

Black holes in rotating, electromagnetic backgrounds, and topological Kerr-Newman-NUT spacetimes

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We observe that a large class of well-behaved stationary and axisymmetric black hole solutions in general relativity and in the Einstein-Maxwell theory can be classified according to the properties of their backgrounds. Indeed, all these backgrounds belong to a unique family, which includes simultaneously all the known axisymmetric and regular backgrounds: the swirling, the Bertotti-Robinson, the Bonnor-Melvin universe, Witten's expanding bubble, and also other novel, regular, rotating gravitational or electromagnetic environments. All these can be, fundamentally, traced back to the double Wick rotation of the topological generalization of the (accelerating) Kerr-Newman-NUT metric. We present a black hole embedded in an unexplored sector of the general background: Schwarzschild inside a generalized rotating (and possibly electromagnetic) universe. These results indicate that basically all the known analytical and exact single black hole solutions in the four-dimensional Einstein-Maxwell theory belong to the (accelerating) Kerr-Newman-NUT family embedded into backgrounds that are a subcase of the conjugated Kerr-Newman-NUT spacetime with an angular manifold of arbitrary topology.

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I. INTRODUCTION

Black holes in vacuum general relativity are very peculiar objects, because they are the most restricted and constrained solutions. Actually, when asymptotic flatness is assumed, the only well-defined¹ black hole is the Kerr one [1] (which, in the static case, reduces to the Schwarzschild black hole). In the presence of the Maxwell fields, it is possible to find its electromagnetic extension, the Kerr-Newman black hole [2], but nothing else.

On the other hand, relaxing the asymptotic flatness, a few more black hole solutions are known in pure general relativity, for instance, the Kerr black hole embedded into a swirling background [3,4] or the Kerr black hole embedded into a bubble of nothing [5]. A few other possibilities can be found if we include the cosmological constant in the action or if gravity is coupled to other forces; however, in this article, we are not taking into account any energy momentum tensor apart from the Maxwell one.

For instance, considering the electromagnetic contribution to the Einstein equations while neglecting the asymptotic flatness condition, the Kerr-Newman-Melvin [6] or the Kerr-Bertotti-Robinson spacetimes [7], or a combination of both [8], can be included in this list. These spacetimes describe black holes embedded in some external back-reacting electromagnetic backgrounds.

However, note that all these asymptotically nonconstant curvature generalizations of the Kerr black hole can hardly be considered completely different black holes with respect to the Kerr family. They can be better physically interpreted as a continuous deformation of the Kerr(-Newman-NUT) black hole class because they are essentially the Kerr-Newman-NUT black hole embedded in some external background. In fact, the Kerr black hole can always be retrieved in some limit, basically vanishing the background. This is in line with the spirit of the uniqueness theorems for axisymmetric and stationary metrics [9].

In this paper, we would like to extend possible backgrounds for embedding black holes by introducing a new rotational feature in the asymptotic geometry with respect to the one of the *swirling universe*. From a mathematical point of view we know (see Ref. [3], for details [10]) that the swirling background can be obtained from a double Wick rotation of a Taub-NUT metric with a flat base manifold. At the same time, it has been shown in [11] that the Bonnor-Melvin universe (with or without the cosmological constant [11–13]) can be obtained again from a double Wick rotation of the Reissner-Nordstrom solution

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¹We are referring only to analytical and exact solutions of the vacuum or electrovacuum Einstein equations, which can be interpreted as black holes, and therefore, without singularities outside the event horizon.

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(with a flat angular surface). Thus, it is natural to ask which is the double Wick rotation of the Kerr black hole, or more generally, of the Kerr-Newman-NUT black hole. If the swirling background and the external electromagnetic Bonnor-Melvin-Bertotti-Robinson field are, respectively, related to the conjugated NUT parameter and monopolar electromagnetic charges, what feature is introduced in the metric by the conjugated angular momentum? We will focus specifically on the possibility that these backgrounds can be used as adequate environments to embed black hole solutions.² We will remain as general as possible to construct possible backgrounds, and thus, not be limited to flat base manifold black hole metrics, because, as shown in [8] and in [16], a curved geometry of the base manifold can be necessary for including the Bertotti-Robinson or the expanding bubble of nothing backgrounds. Although all these backgrounds admit the presence of the cosmological constant, the lack of theoretical tools to deal with solution generating techniques with Λ does not allow us to exploit

these backgrounds to accommodate black holes. In fact, as pointed out in [11], it is not apparent if, in the presence of the cosmological constant, axisymmetric and stationary geometries enjoy integrability properties in general relativity outside the Petrov-D type subclass.

Other possible generalizations of these type-D backgrounds might be obtained through solution generating techniques. However, at the moment, it is not clear if they can be suitable to host legitimate black holes or if they can carry some more fundamental physical properties other than being just a composition of known elementary backgrounds.

II. GENERALIZED ELECTROMAGNETIC AND ROTATING BACKGROUNDS

A. The topological Kerr-Newman-NUT solution

Consider the Kerr-Newman metric with a cosmological constant and an arbitrary curvature in the angular³ section of the metric, i.e., for constant time and radial coordinates (t, r)

$$ds^2 = -\frac{\Delta_r}{\Sigma^2} [d\tau - (2\ell x + ax^2 - \omega_0)d\varphi]^2 + \frac{\Sigma^2}{\Delta_r} dr^2 + \frac{\Sigma^2}{\Delta_x} dx^2 + \frac{\Delta_x}{\Sigma^2} [(r^2 + \ell^2 + a\omega_0)d\varphi + ad\tau]^2, \quad (2.1)$$

where

$$\begin{aligned} \Delta_r(r) &= e^2 + p^2 - k\ell^2 + \Lambda\ell^4 + a^2[1 + k - k^2(1 + \ell^2\Lambda)] - 2mr + \left(k - a^2k^2\frac{\Lambda}{3} - 2\ell^2\Lambda\right)r^2 - \frac{\Lambda}{3}r^4, \\ \Delta_x(x) &= 1 + k_7 + k(1 - k - x^2) + \frac{ak_7x}{\ell} + \frac{ax\Lambda}{3}(k^2 - x^2)(ax + 4\ell), \\ \Sigma(r, x) &= \sqrt{r^2 + (\ell + ax)^2}. \end{aligned} \quad (2.2)$$

The associated electromagnetic vector potential is

$$A_\mu = \left[\frac{er - p(\ell + ax)}{\Sigma^2}, 0, 0, \frac{x[p\ell(ax + \ell) - pr^2 - er(ax + 2\ell)]}{\Sigma^2} + \omega_0 A_t(r, x) \right]. \quad (2.3)$$

This metric⁴ represents the most general nonaccelerating type-D black hole in the theory of general relativity, possibly coupled with standard Maxwell electromagnetism (aligned with the two double-degenerate principal null directions of the Weyl tensor) and eventually also with the cosmological constant Λ . Thus, it is a solution of the standard Einstein-Maxwell- Λ field equations

²This point was preliminarily addressed in [14], while a first study about possible novel backgrounds was carried out in [15].

³We are mainly using, here, spherical-like coordinates (t, r, x, φ) , where the polar angle is related to x by $x = \cos \vartheta$.

⁴A similar solution was presented in [17], but it seems it was not fulfilling the field equations.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 2\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right), \quad (2.4)$$

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0. \quad (2.5)$$

The electromagnetic field strength is defined, as usual, as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The six integrating constants m, a, ℓ, e, p, k are related to physical properties of the solution: the mass, the angular momentum, the NUT parameter, the monopolar electric and magnetic charges, and the curvature of the surface defining the event horizon, respectively. This latter parameter k defines the geometry of the event horizon, which can be of spherical topology for

$k > 0$; flat, cylindrical, or toroidal, depending on the identifications for $k = 0$; hyperbolic or a torus of genus bigger than 1 for $k < 0$. However, note that in the presence of rotation, the multitorus construction may not be compatible with the black hole interpretation [18]. Usually, k is considered to take values in $\{-1, 0, 1\}$. In that case, k_7 can be considered zero, to properly reproduce the possible three topologies of the (x, φ) section. k_7 might become useful in a few circumstances, for instance, to get an easier limit on the Bertotti-Robinson-Bonnor-Melvin background.

B. The general background: Conjugated topological Kerr-Newman-NUT

We note that all the legitimate black hole solutions ever found in vacuum general relativity, possibly coupled with standard Maxwell electromagnetism, are embedded in a background contained in the conjugate [3,19], or equivalently, into the double Wick rotation of the metric (2.1). We are referring especially to black hole solutions that possess no curvature nor conical singularities outside the event horizon.⁵ Therefore, the study of the general case is significant, since it can unveil a number of other possible good gravitational and electromagnetic backgrounds to host black hole metrics, which can be enriched by new integrating constants deforming the black hole, and thus,

new possible physical features. These deformations are always continuous modifications of the Kerr(-Newman) metric, and therefore, are always compatible with the actual phenomenology for small values of the new integrating constants. Similarly, for small values of the deforming background parameters, the deviation from the constant curvature usual backgrounds, that is, Minkowski or (anti-) de Sitter, can be as small as one needs.

So, as a first step, we start with writing the general background as a conjugate solution of the topological Kerr-Newman-NUT solution (2.1)–(2.3). The conjugate transformation can be achieved most easily by the analytical continuation, known as the double Wick rotation, given by

$$\tau \rightarrow i\phi, \quad \varphi \rightarrow it. \quad (2.6)$$

In the presence of an electromagnetic field, this map has to be combined with an imaginary rotation of the electromagnetic parameters in order to remain with a real vector potential. In the case under consideration, it means that $e \rightarrow i\hat{e}$, $p \rightarrow i\hat{p}$ in the Maxwell potential and in Δ_r (so, Δ_x and Σ remain unchanged). Finally, the conjugate solution to the above topological Kerr-Newman-NUT black hole becomes

$$ds^2 = \frac{\Delta_r}{\Sigma^2} [d\phi - (2\ell x + ax^2 - \omega_0)dt]^2 + \frac{\Sigma^2}{\Delta_r} dr^2 + \frac{\Sigma^2}{\Delta_x} dx^2 - \frac{\Delta_x}{\Sigma^2} [(r^2 + \ell^2 + a\omega_0)dt + ad\phi]^2, \quad (2.7)$$

with

$$\Delta_r(r) = -\hat{e}^2 - \hat{p}^2 - k\ell^2 + \Lambda\ell^4 + a^2[1 + k - k^2(1 + \ell^2\Lambda)] - 2mr + \left(k - a^2k^2\frac{\Lambda}{3} - 2\ell^2\Lambda\right)r^2 - \frac{\Lambda}{3}r^4,$$

and the potential one-form

$$A_\mu dx^\mu = -\left[\frac{x(\hat{p}\ell(ax + \ell) - \hat{p}r^2 - \hat{e}r(ax + 2\ell))}{\Sigma^2} + \omega_0 A_\phi(r, x)\right]dt - \left[\frac{\hat{e}r - \hat{p}(\ell + ax)}{\Sigma^2}\right]d\phi. \quad (2.8)$$

The transformation (2.6) is neither a diffeomorphism nor a proper change of coordinates; therefore, the general background, (2.7) and (2.8), is physically inequivalent from the Kerr-Newman spacetime. However, the double Wick rotation does not change the Petrov class; therefore, (2.7) and (2.8) inherit the type D from the conjugate

spacetime (2.1)–(2.3). Therefore, the Klein-Gordon equation is separable in this background. The easiest example consists of vanishing all the physical charges for $k = 1$, remaining with the Minkowski spacetime, which is mapped by (2.6) into itself up to a coordinate transformation. To be more precise, it is the flat spacetime in accelerating coordinates, which is the Rindler spacetime.

This general background solution, (2.7) and (2.8) for $k = 0$ and $\Lambda = 0$, according to the inverse scattering technique [20,21] is related to a background consisting of a complex soliton representing a charged and rotating black hole at infinity of the complex plane of the spectral parameter [22]. In the Weyl coordinates (t, ρ, z, φ) , the non-null metric components are

⁵In the literature, there are known also black holes deformed by external multipolar expansion, which are a reasonable physical model for collapsed stars. In particular, they are not singular outside the event horizon; however, their curvature scalar invariants grow as the radial coordinate in certain directions, so they can be asymptotically unbounded. Thus, we are not considering those backgrounds here.

$$\begin{aligned}
g_{tt} &= \left[2k_2 R - \frac{1}{\kappa \bar{\kappa}} + \kappa \bar{\kappa} (\rho^2 k_1^2 - k_2^2) R^2 \right] \Delta_t^2, \\
g_{t\phi} &= \left[\frac{k_2}{k_1} + \frac{\kappa \bar{\kappa}}{k_1} \right] \Delta_t \Delta_\phi, \\
g_{\rho\rho} &= g_{zz} = \frac{c_0 c_1}{\kappa \bar{\kappa}}, \\
g_{\phi\phi} &= \kappa \bar{\kappa} \left(\rho^2 - \frac{k_2^2}{k_1^2} \right) \Delta_\phi^2,
\end{aligned} \tag{2.9}$$

where k_i and c_i are constants, whereas κ and R are functions defined as follows:

$$\begin{aligned}
\kappa(\rho, z) &= [4k_1^2 \rho^2 - 4k_2 + 1 + i(c_2 + 8k_1 k_2 z)]^{-1}, \\
R(\rho, z) &= 16k_1 c_2 z + 64k_1^2 k_2 z^2 + c_5.
\end{aligned} \tag{2.10}$$

The electromagnetic potential one-form is

$$\begin{aligned}
A &= \left[\frac{c_3}{2} - 4k_1 z \sin \gamma + \frac{k_1 R A_t}{\Delta_\phi} \right] \Delta_t dt \\
&+ \left[\text{Im}(\kappa^{-1}) \sin \gamma - \text{Re}(\kappa^{-1}) \cos \gamma \right] \frac{\kappa \bar{\kappa} \Delta_\phi}{2k_1} d\phi.
\end{aligned} \tag{2.11}$$

The change of coordinates to pass from (2.7), (2.8) to (2.9)–(2.11) is

$$r = -\frac{\Delta_\phi^2 \Delta_t^2 \rho^2 + \hat{e}^2 + \hat{p}^2 - a^2}{m}, \quad x = \frac{\sqrt{c_0 c_1}}{\Delta_t} z, \tag{2.12}$$

while the parametric map is given by

$$\begin{aligned}
a &= \frac{8k_1 k_2 \Delta_t^2}{\sqrt{c_0 c_1}}, & m &= -\frac{8k_1^2 \Delta_t^3}{c_0 c_1}, & \Delta_\phi &= -\frac{8k_1^2 \Delta_t}{\sqrt{c_0 c_1}}, & \ell &= c_2 \Delta_t, \\
\hat{e} &= -\frac{4k_1 \Delta_t^2}{\sqrt{c_0 c_1}} \cos \gamma, & \hat{p} &= -\frac{4k_1 \Delta_t^2}{\sqrt{c_0 c_1}} \sin \gamma, & \omega_0 &= -\frac{\sqrt{c_0 c_1}}{8k_1} c_5.
\end{aligned} \tag{2.13}$$

c_3 is proportional to the gauge additive constant of the electric potential, if present.⁶ Note that for $k \neq 0$, the metric (2.7) and (2.8) is more general, and it represents a larger class of backgrounds particularly suitable to embed black holes in.

Let us give below some examples, both for $k = 0$ and $k \neq 0$, how to recover basically all the regular black hole backgrounds known in the literature at the moment, from the solution (2.7) and (2.8).

1. Swirling Bonnor-Melvin, $k = 0$

Considering $k = 0 = \Lambda$ in (2.7) and (2.8) and the following change of coordinates and parameters redefinition,⁷

$$\begin{aligned}
r &= -\frac{4\rho^2 + \hat{e}^2}{2m}, & x &= -\frac{2z}{m}, & t &= \frac{\tilde{t}}{2} (b^4 + 16J^2)^{1/4}, \\
\phi &= \frac{-\Delta_\phi \tilde{\phi}}{(b^4 + 16J^2)^{1/4}}, & m &= \frac{4}{(b^4 + 16J^2)^{1/4}}, & \ell &= \frac{8J}{(b^4 + 16J^2)^{3/4}}, \\
\hat{e} &= \frac{4b}{(b^4 + 16J^2)^{1/2}}, & \hat{p} &= a = 0, & \omega_0 &= 0,
\end{aligned} \tag{2.14}$$

we obtain the Bonnor-Melvin universe in a swirling background, originally found by Harrison [23]. In a renewed form stemming from symmetry transformations of the Ernst equations, as done in [24], the metric reads

$$ds^2 = \left[\left(1 + \frac{b^2 \rho^2}{4} \right)^2 + J^2 \rho^4 \right] (-d\tilde{t}^2 + dz^2 + d\rho^2) + \frac{\rho^2 (\Delta_\phi d\tilde{\phi} - 4Jz d\tilde{t})^2}{\left(1 + \frac{b^2 \rho^2}{4} \right)^2 + J^2 \rho^4}, \tag{2.15}$$

and its electromagnetic one-form potential⁸ is, up to a additive gauge constant,

⁶In the Kerr-Newman solution above, we have not considered this for brevity; possibly, it can always be added.

⁷Note that this choice is not unique.

⁸Usually, when the Harrison transformation is used to embed a given seed into the swirling Bonnor-Melvin background, an apparently alternative electromagnetic potential is generated: $A = -\frac{2b(4dtjz - \Delta_\phi d\phi)\rho^2(4+b^2\rho^2)}{16+8b^2\rho^2+(b^4+16J^2)\rho^4}$. This one-form is not related by gauge transformations to the potential (2.16); in fact, they produce different electromagnetic Faraday tensors but the same Maxwell energy momentum tensor. Anyway, when b or J vanishes, the two electromagnetic fields coincide. They are related through a duality rotation of the electromagnetic field, that is, the transformation I of [25], where the complex parameter is $\lambda = \frac{b^2 - 4iJ}{b^2 + 4iJ}$.

$$A_\mu dx^\mu = \frac{8b[4b^2 + (b^4 + 16J^2)\rho^2](4Jz\tilde{t} - \Delta_\phi d\tilde{\phi})}{(b^4 + 16J^2)(16 + 8b^2\rho^2 + (b^4 + 16J^2)\rho^4)}. \quad (2.16)$$

A vast group of black holes can be embedded in this background and endowed with electromagnetic charges, acceleration [24], and angular momentum [26].

2. Bonnor-Melvin, $k=0$

When the swirling parameter J becomes zero in the solution (2.15) and (2.16), we have the Bonnor-Melvin universe [12,13],

$$ds^2 = \left(1 + \frac{b^2\rho^2}{4}\right)^2 (-d\tilde{t}^2 + dz^2 + d\rho^2) + \frac{\rho^2 d\tilde{\phi}^2}{\left(1 + \frac{b^2\rho^2}{4}\right)^2}, \quad (2.17)$$

with

$$A = \left(A_{t_0} - \frac{2b\rho^2}{4 + b^2\rho^2}\right) d\tilde{\phi}. \quad (2.18)$$

To get the above solution [(2.17) and (2.18)] from double Wick rotation of the topological Kerr-Newman-NUT [(2.7) and (2.8)], we have to vanish the cosmological constant and make the following change of coordinates and adjustment of the integrating constants⁹:

$$\begin{aligned} r &= 1 + \frac{b^2\rho^2}{4}, \quad x = z, \quad \phi = -\frac{2\tilde{\phi}}{b}, \quad \ell = a = 0, \\ m &= -\frac{b^2}{2}, \quad \hat{e} = -b, \quad \Delta_\phi = -\frac{b^2}{2}, \quad \hat{p} = \omega_0 = 0. \end{aligned} \quad (2.19)$$

A large family of black holes in this background have been found; see, for instance, Refs. [6,27–29].

3. Swirling universe, $k=0$

To reach the swirling universe metric,

$$ds^2 = (1 + J^2\rho^4)(-d\tilde{t}^2 + dz^2 + d\rho^2) + \frac{\rho^2(\Delta_\phi d\tilde{\phi} - 4Jz d\tilde{t})^2}{1 + J^2\rho^4}, \quad (2.20)$$

from (2.7) and (2.8) with $\Lambda = 0$, the following change of coordinates can be used:

⁹This choice of the parametrization is not unique; for instance, it is possible to get the Bonnor-Melvin spacetime from the parametrization used to get the swirling Bonnor-Melvin universe with $J = 0$.

$$\begin{aligned} t &= \sqrt{J}\tilde{t}, \quad r = -\sqrt{J}\rho^2, \quad x = -\sqrt{J}z, \quad \phi = -\frac{\tilde{\phi}}{2\sqrt{J}}, \\ m &= \frac{2}{\sqrt{J}}, \quad \ell = \frac{1}{\sqrt{J}}, \quad \hat{e} = \hat{p} = 0, \quad a = \omega_0 = 0. \end{aligned} \quad (2.21)$$

The accelerating Kerr-Newman black hole class, with all its subcases, has been embedded into the swirling universe [3,4] and even a rotating binary black hole system at equilibrium [30].

4. Minkowski

The Minkowski spacetime can be obtained in several ways, for instance, from the above Bonnor-Melvin universe in Eq. (2.17) switching off the external electromagnetic field, by setting $b = 0$, or by vanishing the swirling parameter J into Eq. (2.20).

5. Levi-Civita, $k=0$

The $\sigma = 1$ Levi-Civita [31] metric,

$$ds^2 = -\rho^{4\sigma} d\tilde{t}^2 + \rho^{4\sigma(2\sigma-1)}(d\rho^2 + dz^2) + \rho^{2(1-2\sigma)} d\tilde{\phi}^2, \quad (2.22)$$

can be obtained from (2.7) and (2.8) with the following changes:

$$r = -\rho^2, \quad x = -z, \quad \phi = -\frac{\tilde{\phi}}{2}, \quad m = 2, \quad (2.23)$$

$$\ell = 0, \quad \hat{e} = \hat{p} = 0, \quad a = 0, \quad \omega_0 = 0. \quad (2.24)$$

6. Bertotti-Robinson-Bonnor-Melvin, $k = -B^2$

When, in the general background [(2.7) and (2.8)], we set $\hat{p} = a = \omega_0 = \Lambda = 0$, we get the Levi-Civita-Bertotti-Robinson-Bonnor-Melvin spacetime, which is a superposition of the Levi-Civita-Bertotti-Robinson¹⁰ and the Bonnor-Melvin universe, recently built in [8]. To write it as in [8],

$$\begin{aligned} ds^2 &= \Xi^2 \left[-(1 + B^2\tilde{r}^2)dt^2 + \frac{d\tilde{r}^2}{1 + B^2\tilde{r}^2} + \frac{\tilde{r}^2 d\tilde{x}^2}{(1 - \tilde{x}^2)} \right] \\ &+ \frac{\tilde{r}^2(1 - \tilde{x}^2)}{\Xi^2[1 + B^2\tilde{r}^2(1 - \tilde{x}^2)]^2} d\tilde{\phi}^2, \end{aligned} \quad (2.25)$$

with

¹⁰Even though Levi-Civita was the first to discover the conformally flat spacetime with a homogeneous electromagnetic field in general relativity [32], often in the literature, his name is dropped, and this solution is known after Bertotti and Robinson.

$$\Xi(\tilde{r}, \tilde{x}) = \frac{-b(b+2B) + (b^2 + 2bB + 2B^2)\sqrt{1 + B^2\tilde{r}^2(1 - \tilde{x}^2)}}{2B^2[1 + B^2\tilde{r}^2(1 - \tilde{x}^2)]}, \quad (2.26)$$

$$A_\mu = \left\{ 0, 0, 0, \frac{2(b+B)\tilde{r}^2(1 - \tilde{x}^2)}{-(b^2 + 2bB + 2B^2)\tilde{r}^2(1 - \tilde{x}^2) + 2\left[1 + \sqrt{1 + B^2\tilde{r}^2(1 - \tilde{x}^2)}\right]} \right\}. \quad (2.27)$$

From (2.7) and (2.8) with $\hat{p} = a = \omega_0 = \Lambda = 0$, one just needs to perform the following change of coordinates:

$$r = 1 - \frac{b(b+2B)[1 - \sqrt{1 + B^2\tilde{r}^2(1 - \tilde{x}^2)}]}{2B^2\sqrt{1 + B^2\tilde{r}^2(1 - \tilde{x}^2)}}, \quad (2.28)$$

$$x = \frac{\tilde{r}\tilde{x}}{\sqrt{1 + B^2\tilde{r}^2(1 - \tilde{x}^2)}}, \quad (2.29)$$

$$\phi = \frac{2}{b^2 + 2bB}\tilde{\phi}, \quad (2.30)$$

and redefinition of the parameters

$$m = -\frac{b^2}{2}, \quad k = -B^2, \quad k_7 = B^2 + B^4, \quad \hat{e} = (b+B). \quad (2.31)$$

Note that from the Bertotti-Robinson-Bonnor-Melvin spacetime, (2.25)–(2.27), when the parameter $B = 0$, we obtain again the Bonnor-Melvin universe:

$$ds^2 = \left[1 + \frac{b^2\tilde{r}^2}{4}(1 - \tilde{x}^2)\right]^2 \left[-dt^2 + d\tilde{r}^2 + \frac{\tilde{r}^2 d\tilde{x}^2}{(1 - \tilde{x}^2)}\right] + \frac{\tilde{r}^2(1 - \tilde{x}^2)}{\left[1 + \frac{b^2\tilde{r}^2}{4}(1 - \tilde{x}^2)\right]^2} d\tilde{\phi}^2, \quad (2.32)$$

$$A_\mu = \left[0, 0, 0, \frac{2b\tilde{r}^2(1 - \tilde{x}^2)}{4 + b^2\tilde{r}^2(1 - \tilde{x}^2)}\right]. \quad (2.33)$$

On the other hand, when $B \neq 0$ and $b = 0$ (or $b = -2B$), we recover the Bertotti-Robinson electromagnetic solution:

$$ds^2 = \frac{1}{1 + B^2\tilde{r}^2(1 - \tilde{x}^2)} \left[-(1 + B^2\tilde{r}^2)dt^2 + \frac{d\tilde{r}^2}{(1 + B^2\tilde{r}^2)} + \frac{\tilde{r}^2 d\tilde{x}^2}{(1 - \tilde{x}^2)} + \tilde{r}^2(1 - \tilde{x}^2)d\tilde{\phi}^2 \right], \quad (2.34)$$

$$A_\mu = \left[0, 0, 0, -\frac{B\tilde{r}^2(1 - \tilde{x}^2)}{1 + B^2\tilde{r}^2(1 - \tilde{x}^2) + \sqrt{1 + B^2\tilde{r}^2(1 - \tilde{x}^2)}}\right]. \quad (2.35)$$

Hence, it is clear that the parameters b and B represent the intensity of the Bonnor-Melvin and Bertotti-Robinson electromagnetic fields, respectively. Several black holes in the Bertotti-Robinson background have been discovered; see Refs. [7,33–35]. To obtain the swirling generalization of the Levi-Civita-Bertotti-Robinson-Bonnor-Melvin background, one needs just to leave the $\ell \neq 0$. Black holes in these backgrounds also have been generated in [8].

7. A hyperbolic pure gravitational background, $k = -B^2$

At the special point where $b = -B$, the Levi-Civita-Bertotti-Robinson-Bonnor-Melvin electrovacuum solution [(2.25)–(2.27)] has a vanishing electromagnetic field [8], so it becomes just a vacuum metric representing a spacetime similar to the Witten bubble of nothing, but instead of being the double Wick rotation ($t \rightarrow i\phi$, $\phi \rightarrow it$) of the standard

Schwarzschild black hole with a spherical event horizon, it is the double Wick rotation of the Schwarzschild metric with negative constant curvature in the section (x, ϕ) , as the negative value of k denotes. The metric reads

$$ds^2 = \left[\frac{1 + \sqrt{1 + B^2\tilde{r}^2(1 - \tilde{x}^2)}}{2(1 + B^2\tilde{r}^2(1 - \tilde{x}^2))} \right]^2 \times \left[-(1 + B^2\tilde{r}^2)dt^2 + \frac{d\tilde{r}^2}{1 + B^2\tilde{r}^2} + \frac{\tilde{r}^2 d\tilde{x}^2}{1 - \tilde{x}^2} \right] + \frac{4\tilde{r}^2(1 - \tilde{x}^2)d\phi^2}{\left(1 + \sqrt{1 + B^2\tilde{r}^2(1 - \tilde{x}^2)}\right)^2}. \quad (2.36)$$

The parameter B in this case represents the intensity of the gravitational field; for $B = 0$, the flat spacetime is recovered. When $B \rightarrow iB$, the sign of the curvature reverses, and the metric describes a usual Witten's bubble of nothing with $k = 1$, described in the next paragraph; for more details, see Ref. [36].

Also in this case, a static black hole can be embedded in this background [36].

8. Expanding bubbles of nothing, $k=1$

When $k=1$, we get a generalization of the expanding bubble of nothing [37,38]. These extensions can include the NUT parameter, the angular momentum, and the charge. The simpler case is the original Witten one that can be obtained from (2.7) and (2.8), when all the charges are set to zero, apart from the mass: $m = r_0/2$, $a = \ell = \hat{e} = \hat{p} = 0$. In this case, thanks to the change of coordinates,

$$\begin{aligned} t &= \operatorname{arcsch} \left[\cos \chi \operatorname{csch} T \sqrt{1 - \sinh^2 T \tan^2 \chi} \right], \\ x &= \cosh T \sin \chi, \end{aligned} \quad (2.37)$$

we remain with the standard form of the Witten bubble of radius r_0 :

$$ds^2 = r^2(-dT^2 + \cosh^2 T d\chi^2) + \frac{dr^2}{1 - \frac{r_0}{r}} + \left(1 - \frac{r_0}{r}\right) d\phi^2. \quad (2.38)$$

Static, rotating, and multi-black holes at equilibrium in this background can be built; see Refs. [5] and [16].

C. A novel rotating background for $k=0$: Bonnor-Melvin swirling curling universe

In all the spacetimes considered above, there is a physical parameter that has been always kept null: a , the reminiscent of the angular momentum of the Kerr black hole before the analytic continuation. It might be interesting to explore the possibility of leaving this rotational parameter, which is independent from the swirling one, switched on, in order to model novel kinds of stationary gravitational backgrounds. This is the aim of this section.

In particular, we present here a three parameter specialization of the general background, which includes the Bonnor-Melvin electromagnetic field and a couple of independent rotational external backgrounds: the swirling one, labeled with the character J , and the new one, which we call *curling*, denoted by the symbol $\aleph$. We write here explicitly the case for $k=0$, which is the direct generalization of the swirling Bonnor-Melvin universe; for more general cases, k can be chosen differently. For this purpose, the following coordinate transformation and redefinition of parameters from (2.7) and (2.8) are needed:

$$\begin{aligned} r &= \frac{(b^4 + 16J^2)^{1/4}}{8} \left(\aleph^2 - 4\rho^2 - \frac{16b^2}{b^4 + 16J^2} \right), & x &= -\frac{z}{2} (b^4 + 16J^2)^{1/4}, \\ \phi &= \frac{-\Delta_\phi \tilde{\phi}}{(b^4 + 16J^2)^{1/4}}, & t &= \frac{\tilde{t}}{2} (b^4 + 16J^2)^{1/4}, \\ \hat{e} &= \frac{4b}{(b^4 + 16J^2)^{1/2}}, & \ell &= \frac{8J}{(b^4 + 16J^2)^{3/4}}, \\ m &= \frac{4}{(b^4 + 16J^2)^{1/4}}, & \hat{p} &= \omega_0 = 0, \\ a &= -\aleph. \end{aligned} \quad (2.39)$$

The resulting metric is

$$ds^2 = \frac{\rho^2 \Delta_\phi^2 d\tilde{t}^2}{f(\rho, z)} + \frac{\Delta_\phi^2 (\aleph^2 - 4\rho^2)}{4f(\rho, z)} (d\rho^2 + dz^2) - f(\rho, z) \left\{ \left[\frac{\Delta_\phi \aleph}{2f(\rho, z)} - \frac{z[b^4 z \aleph + 16J(2 + Jz\aleph)]}{8\Delta_\phi} \right] d\tilde{t} + d\tilde{\phi} \right\}^2, \quad (2.40)$$

with

$$f(\rho, z) = \frac{64\Delta_\phi^2 (\aleph^2 - 4\rho^2)}{256 + 32b^2(4\rho^2 - \aleph^2) + b^4[16z^2\aleph^2 + (\aleph^2 - 4\rho^2)^2] + 16J\{32z\aleph + J[16z^2\aleph^2 + (\aleph^2 - 4\rho^2)^2]\}}. \quad (2.41)$$

Clearly, when the curling parameter $\aleph = 0$, we recover the swirling Bonnor-Melvin background, (2.15) and (2.16).

However, note that the chosen parametrization may be improved because, while the limits to the swirling Bonnor-Melvin background are straightforward and well defined, for $\aleph = 0$, the curling Bonnor-Melvin and the curling swirling

backgrounds go directly to the Minkowski spacetime for vanishing b and J ; more details about that appear in Sec. III.

Note that all the above background subcases can be straightforwardly extended in the presence of the cosmological constant, just leaving the non-null Λ form of the general background (2.7) and (2.8), as has been noticed for

the Bonnor-Melvin and the swirling cases in [11] and [3], respectively. However, in the presence of the cosmological constant, the superposition of a black hole with any of these backgrounds is difficult because some symmetries of the gravitational field equations, fundamental for developing useful solution generating techniques, are broken [11].

III. EMBEDDING SCHWARZSCHILD IN THE NOVEL ROTATING BACKGROUND

A. Schwarzschild in the curling, Bonnor-Melvin universe

Having unified and clarified the relation between the black hole backgrounds in general relativity, it would be

interesting to embed inside these spacetimes a black hole metric. We will insert the Schwarzschild black hole into the above background solution with $j = 0$, just to keep the computations simple. Nevertheless, there are no conceptual or computational obstacles in considering charged and rotating black holes or keeping the background more general.

Thanks to the inverse scattering technique [21], we can build the metric and the electromagnetic potential representing the superposition of the Schwarzschild black hole and the Bonnor-Melvin electromagnetic background, which, in spherical coordinates, read¹¹

$$ds^2 = -f(r, x)[d\varphi - \omega(r, x)dt]^2 + \frac{1}{f(r, x)} \left\{ \rho^2(r, x)dt^2 - e^{2\gamma(r, x)} \left[\frac{dr^2}{\Delta_r(r)} + \frac{dx^2}{\Delta_x(x)} \right] \right\} \quad (3.1)$$

$$\begin{aligned} f(r, x) &= -[256r^4(1-x^2) - 64r(r-2m)\aleph^2]/D(r, x) \\ \omega(r, x) &= \frac{b^4\aleph}{32} \left\{ r(r-6m)(1-x^2) - 12m^2(1+x^2) + \aleph^2 + \frac{16[8m-4r+b^4r^3(r-3m)^2(x^4-x^2)]}{b^4[(r-2m)\aleph^2 - 4r^3(1-x^2)]} \right\} + \omega_0 \\ \gamma(r, x) &= \frac{1}{2} \log \left\{ \frac{r}{4} [4r^3(1-x^2) - (r-2m)\aleph^2] \right\} + \gamma_0, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \rho(r, x) &= \sqrt{\Delta_r(r)\Delta_x(x)}, \quad \Delta_r(r) = r^2 - 2mr, \quad \Delta_x(x) = 1 - x^2, \\ D(r, x) &= 16b^4r^2(r-3m)^2x^2\aleph^2 + [b^2\aleph^2(r-2m) - 4r(4 + b^2r^2(1-x^2))]^2, \\ A_t(r, x) &= \{32br[-4 - b^2r^2(1-x^2)]\{4r + 2b^2r^3 + 6b^2m^2r(2x^2-1) + m[4 - 3b^2r^2(1+3x^2)]\}\aleph \\ &\quad + 4b^3\{4r^2 + 3b^2r^4(1-5x^2) + 24b^2m^3r(1+x^2) - 4m^2(4 + 33b^2r^2x^2)\}\aleph^3 \\ &\quad - 16b^3mr[8 - b^2r^2(5-23x^2)]\aleph^3 + b^5(r-2m)(6m+r)\aleph^5\}/[4D(r, x)] + A_{t0}, \\ A_\varphi(r, x) &= \{32r[16r + 4b^2r^3(1-x^2) - b^2(r-2m)\aleph^2]\}/[bD(r, x)] + A_{\varphi0}. \end{aligned} \quad (3.3)$$

We recall that A_{t0} , $A_{\varphi0}$, γ_0 , and ω_0 are just gauge constants¹² (always present in general relativistic metrics written in Lewis-Weyl-Papapetrou form), which can be useful to regularize a possible conical singularity or to set the angular velocity of an observer at spatial infinity, respectively.

Since the metric contains the Schwarzschild-Bonnor-Melvin, it must be of general type I. Possible divergences of the Kretschmann scalar invariant for the above solution are located where its denominator,

$$\frac{16b^4r^2x^2(r-3m)^2}{\aleph^2} + \left\{ b^2(r-2m) - \frac{4r[b^2r^2(1-x^2)+4]}{\aleph^2} \right\}^2,$$

¹¹A *Mathematica* notebook containing this solution is available [39].

¹²In general, these quantities can be thought null, unless elsewhere specified.

is null. Since it is the sum of two squares, it can be zero only if both the two squares are simultaneously vanishing. This cannot occur for a significant portion of the parametric space. To be more precise, note that this condition is necessary, but not sufficient to guarantee a divergence of the scalar invariant. For instance, the Kretschmann denominator is zero for $b = 0$ or for $m = 0$ in $r = 0$; nevertheless, the full scalar invariant is finite at $b = 0$ or at $m = 0$ in $r = 0$ (apart in a point of the parametric space where simultaneously $m = 0$, $r = 0$, and $\aleph b = \pm 4$). In the general case, when the physical parameters are not trivial, the spacetime is not singular when the physical parameters fulfil one of the following two conditions:

$$\begin{aligned} (1) \quad & b^2\aleph^2 < 16, \\ (2) \quad & 16 < b^2\aleph^2 < 48 \quad \text{and} \quad 108m^2b^6\aleph^4 > (b^2\aleph^2 - 16)^3. \end{aligned} \quad (3.4)$$

Obviously, when one of the above conditions is not fulfilled, the quadratic scalar invariant diverges on the ring $r = 3m$ or in the equatorial plane $x = 0$, and thus, spacetime represents a naked singularity. Therefore, the background geometry, in particular the novel features coming from the presence of b and $\aleph$, can smear the typical curvature singularity in $r = 0$ of the Schwarzschild black hole in a certain sector of the parametric space. So,

$$ds^2 = \left[1 + \frac{b^2 r^2}{4} (1 - x^2)\right]^2 \left[-\left(1 - \frac{2m}{r}\right) dt^2 + \frac{e^{2\gamma_0} dr^2}{1 - \frac{2m}{r}} + \frac{e^{2\gamma_0} r^2 dx^2}{(1 - x^2)}\right] + \frac{r^2 (1 - x^2) d\varphi^2}{\left[1 + \frac{b^2 r^2}{4} (1 - x^2)\right]^2}, \quad (3.5)$$

$$A_\mu = \left[0, 0, 0, \frac{2br^2(1 - x^2)}{4 + b^2 r^2 (1 - x^2)} + A_{\varphi 0}\right]. \quad (3.6)$$

In this $\aleph = 0$ subcase, the usual curvature singularity of the static black hole is also retrieved. Also, the electromagnetic background is completely regular. Clearly, the above solution (3.5) and (3.6) reduces to the magnetic Bonnor-Melvin universe when we further switch off the black hole mass m , as written in Eqs. (2.32) and (2.33). On the other hand, the limit for $b \rightarrow 0$ gives the Schwarzschild black hole (in a rotating frame of reference; to have it in the usual static gauge, an extra gauge constant has to be set to zero). This fact is independent of whether we perform the limit starting from (3.5) and (3.6) or from (3.1)–(3.3), because at this level, the parameters b and $\aleph$ are somehow intertwined. Obviously, when the black hole is removed ($m = 0$) from the spacetime (3.1)–(3.3), we have the Bonnor-Melvin curling background, which is a nonswirling subcase of the spacetime of Sec. II C. The background for $m = 0$ is free of curvature singularities when $b^2 \aleph^2 < 16$.

we can have a black hole geometry without curvature singularities, not even inside the event horizon. This is not a complete novelty, because it can occur also in the context of Bonnor-Melvin and swirling backgrounds [26].

The $\aleph \rightarrow 0$ limit (adjusting properly the gauge constants) of the above solution recovers the Schwarzschild black hole embedded into the magnetic Bonnor-Melvin universe:

So, the solution (3.1)–(3.3) clearly contains the Schwarzschild and the Schwarzschild-Bonnor-Melvin metric. The black hole interpretation for this spacetime is valid in almost all the range of the physical parameters.

B. Schwarzschild embedded in a curling background

The removal of the Bonnor-Melvin background is not so direct in this parametrization because of the abovementioned entanglement between b and $\aleph$. However, it can be done in two different approaches, which give an equivalent result: by applying the Harrison transformation to solution (3.1)–(3.3) or by taking its limit for $b \rightarrow \infty$.¹³ In any case, the result represents a Schwarzschild metric embedded only in the novel rotating background, which was the last remaining unexplored background related to the Kerr-Newman metric with a flat base angular manifold,¹⁴ the one related to the angular momentum parameter a . Notice that the electromagnetic field completely vanishes in this limit. It can be written¹⁵ from the general metric (3.1) and (3.2) and

$$f(r, x) = \frac{-16r^4(1 - x^2) + 4r(r - 2m)\aleph^2}{16r^6(1 - x^2)^2 + 8r^2[r(2m - r) + (18m^2 - 14mr + 3r^2)x^2]\aleph^2 + (r - 2m)^2\aleph^4}, \quad (3.7)$$

$$\omega(r, x) = \left[r\left(\frac{r}{2} - 3m\right)(1 - x^2) - 6m^2(1 + x^2)\right]\aleph + \frac{8r^3(r - 3m)^2 x^2 (x^2 - 1)\aleph}{4r^3(x^2 - 1) + (r - 2m)\aleph^2} + \omega_0. \quad (3.8)$$

The effect of the external background is to remove curvature singularity, typical of the Schwarzschild black hole in $r = 0$. The Kretschmann scalar invariant for this metric is finite because for large radial distance it goes to zero; however, for finite r , its denominator,¹⁶

$$\{16r^6(1 - x^2)^2 + 8r^2[(2m - r)r + (18m^2 - 14mr + 3r^2)x^2]\aleph^2 + (r - 2m)^2\aleph^4\}^6, \quad (3.9)$$

¹³To select the right leading order in the limiting procedure, we rescale properly the Killing coordinates and the gauge constants. More specifically, $\omega_0 \rightarrow \omega_0 b^4$, $t \rightarrow 4t/b^2$, $\varphi \rightarrow \varphi b^2/4$, $\gamma_0 = 2 \log(2/b)$.

¹⁴We recall that the background related to the conjugate solution with respect to the flat (extremal) Reissner-Nordstrom is the Bonnor-Melvin background [12, 13] and the swirling background of [23]. The black hole solutions in these backgrounds were discovered in [27] and [3], respectively.

¹⁵A *Mathematica* notebook containing this solution is available [39].

¹⁶The full expression for the quadratic Riemann invariant is given in the *Mathematica* notebook [39].

can be zero for any value of the curling parameter. Actually, outside the known cases (i.e., $\aleph = 0, \aleph = \infty$; see below), there are two different regimes, and both are plagued by some curvature singularity: (i) $0 < \aleph^2 \leq 108m^2$ is singular on the $r = 3m$ ring, and (ii) $\aleph^2 > 108m^2$ is singular on the equatorial plane for $x = 0$.

The above metric gives the pure curling background when $m = 0$. To write in Weyl coordinates (ρ, z) ,

$$ds^2 = -f(d\varphi - \omega dt)^2 + \frac{1}{f}[-e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \quad (3.10)$$

where

$$f = \frac{-4(4\rho^2 - \aleph^2)}{(4\rho^2 - \aleph^2)^2 + 16z^2\aleph^2}, \quad \omega = \frac{\rho^2\aleph(16z^2 + 4\rho^2 - \aleph^2)}{8\rho^2 - 2\aleph^2},$$

$$\gamma = \frac{1}{2} \log\left(\rho^2 - \frac{\aleph^2}{4}\right),$$

the following changes in coordinates have been used:

$$r = \sqrt{\rho^2 + z^2}, \quad x = \frac{z}{\sqrt{\rho^2 + z^2}}. \quad (3.11)$$

Note that the Weyl coordinates are not cylindrical for this metric; in order to put it in cylindrical coordinates, we can further shift the ρ coordinate: $\rho \rightarrow \tilde{\rho} = \sqrt{\rho^2 + \aleph^2}/2$.

The background, however, is not more regular with respect to the $m \neq 0$ spacetime; in fact, scalar invariants diverge on the equatorial plane ($x = 0$) for $\rho = \aleph/2$. So, in this respect, the above section background in the presence of the Bonnor-Melvin electromagnetic field behaves better, for instance, for small values of the magnetic field parameter b .

The spacetime (3.1), (3.7), and (3.8) belongs to the general Petrov type, and its behavior for large values of the coordinate r is not dependent on the mass parameter m , so it is shared by the curling background [(3.10) and (3.11)]. At the first leading order, it is given by¹⁷

$$ds^2 = (1 - x^2)^2 r^4 \left[-dt^2 + dr^2 + \frac{r^2}{1 - x^2} \right] + \frac{d\varphi^2}{r^2(1 - x^2)} - \frac{1 + 3x^2}{1 - x^2} \aleph dt d\varphi. \quad (3.12)$$

In terms of the coordinates defined in (3.11), the above metric can be written as

¹⁷Of course, this metric should not satisfy the Einstein equations, unless $\aleph = 0$.

$$ds^2 = \rho^4(-dt^2 + d\rho^2 + dz^2) + \rho^{-2}d\varphi^2 - \aleph(1 + 4z^2\rho^{-2})dt d\varphi.$$

Thus, it resembles a rotating Levi-Civita background.

The Schwarzschild black hole,

$$ds^2 = -\left(1 - \frac{2m}{\bar{r}}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{\bar{r}}} + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta d\varphi^2, \quad (3.13)$$

can be retrieved from (3.1), (3.7), and (3.8) for $\aleph = \infty$.¹⁸ Before taking that limit, to get a clearer result, the following rescaling of the Killing coordinates, shift of the radial coordinate, and proper fixing of the gauge parameters have been performed:

$$t = \frac{2}{\aleph} t, \quad \varphi = \frac{\aleph}{2} \varphi, \quad \omega_0 = \frac{3m^2\aleph}{8} - \frac{\aleph^3}{32},$$

$$\gamma_0 = \frac{1}{2} \log\left(\frac{16}{\aleph^4}\right), \quad r = 2m - \bar{r}. \quad (3.14)$$

On the other hand $\aleph \rightarrow 0$ limit gives the Schwarzschild solution into the Levi-Civita background [40]:

$$ds^2 = r^4(1 - x^2)^2 \left[-\left(1 - \frac{2m}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + \frac{r^2 dx^2}{1 - x^2} \right] + \frac{d\varphi^2}{r^2(1 - x^2)}. \quad (3.15)$$

Note, however, that this metric is not suitable for black hole interpretation because it displays naked, unavoidable curvature singularities on the axis of symmetry ($x = \pm 1$). Actually, it represents a physically pathological limit of the Schwarzschild swirling black hole solution that contains it [8]. We recall that the Schwarzschild swirling does not present curvatures or conical singularities outside the event horizon.

More generally, black holes embedded into the Levi-Civita background can be retrieved from any black hole of this class, that is, black holes in Bonnor-Melvin, swirling, or curling background. For more details about that, see the Appendix. This is fundamentally due to the common asymptotic behavior this class of background possesses.

¹⁸Obviously, defining a new parameter related to the rotation as $\Xi = 1/\aleph$ for the Schwarzschild curling metric [(3.1), (3.7), and (3.8)] would allow one to recover the static, spherically symmetric black hole in the more intuitive notation $\Xi \rightarrow 0$.

IV. INCLUDING ACCELERATION

A. Self-duality of the C metric

The only parameter we have not explicitly considered yet is the acceleration. That is because it behaves differently with respect to the other parameters of the black hole solution. In fact, while the external properties of the backgrounds (2.7) are related to the associated double Wick rotated black hole charges, accelerating black holes (and the Rindler background) are self-dual under the conjugation transformation of the metric. This is quite obvious from the rod diagram representation of the metrics. For instance, if we write the rod representation of the accelerating black hole in Weyl coordinates (ρ, z) [41] as in Fig. 1 and their conjugate metrics as in Fig. 2, we can clearly see how the two diagrams are identical upon the inversion of the z axis.

In fact, the coordinate transformation $z \rightarrow -z$ is exactly the diffeomorphism that maps the Rindler metric

$$ds^2 = -\mu_3 dt^2 + \frac{\mu_3}{\rho^2 + \mu_3^2} (d\rho^2 + dz^2) + \frac{\rho^2}{\mu_3} d\varphi^2$$

and the neutral C metric

$$ds^2 = -\frac{\mu_1 \mu_3}{\mu_2} dt^2 + \frac{16C_f \mu_1^3 \mu_2^3 \mu_3^3 (d\rho^2 + dz^2)}{\mu_{12} \mu_{23} W_{13}^2 W_{11} W_{22} W_{33}} + \frac{\rho^2 \mu_2}{\mu_1 \mu_3} d\varphi^2$$

to their respective conjugate metrics

$$ds^2 = -\frac{\rho^2}{\mu_3} dt^2 + \frac{\mu_3}{\rho^2 + \mu_3^2} (d\rho^2 + dz^2) + \mu_3 d\varphi^2,$$

$$ds^2 = -\frac{\rho^2 \mu_2}{\mu_1 \mu_3} dt^2 + \frac{16C_f \mu_1^3 \mu_2^3 \mu_3^3 (d\rho^2 + dz^2)}{\mu_{12} \mu_{23} W_{13}^2 W_{11} W_{22} W_{33}} + \frac{\mu_1 \mu_3}{\mu_2} d\varphi^2.$$

Here, we are using the standard solitonic inverse scattering notation [21,42], where the basic building blocks are

$$\mu_i = w_i - z + \sqrt{\rho^2 + (z - w_i)^2}, \quad \mu_{ij} = (\mu_i - \mu_j)^2,$$

$$W_{ij} = \rho^2 + \mu_i \mu_j.$$

For completeness, we recall the standard physical parametrization

$$w_1 = -m, \quad w_2 = m, \quad w_3 = \frac{1}{2A}, \quad C_f = \frac{m^2}{A^3},$$

that gives, thanks to the change in coordinates,

$$\rho(r, \theta) = \frac{\sqrt{(r^2 - 2mr)(1 - A^2 r^2)(1 + 2mA x)(1 - x^2)}}{(1 - A r x)^2}, \quad z(r, \theta) = \frac{(Ar + x)[r - m(1 - A r x)]}{(1 - A r x)^2},$$

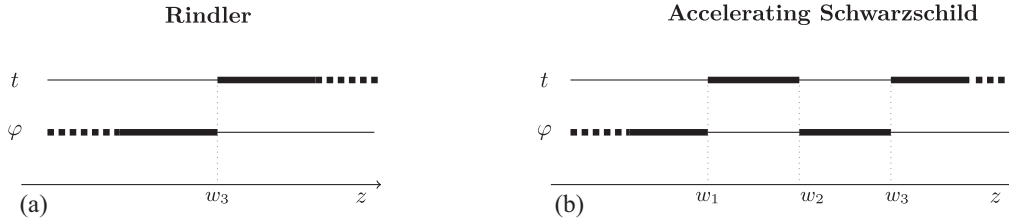


FIG. 1. Rods diagram representing the conjugate Rindler metric (a) and the conjugate accelerating Schwarzschild black hole or the uncharged c metric (b). The finite timelike segment represents the event horizon, while the accelerating horizon is represented by a infinite timelike semiline.

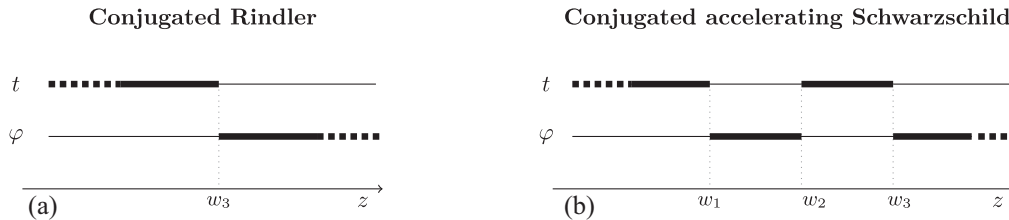


FIG. 2. Rods diagram representing the Rindler metric (a) and the accelerating Schwarzschild black hole or the uncharged c metric (b). The diagrams coincide with the nonconjugated ones of Fig. 1 just by inverting the z coordinate.

the accelerating Schwarzschild black hole, also known as the uncharged C metric, in spherical coordinates:

$$ds^2 = \frac{\left(1 - \frac{2m}{r}\right)(A^2 r^2 - 1)dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)(1 - A^2 r^2)} + \frac{r^2 dx^2}{(1 - 2mA x)(1 - x^2)} + r^2(1 - 2mA x)(1 - x^2)d\varphi^2}{(1 - A r x)^2}.$$

The Rindler parametrization is clearly achieved from the C metric by removing the black hole for $w_1 \rightarrow w_2$, or equivalently, vanishing the black hole mass parameter $m \rightarrow 0$.

We have seen that in the presence of the acceleration, the conjugation operation maps accelerating black holes into accelerating black holes; thus, conjugated metrics of accelerating black holes are not always suitable to model gravitational backgrounds, in the same spirit of what we have done in Sec. II B.

So, to include the acceleration in this picture and enlarge this framework to all¹⁹ known stationary and axisymmetric black holes in the theory of general relativity, possibly coupled with the Maxwell electromagnetism, we have two possibilities. The first is to embed into the background class in (2.7) accelerating black holes, while the second, as detailed in the following section, can be to extend the

conjugated topological Kerr-Newman-NUT metric (2.7) to the accelerating parameter. A composition of these two accelerating effects, while it is possible, might be redundant.

B. Accelerating backgrounds as the conjugated Plebanski-Demianski metric

The accelerating generalization of the Kerr-Newman black hole is still a type-D spacetime, according to the Petrov classification, belonging to the Plebanski-Demianski metric family. For describing black holes, the best parametrization of the Plebanski-Demianski spacetime is given in [42,43]; see also Ref. [44] for a relation with the previous parametrizations. Anyway, for describing the background here, we can use the parametrization of [45], because it is more similar to the Sec. II notation, whose metric and potential are

$$ds^2 = \frac{[\omega dt + (r^2 + \omega \hat{\omega}_0)d\varphi]^2 P(x) - [dt + (-x^2 \omega + \hat{\omega}_0)d\varphi]^2 Q(r)}{R^2(r, x)\Omega^2(r, x)} + \frac{[dr^2 P(x) + dx^2 Q(r)]R^2(r, x)}{P(x)Q(r)\Omega^2(r, x)},$$

$$A_\mu dx^\mu = -\frac{er + \omega px}{R^2(r, x)}dt + \left[\frac{e\omega r x^2 - p x r^2}{R^2(r, x)} + \hat{\omega}_0 A_t(r, x)\right]d\varphi, \quad (4.1)$$

where

$$Q(r) = \hat{k}\omega^2 + p^2 + e^2 - 2mr + r^2\epsilon - \frac{2nr^3 A}{\omega} - \frac{r^4 \Lambda}{3} - \hat{k}r^4 A^2,$$

$$P(x) = \hat{k} + \frac{2nx}{\omega} - x^2\epsilon + 2mx^3 A + x^4 \left[-\frac{\Lambda\omega^2}{3} - A^2 \left(p^2 + e^2 + \hat{k}\omega^2\right)\right],$$

$$R^2(r, x) = r^2 + \omega^2 x^2,$$

$$\Omega(r, x) = 1 - A r x. \quad (4.2)$$

Here, $m, \omega, e, p, n, A, \hat{k}, \epsilon, \hat{\omega}_0$ are integrating constants related to the mass, the electric and magnetic monopolar charges, the nut parameter, the acceleration, the topology, and the asymptotic angular speed of a rotating observer. In the nonaccelerating case $A = 0$, we recover the topological Kerr-Newman solutions, (2.1)–(2.3), for

¹⁹We are thinking of black holes that are not singular outside their simply connected event horizons. However, the general class of backgrounds, as in (2.7), is suitable to host multi-black-hole systems [5,30]. Accelerating black holes can present conical singularities, but thanks to the interaction with Bonnor-Melvin or swirling backgrounds, they can be regularized [4,28].

$$n \rightarrow k\ell - \frac{4\ell^3 \Lambda}{3} + a^2 \left(\frac{k_7}{2\ell} + \frac{k^2 \ell \Lambda}{3}\right),$$

$$\epsilon \rightarrow k - \frac{\Lambda}{3} a^2 k^2 - 2\ell^2 \Lambda,$$

$$\hat{k} \rightarrow \frac{-k\ell^2 + \ell^4 \Lambda + a^2[1 + k - k^2(1 + \ell^2 \Lambda)]}{\omega^2},$$

$$\hat{\omega}_0 \rightarrow \frac{\ell^2}{a} + \omega_0, \quad \omega \rightarrow a,$$

$$x \rightarrow \frac{ax + \ell}{\omega}.$$

On the other hand, in the presence of the acceleration, the above map should be extended.²⁰

In any case, the generalized background, also in the presence of the acceleration that is the natural accelerating extension of (2.7) and (2.8), is contained in the conjugation or the double Wick rotation (2.6) of the solution [(4.1) and (4.2)].

V. CONCLUSIONS

In this article, we have taken into consideration all the known exact, analytical, and physical well-behaved black hole solutions in four-dimensional general relativity (possibly coupled with standard electromagnetism). Those metrics are not necessarily asymptotically flat, but their geometry does not present conical, or curvature singularities outside the event horizon. We have noticed that these spacetimes belong to a common family and that they can be comprehensively classified according to the properties of their backgrounds. This general class has an unexplored sector. We have identified the background characterizing this unknown sector, and we have built new black hole solutions whose backgrounds possess these novel physical features. In this way, we can provide a more detailed and rich black hole picture in the Einstein-Maxwell theory of gravitation. The last unexplored parameter stems from the conjugated general black hole metric of type D, and it is directly related to the angular momentum in the black hole metric (before the conjugation). This integration constant has been, so far, overlooked, probably because it does not directly descend from Lie-point transformations such as the electromagnetic charges or the NUT parameter, which are strictly connected to the Harrison and Ehlers symmetries of the Ernst equations. Nevertheless, the inverse scattering technique has been able to take into account this extra parameter. This new rotational feature naturally can be superposed to the already known ones such as the Bonnor-Melvin and the swirling parameters.

These results point to a remarkable insight: basically *all* the known legitimate axisymmetric and stationary black hole solutions in the Einstein-Maxwell theory are (accelerating) Kerr-Newman-NUT black holes embedded into the double Wick rotated (accelerating) Kerr-Newman-NUT background.

All these metrics can be built thanks to the axisymmetric and stationary solution generating techniques. The inverse scattering method also suggests a physical interpretation for these backgrounds as the gravitational and electromagnetic fields of far away black holes. This is coherent with the interpretation of the Bonnor-Melvin universe as a couple of Reissner-Nordstrom black holes at infinity [46] or the expanding bubble of nothing as a couple of big

Schwarzschild black holes that extend to spatial infinity [16].

Now, the bigger open question is about the uniqueness or eventual further generalization of these regular backgrounds. In general, according to the inverse scattering method, it would be possible to add extra solitons at infinity to add extra integrating constants. However, it is not clear if these possible extra degrees of freedom might be reabsorbed in the previous ones.

What we have done here can be promptly adapted to other gravitational theories closely related to general relativity, such as minimally and conformally coupled scalar fields, nonlinear sigma models, or some Brans-Dicke and $f(R)$ theories, as noticed in [47] and [48].

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DATA AVAILABILITY

The data that support the findings of this article are openly available [39].

APPENDIX: ABOUT BLACK HOLES IN THE LEVI-CIVITA BACKGROUND

Recently, there has been some interest in black holes in a Levi-Civita background, Schwarzschild, or Kerr with an asymptotic that is not Minkowskian but is a special subclass of the Levi-Civita metric; see, for instance, Refs. [40,49]. As noticed in [8], these solutions belong to the more general class of black holes in the swirling background. The rationale is quite straightforward: black holes in the Levi-Civita background can be generated from the (magnetic) inversion transformation of the Ernst equations, but the inversion transformation is contained in the (magnetic) Ehlers transformation; hence, the black holes in the Levi-Civita background are contained in the Ehlers transformed Schwarzschild or Kerr black hole. Unfortunately, misleading and incorrect literature is continuing to emerge ignoring this inclusion; therefore, in this appendix, we further clarify this point, explicitly verifying the relation between the swirling and Levi-Civita black holes for some particular cases.

1. Kerr-Levi-Civita from Kerr swirling

We want to prove that the inversion transformation of the Kerr black hole is contained in the Ehlers transformed Kerr metric for a specific value of the parameter, j , introduced by the Ehlers transformation. The Kerr metric deformed by a magnetic Ehlers transformation has been found in [3] and describes a Kerr black hole embedded into a rotating background (dubbed swirling universe for its peculiar

²⁰For instance, $\epsilon \rightarrow 1 - (a^2 - e^2 - p^2)A^2 - 2\ell^2\Lambda$, $k_7 = -\ell[k\ell + 2a^2mA + a^2k\ell\lambda - \ell^3\Lambda\ell]/a^2$ gives the accelerating black hole in the parametrization [42].

properties). Following the notation of [3] (with $x = \cos \theta$), the metric can be cast in the following Lewis-Papapetrou metric,

$$ds^2 = f(d\varphi - \omega dt)^2 + \frac{1}{f} \left[-\rho^2 dt^2 + \Sigma(1-x^2) \left(\frac{dr^2}{\Delta_r} + \frac{dx^2}{1-x^2} \right) \right], \quad (\text{A1})$$

with

$$\begin{aligned} f(r, x) &= \frac{\Sigma(1-x^2)R^2}{(R^2 + 2Jma\Xi x)^2 + J^2\Sigma^2(1-x^2)^2}, \\ \omega(r, x) &= \bar{\omega}_0 - \frac{2mar}{\Sigma} + J \frac{4x(r^3 - a^2(m + (m-r)x^2))\Delta_r}{\Sigma} \\ &\quad + J^2 2am \frac{(a^2 - 3r^2)(r^3 + a^2(r+2m)) + x^2(r^3(x^2 - 6) + a^2(2m(3+x^2)) - 3r(2+x^2))\Delta_r}{\Sigma}, \end{aligned}$$

where

$$\begin{aligned} \Delta_r &= r^2 - 2mr + a^2, \quad \Xi = r^2(x^2 - 3) - a^2(1+x^2), \\ \rho^2 &= \Delta_r(1-x^2), \quad R = r^2 + a^2x^2, \\ \Sigma &= (r^2 + a^2)^2 + a^2\Delta_r(1-x^2), \quad \bar{\omega}_0 = \omega_0 + -\frac{4c^2m^2(3a^4 + 16a^2m^2 - 16m^4)}{a^3}. \end{aligned}$$

To pick the leading order in the limiting procedure, we rescale some parameters and coordinates and choose the rotating constant gauge ω_0 as follows:

$$m \rightarrow \frac{m}{J}, \quad a \rightarrow -\frac{a}{J}, \quad t \rightarrow \frac{t}{J}, \quad \varphi \rightarrow \varphi J^2, \quad r \rightarrow \frac{r}{J}, \quad J \rightarrow J^3, \quad \omega_0 \rightarrow \frac{64m^4(m^2 - a^2)}{a^3} J^3.$$

Then, taking the limit for large swirling parameter J , we get

$$ds^2 = \tilde{f}(d\varphi - \tilde{\omega} dt)^2 - \frac{\Delta_r(1-x^2)}{\tilde{f}} dt^2 + \frac{\epsilon^{2\tilde{r}}}{\tilde{f}} \left(\frac{dr^2}{\Delta_r} + \frac{dx^2}{1-x^2} \right), \quad (\text{A2})$$

where

$$\begin{aligned} \tilde{f} &= \frac{R^2(1-x^2)\Sigma}{4m^2a^2x^2[a^2(1-x^2)^2 + R^2(3-x^2)]^2(1-x^2)^2\Sigma} \\ \tilde{\omega} &= -2ma \frac{(2a^2m - 3a^2r + r^3)\Delta_r x^4 - 6r(a^2 + r^2)\Delta_r x^2 + (2a^2m + a^2r + r^3)(a^2 - 6mr - 3r^2)}{\Delta_r a^2 x^2 + r(2a^2m + a^2r + r^3)}, \end{aligned} \quad (\text{A3})$$

which is precisely the solution obtained with the inversion transformation of the Kerr seed [49], basically even in the same set of coordinates.

Actually, using the gauge freedom typical of axisymmetric and stationary vacuum spacetimes in general relativity, $\Sigma \rightarrow \Sigma/J^2$, we can alternatively get the same result more easily, only by setting

$$a \rightarrow -a, \quad t \rightarrow \frac{t}{J}, \quad \varphi \rightarrow \varphi J, \quad \omega_0 \rightarrow \frac{64m^4(m^2 - a^2)}{a^3} J^2 \quad (\text{A4})$$

before performing the limit.

So, we have shown that the Kerr-Levi-Civita solution [49] is a subcase of the Kerr swirling [3] for a specific value of the swirling parameter J . All the information of the Kerr-Levi-Civita metric is yet encoded, by construction, into the swirling Kerr solution. Thus, the local equivalence is proven, and the limiting procedure can recover also the global properties of the Kerr-Levi-Civita metric by considering a proper range for the azimuthal coordinate before taking the large J limit.

In case one prefers to take the limit to the Kerr-Levi-Civita spacetime while preserving the no conical singularity condition at any point of the limit, one has to start from a metric with a regular axis, for instance, to consider the regularized swirling Kerr solution (2.13) of [4]. In that case, the constrained relation between the accelerating parameter and the swirling parameter, Eqs. (3.3) and (3.4) of [4], allowed us to remove the conical singularity and automatically fix a proper range for the ϕ coordinate that removes any axial conicity. The limit for zero acceleration has two distinct branches. The first forces j to be small and gives the Kerr metric, while the second forces j to be large, giving the regularized Kerr-Levi-Civita spacetime, from both a local and a global point of view. The coordinate transformation,

$$\begin{aligned} t &\rightarrow 4aAm^2t, & \varphi &\rightarrow -\Delta_\varphi\varphi, & \gamma_0 &\rightarrow \log(4aAm^2), \\ \omega_0 &\rightarrow \frac{1 + A^2[3a^4A^2 + 2m^2 - 9A^2m^4 + a^2(3 - 14A^2m^2)]}{16a^3A^6m^4}, \end{aligned} \quad (\text{A5})$$

is well defined, also in the vanishing acceleration limit, as can be seen from the Jacobian determinant of the transformation.²¹ Note that the map (A4) is well defined as well. This procedure does not depend on the regularizing scheme used to remove the conical singularity, and it works also with other regularizations different from the acceleration.

When the same vanishing accelerating limit is applied to the accelerating swirling Kerr-Newman solution (see Ref. [4], Sec. 6.1), the Kerr-Newman-Levi-Civita can be retrieved. That solution can be obtained alternatively from the Kerr-Newman in Bonnor-Melvin universe, as explained in the next section, and it is explicitly given, for the interested reader, inside a *Mathematica* notebook in the data files of the paper [39].

On the other hand, considering the large j limit of the accelerating swirling Kerr-Newman [4], for finite accelerating parameter A (that is, not considering the regularizing constraint), gives the accelerating Kerr-Newman-Levi-Civita.

2. Schwarzschild-Levi-Civita from Schwarzschild swirling

Similarly, the metric found in [40] stems from the Schwarzschild in the swirling background [3]. Clearly, it would be sufficient to switch off the angular momentum parameter a in the above section; however, for maximum clarity, let us derive it directly from the swirling Schwarzschild metric. To get the Schwarzschild-Levi-Civita metric

from the swirling Schwarzschild, we have just to take the limit for $c = \infty$ of the swirling metric. To take the limit properly, the following rescaling of some coordinates or parameters has to be considered:

$$m \rightarrow \frac{m}{j}, \quad t \rightarrow \frac{t}{j}, \quad r \rightarrow \frac{r}{j}, \quad \varphi \rightarrow \varphi j^2, \quad j \rightarrow j^3. \quad (\text{A6})$$

As in the rotating case, a quicker path for the same result comes from taking advantage of the gauge freedom of the γ functions, as in (3.1). In this case, one needs to rescale only $t \rightarrow t/j$ and $\varphi \rightarrow \varphi j$.²²

3. Schwarzschild swirling in an external gravitational background

Another example where the large swirling parameter limit gives a solution obtained through the inversion transformation comes from the Schwarzschild-swirling black hole embedded in an external gravitational field, recently found in Eqs. (4.12), (4.14), and (4.15) of [8]. This metric specializes into the Schwarzschild-Levi-Civita in the external gravitational background for $c = \infty$. To evaluate the metric in $c = \infty$, we preliminarily set the gauge freedom by

$$t \rightarrow \frac{t}{c}, \quad \varphi \rightarrow c\varphi, \quad \gamma_0 \rightarrow -\log c.$$

The resulting spacetime coincides with the nonaccelerating case studied in Sec. III.B of [50].

Moreover, note that the Ehlers transformation (for big values of its Lie parameter) can remove the Bertotti-Robinson electromagnetic field surrounding the black hole for $c = \infty$. For instance, if we consider the large swirling regime of the Schwarzschild-swirling-Bertotti-Robinson of Appendix B of [8], we get again the Schwarzschild-Levi-Civita into the external gravitational background, so the electromagnetic field vanishes. However, we note that at this scope, the Harrison transformation has to be preferred, because it is more precise and gives more general and physical results. In fact, the metric obtained from the Harrison transformation contains not only the Levi-Civita background but also the Minkowski ground state, moreover, it is continuously connected to the seed.

4. Inversion, Ehlers, and Harrison transformations

It may be worth emphasizing that the above direct verifications are completely unnecessary because conceptually, they stem just from the properties of the Lie-point

²¹The value of the constant Δ_φ is given in Eq. (3.4) of [4]. The map (A5) represents maybe the cleaner transformation; however, other parametrizations can be found, including one with the gauge constant $\gamma_0 = 0$, analogously to the nonaccelerating case.

²²Of course, these are not the only paths to get the Schwarzschild-Levi-Civita spacetime; for instance, another way, not involving the rescaling of the azimuthal angle φ , is given by setting $m \rightarrow jm, t \rightarrow t/c, \gamma_0 \rightarrow \log(1/c^2), j \rightarrow 1/j, r \rightarrow rj$ before taking the large j limit. Similar limits are possible for more general metrics.

transformations. In fact, as mentioned, the inversion transformation is not an independent symmetry of the Ernst equations, but it is contained in other transformations, such as the Ehlers transformation; see Ref. [31] or footnote 5 of [25]. To put things into context, we can say that the axisymmetric and stationary space of solutions of Einstein-Maxwell theory enjoys an eight parameter

continuous group of transformations. These transformations allow us to move in the space of axisymmetric and stationary solutions of the theory, thus generating new solutions starting from old ones (called seeds). These $SU(2, 1)$ symmetries of the system can be arranged into the following five Lie-point transformations:

$$\begin{aligned}
 \text{(I)} \quad & \mathcal{E} \longrightarrow \mathcal{E}' = \lambda \lambda^* \mathcal{E}, & \Phi \longrightarrow \Phi' = \lambda \Phi, \\
 \text{(II)} \quad & \mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E} + ib, & \Phi \longrightarrow \Phi' = \Phi, \\
 \text{(III)} \quad & \mathcal{E} \longrightarrow \mathcal{E}' = \frac{\mathcal{E}}{1 + ij\mathcal{E}}, & \Phi \longrightarrow \Phi' = \frac{\Phi}{1 + ij\mathcal{E}}, \\
 \text{(IV)} \quad & \mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E} - 2\beta^* \Phi - \beta\beta^*, & \Phi \longrightarrow \Phi' = \Phi + \beta, \\
 \text{(V)} \quad & \mathcal{E} \longrightarrow \mathcal{E}' = \frac{\mathcal{E}}{1 - 2\alpha^* \Phi - \alpha\alpha^* \mathcal{E}}, & \Phi \longrightarrow \Phi' = \frac{\Phi + \alpha\mathcal{E}}{1 - 2\alpha^* \Phi - \alpha\alpha^* \mathcal{E}}
 \end{aligned} \tag{A7}$$

where the parameters $b, j \in \mathbb{R}$ and $\alpha, \lambda, \beta \in \mathbb{C}$. The most interesting transformations are the (I) and the (V), called the Ehlers transformation and the Harrison transformation. The (I) can act as a duality transformation between the electric and the magnetic field, while the others can be reabsorbed by gauge and coordinate transformations; for more information, see Refs. [25,31] or [51].

Composing the elements of this group of transformations, it is possible to build other transformations. Of course, we cannot expect that the composite transformation can extend the group and generate novel solutions with respect to the given set (A7). A well-known discrete transformation that can be derived from the set (A7) is the inversion transformation²³:

$$(inv) \quad \bar{\mathcal{E}} = \frac{1}{\mathcal{E}}, \quad \bar{\Phi} = \frac{\Phi}{\mathcal{E}}.$$

It can be obtained in several ways, for instance, by combining (I) $\circ$ (III) $\circ$ (II) for $j = 1/b$ and $\lambda = i/b$ and then taking the limit for $b \rightarrow \infty$. This point is even clearer if we consider the enhanced Ehlers transformation, as defined in [25],

$$\mathcal{E} \mapsto \bar{\mathcal{E}} = \frac{\mathcal{E} + ij}{1 + ij\mathcal{E}}, \quad \Phi \mapsto \bar{\Phi} = \frac{\Phi(1 + ij)}{1 + ij\mathcal{E}}, \tag{A8}$$

which is a variation of the Ehlers transformation build²⁴ to preserve some asymptotic properties of the asymptotically flat seeds. The enhanced Ehlers transformation is built from the standard Ehlers transformation combined with some

gauge transformations, and therefore, its effect on the seed metric is physically equivalent. As noted in [52], the large j limit of the enhanced Ehlers transformation (A8) is the inversion transformation (A8). That is the reason why we are taking the large j limit to get the Kerr-Levi-Civita metric from the Kerr swirling solution.²⁵

So, by construction, the Ernst fields of the transformed seed according to the inversion transformation are a special case of the ones obtained with the Ehlers transformation (both the standard and the enhanced) up to gauge transformations. Since the Ernst complex potentials determine univocally the metric and electromagnetic potential solution up to gauge transformations, also at the metric level, solutions obtained from the inversion transformation are a subclass of the ones obtained from the Ehlers transformation acting on the same seed.

This is true for any seed, not just for the explicit examples we treated above and for non-black-hole spacetimes. So the inversion transformation cannot really provide new solutions with respect to the Ehlers transformation. Moreover, generating a solution with a Levi-Civita background from already known swirling ones is faster than using the symmetries of the Ernst equations. But it is not just a matter of generality. The Ehlers, as other nontrivial Lie-point transformations, contrary to the inversion transformation, which is discrete, brings into the initial solution a new integrating constant, continuously deforming the seed. This means that the Lie-point

²³In the vacuum subcase, the inversion transformation is sometimes called a Buchdahl transformation.

²⁴To obtain the enhanced Ehlers transformation, just compose (II) $\circ$ (I) $\circ$ (III), with $b = j$ and $\lambda = 1 + ij$.

²⁵Note that if, contrary to the standard conventions, we define $j = 1/c$, the enhanced Ehlers transformation becomes the identity for $c = \infty$, and it would suggest to us to take the $c = 0$ to get Kerr-Levi-Civita from Kerr swirling. So, in general, depending on how we parametrize the Ehlers transformation, we would have a different limiting value for the swirling parameter to obtain the Levi-Civita background.

transformations can generate solutions as close as necessary to the seed to fit the actual phenomenology, still carrying new physical features.

a. Schwarzschild-Levi-Civita from Schwarzschild-Bonnor-Melvin

The fact that in Sec. III B we obtained the Schwarzschild-Levi-Civita black hole as a limit of the curling Schwarzschild metric makes us question if the superposition of black holes (or any metric) into the Levi-Civita background can stem from other spacetimes apart from the black hole solutions in the curling or in the swirling background. Maybe it is a common feature with a larger class that shares a similar Levi-Civita asymptotic. We know also that black holes embedded into the Bonnor-Melvin universe have a similar falloff for large radial distances, so it could be interesting to verify if this family of solutions can also contain Schwarzschild-Levi-Civita or Kerr-Levi-Civita.

At this scope, consider the Schwarzschild solution embedded into the external magnetic field of Bonnor-Melvin, as written in Eqs. (3.5) and (3.6), and rescale some coordinates and parameters to execute the limit properly:

$$t \rightarrow t \frac{4}{b^2}, \quad \varphi \rightarrow \varphi \frac{b^2}{4}, \quad \gamma_0 \rightarrow \log \frac{4}{b^2}, \quad A_{\varphi_0} \rightarrow \frac{2}{b}.$$

Then, taking the limit for $b \rightarrow \infty$, we get again exactly the Schwarzschild-Levi-Civita solution (3.15).

b. Kerr-Newman-Levi-Civita from Kerr-Newman-Bonnor-Melvin

Similarly to the above cases, we can obtain the rotating and charged black hole with Levi-Civita background as a special case of the Kerr-Newman-Bonnor-Melvin solution build in [6]. Thus, we consider it in the notation²⁶ of [53] and take the leading order of large b regime for the solution; this automatically selects the needed rescaling of the Killing coordinates and the gauge constants γ_0 , A_{t_0} , and A_{φ_0} as

$$t \rightarrow t \frac{4}{b^2}, \quad \varphi \rightarrow \varphi \frac{4}{b^2}, \quad \gamma_0 \rightarrow \log \frac{4}{b^2},$$

$$A_{t_0} \rightarrow \frac{3}{2} am^2 b^3, \quad A_{\varphi_0} \rightarrow \left[e^4 \left(\frac{1}{8} - \frac{1}{128 a^2 m^2} - \frac{1}{8} + 2a^2 m^2 \right) \right] b^3,$$

while the other arbitrary constants, such as the rotational speed at infinity or the dilatation of the azimuthal angle, are chosen just to make contact with the above uncharged metric,

²⁶Note that the parameter denoting the intensity of the external Bonnor-Melvin electromagnetic field, B in [53], is relabeled here as b to homogenize the notation and avoid confusion with the Bertotti-Robinson field denoted by B in this article.

$$\Delta_\varphi = \frac{1}{16a^2 m^2}, \quad \omega_0 = \frac{3am^2 b^4}{4},$$

even though, to avoid conical singularities, Δ_φ should take a unitary value.

The resulting metric is precisely the generalization of the Kerr-Levi-Civita (A2) and (A3) in the presence of monopolar electric charge, so the Kerr-Newman-Levi-Civita. We leave this solution in a *Mathematica* notebook²⁷ for the interested reader. From this procedure, it can be appreciated that spacetimes in the Levi-Civita background can be obtained in a simple way, without using the symmetry transformations of the Ernst equations or passing through the complex Ernst potential, in the case that these solutions are already available, since they have generated through the Ehlers, the Harrison, or other generating techniques (as the solution of Sec. III).

c. The Schwarzschild-Levi-Civita into the external gravitational background from Schwarzschild Bertotti-Robinson-Bonnor-Melvin

The Schwarzschild-Levi-Civita metric embedded in the external gravitational background described in [50] is also contained in the Schwarzschild Bertotti-Robinson-Bonnor-Melvin, i.e., the solution in Eqs. (2.18)–(2.20) of [8], for $b = \infty$. To verify explicitly, before evaluating the Schwarzschild Bertotti-Robinson-Bonnor-Melvin metric in $b = \infty$, set the gauge freedom as follows:

$$t \rightarrow t \frac{4}{b^2}, \quad \varphi \rightarrow \varphi \frac{b^2}{4}, \quad \gamma_0 \rightarrow \log \frac{4}{b^2}. \quad (\text{A9})$$

Note that automatically, the electromagnetic field vanishes. Exactly the same procedure works well in the presence of acceleration. Therefore, the accelerating Schwarzschild-Levi-Civita into the external gravitational background is a special case of the accelerating Schwarzschild into the Bertotti-Robinson-Bonnor-Melvin electromagnetic field. There is no need to generate it twice, as they are independent, as done in [50].

This kind of observation can be useful to deduce properties of the black hole. For instance, since the thermodynamics of the Schwarzschild Bertotti-Robinson-Bonnor-Melvin black hole does not depend on the Bonnor-Melvin parameter b , we expect that the thermodynamics of the Schwarzschild-Levi-Civita-Bertotti-Robinson can stem from that of Schwarzschild Bertotti-Robinson-Bonnor-Melvin. This is an advantage, not only from a phenomenological point of view, but also from a theoretical perspective because, in the framework of the phase-space formalism, the correct integrating factor that determines the conserved charges, the thermodynamics quantities, and the first law can be uniquely determined only when the black

²⁷The *Mathematica* file can be found at [39].

hole is continuously connected with known subcases, as explained in [54]. This can avoid ambiguities and mistakes.

Now that the nature of these solutions obtained with the inversion transformation has been elucidated, it is clear how to write fundamentally any generalization. For instance, since the accelerating Kerr-Newman-Levi-Civita is a subcase of the swirling accelerating Kerr-Newman solution of [4], if we want to write the metric, we have just to rescale $t \rightarrow t/J$, $\varphi \rightarrow \varphi/J$ and set $\gamma_0 \rightarrow -\log J$ before evaluating the swirling solution for large values of the swirling parameter J . The full solution is provided in the *Mathematica* notebook [39] for the interested reader.

d. Inversion from the Harrison transformation

It is interesting to understand why it is possible to recover solutions in the Levi-Civita background also as a special case of metrics in the Bonnor-Melvin background. From a solution generating perspective, the motivation lies in the fact that the inversion transformation is a particular subcase, not only of the Ehlers, but also of the Harrison transformation.

Indeed, by composing the real Harrison transformation with two gauge transformations and taking the limit for $\beta \rightarrow \infty$, we get the inversion transformation:

$$(I) \circ (V) \circ (IV) \Big|_{\substack{\lambda=1/\beta \\ \alpha=1/\beta}} \longrightarrow [\beta \rightarrow \infty] \quad (inv). \quad (A10)$$

In case we want to get the inversion transformation without applying the limiting procedure, it is possible to compose the above combination (A10) with another

real (IV) transformation:

$$(I) \circ (IV) \circ (V) \circ (IV) \Big|_{\substack{\lambda=1/\beta \\ \alpha=1/\beta}} = (inv). \quad (A11)$$

However, especially at the metric level, the limit representation seems more insightful and practical to reproduce the inversion transformation result from a given solution that already possesses a swirling or Bonnor-Melvin background.

Of course, this shortcut to reproduce the action of the inversion transformation from the Ehlers or the Harrison transformation is not limited to the magnetic version of the transformation seen above, but it is possible to use it to reproduce the action of the electric inversion transformation from metrics that possess electromagnetic monopolar charges or the NUT parameter: Reissner-Nordstrom or Taub-NUT. For instance, the Schwarzschild-NUT metric in the large NUT parameter regime (large ℓ) gives²⁸ the electrically²⁹ inverted Schwarzschild metric (which is, again, the Schwarzschild black hole).

²⁸To verify it explicitly, just choose $\gamma_0 = 1/\ell$, and rescale the coordinates and the parameters as follows: $\varphi \rightarrow \varphi/\ell$, $r \rightarrow \ell r$, $m \rightarrow \ell m$, $\ell \rightarrow \sqrt{\ell}$ before taking the $\ell \rightarrow \infty$ limit. Incidentally, in this case, the same limit to the Schwarzschild spacetime can be obtained also for $\ell \rightarrow 0$ from the Schwarzschild-NUT, but this is a peculiarity of the seed considered in this simple example related to the uniqueness of asymptotically flat black holes.

²⁹Regarding the electric and magnetic symmetries transformations, see Refs. [3,51].

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