thermalization, thus providing a thermal bath for the hard gluons that are formed due to the perturbative decay of the massive inflaton [35]. The hard gluons further split into daughter gluons which carry lower momenta than their parent, within a typical timescale t_{hard} which can be obtained [35] by matching the splitting rate Γ_{split} with the Hubble parameter H ,

$$t_{\text{hard}} \simeq \frac{(0.2 - 0.8)}{N^2} \left(\frac{g^2}{4\pi} \right)^{-16/5} \left(\frac{\Gamma_{\Phi}}{m_{\Phi}^3/M_{\text{Pl}}^2} \right)^{-3/5} m_{\Phi}^{-1}. \quad (17)$$

Here, Γ_{Φ} and m_{Φ} are the decay width and mass of the inflaton, respectively, whose typical estimates are $\Gamma_{\Phi} \simeq 5.68 \times 10^{-18} M_{\text{Pl}}$ [40–42] and $m_{\Phi} = 0.51 \times 10^{-5} M_{\text{Pl}}$ [43] in units of the Planck mass M_{Pl} . The mass of the inflaton is chosen to roughly match the COBE normalization for the power spectrum of curvature perturbation. If the temperature during reheating is $T = 10^{13}$ GeV, then $t_{\text{hard}} \simeq (2 - 8)^{-1} \times 10^{-8}$ GeV $^{-1}$. The time required for the ultrasoft SU(3) gluons to thermalize is less than t_{hard} required by the hard gluons to entirely split into softer ones thereby acquiring a thermal distribution. Our findings are thus consistent with the perturbative reheating scenario. However, if the temperatures are on the lower side $T = 10^9$ GeV, then typical estimates of $t_{\text{hard}} \simeq (5.7 - 22.8)^{-1} \times 10^{-8}$ GeV $^{-1}$ are such that $t_{\text{hard}} < t_{\text{th}}$, which is not feasible. Hence, for the perturbative reheating scenario to work well, which requires $t_{\text{th}} < t_{\text{hard}}$, typical temperatures during reheating should be at least $\gtrsim 10^{10}$ GeV, which is typically favored in the Higgs inflation scenario [44].

B. Nonperturbative thermal axion production rate and its consequences

Explaining the strong CP problem is one of the long-standing challenges for physics beyond the Standard Model (SM). Out of many proposed solutions, the Peccei-Quinn (PQ) mechanism [45,46] turns out to be one of the most attractive ways out since it also provides a plausible explanation of the abundance of dark matter in the present Universe. The PQ mechanism involves invoking a global $U_{\text{PQ}}(1)$ symmetry which is anomalous under the SU(3) color group of the SM and is spontaneously broken at some high scale. The remnant due to this spontaneously broken $U_{\text{PQ}}(1)$ symmetry is a pseudo-Nambu-Goldstone mode known as an axion, which has an anomalous coupling to the

SU(3) color fields given by the interaction term in the Lagrangian $\mathcal{L}_{\text{int}} = g^2/(32\pi^2 f_a) a(x) \cdot G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a(x)$. Here, f_a is the axion decay parameter, which is constrained from astrophysical and cosmological observations to be $4 \times 10^8 < f_a < 10^{12}$ GeV [47]. Formation of an axion condensate provides an explanation of the cold dark matter abundance produced nonthermally through the misalignment mechanism [48–50]. When the primordial plasma attained a temperature ~ 1 GeV during cosmological evolution, the axion started to feel the QCD vacuum potential and underwent damped oscillations around its minima as a result of which it acquired a mass whose typical values were $m_a = 5.7 \text{ eV} \times (10^6 \text{ GeV}/f_a)$ [51–53].

However, there can be another mechanism of production of axions as well. Once the primordial plasma formed during initial stages of the evolution of our Universe thermalizes, relativistic axions can be produced through scattering among SM particles even though they might not immediately thermalize [54]. Eventually, when the produced hot axions acquire a thermal distribution, these contribute to the radiation budget of the Universe, quantified in terms of the total number of neutrinos. The effective number of neutrinos N_{eff} quantifies how many massless relativistic matter degrees of freedom contribute to the total radiation density. Any relativistic particle with a substantial energy density, e.g., axions will contribute to N_{eff} . A deviation from the standard cosmological value for the number of neutrino species $N_\nu \simeq 3.044$ [55–57] can be estimated to be

$$\Delta N_{\text{eff}} = N_{\text{eff}} - N_\nu = \frac{8}{7} \left(\frac{11}{4} \right)^{4/3} \left(\frac{\rho_a}{\rho_\gamma} \right)_{\text{CMB}}. \quad (18)$$

Here, ρ_a and ρ_γ are the energy densities of the axions and photons, respectively, during the recombination era. Any additional source of radiation should be thus detectable in the cosmic microwave background (CMB). Furthermore, under the assumption of instantaneous decoupling of the thermal population of axions, the above equation can be written as [58]

$$\Delta N_{\text{eff}} \simeq \frac{4}{7} \left(\frac{11}{4} \right)^{4/3} \left[\frac{g_{*,s}(T_{\text{CMB}})}{g_{*,s}(T_D)} \right]^{4/3}. \quad (19)$$

The decoupling temperature T_D defines the epoch at which the thermal axion production rate $R(T_D)$ matches the Hubble expansion parameter. Thus, in order to estimate the decoupling temperature and ΔN_{eff} very precisely, one needs to accurately calculate the thermal production rate of axions. We consider the coupling of the axions to SU(N) gluons, where we have studied $N = 2, 3$, respectively. It is well known that even at asymptotically high temperatures, the soft gluons in non-Abelian gauge theories whose momenta are at and below the magnetic scale $g^2 T/\pi$, interact nonperturbatively [59]. We thus calculate the axion

production rate on a lattice to accurately take into account the contribution of magnetic gluons, within the effective theory.

In thermal equilibrium, the creation and annihilation rates for axions at a temperature T denoted as $\Gamma^<(T)$ and $\Gamma^>(T)$, respectively, are related through

$$\Gamma^>(\mathbf{k}, T) = e^{\omega/T} \Gamma^<(\mathbf{k}, T) = \frac{\Gamma_{\text{top}}^>(\mathbf{k}, T)}{2\omega f_a^2}, \quad (20)$$

where $\Gamma_{\text{top}}^>$ is extracted from the two-point correlation function of the topological charge,

$$\Gamma_{\text{top}}^>(\mathbf{k}, T) = \left(\frac{g^2}{32\pi^2} \right)^2 \int d^4x e^{ik^\mu x_\mu} \langle G\tilde{G}(x) \cdot G\tilde{G}(0) \rangle. \quad (21)$$

Here, $G\tilde{G}(x) = G_{\mu\nu}^a(x) \tilde{G}_{\mu\nu}^a(x)$ and the energy of the axion ω , or equivalently, $k_0 = \sqrt{\mathbf{k}^2 + m_a^2}$. We consider the on-shell axions and calculate their production rate. Although we are using an effective theory of soft gluons where the typical momenta and frequencies are $|\mathbf{k}| \lesssim g^2 T/\pi$, $\omega \lesssim g^4 T$, respectively, the dominant contribution to the axion production rate due to gluons at temperatures lower than the electroweak scale, indeed comes from the soft magnetic sector [60]. We have calculated $\Gamma_{\text{top}}^>(\mathbf{k}, T)$ for a wide range of temperatures for both SU(2) and SU(3) gauge theories on an $N_s = 32$ lattice, performing a thermal average over 200 configurations. Next, we calculate the average thermal production rate of axions defined as

$$R(T) = \frac{1}{n_a^{\text{eq}}} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Gamma^<(\mathbf{k}, T) \quad (22)$$

by performing the momentum integration of $\Gamma^<(\mathbf{k}, T)$ numerically. The lower and upper limits of the integral are given by $|\mathbf{k}|/T = 0, \sqrt{3}\pi/1.45$, respectively; the latter is the maximum allowed magnitude of momentum for the particular lattice spacing that we have considered. The integration was performed numerically using Simpson's rule. The quantity $n_a^{\text{eq}} = \zeta(3)T^3/\pi^2$ is the number density of a noninteracting gas of axions at a temperature T . Results for the rate $R(T)$ obtained from our lattice computations are compiled in Fig. 11.

Nevertheless, our results reinforces a previous study of the momentum and frequency dependence of $\Gamma_{\text{top}}^>$, where it was observed that the maximum contribution to the production rate of lightlike axions characterized by $|\mathbf{k}| \sim \omega$ comes from the soft magnetic sector [60].

We also compare our results with the perturbative estimates $R(T) = \frac{\zeta(3)g^6 T^3}{64\pi^7 f_a^2} [\log \frac{3T^2}{m_D^2} + 0.406]$, where $m_D = gT \sqrt{\frac{N+n_f/2}{3}}$, $n_f = 6$, $N = 2-3$ calculated [54] within the hard-thermal loop (HTL) resummation scheme shown as

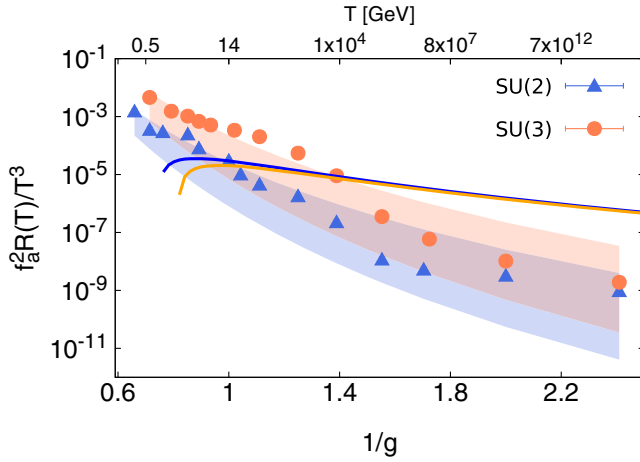


FIG. 11. The thermal axion production rate in non-Abelian gauge theory shown as a function of the inverse gauge coupling. The solid lines correspond to the rates calculated perturbatively within the hard-thermal loop resummation scheme. The shaded bands correspond to the axion production rates only due to sphaleron transitions.

solid lines in the same figure. At very high temperatures, i.e., $g < 0.66$, the perturbative HTL result is significantly higher than the axion production rate due to the magnetic gluons. But at comparatively lower temperatures, the dominant contribution to the production rate comes from the soft magnetic sector. Even at the electroweak scale, the magnetic gluons contribute to almost 50% of the axion production rate justifying the necessity of a nonperturbative calculation. The bands in Fig. 11 denote the axion production rate calculated by substituting $\Gamma_{\text{top}}^> = \Gamma_{\text{sph}}$ in Eq. (22) and integrating over all momenta lower than a maximum cutoff $|\mathbf{k}| < |\mathbf{k}_s|$. The lower boundary of the shaded band is obtained for a typical choice $|\mathbf{k}_s|/T = 3g^2/4\pi$, whereas the upper boundary is obtained when the cutoff $|\mathbf{k}_s|/T = \sqrt{3}\pi/1.45$. The nonperturbative axion production rate only due to sphalerons explains our lattice data quite well for $g < 0.7$, above which the agreement is barely visible at the upper edge of the error band. This also provides a compelling motivation for our calculations.

Our computation of the axion rate does not capture the contributions due to semihard and hard gluons. We thus add the hard-thermal loop contribution to our results for the rate arising due to the soft magnetic gluons for $g \lesssim 1$. For $g \gtrsim 1$, we consider only the production rate obtained from our lattice calculation as the HTL approximation breaks down. We can now estimate the decoupling temperature T_D at which the produced axions are no longer in thermal equilibrium with the bath consisting of soft SU(3) gluons by matching $R(T_D) = H(T_D)$, which comes out to $T_D = 8 \times 10^4$ GeV for $f_a = 4 \times 10^8$ GeV. At this temperature, the perturbative HTL rate is dominantly larger compared to the contribution only from the soft sector. As the Universe evolves toward lower temperatures, the nonperturbative

production rates start to dominate, and the produced axions again attain a thermal distribution.

Having obtained the thermal axion rate, we can now study the time evolution of the axion yields $Y(x) = n_a(x)/s(x)$, where $n_a(x), s(x)$ are the number density of axions and total entropy density, respectively, at a scaled temperature $x = T_f/T$ by solving the Boltzmann equation

$$\frac{dY(x)}{d \log x} = [Y^{\text{eq}} - Y(x)] \frac{R(x)}{H} \left[1 - \frac{1}{3} \frac{d \log g_*}{d \log x} \right] \quad (23)$$

in the universe expanding with a Hubble rate H . Here, T_f is the final temperature up to which this evolution is performed. We have considered the fact that the number of relativistic degrees of freedom g_* also evolve as a function of x . Note that the yield of the thermal relic axion in the present Universe is $Y_{\text{eq}} = n_a^{\text{eq}}/s \simeq 2.6 \times 10^{-3}$ [54]. This evolution equation of the axion yields is obtained [61–63] by integrating the kinetic Boltzmann equation over the allowed momenta whose interaction kernel has contributions from both production and decay rates, under the assumption that phase-space distribution of axions follows $f(\mathbf{k}) = \frac{n_a}{n_a^{\text{eq}}} f^{\text{eq}}(\mathbf{k})$ [64]. We start this nonequilibrium evolution at T_D and evolve down to $T_f = 600$ MeV up to which our effective theory calculations of $R(T)$ remain valid. The axion energy density at this final scaled temperature x_f is $\rho_a(x = x_f) = \frac{\pi^{14/3}}{30} \left(\frac{sY(x_f)}{\zeta_3} \right)^{4/3}$ where $s = \frac{2\pi^2 g_* T^3}{45}$. Now substituting $(\rho_a)_{\text{CMB}} = \rho_a(x_f) \left(\frac{g_{*,\text{CMB}}}{g_{*,f}} \right)^{4/3}$ in Eq. (18), one obtains $\Delta N_{\text{eff}} = 0.034$ for $f_a = 4 \times 10^8$ GeV. Here the effective degrees of freedom are $g_{*,\text{CMB}} = 3.93$ [65] at the CMB epoch and $g_{*,f} = 61.75$ at $T_f = 600$ MeV [53]. The value of ΔN_{eff} that we obtain is well below the bound $\Delta N_{\text{eff}} \leq 0.28$ from Planck [66]. Note that our estimate of ΔN_{eff} is slightly lower than the result reported in [60]. Our determination of the axion production rate is not available at lower temperatures, where a clear separation of scales no longer exists, and thus the effective theory is not valid. As the temperature decreases further, strong interactions become increasingly important, leading to an enhancement of ΔN_{eff} . Thus, an accurate determination of the axion production rate for on-shell axions near the QCD crossover will be important in the future to tightly constrain the value of ΔN_{eff} .

The ratios of the relic axion yields with respect to their values in thermal equilibrium in the present Universe can be obtained by solving Eq. (23), starting from different initial temperatures, T_R and allowed values of f_a , the results of which are compiled in Fig. 12. We can now estimate the typical decoupling temperatures T_D from the onset of a kink in Y/Y_{eq} in Fig. 12, which signals a departure of the axion yields from their equilibrium values. The decoupling temperatures are observed to vary between

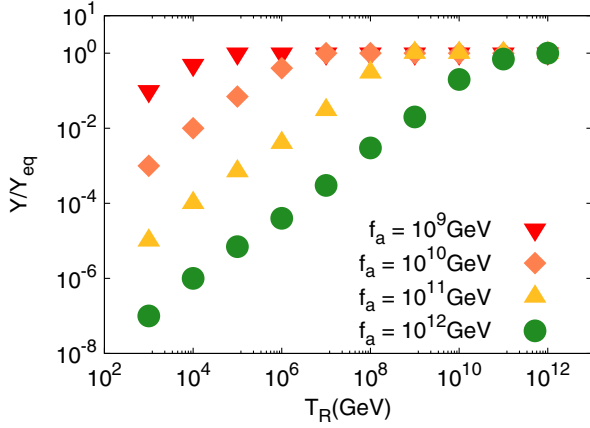


FIG. 12. Present-day relic axion yields normalized by its thermal value in an SU(3) plasma obtained after solving Eq. (23), shown as a function of different initial choices of reheating temperatures T_R and for different allowed values of the axion decay parameter f_a .

$T_D \sim 10^5$ and 10^{12} GeV depending on the values of $f_a \sim 10^9$ and 10^{12} GeV. The axions thus never attain thermal equilibrium if initial reheating temperatures are $T_R < 10^5$ GeV for any allowed values of f_a . From our previous discussion, it turns out that the correct hierarchy between thermalization times of the hard and soft gluons is ensured if $T_R > 10^{10}$ GeV. Given this scenario, the thermally produced axions will fall out of equilibrium much faster for f_a in the higher range of the allowed values compared to a lower range.

V. SUMMARY AND OUTLOOK

In this work, we have calculated the sphaleron transition rate in SU(2) and SU(3) gauge theory under both in- and out-of-thermal-equilibrium conditions using lattice gauge theory techniques. At very high temperatures where the hard, electric, and magnetic scales are well separated, we have used an effective theory for the soft gluons to calculate the sphaleron rates for a wide range of temperatures spanning from 0.6 to 10^{15} GeV, without being beset by finite volume corrections. The sphaleron rates start to agree with their parametric estimates at weak couplings only at temperatures beyond the electroweak scale. Incidentally, sphaleron rates in a nonthermal plasma are higher than in a thermal plasma with similar energy densities, where the nonthermal plasma consists of gluons whose occupation numbers exhibit a self-similar scaling. At high temperatures, where the hard and soft scales are sufficiently wide apart, one would expect little effect on the sphaleron rate, which is driven primarily by soft gluons, due to dynamical quarks whose momenta are hard. In the context of the thermal effective theory we are studying, effects due to dynamical quarks enter through color conductivity, which dampens the random motion of soft gluons. Clearly, the

contribution due to quarks in the sphaleron rate will become significant as one approaches the crossover temperature in QCD where the separation of scales is no longer valid.

By comparing the sphaleron rates in a thermal versus a nonthermal plasma at similar energy densities, we have estimated typical thermalization times for these ultrasoft magnetic modes during the early stages of the reheating epoch. For temperatures $T \geq 10^{10}$ GeV, our results demonstrate that the thermalization time for ultrasoft magnetic gluons is much lower compared to the timescale required for the splitting of hard gluons into softer counterparts obtained within the kinetic theory framework. Our results thus support a scenario where the soft gluons formed due to the decay of inflaton thermalizes very quickly as a result of nonperturbative interactions which then forms a thermal bath for hard gluons to interact with and eventually thermalize on a longer timescale. Our study also allows us to put a lower bound on the reheating temperature $T_R \geq 10^{10}$ GeV for this perturbative reheating scenario to be a valid physical description of the early Universe.

We have also calculated the axion production rate at high temperatures in SU(2) and SU(3) thermal plasma. We observe that the soft magnetic sector contributes significantly to the axion production rate, even at the electroweak scale, where our lattice results exceed the perturbative HTL prediction by a substantial margin. This is a generic feature of all physical observables which have a notable contribution from the nonperturbatively interacting magnetic gluons. We could also uncover the reason behind the large enhancement of our estimates for the axion production rate compared to HTL predictions at couplings $g \gtrsim 2/3$, arising due to a sizeable contribution from sphaleron transitions. By solving the Boltzmann equation with the nonperturbative axion production rates as an input, we could calculate the yields for relic axions. For the allowed range of values of f_a , the axions typically remain in thermal equilibrium with the bath consisting of soft gluons for a longer time if the values of f_a are on the lower side. Moreover, for values of f_a even in the higher side of its allowed range, the decoupling temperature of axions is consistent with our estimated lower bound on the reheating temperature. In the future, we would like to develop new lattice techniques to calculate the axion production rate at lower temperatures, closer to the QCD crossover.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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