

Fluctuations and correlations of conserved charges in the Polyakov chiral SU(3) quark mean field model

Dhananjay Singh^{*} and Arvind Kumar[†]

Department of Physics, Dr. B. R. Ambedkar National Institute of Technology, Jalandhar 144008, India



(Received 10 June 2026; accepted 11 September 2026; published 1 October 2026)

We compute generalized susceptibilities of conserved charges in the Polyakov chiral SU(3) quark mean field (PCQMF) model with the fermion vacuum term. At $\mu_B = 0$ MeV, the calculation covers the diagonal $\chi_n^{B,Q,S}$ through eighth order and all twelve independent fourth-order off-diagonal correlators. Extending to finite μ_B at $\mu_Q = \mu_S = 0$, we compute χ_n^B through eighth order, $\chi_n^{Q,S}$ through fourth order, the second-order off-diagonals, all twelve fourth-order off-diagonal correlators, and the odd-order baryon susceptibilities χ_1^B , χ_3^B , χ_5^B . The calculation includes the vacuum term (vac = 1) and is repeated for an independently refitted no-sea variant (vac = 0). At $\mu_B = 0$ MeV, the chiral pseudocritical temperature is $T_{pc} = 170.5$ MeV (vac = 1) and 166.4 MeV (vac = 0), while the Polyakov-loop deconfinement temperature is $T_{dec} = 144.4$ MeV (vac = 1) and 146.6 MeV (vac = 0). In vac = 1, the derivative $-d\Delta_{l,s}/dT$ of the subtracted chiral condensate develops an inflection near T_{dec} . Higher derivative orders resolve the chiral-deconfinement splitting as twin maxima in χ_4^B and χ_6^Q , twin minima in χ_8^B and χ_8^Q , and multiple zero crossings in χ_6^B . Among the fourth-order off-diagonal correlators, vac = 1 amplitudes exceed vac = 0 in the baryon-charge channel across the chiral crossover. The baryon-strangeness, charge-strangeness, and mixed baryon-charge-strangeness components peak near the strange-melting temperature, where vac = 0 dominates. Along $T_{pc}(\mu_B)$, the kurtosis ratio $R_{42}^B \equiv \chi_4^B/\chi_2^B$ of vac = 1 crosses zero at $\mu_B/T_{pc} \approx 2.15$ for $\mu_Q = \mu_S = 0$ and at ≈ 2.47 under the heavy-ion constraints of strangeness neutrality and $n_Q/n_B = 0.4$, while vac = 0 stays positive in both cases over $\mu_B \leq 500$ MeV. The higher-order ratios $R_{51}^B \equiv \chi_5^B/\chi_1^B$ and $R_{62}^B \equiv \chi_6^B/\chi_2^B$ start negative in vac = 1 and grow more negative as μ_B increases.

DOI: [10.1103/6knj-rwl8](https://doi.org/10.1103/6knj-rwl8)

I. INTRODUCTION

Quantum chromodynamics (QCD) at finite temperature T and baryon chemical potential μ_B describes matter under conditions realized in the early universe and in relativistic heavy-ion collisions. At high temperature, asymptotic freedom implies deconfinement into a quark-gluon plasma, and chiral symmetry is expected to be approximately restored. Lattice simulations at $\mu_B = 0$ MeV have established that the two transitions (chiral and deconfinement) proceed as a single smooth crossover [1–3]. The deconfinement crossover lies a few MeV below the chiral one [4]. The pseudocritical temperature at $\mu_B = 0$ is $T_{pc} = 158.0 \pm 0.6$ MeV [5]. At finite μ_B , effective models and universality arguments suggest a first-order phase boundary that

terminates at a critical end point (CEP) [6–8]. The fermion sign problem prevents direct lattice simulations at finite μ_B . The lattice equation of state (EoS) now extends to $\mu_B/T \approx 3.5$ [9], and recent Padé and Lee-Yang analyses find a CEP unlikely below that reach [10,11]. On the experimental side, the RHIC Beam Energy Scan program has measured net-proton cumulants at $\sqrt{s_{NN}} = 7.7\text{--}200$ GeV, where a nonmonotonic $\kappa\sigma^2$ was first seen in BES-I [12] and confirmed in BES-II [13], which also reported the first fifth- and sixth-order results [14]. Fixed-target measurements extend the reach to $\sqrt{s_{NN}} = 3$ GeV [15]. Upcoming programs at FAIR (CBM) and NICA will extend the reach to higher μ_B (lower $\sqrt{s_{NN}}$) [16,17]. Recent reviews are given in [18,19]. These data call for theoretical predictions for the full set of generalized baryon-charge-strangeness (BQS) susceptibilities, both diagonal and off-diagonal. Cumulants of net-proton, net-kaon, and net-charge distributions measured in heavy-ion collisions serve as experimental proxies for specific components of this set [20,21].

The generalized susceptibilities χ_{ijk}^{BQS} are derivatives of the scaled pressure with respect to $\hat{\mu}_{B,Q,S} = \mu_{B,Q,S}/T$. They can be computed in both lattice QCD and effective models,

^{*}Contact author: snaks16aug@gmail.com

[†]Contact author: kumara@nitj.ac.in

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

and correspond directly to the cumulants of event-by-event net-particle distributions [20,22–24]. At second order, the diagonal $\chi_2^{B,Q,S}$ have been obtained on the lattice at $\mu_B = 0$ MeV [25–28], at finite μ_B via Taylor expansion [29,30], and across the full (T, μ_B, μ_Q, μ_S) plane [9]. In the Polyakov Nambu Jona Lasinio (PNJL) model [31,32], diagonal susceptibilities were computed with multiquark interactions [33,34], where eight-quark terms improved agreement with lattice data below T_{pc} . Explicit chiral symmetry breaking interactions in the same eight-quark framework give the second-order diagonal susceptibilities and the baryon-charge (BQ), baryon-strangeness (BS) and charge-strangeness (QS) correlators $\chi_{11}^{BQ,BS,QS}$ at $\mu_B = 0$, with and without the Polyakov loop and for two Polyakov-loop potentials [35]. Later PNJL studies covered net-baryon higher moments [36] and net-strangeness fluctuations [37] at BES energies. Off-diagonal correlators and the C_{BS} ratio at finite μ_B have also been computed [38,39]. In the Polyakov quark meson (PQM) model [40,41], diagonal susceptibilities and their ratios were computed with and without the fermion vacuum term [42,43]. In the hadron resonance gas (HRG) model, repulsive mean-field interactions are essential for reproducing higher-order susceptibilities near T_{pc} [44].

The second-order off-diagonal correlators $\chi_{11}^{BQ,BS,QS}$ have received much less attention. They probe correlated fluctuations between conserved charges and are sensitive to the quantum numbers of the dominant carriers. For instance, the baryon-strangeness correlation coefficient C_{BS} distinguishes whether strangeness is carried by hadrons or by quarks [22]. It has been computed in the PNJL model and on the lattice along the pseudocritical line [45,46], and baryon-charge correlations have been examined in nuclear matter [47]. A persistent deficiency of these mean-field chiral models is the underestimation of χ_2^Q below T_{pc} [35,43], since pions in mean-field PCQMF are frozen at their classical value and contribute no fluctuations.

Higher-order susceptibilities are directly sensitive to critical dynamics. The kurtosis χ_4^B/χ_2^B develops nonmonotonic structure near the chiral crossover [48,49], and χ_6^B changes sign near T_{pc} , a feature linked to $O(4)$ universality [23,50]. Continuum-extrapolated lattice results for χ_6^B and χ_8^B now exist [51], and χ_6 through χ_{10} have been measured on finite-volume lattices [52]. Lattice skewness and kurtosis ratios at finite μ_B have been compared with STAR BES-I data [30]. More recently, higher-order cumulants have been evaluated along the pseudocritical line [46]. In effective models, the scope of existing calculations varies considerably. In the PQM model, all twelve fourth-order off-diagonal correlators and diagonal susceptibilities through sixth order were computed with the vacuum term at $\mu_B = 0$ MeV [43]. A separate calculation extended baryon-number ratios to twelfth order at finite μ_B along a single quark chemical potential [42]. In the NJL model,

diagonal BQS susceptibilities were computed through fourth order [53] and baryon-number susceptibilities through eighth order [54]. Beyond mean field, functional renormalization group (fRG) calculations resum mesonic and fermionic quantum fluctuations and have produced baryon fluctuations through tenth order [55–57]. The same approach has been applied to strangeness neutrality and baryon-strangeness correlations [58,59]. Holographic QCD has produced baryon susceptibilities through twelfth order [60]. No existing effective-model study provides all twelve fourth-order off-diagonal BQS correlators together with diagonal susceptibilities through eighth order at $\mu_B = 0$.

The vacuum (Dirac-sea) term, the zero-temperature one-loop quark contribution to the thermodynamic potential, significantly affects the crossover. In the two-flavor PQM model, the vacuum term restores second-order $O(4)$ universality in the chiral limit [61]. In the $(2+1)$ -flavor extension, it smooths the crossover and produces oscillatory sign structure in $\chi_6^{B,Q,S}$ [42,43], and improves agreement with lattice thermodynamics [62]. It also shifts the CEP to larger μ_B and lower T [42,63], though the magnitude of the shift depends on the renormalization scheme [64]. In the Polyakov chiral SU(3) quark mean field (PCQMF) model, the vacuum term has only been used for transport coefficients [65] and anisotropic quark matter [66], while all existing thermodynamic and susceptibility studies [67–69] omitted it. Its effect on the full set of generalized BQS susceptibilities has not been studied.

The PCQMF model includes the nonstrange and strange scalar fields σ and ζ as order parameters for chiral symmetry breaking, and the Polyakov loop Φ as an order parameter for deconfinement. Vector mesons ω , ρ , ϕ provide short-range repulsion, and a dilaton field χ implements the QCD trace anomaly. An isovector scalar δ is included for isospin-asymmetric matter. The only existing susceptibility calculation in this model [68] omitted the vacuum term and computed only diagonal susceptibilities, through sixth order for charge and strangeness and eighth order for baryon. No off-diagonal correlators were reported. We report the first computation of the full set of generalized BQS susceptibilities in the PCQMF model with the fermion vacuum term. At $\mu_B = 0$ MeV, the calculation covers all diagonal $\chi_n^{B,Q,S}$ for $n = 2, 4, 6, 8$ and all twelve independent fourth-order off-diagonal correlators, namely the nine two-charge correlators $\chi_{ij}^{BQ,BS,QS}$ with $i + j = 4$ and the three mixed correlators χ_{ijk}^{BQS} with $i + j + k = 4$. The extension to finite μ_B up to 500 MeV at $\mu_Q = \mu_S = 0$ covers $\chi_2^{B,Q,S}, \chi_4^{B,Q,S}, \chi_6^B, \chi_8^B$, the second-order off-diagonals $\chi_{11}^{BQ,BS,QS}$, and the odd-order baryon susceptibilities $\chi_1^B, \chi_3^B, \chi_5^B$. The calculation is repeated with an independently refitted no-sea ($\text{vac} = 0$) parameter set for comparison. We compare with continuum-extrapolated lattice data [9,27,28,51] where benchmarks exist.

The article is organized as follows. Section II introduces the chiral SU(3) Lagrangian, the PCQMF grand potential

with the vacuum term, the coupled gap equations, and the susceptibility definitions. Section III presents results at $\mu_B = 0$ MeV, including order parameters, bulk thermodynamics, diagonal susceptibilities through eighth order, all twelve fourth-order off-diagonal correlators, and correlation ratios. Section IV extends the analysis to finite μ_B , covering diagonal and off-diagonal susceptibilities at second and fourth order, diagonal baryon susceptibilities through eighth order, and cumulant ratios along $T_{pc}(\mu_B)$. Section V discusses the vacuum-term effects and mean-field limitations. We conclude in Sec. VI with a summary and outlook. Stefan-Boltzmann limits for the quantities computed in this work are collected in Appendix.

II. MODEL AND FORMALISM

A. Chiral SU(3) Lagrangian

We employ the Polyakov chiral SU(3) quark mean field (PCQMF) model. The chiral SU(3) Lagrangian [70,71] uses a nonlinear realization of chiral symmetry. Baryons couple to scalar mesons σ , ζ , δ and vector mesons ω , ρ , ϕ . The dilaton χ accounts for broken scale invariance. A later reformulation at the quark level [72,73] replaced the baryons with constituent quarks, and the Polyakov-loop variables (Φ , $\bar{\Phi}$) [31,32] were added in Ref. [67]. The fermion vacuum (Dirac-sea) term, first included in the PCQMF model in Ref. [65], is retained throughout the present work.

The Lagrangian comprises scalar and vector meson self-interactions, explicit chiral symmetry breaking, and a strange-quark mass term:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\Sigma\Sigma} + \mathcal{L}_{VV} + \mathcal{L}_{SB} + \mathcal{L}_{\Delta m}. \quad (1)$$

The scalar sector reads

$$\begin{aligned} \mathcal{L}_{\Sigma\Sigma} = & -\frac{1}{2}k_0\chi^2(\sigma^2 + \zeta^2 + \delta^2) + k_1(\sigma^2 + \zeta^2 + \delta^2)^2 \\ & + k_2\left(\frac{\sigma^4}{2} + \frac{\delta^4}{2} + 3\sigma^2\delta^2 + \zeta^4\right) \\ & + k_3\chi(\sigma^2 - \delta^2)\zeta - k_4\chi^4 - \frac{1}{4}\chi^4 \ln \frac{\chi^4}{\chi_0^4} \\ & + \frac{d}{3}\chi^4 \ln \left[\frac{(\sigma^2 - \delta^2)\zeta}{\sigma_0^2\zeta_0} \cdot \frac{\chi^3}{\chi_0^3} \right], \end{aligned} \quad (2)$$

where σ_0 and ζ_0 are the vacuum expectation values of σ and ζ and given by $\sigma_0 = -f_\pi$ and $\zeta_0 = (f_\pi - 2f_K)/\sqrt{2}$. The dilaton vacuum value χ_0 is fitted to vacuum observables and listed in Table I. The pion and kaon decay constants are $f_\pi = 93$ MeV and $f_K = 115$ MeV. The logarithmic and k_4 terms together implement broken scale invariance and generate the trace anomaly [71,72].

TABLE I. PCQMF model parameters for the variant with the fermion vacuum term (vac = 1) and without it (vac = 0). The explicit symmetry-breaking parameters h_x , h_y are common to both variants, as they are fixed by the π and K mass constraints.

Parameter	Symbol	vac = 1	vac = 0
Scalar potential couplings	k_0	2.002×10^{-1}	3.881
	k_1	2.388	1.855
	k_2	-19.50	-10.19
	k_3	-4.733	-4.443
	k_4	-0.060	-0.060
	d	0.002	0.182
Dilaton scale (MeV)	χ_0	327.8	278.2
Explicit breaking (MeV ³)	h_x	1.797×10^6	1.797×10^6
	h_y	3.874×10^7	3.874×10^7
UV cutoff (MeV)	Λ	600	—
Vector sector	m_v	673.6 MeV	673.6 MeV
	μ_v	2.34 fm ²	2.34 fm ²
	g_4	37.5	37.5
	g_v	0	0
	Δm_s	29	29
Polyakov potential	a_0	1.81	1.81
	a_1	-2.47	-2.47
	a_2	15.2	15.2
	b_3	-1.75	-1.75
	T_0 (MeV)	200	200

The vector meson self-interaction reads

$$\begin{aligned} \mathcal{L}_{VV} = & \frac{1}{2}\frac{\chi^2}{\chi_0^2}(m_\omega^2\omega^2 + m_\rho^2\rho^2 + m_\phi^2\phi^2) \\ & + g_4(\omega^4 + 6\omega^2\rho^2 + \rho^4 + 2\phi^4), \end{aligned} \quad (3)$$

with $m_\omega^2 = m_\rho^2 = m_v^2/(1 - \frac{1}{2}\mu_v\sigma^2)$ and $m_\phi^2 = m_v^2/(1 - \mu_v\zeta^2)$, where m_v and μ_v are chosen to reproduce $m_\omega = 783$ MeV and $m_\phi = 1020$ MeV [67]. Explicit chiral symmetry breaking enters through

$$\mathcal{L}_{SB} = -\frac{\chi^2}{\chi_0^2}(h_x\sigma + h_y\zeta), \quad (4)$$

where $h_x = m_\pi^2 f_\pi$ and $h_y = \sqrt{2}m_K^2 f_K - m_\pi^2 f_\pi/\sqrt{2}$ are fixed by the physical pion and kaon masses. A strange-quark mass term

$$\mathcal{L}_{\Delta m} = -\Delta m_s \bar{\psi} S \psi, \quad (5)$$

with $S = \text{diag}(0, 0, 1)$ and $\Delta m_s = 29$ MeV, supplies a bare mass to the strange quark.

The quark-meson couplings follow from the SU(3) flavor structure,

$$g_{\sigma u} = g_{\sigma d} = g_{\delta u} = -g_{\delta d} = \frac{g_s}{\sqrt{2}}, \quad g_{\zeta s} = g_s, \quad (6)$$

with all other scalar couplings equal to zero and $g_s = 4.76$ as the single independent coupling. The vector coupling is set to $g_v = 0$.

Numerical values of the parameters for both variants are collected in Table I. The fit imposes physical π and K masses, the combined η - η' mass, the scalar σ -meson mass, and the vacuum stationarity conditions. The procedure for this model was established in Refs. [67,72]. The PCAC relations fix h_x and h_y to common values in both variants. The resulting vacuum constituent quark masses are $m_u^* = m_d^* = 313$ MeV and $m_s^* = 490$ MeV.

B. Thermodynamic potential

In the mean-field approximation, the thermodynamic potential density reads

$$\Omega = \Omega_{\text{mes}} + \Omega_{\bar{q}q}^{\text{vac}} + \Omega_{\bar{q}q}^{\text{th}} + \mathcal{U}(\Phi, \bar{\Phi}; T), \quad (7)$$

with Ω_{mes} the mesonic potential, $\Omega_{\bar{q}q}^{\text{vac}}$ the vacuum contribution, $\Omega_{\bar{q}q}^{\text{th}}$ the thermal quark-antiquark contribution, and $\mathcal{U}(\Phi, \bar{\Phi}; T)$ the Polyakov-loop effective potential. The constituent quark masses entering $\Omega_{\bar{q}q}^{\text{vac}}$ and $\Omega_{\bar{q}q}^{\text{th}}$ are

$$m_i^* = -g_{\sigma i}\sigma - g_{\zeta i}\zeta - g_{\delta i}\delta + m_{i0}, \quad (8)$$

with $m_{u0} = m_{d0} = 0$ and $m_{s0} = \Delta m_s$. The quark chemical potentials are

$$\mu_i = B_i\mu_B + Q_i\mu_Q + S_i\mu_S, \quad (9)$$

with $B_i = 1/3$ for all flavors, $Q_u = 2/3$, $Q_d = Q_s = -1/3$, $S_s = -1$, and $S_u = S_d = 0$, giving

$$\begin{aligned} \mu_u &= \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q, \\ \mu_d &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q, \\ \mu_s &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S. \end{aligned} \quad (10)$$

Coupling to the vector mean fields shifts the chemical potentials to

$$\mu_i^* = \mu_i - g_\omega^i \omega - g_\rho^i \rho - g_\phi^i \phi, \quad (11)$$

with the vector couplings $g_\omega^u = g_\omega^d = g_v/\sqrt{2}$, $g_\rho^u = -g_\rho^d = g_v/\sqrt{2}$, and $g_\phi^s = g_v$. All other vector couplings are set to zero.

The mesonic potential follows from Eq. (1) as

$$\Omega_{\text{mes}} = -\mathcal{L}_{\text{eff}} - \mathcal{V}_{\text{vac}}, \quad (12)$$

where $\mathcal{V}_{\text{vac}}$ is the additive constant enforcing $\Omega = 0$ in vacuum.

The thermal quark-antiquark contribution in Eq. (7), dressed by the Polyakov-loop variables, reads

$$\Omega_{\bar{q}q}^{\text{th}} = - \sum_{i=u,d,s} \frac{\gamma_i T}{(2\pi)^3} \int_0^\infty d^3k \{ \ln \mathcal{N}_i^- + \ln \mathcal{N}_i^+ \}, \quad (13)$$

with spin degeneracy $\gamma_i = 2$ and

$$\mathcal{N}_i^- = 1 + e^{-3E_-} + 3\Phi e^{-E_-} + 3\bar{\Phi} e^{-2E_-}, \quad (14)$$

$$\mathcal{N}_i^+ = 1 + e^{-3E_+} + 3\bar{\Phi} e^{-E_+} + 3\Phi e^{-2E_+}, \quad (15)$$

where $E_\pm = (E_i^*(k) \pm \mu_i^*)/T$ and $E_i^*(k) = \sqrt{k^2 + m_i^{*2}}$ is the effective single-particle energy.

The Polyakov-loop effective potential in Eq. (7) is taken in the logarithmic form [74]

$$\begin{aligned} \frac{\mathcal{U}(\Phi, \bar{\Phi}; T)}{T^4} &= -\frac{a(T)}{2} \Phi \bar{\Phi} + b(T) \ln[1 - 6\Phi \bar{\Phi} \\ &\quad + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi \bar{\Phi})^2], \end{aligned} \quad (16)$$

with $a(T) = a_0 + a_1(T_0/T) + a_2(T_0/T)^2$ and $b(T) = b_3(T_0/T)^3$. The logarithm argument is proportional to the SU(3) Haar measure, restricting Φ and $\bar{\Phi}$ to $|\Phi|, |\bar{\Phi}| \leq 1$. The corresponding parameters are collected in Table I [67]. To account for the backreaction of dynamical quarks on the gluonic sector, the temperature argument of a and b is replaced by an effective Yang-Mills temperature T_{YM} . The glue temperature of the full theory T_{glue} is identified with the matter temperature T , and T_{YM} follows from the mapping [75]

$$\frac{T_{\text{YM}} - T_0^{\text{YM}}}{T_0^{\text{YM}}} = 0.57 \left(\frac{T_{\text{glue}} - T_0^{\text{glue}}}{T_0^{\text{glue}}} \right), \quad (17)$$

with $T_0^{\text{YM}} = T_0^{\text{glue}} = T_0 = 200$ MeV.

One of the parameter variations of Sec. III C replaces Eq. (16) by the polynomial form [32]

$$\begin{aligned} \frac{\mathcal{U}_{\text{poly}}(\Phi, \bar{\Phi}; T)}{T^4} &= -\frac{b_2(T)}{2} \Phi \bar{\Phi} - \frac{b_3}{6} (\Phi^3 + \bar{\Phi}^3) \\ &\quad + \frac{b_4}{4} (\Phi \bar{\Phi})^2. \end{aligned} \quad (18)$$

Here $b_2(T) = a_0 + a_1(T_0/T) + a_2(T_0/T)^2 + a_3(T_0/T)^3$, with $a_0 = 6.75$, $a_1 = -1.95$, $a_2 = 2.625$, $a_3 = -7.44$, $b_3 = 0.75$ and $b_4 = 7.5$ [32]. The T_{YM} substitution of Eq. (17) applies to b_2 as well, at the same T_0 .

C. Fermion vacuum contribution

The zero-temperature one-loop quark contribution is

$$\Omega_{\bar{q}q}^{\text{vac}} = -2N_c \sum_{i=u,d,s} \int_0^\infty \frac{d^3k}{(2\pi)^3} E_i^*(k), \quad (19)$$

which is ultraviolet divergent. Dimensional regularization of Eq. (19) yields [42,61]

$$\Omega_{\bar{q}q}^{\text{vac}} = -\frac{N_c}{8\pi^2} \sum_{i=u,d,s} m_i^{*4} \ln \frac{m_i^*}{\Lambda}, \quad (20)$$

where $\Lambda = 600$ MeV is the regularization scale. A change in Λ is absorbed by refitting the scalar-sector parameters to the same vacuum observables.

Because m_i^* depends on the scalar fields through Eq. (8), $\Omega_{\bar{q}q}^{\text{vac}}$ modifies the curvature of the effective potential in field space and contributes to the scalar gap equations derived in Sec. II D. For comparison, we consider a no-sea variant (vac = 0) in which $\Omega_{\bar{q}q}^{\text{vac}}$ is dropped, and the mesonic parameters are refitted to the same vacuum observables listed in Sec. II A. The two parameter sets are listed in Table I.

D. Gap equations

The equilibrium mean fields at given T , μ_B , μ_Q , μ_S are obtained by minimizing Ω with respect to all dynamical variables,

$$\frac{\partial \Omega}{\partial \varphi_a} = 0, \quad \varphi_a \in \{\sigma, \zeta, \delta, \chi, \omega, \rho, \phi, \Phi, \bar{\Phi}\}. \quad (21)$$

The gap equations read

$$\begin{aligned} \frac{\partial \Omega}{\partial \sigma} &= k_0 \chi^2 \sigma - 4k_1(\sigma^2 + \zeta^2 + \delta^2) \sigma - 2k_2(\sigma^3 + 3\sigma \delta^2) \\ &\quad - 2k_3 \chi \sigma \zeta - \frac{d}{3} \chi^4 \frac{2\sigma}{\sigma^2 - \delta^2} + \frac{\chi^2}{\chi_0^2} h_x \\ &\quad - \frac{\chi^2}{\chi_0^2} m_\omega \omega^2 \frac{\partial m_\omega}{\partial \sigma} - \frac{\chi^2}{\chi_0^2} m_\rho \rho^2 \frac{\partial m_\rho}{\partial \sigma} \\ &\quad - \sum_{i=u,d} g_\sigma^i \rho_i^s = 0, \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{\partial \Omega}{\partial \zeta} &= k_0 \chi^2 \zeta - 4k_1(\sigma^2 + \zeta^2 + \delta^2) \zeta - 4k_2 \zeta^3 \\ &\quad - k_3 \chi(\sigma^2 - \delta^2) - \frac{d}{3} \frac{\chi^4}{\zeta} + \frac{\chi^2}{\chi_0^2} h_y \\ &\quad - \frac{\chi^2}{\chi_0^2} m_\phi \phi^2 \frac{\partial m_\phi}{\partial \zeta} - g_\zeta^s \rho_s^s = 0, \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{\partial \Omega}{\partial \delta} &= k_0 \chi^2 \delta - 4k_1(\sigma^2 + \zeta^2 + \delta^2) \delta - 2k_2(\delta^3 + 3\sigma^2 \delta) \\ &\quad + 2k_3 \chi \delta \zeta + \frac{2d}{3} \chi^4 \frac{\delta}{\sigma^2 - \delta^2} - \sum_{i=u,d} g_\delta^i \rho_i^s = 0, \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{\partial \Omega}{\partial \chi} &= k_0 \chi(\sigma^2 + \zeta^2 + \delta^2) - k_3(\sigma^2 - \delta^2) \zeta \\ &\quad + \chi^3 \left[1 + \ln \frac{\chi^4}{\chi_0^4} \right] + (4k_4 - d) \chi^3 \\ &\quad - \frac{4d}{3} \chi^3 \ln \left[\frac{(\sigma^2 - \delta^2) \zeta}{\sigma_0^2 \zeta_0} \cdot \frac{\chi^3}{\chi_0^3} \right] + \frac{2\chi}{\chi_0^2} (h_x \sigma + h_y \zeta) \\ &\quad - \frac{\chi}{\chi_0^2} (m_\omega^2 \omega^2 + m_\rho^2 \rho^2 + m_\phi^2 \phi^2) = 0, \end{aligned} \quad (25)$$

$$\frac{\partial \Omega}{\partial \omega} = \frac{\chi^2}{\chi_0^2} m_\omega^2 \omega + 4g_4 \omega^3 + 12g_4 \omega \rho^2 - \sum_{i=u,d} g_\omega^i \rho_i = 0, \quad (26)$$

$$\frac{\partial \Omega}{\partial \rho} = \frac{\chi^2}{\chi_0^2} m_\rho^2 \rho + 4g_4 \rho^3 + 12g_4 \omega^2 \rho - \sum_{i=u,d} g_\rho^i \rho_i = 0, \quad (27)$$

$$\frac{\partial \Omega}{\partial \phi} = \frac{\chi^2}{\chi_0^2} m_\phi^2 \phi + 8g_4 \phi^3 - g_\phi^s \rho_s = 0, \quad (28)$$

$$\begin{aligned} \frac{\partial \Omega}{\partial \Phi} &= -T^4 \left[\frac{a(T)}{2} \bar{\Phi} + \frac{6b(T)(\bar{\Phi} - 2\Phi^2 + \bar{\Phi}^2 \Phi)}{D} \right] \\ &\quad - \sum_{i=u,d,s} \frac{2TN_c}{(2\pi)^3} \int_0^\infty d^3k \left[\frac{e^{-E_-}}{\mathcal{N}_i^-} + \frac{e^{-2E_+}}{\mathcal{N}_i^+} \right] = 0, \end{aligned} \quad (29)$$

$$\begin{aligned} \frac{\partial \Omega}{\partial \bar{\Phi}} &= -T^4 \left[\frac{a(T)}{2} \Phi + \frac{6b(T)(\Phi - 2\bar{\Phi}^2 + \Phi^2 \bar{\Phi})}{D} \right] \\ &\quad - \sum_{i=u,d,s} \frac{2TN_c}{(2\pi)^3} \int_0^\infty d^3k \left[\frac{e^{-2E_-}}{\mathcal{N}_i^-} + \frac{e^{-E_+}}{\mathcal{N}_i^+} \right] = 0, \end{aligned} \quad (30)$$

with

$$D \equiv 1 - 6\Phi \bar{\Phi} + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi \bar{\Phi})^2 \quad (31)$$

the Haar-measure argument appearing in Eq. (16). The scalar density ρ_i^s of flavor i is

$$\begin{aligned} \rho_i^s &= -\frac{N_c}{2\pi^2} m_i^{*3} \left(\ln \frac{m_i^*}{\Lambda} + \frac{1}{4} \right) \\ &\quad + \gamma_i N_c \int_0^\infty \frac{d^3k}{(2\pi)^3} \frac{m_i^*}{E_i^*(k)} (f_i + \bar{f}_i), \end{aligned} \quad (32)$$

and the net quark number density of flavor i

$$\rho_i = \gamma_i N_c \int_0^\infty \frac{d^3k}{(2\pi)^3} (f_i - \bar{f}_i), \quad (33)$$

with Polyakov-loop-modified distributions

$$f_i = \frac{\Phi e^{-E_-} + 2\bar{\Phi} e^{-2E_-} + e^{-3E_-}}{\mathcal{N}_i^-}, \quad (34)$$

$$\bar{f}_i = \frac{\bar{\Phi} e^{-E_+} + 2\Phi e^{-2E_+} + e^{-3E_+}}{\mathcal{N}_i^+}. \quad (35)$$

The first term in Eq. (32) is the vacuum contribution and vanishes in the $\text{vac} = 0$ variant. At $g_v = 0$, the source terms in Eqs. (26)–(28) vanish and the remaining homogeneous equations admit only the trivial solution $\omega = \rho = \phi = 0$. The effective chemical potentials in Eq. (11) then reduce to the bare quark chemical potentials, $\mu_i^* = \mu_i$. The effect of nonzero g_v on the finite- μ_B results is examined in Sec. IV F.

E. Conserved-charge susceptibilities

The generalized susceptibilities of conserved charges are defined as derivatives of the scaled pressure with respect to the reduced chemical potentials $\hat{\mu}_X \equiv \mu_X/T$ [25,26],

$$\chi_{ijk}^{\text{BQS}} = \frac{\partial^{i+j+k}(p/T^4)}{\partial \hat{\mu}_B^i \partial \hat{\mu}_Q^j \partial \hat{\mu}_S^k}, \quad (36)$$

where $p = -\Omega$. Diagonal susceptibilities are denoted $\chi_n^B \equiv \chi_{n00}^{\text{BQS}}$, $\chi_n^Q \equiv \chi_{0n0}^{\text{BQS}}$, and $\chi_n^S \equiv \chi_{00n}^{\text{BQS}}$. Off-diagonal correlators between two charge sectors are denoted χ_{ij}^{XY} , with $X, Y \in \{B, Q, S\}$ and $i + j$ equal to the total order. Mixed three-sector correlators retain the full χ_{ijk}^{BQS} notation. The conserved-charge densities follow from the flavor densities of Eq. (33) as $n_B = (\rho_u + \rho_d + \rho_s)/3$, $n_Q = (2\rho_u - \rho_d - \rho_s)/3$ and $n_S = -\rho_s$. Each satisfies $n_X = T^3 \chi_1^X$ for $X = B, Q, S$. At the stationary point of Ω , first derivatives with respect to μ_X reduce to partial derivatives at fixed fields [43]. At second and higher orders, the implicit response of the mean fields to chemical-potential variation contributes and must be tracked through Eqs. (22)–(30). Lattice and experimental results set the highest total order computed in the baryon channel and in the off-diagonal tensor. The diagonal baryon channel is carried to eighth order. Continuum lattice results reach sixth order there, with a single continuum point at eighth [51], and STAR reports net-proton cumulants through sixth order [14]. The diagonal charge and strangeness channels follow the baryon channel to eighth order at $\mu_B = 0$. Their benchmarks stop at fourth order, so their sixth- and eighth-order results are model predictions. The off-diagonal tensor stops at fourth order, the highest order with continuum lattice results for the mixed correlators [9]. At finite μ_B , the baryon channel is carried to eighth order and the diagonal charge and strangeness channels to fourth, because the STAR comparison of Sec. IV E is of the baryon cumulant ratios. The susceptibilities are evaluated directly at each (T, μ_B) rather than from a Taylor expansion in μ_B , so the truncation order

is not limited by a radius of convergence. The derivatives of Eq. (36) are taken by automatic differentiation in JAX [76].

In the high-temperature limit, where all flavors become massless and noninteracting, the second-order susceptibilities approach their Stefan-Boltzmann (SB) values

$$\chi_2^X|_{\text{SB}} = \sum_{i=u,d,s} X_i^2, \quad (37)$$

yielding $\chi_2^B|_{\text{SB}} = 1/3$, $\chi_2^Q|_{\text{SB}} = 2/3$, and $\chi_2^S|_{\text{SB}} = 1$. Stefan-Boltzmann limits for the higher-order and off-diagonal quantities computed in this work are collected in Appendix.

III. RESULTS AT $\mu_B = 0$

Throughout this section, results are reported for the vacuum-included ($\text{vac} = 1$) and the independently refitted no-sea ($\text{vac} = 0$) parameter sets of Table I. Solid curves denote $\text{vac} = 1$ and dashed curves $\text{vac} = 0$ in every figure. The diagonal susceptibilities through eighth order and the complete fourth-order off-diagonal tensor are evaluated with derivatives in μ_B , μ_Q , and μ_S taken at $\mu_B = \mu_Q = \mu_S = 0$.

A. Order parameters and bulk thermodynamics

Figure 1 shows the chiral and deconfinement order parameters as functions of temperature at $\mu_B = 0, 100, 200$, and 300 MeV. The nonstrange scalar field σ/σ_0 in panel (a) undergoes a smooth crossover that shifts to lower temperature with increasing μ_B , and the $\text{vac} = 1$ drop is steeper than the $\text{vac} = 0$ drop at every μ_B slice. The strange scalar field ζ/ζ_0 in panel (b) shifts similarly to lower temperature with increasing μ_B , but the four slices stay much closer together than the σ/σ_0 slices because $\mu_s = \mu_B/3$ remains small compared to m_s^* . The $\text{vac} = 1$ drop is again steeper than $\text{vac} = 0$ at every slice. At fixed μ_B , ζ/ζ_0 falls more slowly with T than σ/σ_0 because m_s^* is heavier than $m_u^* = m_d^*$. The Polyakov loop Φ in panel (c) rises through the crossover region with weak μ_B dependence in both variants. Because the Polyakov-loop potential $\mathcal{U}(\Phi, \bar{\Phi}, T)$ carries no explicit chemical-potential dependence, μ_B enters Φ only indirectly through the scalar fields in the gap equations. Panel (d) shows the subtracted chiral condensate $\Delta_{l,s}$ as a function of temperature. In the chiral limit, the quark condensate vanishes at the chiral transition. At a nonzero quark mass, it does not, and a subtraction is used to remove the explicit mass contribution. The lattice definition reads [77]

$$\Delta_{l,s}(T) = \frac{\langle \bar{\psi}\psi \rangle_{l,T} - (\hat{m}_l/\hat{m}_s) \langle \bar{\psi}\psi \rangle_{s,T}}{\langle \bar{\psi}\psi \rangle_{l,0} - (\hat{m}_l/\hat{m}_s) \langle \bar{\psi}\psi \rangle_{s,0}}. \quad (38)$$

In the PCQMF model, σ and ζ replace the light and strange condensates and $r = h_x/h_y$ replaces the bare-mass ratio [41,65], giving

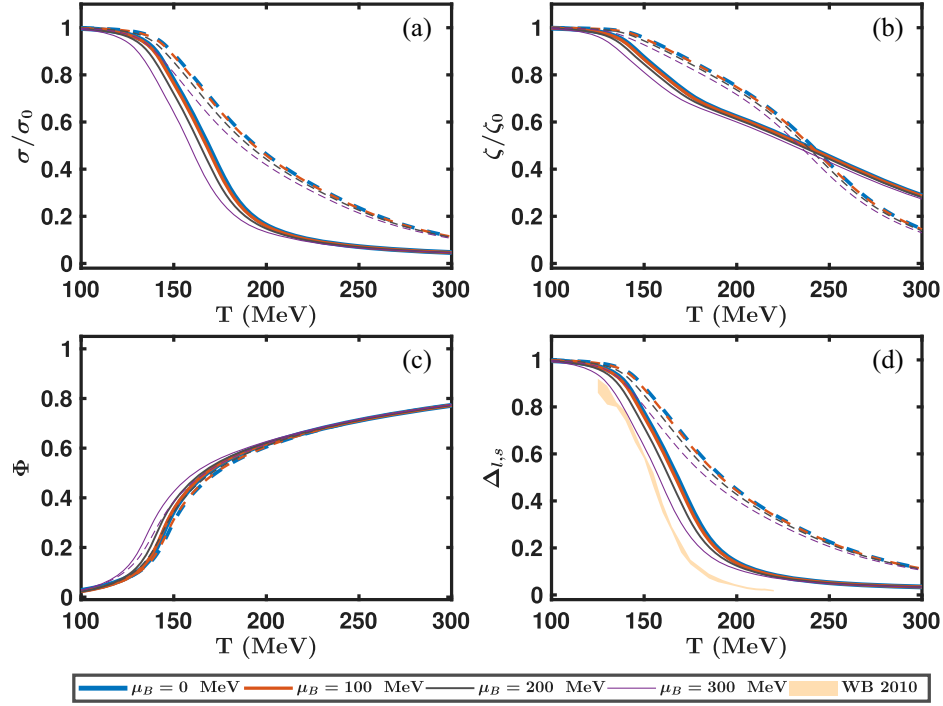


FIG. 1. Order parameters vs temperature at $\mu_B = 0, 100, 200$, and 300 MeV. (a) nonstrange scalar field σ/σ_0 , (b) strange scalar field ζ/ζ_0 , (c) Polyakov loop Φ , (d) subtracted condensate $\Delta_{l,s}$. Each μ_B slice has its own color, and thicker lines mark lower μ_B . Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. The shaded band in panel (d) is the WB 2010 [3] continuum-extrapolated lattice band for $\Delta_{l,s}$.

$$\Delta_{l,s}(T) = \frac{\sigma - r\zeta}{\sigma_0 - r\zeta_0}. \quad (39)$$

$\Delta_{l,s}(0) = 1$ by construction, and $\Delta_{l,s}$ decreases with temperature. Both variants lie above the WB 2010 continuum-extrapolated band [3] across the crossover region. The $\text{vac} = 1$ curve sits closer to the band. The band is a $\mu_B = 0$ result and the present model $T_{pc}(0)$ lies above the lattice value. T_{pc} falls with μ_B , so the higher- μ_B curves lie closer to the band than the $\mu_B = 0$ curve. In $\text{vac} = 1$ the crossover sits at 169.4, 165.6 and 159.5 MeV for $\mu_B = 100, 200$ and 300 MeV, bringing the $\mu_B = 300$ MeV curve within 1.5 MeV of the lattice value.

In Fig. 2, the scaled pressure p/T^4 , energy density ϵ/T^4 , entropy density s/T^3 , and squared speed of sound c_s^2 are compared with continuum-extrapolated lattice data [78,79]. The PCQMF model treats quarks as the active degrees of freedom at all temperatures and underpredicts these quantities at low temperatures, where the lattice EoS is dominated by hadronic contributions. At higher temperatures, the model values of ϵ/T^4 , s/T^3 , and c_s^2 lie within $\sim 5\%$ of the lattice bands, while p/T^4 remains 15 – 25% below. All four quantities approach the Stefan-Boltzmann (SB) limit from below at high temperature. The minimum value of the speed of sound is 0.057 ($\text{vac} = 1$) and 0.062 ($\text{vac} = 0$), well below the lattice value of 0.14 [78].

Figure 3(a) shows the pseudocritical line $T_{pc}(\mu_B)$ for $\mu_B \leq 500$ MeV. The chiral pseudocritical temperature T_{pc}

is the temperature at which $-d\Delta_{l,s}/dT$ peaks [Fig. 3(b)], and the deconfinement pseudocritical temperature T_{dec} is the temperature at which $d\Phi/dT$ peaks [Fig. 3(c)]. At $\mu_B = 0$ MeV, $T_{pc} = 170.5$ MeV ($\text{vac} = 1$) and 166.4 MeV ($\text{vac} = 0$), and $T_{dec} = 144.4$ MeV ($\text{vac} = 1$) and 146.6 MeV ($\text{vac} = 0$). The chiral-deconfinement splitting is 26.1 MeV for $\text{vac} = 1$ and 19.8 MeV for $\text{vac} = 0$. The same glue-to-Yang-Mills mapping of Eq. (17) [75] generates the splitting in both variants. Earlier PCQMF works without this mapping [67,68] placed both crossovers near 165 MeV. The model $T_{pc}(\mu_B)$ lies above the continuum-extrapolated chiral crossover band [5]. The offset comes from the mean-field $T_{pc}(0)$ overestimate. The chemical freeze-out parametrization [80] and the STAR BES-I freeze-out points [81] are overlaid. Each sits below $T_{pc}(\mu_B)$ because chemical freeze-out occurs after the partonic phase has cooled through the crossover.

Figure 3(b) shows the temperature derivative of the subtracted condensate, $-d\Delta_{l,s}/dT$, as a function of T at $\mu_B = 0$ MeV for both variants. The $\text{vac} = 1$ curve rises from below, inflects near T_{dec} , and continues into the main chiral peak at T_{pc} . The $\text{vac} = 0$ curve rises monotonically into its chiral peak with no such inflection. The splitting from the glue mapping is 26.1 MeV in $\text{vac} = 1$ and 19.8 MeV in $\text{vac} = 0$. The deconfinement-driven inflection in $\Delta_{l,s}(T)$ near T_{dec} becomes visible in $-d\Delta_{l,s}/dT$ in $\text{vac} = 1$ through the coupled gap equations, and is absent

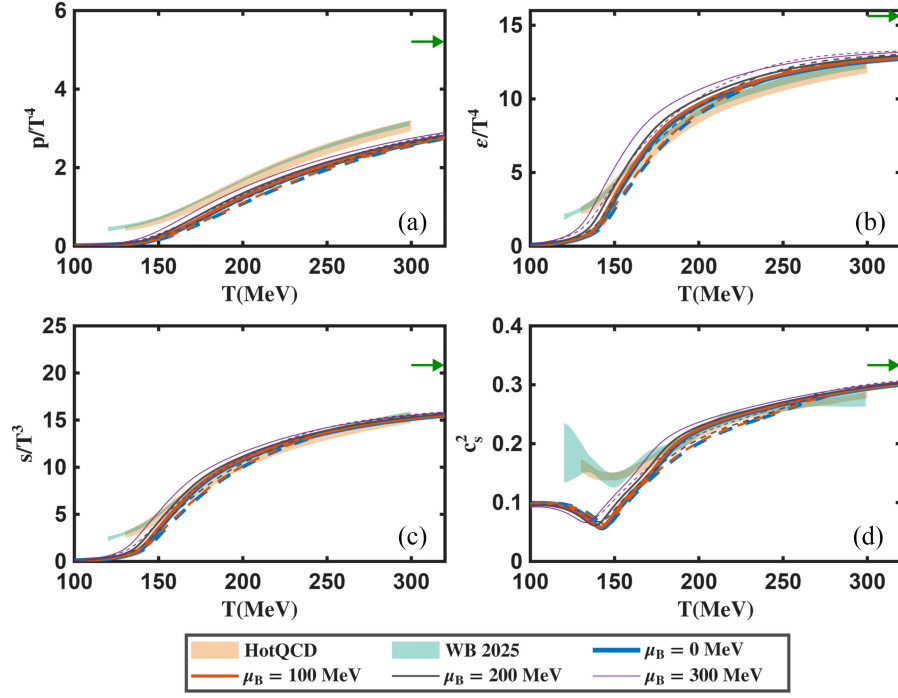


FIG. 2. Bulk thermodynamics at $\mu_B = 0, 100, 200$, and 300 MeV. (a) Pressure p/T^4 , (b) energy density ϵ/T^4 , (c) entropy density s/T^3 , (d) speed of sound squared c_s^2 . Each μ_B slice has its own color, and thicker lines mark lower μ_B . Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Lattice bands are HotQCD 2014 [78] (orange) and Borsanyi *et al.* 2025 [79] (teal). Forest-green right-edge arrows mark the SB limits (Appendix).

in $\text{vac} = 0$. Figure 3(c) shows $d\Phi/dT$ versus T at $\mu_B = 0$ MeV for both variants. The $\text{vac} = 1$ curve peaks at $T_{\text{dec}} = 144.4$ MeV and the $\text{vac} = 0$ curve peaks at $T_{\text{dec}} = 146.6$ MeV. The minimum of c_s^2 in Fig. 2(d) lies at $T = 142$ MeV ($\text{vac} = 1$) and 145 MeV ($\text{vac} = 0$), and it tracks T_{dec} rather than T_{pc} . The lattice places it at 145 – 150 MeV [78], below its own T_{pc} as well. The minimum moves further below that window as μ_B rises, reaching

133 MeV at $\mu_B = 300$ MeV in $\text{vac} = 1$, while T_{pc} falls toward the lattice value over the same range.

The lattice value $T_{\text{pc}} = 158.0 \pm 0.6$ MeV [5] lies below both variants. The $\text{vac} = 0$ result is closer to the lattice, but this is accidental. The vacuum term is required for the correct universality class in the chiral limit [61]. The $(2 + 1)$ -flavor PQM model with the vacuum term [43,62] reproduces lattice thermodynamics. PCQMF and PQM

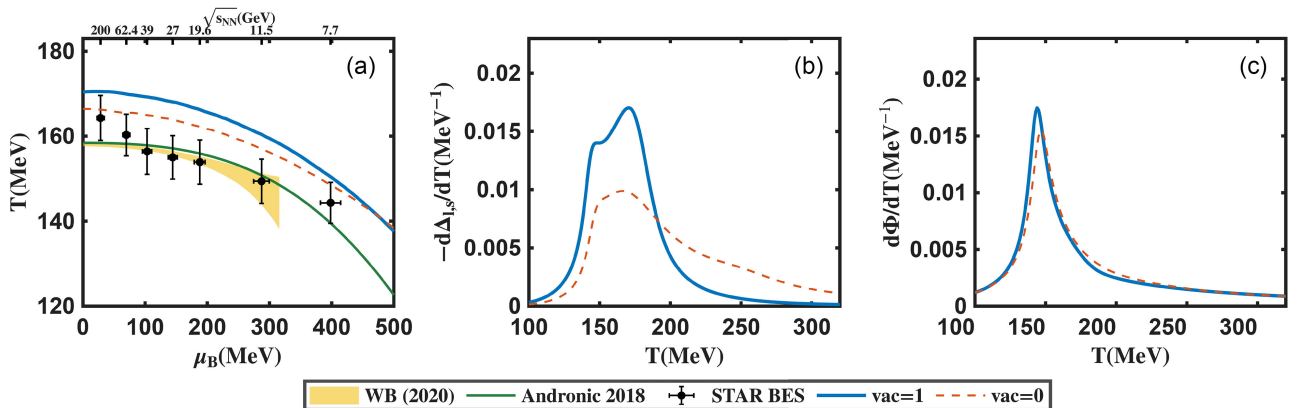


FIG. 3. (a) Pseudocritical temperature $T_{\text{pc}}(\mu_B)$ from PCQMF gap-equation solutions for $\mu_B \leq 500$ MeV. (b) Temperature derivative of the subtracted condensate $-d\Delta_{l,s}/dT$ at $\mu_B = 0$ MeV. (c) Temperature derivative of the Polyakov-loop expectation value $d\Phi/dT$ at $\mu_B = 0$ MeV. Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Panel (a) also shows the WB 2020 [5] continuum-extrapolated chiral crossover band (gold), the Andronic *et al.* 2018 [80] chemical freeze-out parametrization (green), and STAR BES-I freeze-out points [81] (black markers with error bars). Upper axis tick marks give $\sqrt{s_{NN}}$ for the STAR points.

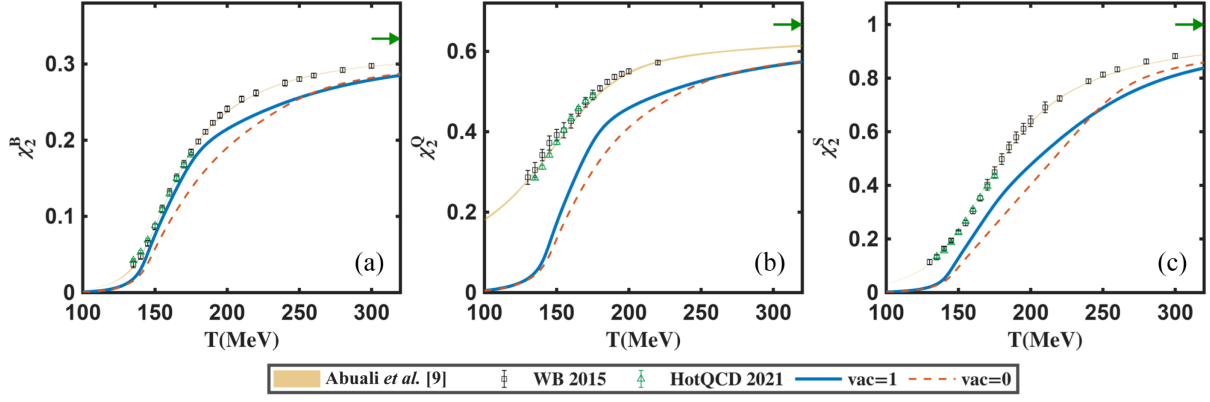


FIG. 4. Diagonal second-order susceptibilities (a) baryon-number χ_2^B , (b) electric-charge χ_2^Q , (c) strangeness χ_2^S vs temperature at $\mu_B = 0$ MeV. Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Lattice data are WB 2015 [27] (black squares), HotQCD 2021 [28] (dark green triangles), and Abuali *et al.* [9] (amber bands). Forest green right-edge arrows mark the SB limits (Appendix).

with the vacuum term both place T_{pc} above the lattice value [63]. The PCQMF T_{pc} difference between variants, approximately 4 MeV, is smaller than the vacuum-term shifts reported for PQM [61,62]. PNJL calculations [34] also place T_{pc} above the lattice value. In all three model classes, the overshoot magnitude varies with the parameter set. Adding mesonic fluctuations beyond mean field via the fRG lowers T_{pc} toward the lattice value at $\mu_B = 0$ MeV [55,56,58,82]. A beyond-mean-field PCQMF treatment of these fluctuations is left for future work.

B. Second-order susceptibilities

Figure 4 shows the diagonal second-order susceptibilities χ_2^B , χ_2^Q , and χ_2^S at $\mu_B = 0$ MeV. Three lattice datasets are overlaid for comparison [9,27,28]. The SB limits $\chi_2^B|_{SB} = 1/3$, $\chi_2^Q|_{SB} = 2/3$, and $\chi_2^S|_{SB} = 1$ are marked by forest-green right-edge arrows. All three model curves lie below the lattice bands across the full temperature range and approach the SB limits from below at high T . The deficit arises from the absence of hadronic contributions at low T and the absence of bosonic fluctuations beyond the mean field at high T . The same mean-field limitations cause the pressure deficit (Fig. 2). The inflection of χ_2^B tracks T_{dec} rather than T_{pc} . In $\text{vac} = 1$, this inflection lies ≈ 2 MeV below the $\text{vac} = 0$ value, and the rise is 25% steeper.

The strangeness susceptibility χ_2^S approaches its SB limit more slowly than χ_2^B and χ_2^Q because the strange constituent mass remains above the light-quark mass through the crossover region. All three model curves underpredict the lattice at low T , but the deficit is most severe for χ_2^Q and smallest for χ_2^B . Pions, the lightest charged hadrons, dominate χ_2^Q in the hadronic phase but are frozen at their mean-field classical value in PCQMF and do not contribute to the fluctuations. The same deficit was identified in PQM with the vacuum term [43]. The χ_2^B and χ_2^S deficits are

smaller because their dominant hadronic carriers are nucleons and kaons, both much heavier than pions. The same hierarchy appears in PNJL [34] and NJL [53] mean-field calculations.

Figure 5 shows second-order off-diagonal correlators at $\mu_B = 0$ MeV. Panel (b) plots $-\chi_{11}^{BS}$ because χ_{11}^{BS} is negative across the full T range. In the hadronic phase, hyperons carry $B = +1$ and $S < 0$. In the partonic phase, the strange quark carries $B = +1/3$ and $S = -1$. Hence, χ_{11}^{BS} remains negative in both regimes. χ_{11}^{BQ} peaks near the crossover and decays toward zero on both sides. At low T , the relevant (B, Q) degrees of freedom are heavy and suppress the correlator. At high T , the up-quark contribution to χ_{11}^{BQ} is canceled by the combined down and strange contributions, so the correlator vanishes in the SB limit [43]. By contrast, $-\chi_{11}^{BS}$ and χ_{11}^{QS} rise monotonically toward the SB limit $+1/3$. The strange quark is the only flavor carrying S , so its liberation through the crossover drives both correlators upward. χ_{11}^{BQ} in panel (a) tracks the lattice data [9,27,28] through the peak. In panels (b) and (c), the lattice data lie above both model variants. Hyperons, which carry both B and S , are absent from the PCQMF spectrum, since the model contains only quark and meson degrees of freedom. Kaons, which carry Q and S , are frozen at their mean-field classical value.

C. Higher-order baryon susceptibilities

Figure 6 shows the higher-order diagonal baryon susceptibilities χ_4^B , χ_6^B , and χ_8^B (top row) and the cumulant ratios $R_{n2}^B \equiv \chi_n^B / \chi_2^B$ for $n = 4, 6, 8$ (bottom row) at $\mu_B = 0$ MeV. Continuum lattice bands [51] appear in panels (a), (b), (d), and (e), the eighth-order interval at $T = 145$ MeV in panels (c) and (f), and additional bands [9] in panels (a) and (d). The HRG baseline $R_{n2}^B = 1$ provides a low-temperature reference for the bottom row.

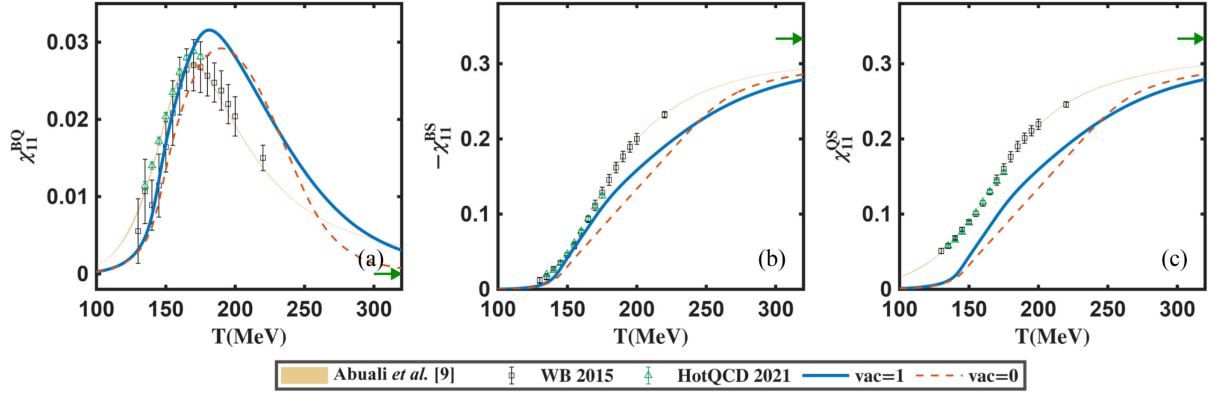


FIG. 5. Second-order off-diagonal correlators (a) baryon-charge χ_{11}^{BQ} , (b) baryon-strangeness $-\chi_{11}^{BS}$, (c) charge-strangeness χ_{11}^{QS} vs temperature at $\mu_B = 0$ MeV. Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Lattice data as in Fig. 4. Forest green right-edge arrows mark the SB limits (Appendix).

The χ_4^B curve develops a double-peak structure in both variants. The maxima lie near T_{dec} and T_{pc} . Table II shows that the splitting $\Delta = T_{\text{pc}} - T_{\text{dec}}$ controls the structure. The second maximum and the two extra χ_6^B sign changes are present in every parameter set with $\Delta \geq 19.8$ MeV and absent in every set with $\Delta \leq 16$ MeV, across two independent refits of the chiral sector, three values of T_0 , and both functional forms of the Polyakov potential. The dominant χ_6^B minimum behaves differently in the two refits. In $\text{vac} = 1$, it lies above T_{pc} in all four sets, between

1.056 and 1.058 of it, while Δ decreases from 31.9 to 8.2 MeV. In $\text{vac} = 0$, it lies below T_{pc} at $\Delta = 27.0$ and 19.8 MeV and above it at $\Delta = 13.8$ and 3.9 MeV. The variations that remove the extra structure carry costs elsewhere. $T_0 = 270$ MeV pushes $T_{\text{pc}}(0)$ to 182.6 MeV and the c_s^2 minimum to 172 MeV, both further from the lattice. The polynomial potential leaves no c_s^2 minimum above $T = 100$ MeV. We keep $T_0 = 200$ MeV. The double peak is the higher-derivative resolution of the chiral-deconfinement splitting. Each crossover contributes its own

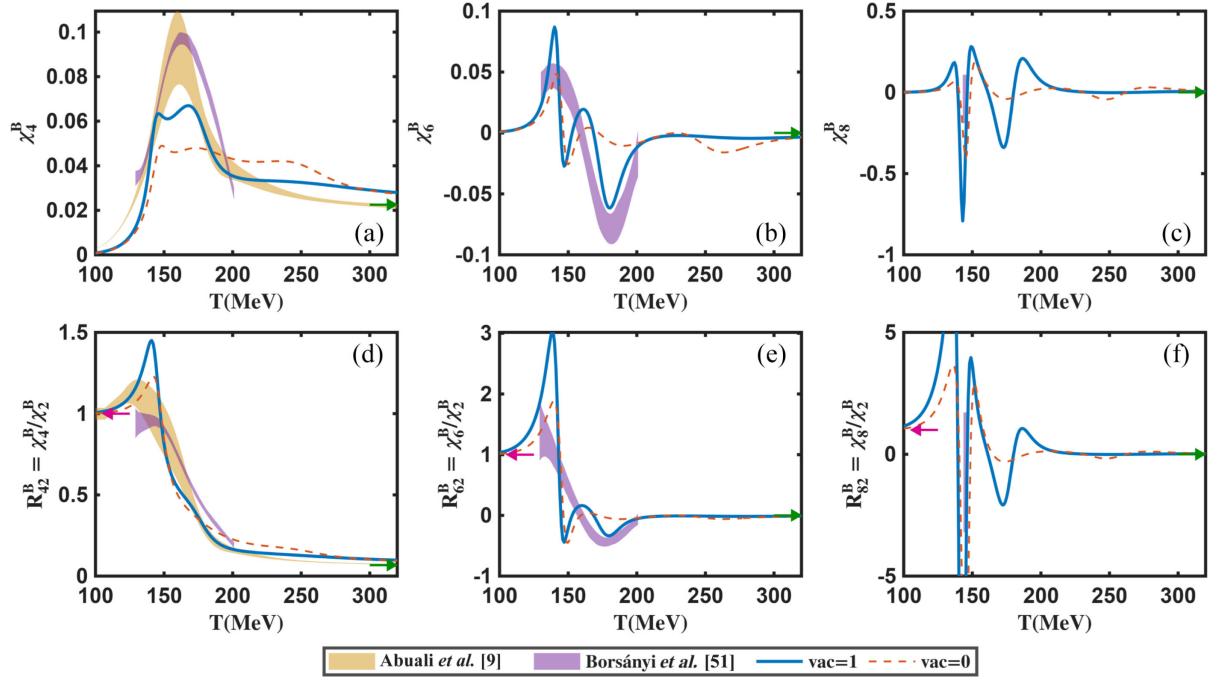


FIG. 6. Higher-order diagonal baryon susceptibilities and their ratios at $\mu_B = 0$ MeV. The top row shows (a) fourth-order χ_4^B , (b) sixth-order χ_6^B , and (c) eighth-order χ_8^B . The bottom row shows the cumulant ratios (d) R_{42}^B , (e) R_{62}^B , and (f) R_{82}^B . Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Lattice bands are Borsanyi *et al.* 2024 [51] (deep purple) and Abuali *et al.* [9] (amber). Forest green right-edge arrows mark the SB limits (Appendix). Magenta left-edge arrows in the bottom row mark the HRG baseline $R_{n2}^B = 1$.

TABLE II. Higher-order baryon structure and the c_s^2 minimum under variations of the Polyakov sector. All temperatures in MeV. $\Delta = T_{\text{pc}} - T_{\text{dec}}$ is the chiral-deconfinement splitting, so $T_{\text{dec}} = T_{\text{pc}} - \Delta$. “Peaks” counts the χ_4^B maxima and “zeros” the χ_6^B sign changes over $T = 125\text{--}200$ MeV. $T_{\text{dip}}/T_{\text{pc}}$ locates the dominant χ_6^B minimum over the same window, and T_{min} the minimum of c_s^2 . T_0 spans the glue critical temperature from its 2 + 1-flavor value to its pure-gauge value [75], with 200 MeV used throughout this work. The polynomial rows replace Eq. (16) by Eq. (18) and give no c_s^2 minimum above $T = 100$ MeV. The vac = 1 and vac = 0 rows are independent refits. The lattice $T_{\text{dip}}/T_{\text{pc}}$ is read from the published band, since no numeric value is quoted. The lattice splitting is a few MeV [4], at a single lattice spacing and with no uncertainty quoted.

Polyakov sector	vac	T_{pc}	Δ	Peaks	Zeros	$T_{\text{dip}}/T_{\text{pc}}$	T_{min}
$T_0 = 182$ MeV	1	167.5	31.9	2	3	1.057	134
$T_0 = 200$ MeV	1	170.5	26.1	2	3	1.057	142
Polynomial	1	165.6	16.0	1	1	1.058	...
$T_0 = 270$ MeV	1	182.6	8.2	1	1	1.056	172
$T_0 = 182$ MeV	0	164.6	27.0	2	3	0.853	136
$T_0 = 200$ MeV	0	166.4	19.8	2	3	0.902	145
Polynomial	0	166.4	13.8	1	1	1.173	...
$T_0 = 270$ MeV	0	182.6	3.9	1	1	1.035	176
Continuum lattice	...	158.0 [5]	...	1 [9]	1 [51]	≈ 1.15 [51]	145–150 [78]

oscillatory feature to the pressure derivatives, and the splitting separates the two contributions. At higher derivative orders, χ_6^B and χ_8^B exhibit multiple zero crossings. Dominant χ_8^B minima also lie near T_{dec} and T_{pc} . By contrast, the continuum lattice resolves a single crossover region at $\mu_B = 0$ MeV and shows only a single peak in χ_4^B [9] and a single zero crossing in χ_6^B [51]. The additional sign changes in PCQMF result from the resolved chiral-deconfinement splitting. The lattice splitting of a few MeV [4] is too small to resolve the two crossovers into separate features. The high-temperature negative dip of χ_6^B is common to PCQMF and lattice. This dip is the universal χ_6^B sign change predicted by $O(4)$ scaling [50]. In Fig. 6(b), the vac = 1 dominant χ_6^B minimum lies close to the lattice one. Its position tracks T_{pc} across the four vac = 1 sets, so it inherits the T_{pc} overestimate. A model reproducing the lattice T_{pc} would place the minimum about 13 MeV lower.

In Fig. 6(d), both PCQMF curves of R_{42}^B start at the HRG baseline at low temperature. They rise to a single peak in the crossover region and decline toward the SB limit $6/(9\pi^2)$ at high temperature. The double peak of χ_4^B is suppressed in the ratio because χ_2^B is a smoothly rising denominator that absorbs the shallow dip between the two peaks. The lattice R_{42}^B shows a single peak [9,51]. The vac = 0 curve sits within the lattice envelope, and the vac = 1 curve overshoots it. In Figs. 6(e) and 6(f), the PCQMF R_{62}^B and R_{82}^B carry the multicrossing structure of χ_6^B and χ_8^B . The lattice R_{62}^B in panel (e) has a single sign change [51]. The PCQMF R_{62}^B peak overshoots the lattice peak. Panel (f) shows the R_{82}^B oscillation across the crossover region. The vac = 1 amplitudes substantially exceed vac = 0. Large amplitudes for higher-order baryon ratios also appear in the three-flavor PQM model with R_{n2}^B extended to twelfth order [42], in the three-flavor PNJL model with quark-number cumulants extended to eighth

order [34], and in the three-flavor NJL model with χ_n^B extended to eighth order [54]. An fRG calculation that resums quantum, thermal, and density fluctuations recovers the lattice R_{42}^B and R_{62}^B quantitatively at $\mu_B = 0$ MeV [55].

D. Charge and strangeness higher-order susceptibilities

Figure 7 presents the diagonal charge and strangeness susceptibilities χ_n^Q and χ_n^S for $n = 4, 6, 8$ at $\mu_B = 0$ MeV. The fourth-order panels carry continuum-extrapolated lattice bands [9]. The sixth- and eighth-order panels are model predictions without continuum lattice benchmarks.

χ_4^Q exceeds χ_4^B in peak amplitude. The peak amplitudes track the SB-limit inequality $\sum_f Q_f^4 > \sum_f B_f^4$. The up quark dominates the Q-channel sum, and this flavor hierarchy survives in the crossover region. In Fig. 7(a), the vac = 1 curve is taller, narrower, and centered at slightly lower temperature than vac = 0. At higher T , the vac = 1 curve matches the lattice peak amplitude [9], while vac = 0 underestimates it. In Fig. 7(d), χ_4^S peaks at higher T than χ_4^Q because the strange chiral condensate melts later than the light condensate. The amplitude ordering of vac = 1 and vac = 0 reverses for χ_4^S . The vac = 0 curve peaks above vac = 1, opposite to Fig. 7(a). The lattice band peaks at a lower temperature and lower amplitude than the PCQMF curves.

In Fig. 7(b), χ_6^Q oscillates across the chiral crossover region. The free massless quark pressure is a polynomial of degree four in μ_X/T , so $\chi_n^X|_{\text{SB}} = 0$ for all $n \geq 6$. The vac = 1 curve shows two positive peaks separated by a shallow dip. A deep negative trough follows at higher T . These two maxima resolve the chiral-deconfinement splitting at sixth-derivative order, the same pattern as in χ_4^B (Sec. III C). The vac = 1 amplitudes substantially exceed vac = 0. In Fig. 7(e), χ_6^S oscillates at higher T , where the strange condensate proportional to ζ is melting. The vac = 1 curve

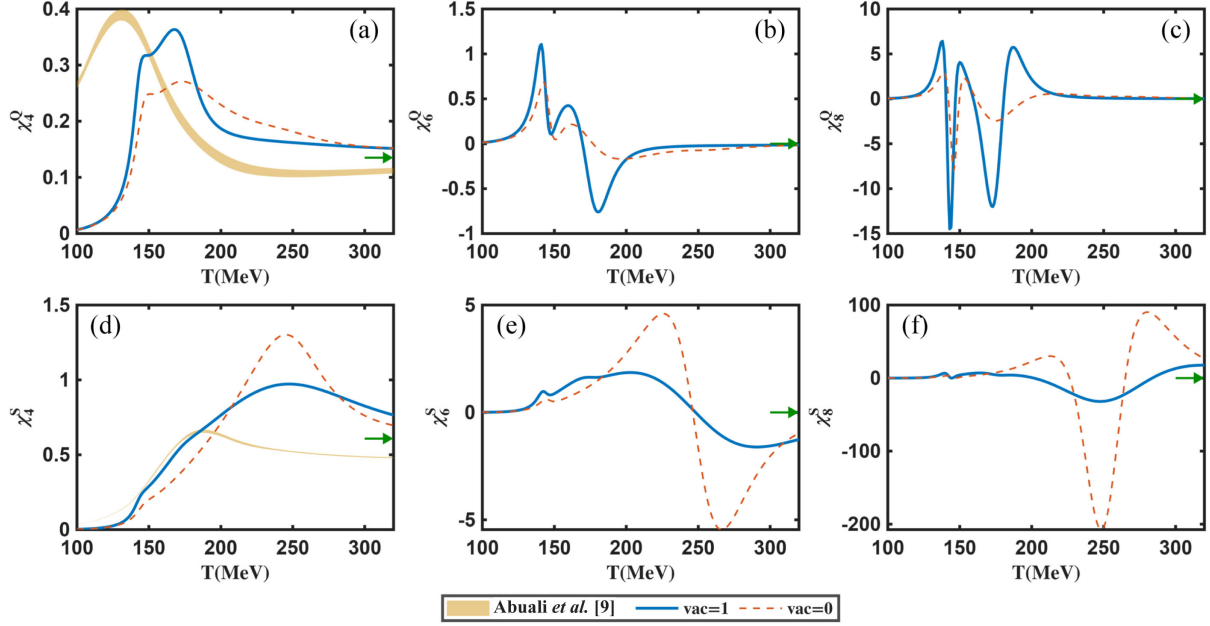


FIG. 7. Higher-order diagonal charge and strangeness susceptibilities at $\mu_B = 0$ MeV. The top row shows (a) fourth-order χ_4^Q , (b) sixth-order χ_6^Q , and (c) eighth-order χ_8^Q . The bottom row shows (d) fourth-order χ_4^S , (e) sixth-order χ_6^S , and (f) eighth-order χ_8^S . Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Lattice bands in panels (a) and (d) are Abuali *et al.* [9] (amber). Forest-green right-edge arrows mark the SB limits (Appendix).

carries the χ_6^Q chiral-deconfinement signature near T_{dec} and T_{pc} , plus a much larger peak and trough in the strange-melting region. The amplitude ordering reverses for χ_6^S . The $\text{vac} = 0$ amplitudes substantially exceed $\text{vac} = 1$ in the strange-melting region. The basic sign-change pattern of $\chi_6^{Q,S}$ was reported earlier in $(2+1)$ -flavor PQM with the vacuum term [43] and in PCQMF without it [68]. Those calculations used parameter sets where the chiral and deconfinement crossovers nearly coincide.

In Fig. 7(c), χ_8^Q shows two deep negative minima near T_{dec} and T_{pc} . The two minima are the eighth-derivative analog of the χ_6^Q double maximum. The $\text{vac} = 1$ amplitudes substantially exceed $\text{vac} = 0$, with the disparity larger than at sixth order. In Fig. 7(f), χ_8^S shows a similar structure to χ_6^S . The $\text{vac} = 1$ curve carries a large negative dip in the strange-melting region. As in χ_4^S and χ_6^S , $\text{vac} = 0$ dominates the strange-melting region. The dominance is more pronounced at this order.

E. Complete fourth-order off-diagonal tensor

Figure 8 presents the twelve independent fourth-order off-diagonal correlators at $\mu_B = 0$ MeV. Rows 1–3 hold the nine two-charge components $\chi_{ij}^{BQ,BS,QS}$ with $i+j=4$. Row 4 holds the three mixed components χ_{ijk}^{BQS} with $i+j+k=4$. Panels (d), (e), (k), (l) carry an overall minus sign to match the convention of [9], so that all twelve plotted quantities have positive SB limits. The same twelve correlators have been computed in $(2+1)$ -flavor PQM

with the vacuum term at $\mu_B = 0$ MeV [43]. The complete fourth-order off-diagonal tensor in PCQMF with the fermion vacuum term has not been reported elsewhere. The $\text{vac} = 1$ peaks exceed $\text{vac} = 0$ in the BQ row, and at high T the $\text{vac} = 0$ peaks exceed $\text{vac} = 1$ in the BS, QS, and BQS rows.

All but one of the twelve plotted quantities remain positive throughout the figure. Panel (a) carries the lone sign change, the SB limit vanishes there, and the curve crosses zero near $T \approx 180$ MeV. The cancellation follows from $\sum_f Q_f = 2/3 - 1/3 - 1/3 = 0$ across the (u, d, s) triplet. The three baryon derivatives contribute only a flavor-independent prefactor $(1/3)^3$. The high- T residual in panel (a) therefore probes quark-mass effects rather than free-quark counting. χ_{22}^{BQ} and χ_{13}^{BQ} carry comparable $\text{vac} = 1$ peaks of 0.081 and 0.074, well above χ_{31}^{BQ} at 0.015. Peaks in panels (b) and (c) track the light chiral crossover at $T \approx 170$ MeV. The peak in panel (a) shifts to $T \approx 145$ MeV, near T_{dec} . The $\text{vac} = 1$ peaks exceed $\text{vac} = 0$ by about 20% across the BQ row. In panels (b) and (c), the lattice central values sit well below the model peaks. Panel (a) shows the closest lattice-model agreement in the BQ row. The band overlaps the $\text{vac} = 1$ peak in temperature and amplitude.

χ_{13}^{BS} is the dominant BS correlator, well above χ_{22}^{BS} and χ_{31}^{BS} in peak amplitude. Dominant peaks of panels (e) and (f) lie at $T \approx 245$ MeV, in the strange-melting region. Panels (d) and (f) carry a double-peak fine structure across $T \approx 150$ – 170 MeV that resolves the chiral-deconfinement splitting. $\text{vac} = 0$ peaks exceed $\text{vac} = 1$ across the BS row.

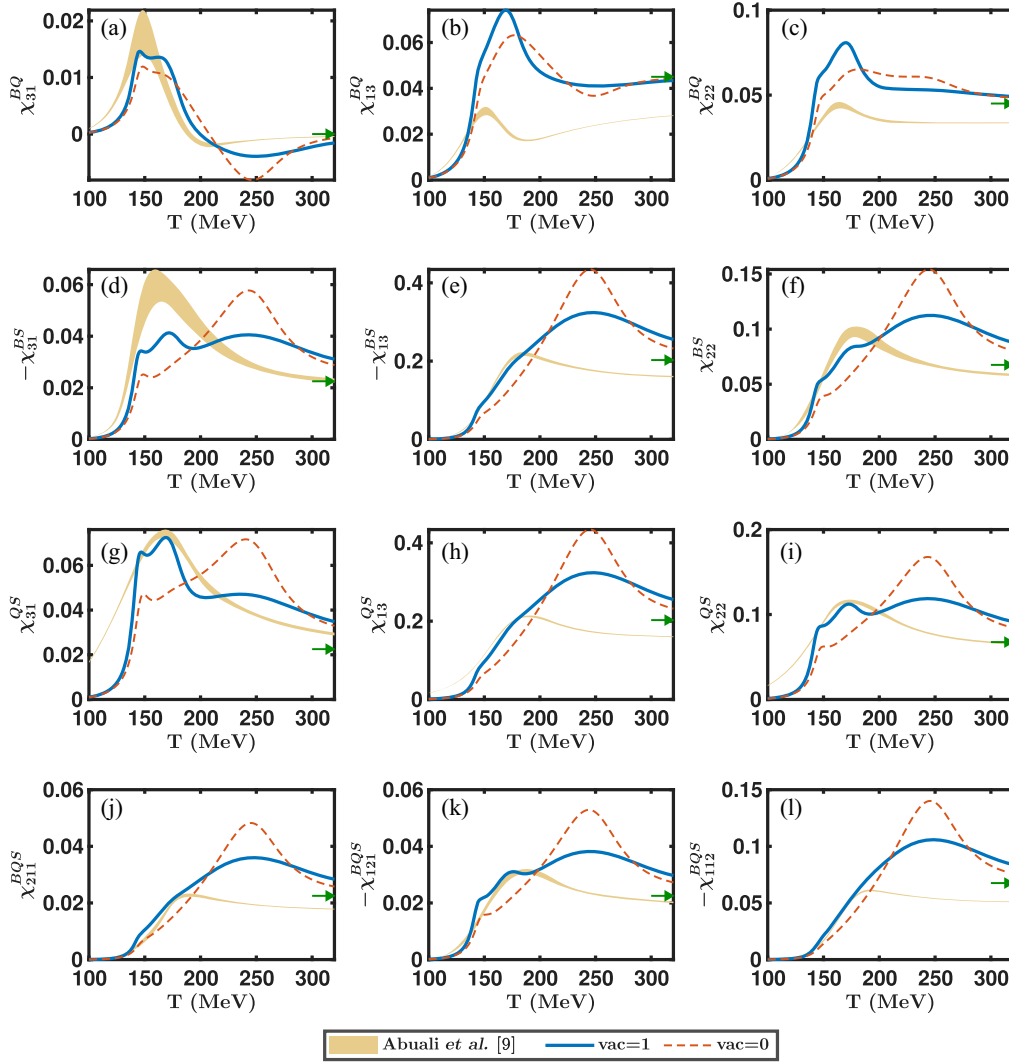


FIG. 8. Complete fourth-order off-diagonal BQS tensor vs temperature at $\mu_B = 0$ MeV. Rows 1–3 show the two-charge correlators $\chi_{ij}^{BQ,BS,QS}$ with $i + j = 4$, columns ordered as $\chi_{31}, \chi_{13}, \chi_{22}$. Row 4 shows the mixed correlators χ_{ijk}^{BQS} with $i + j + k = 4$, columns ordered as $\chi_{211}, -\chi_{121}, -\chi_{112}$. Panels (d), (e), (k), (l) carry an overall minus sign to match the convention of Abuali *et al.* [9]. Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Amber bands are Abuali *et al.* [9]. Forest green right-edge arrows mark the SB limits (Appendix).

The lattice band in panel (d) sits above both model variants in the crossover region. In panel (f), the lattice exceeds both model variants in the crossover region, then falls below them at high T .

The QS row repeats the BS hierarchy. χ_{13}^{QS} dominates, well above χ_{22}^{QS} and χ_{31}^{QS} . Dominant peaks of panels (h) and (i) lie at $T \approx 245$ MeV, in the strange-melting region. Panels (g) and (i) carry a double-peak fine structure across $T \approx 150$ – 170 MeV that resolves the chiral-deconfinement splitting. At high T , the $\text{vac} = 0$ peaks exceed the $\text{vac} = 1$ peaks across all three panels. At low T , the $\text{vac} = 1$ peak in panels (g) and (i) exceeds $\text{vac} = 0$ and matches the lattice peak in temperature and magnitude. In panel (h), the model overshoots the lattice throughout.

χ_{112}^{BQS} in panel (l) dominates the BQS row. χ_{121}^{BQS} in panel (k) and χ_{211}^{BQS} in panel (j) are much smaller and comparable to each other. Dominant peaks across the BQS row lie at $T \approx 245$ MeV, in the strange-melting region. The chiral-deconfinement double-peak across $T \approx 150$ – 170 MeV appears in panel (k). The $\text{vac} = 0$ peaks exceed $\text{vac} = 1$ across the BQS row. In all three panels, the $\text{vac} = 1$ curve shows reasonable agreement with the lattice band for $T < 200$ MeV.

The same qualitative pattern appears in $(2+1)$ -flavor PQM with the vacuum term [43]. There, the BQ correlators peak sharply at the crossover, and the BS, QS, and BQS correlators rise steeply and melt slowly toward the SB limit. A double peak appears only as a hint in that calculation, in

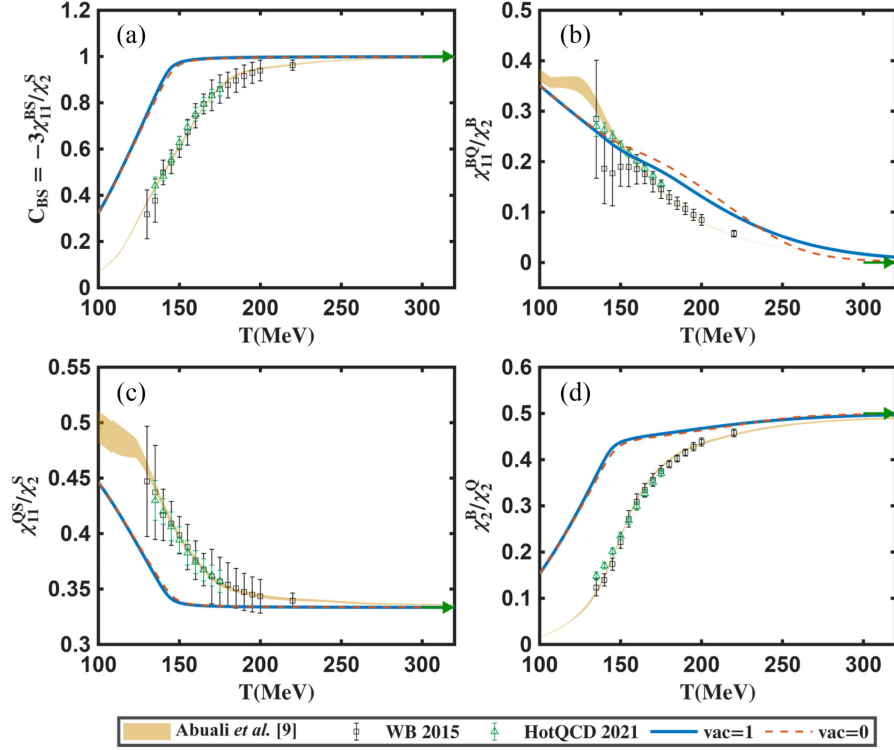


FIG. 9. Susceptibility ratios (a) $C_{BS} = -3\chi_{11}^{BS}/\chi_2^S$, (b) χ_{11}^{BQ}/χ_2^B , (c) χ_{11}^{QS}/χ_2^S , and (d) χ_2^B/χ_2^Q vs temperature at $\mu_B = 0$ MeV. Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Lattice data are WB 2015 [27] (black squares), HotQCD 2021 [28] (dark green triangles), and Abuali *et al.* [9] (amber bands). The Abuali *et al.* bands are constructed by uncorrelated error propagation, which overestimates the true uncertainty. Forest green right-edge arrows mark the SB limits (Appendix).

the χ_{22} and χ_{31} components of the BS and QS rows and in χ_{121}^{BQS} , where the chiral and deconfinement crossovers nearly coincide. The 26 MeV splitting in PCQMF resolves the same components into a clear double peak. PCQMF also provides the explicit vacuum-term comparison and the lattice confrontation that are absent from that work. In the BS, QS, and BQS rows, the model curves overshoot the lattice peak and approach the SB limit from above, while the lattice band approaches from below. The mean-field treatment of mesonic fluctuations places the model peaks above the SB limit. The fRG resums these fluctuations and softens the peaks in the diagonal sector [55].

The off-diagonal peaks in the crossover region track the lattice more closely than χ_4^Q does (Sec. III D). Each carries a baryon or strangeness derivative and projects onto nucleons and kaons, both far heavier than the pions. They contribute little below the crossover, so their absence barely shifts these peaks. Only χ_4^Q shifts, because the light pions that would carry it below the crossover are frozen (Sec. III B).

F. Correlation ratios

Figure 9 shows the temperature dependence of four susceptibility ratios at $\mu_B = 0$ MeV, namely (a) $C_{BS} = -3\chi_{11}^{BS}/\chi_2^S$, (b) χ_{11}^{BQ}/χ_2^B , (c) χ_{11}^{QS}/χ_2^S , and (d) χ_2^B/χ_2^Q . The ratio C_{BS} serves as a diagnostic of the strangeness carrier

across the deconfinement transition [22]. Three lattice datasets are overlaid [9,27,28].

In Fig. 9(a), C_{BS} rises monotonically and approaches unity above $T \approx 170$ MeV. The rise tracks the change in strangeness carriers across the crossover. In the hadronic phase, kaons with $B = 0$ dominate χ_2^S while χ_{11}^{BS} receives contributions only from strange baryons. In the partonic phase, the strange quark contributes to both χ_{11}^{BS} and χ_2^S . The $\text{vac} = 1$ and $\text{vac} = 0$ curves nearly coincide. Both χ_{11}^{BS} and χ_2^S are strange-sector quantities, and the ratio divides out their common vacuum-term dependence. Their maximum separation is ~ 0.01 near $T = 150$ MeV. The PCQMF rise is steeper than the lattice. At $T = 150$ MeV, the model curves are near unity while the lattice datasets remain near 0.6.

The remaining three ratios approach their SB limits monotonically. In Fig. 9(b), χ_{11}^{BQ}/χ_2^B decays from above through the crossover toward zero. The PCQMF curves track the lattice data within errors across the crossover region, unlike panel (a). Within the same range, the $\text{vac} = 0$ curve lies above the $\text{vac} = 1$ curve. The ordering reverses above $T \approx 250$ MeV. In Fig. 9(c) and (d), the ratios χ_{11}^{QS}/χ_2^S and χ_2^B/χ_2^Q approach their SB limits from above and below, respectively. The $\text{vac} = 1$ and $\text{vac} = 0$ curves nearly coincide in both panels. The PCQMF

approach to SB is sharper than the lattice in both. The same qualitative ratio behavior is found in (2 + 1)-flavor PQM with the vacuum term [43]. These ratios are accessible through net-particle cumulants measured in heavy-ion collisions. Section IV E carries out the corresponding comparison for the baryon cumulant ratios along $T_{pc}(\mu_B)$ under the heavy-ion constraints.

IV. RESULTS AT FINITE μ_B

In this section, the susceptibilities of Sec. III are extended to finite baryon chemical potential up to $\mu_B = 500$ MeV. The chemical potentials μ_Q and μ_S are held at zero, which isolates the response of the diagonal susceptibilities to baryon density alone. Section IV E is the exception. It reports the baryon cumulant ratios at $\mu_Q = \mu_S = 0$ and under the heavy-ion constraints. The constraints fix μ_Q and μ_S to nonzero functions of T and μ_B . Both sets of curves are partial derivatives of Eq. (36) at fixed μ_Q and μ_S . The constraint moves the evaluation point and leaves the derivatives unchanged. The $\text{vac} = 1$ and $\text{vac} = 0$ comparison and the solid/dashed convention of Sec. III carry over to every figure.

A. Second- and fourth-order diagonal susceptibilities at finite μ_B

Figure 10 presents the diagonal susceptibilities at $\mu_B = 100, 300, 500$ MeV. The top row [panels (a)–(c)] shows $\chi_2^B, \chi_2^Q, \chi_2^S$ and the bottom row [panels (d)–(f)] shows $\chi_4^B, \chi_4^Q, \chi_4^S$. Each μ_B slice carries a distinct color, and line thickness decreases as μ_B grows. SB limits appear as forest green arrows on the right edge of each panel.

In Fig. 10(a), the χ_2^B curves are monotonic at $\mu_B = 100$ and 300 MeV. At $\mu_B = 500$ MeV, the $\text{vac} = 1$ curve develops a sharp interior peak at $T \approx 137$ MeV. The peak amplitude reaches 0.475, well above the SB limit $\chi_2^B|_{\text{SB}} = \frac{1}{3} + \frac{\hat{\mu}_B^2}{9\pi^2}$ (Appendix). The $\text{vac} = 0$ curve at the same slice peaks at a lower amplitude and has a broader width. In Fig. 10(b), χ_2^Q stays monotonic across all slices and rises toward the SB limit $\chi_2^Q|_{\text{SB}} = \frac{2}{3} + \frac{2\hat{\mu}_B^2}{9\pi^2}$. No analogous interior peak develops in the Q channel because $\mu_Q = 0$ removes the direct chemical-potential drive. In Fig. 10(c), χ_2^S is likewise monotonic and stays below $\chi_2^S|_{\text{SB}} = 1 + \frac{\hat{\mu}_B^2}{3\pi^2}$. The strangeness sector responds weakly to μ_B at $\mu_S = 0$, and the relevant crossover scale is set by the strange-quark constituent mass.

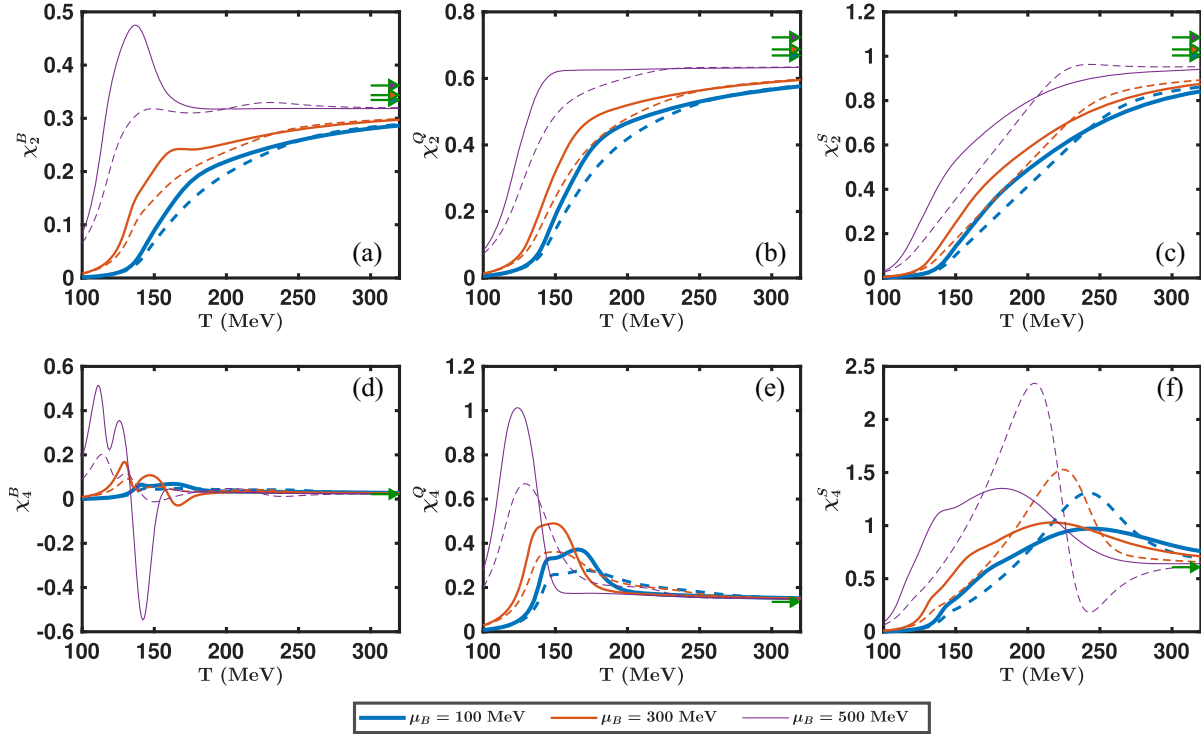


FIG. 10. Second- and fourth-order diagonal susceptibilities at finite baryon chemical potential. The top row shows (a) χ_2^B , (b) χ_2^Q , (c) χ_2^S . The bottom row shows (d) χ_4^B , (e) χ_4^Q , (f) χ_4^S . Three slices are overlaid in each panel, $\mu_B = 100, 300$, and 500 MeV. Each μ_B slice has its own color, and thicker lines mark lower μ_B . Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Forest-green right-edge arrows mark the SB limits. In the top row, the SB limit is μ_B -dependent at finite μ_B , so each slice carries its own arrow with the arrowhead filled by slice color (Appendix).

In Fig. 10(d), χ_4^B develops multipeak structure that strengthens and broadens with μ_B . At $\mu_B = 100$ MeV, both curves show close double peaks near $T = 141$ and 163 MeV separated by a shallow trough. At $\mu_B = 300$ MeV, the $\text{vac} = 1$ curve dips below zero between two zero crossings at $T = 162$ and 174 MeV, with a minimum at -0.029 . At $\mu_B = 500$ MeV, the $\text{vac} = 1$ peak reaches 0.515 at $T = 111$ MeV, and the trough drops to -0.547 at $T = 142$ MeV. The $\text{vac} = 0$ curve follows the same sequence, but the trough reaches only -0.012 . The double-peak fine structure resolves the chiral-deconfinement splitting established in Sec. III C. In Fig. 10(e), χ_4^Q shows a single dominant peak in each slice that shifts to lower T and grows in amplitude with μ_B . The inflection near $T \approx 148$ MeV at $\mu_B = 100$ MeV carries the same chiral-deconfinement signature as the χ_4^B fine structure. In Fig. 10(f), χ_4^S shows a single peak that shifts from $T \approx 244$ MeV at $\mu_B = 100$ MeV to $T \approx 182$ MeV at $\mu_B = 500$ MeV. The $\text{vac} = 0$ peak exceeds the $\text{vac} = 1$ peak in amplitude across all slices, and the gap widens with μ_B . This ordering is the opposite of that in the baryon and charge channels.

Susceptibility amplitudes grow with μ_B in every channel, but the fourth-order curves grow far more steeply than the second-order curves. Between $\mu_B = 100$ and 500 MeV, χ_2^B peak amplitudes grow by a factor of 1.7, while χ_4^B peak amplitudes grow by a factor of 7.5. As μ_B grows, the peak locations shift to lower T along $T_{\text{pc}}(\mu_B)$. The μ_B sensitivity decreases from baryon to charge to strangeness at both second and fourth order. The baryon channel receives direct chemical-potential drive, while charge and strangeness respond only through the implicit μ_B dependence of the scalar fields and Polyakov loop. The amplitude separation between $\text{vac} = 1$ and $\text{vac} = 0$ observed at $\mu_B = 0$ persists into the finite-density region. The $\text{vac} = 1$ curves retain their enhanced peak amplitudes for χ_2^B , χ_4^B , and χ_4^Q , while χ_4^S

shows the $\text{vac} = 0$ dominance established in Sec. III D. The χ_4^B trough also falls deeper in $\text{vac} = 1$. Similar finite- μ_B growth of baryon-channel susceptibilities has been reported in PQM [42], PNJL [38,39], and fRG [55]. The diagonal χ_2 and χ_4 in all three BQS channels at finite μ_B have been computed in NJL at the same constraint surface $\mu_Q = \mu_S = 0$ [53], which provides the closest content-level comparison to Fig. 10.

B. Second-order off-diagonal correlators at finite μ_B

Figure 11 presents the second-order off-diagonal correlators χ_{11}^{BQ} , $-\chi_{11}^{BS}$, and χ_{11}^{QS} at $\mu_B = 100, 300, 500$ MeV. Each μ_B slice carries a distinct color, and line thickness decreases as μ_B grows. SB limits appear as forest-green arrows on the right edge of each panel.

In panel (a), χ_{11}^{BQ} peaks near the chiral crossover at every slice, and the peak position shifts to lower T with increasing μ_B . At $\mu_B = 500$ MeV, the $\text{vac} = 1$ peak reaches 0.125 at $T \approx 133$ MeV. The $\text{vac} = 0$ curve follows the same pattern with smaller amplitudes. Light quarks carry both B and Q , so the correlator couples directly to μ_B and the peak sharpens as the crossover strengthens. The $\text{vac} = 1$ peak grows by a factor of 3.8 between $\mu_B = 100$ and 500 MeV, more than double the χ_2^B growth of 1.7 (Fig. 10).

In panels (b) and (c), $-\chi_{11}^{BS}$ and χ_{11}^{QS} rise through the crossover toward the SB limit $\frac{1}{3} + \frac{\mu_B^2}{9\pi^2}$. $-\chi_{11}^{BS}$ develops increasing structure at larger μ_B . At $\mu_B = 500$ MeV, the $\text{vac} = 0$ curve shows a broad maximum near $T \approx 237$ MeV with a weaker structure at lower T . In $\text{vac} = 1$, the low- T structure near the chiral crossover dominates the curve. The chiral crossover affects the baryon sector more strongly, so the $\text{vac} = 1/\text{vac} = 0$ separation is larger in χ_{11}^{BQ} than in $-\chi_{11}^{BS}$. Unlike χ_{11}^{BQ} , which vanishes in the SB limit and grows as a pure crossover feature, $-\chi_{11}^{BS}$ and χ_{11}^{QS} approach a nonzero SB limit. The μ_B dependence therefore appears

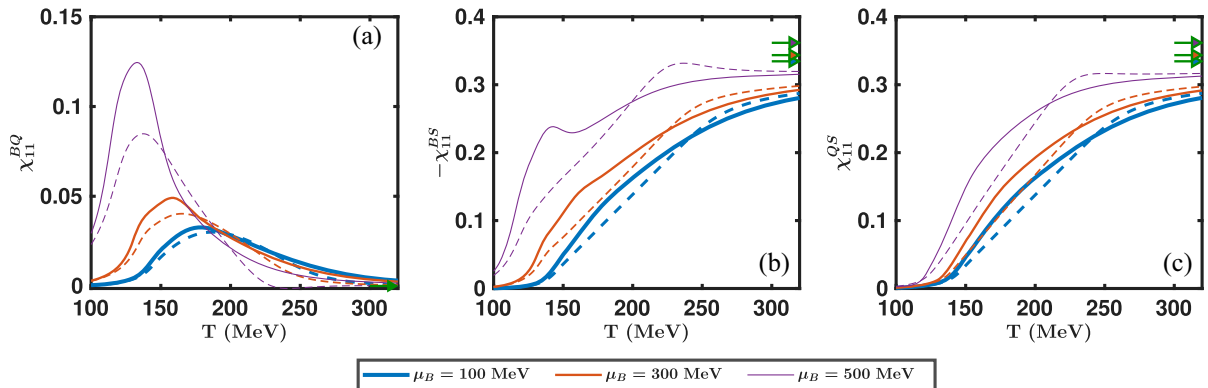


FIG. 11. Second-order off-diagonal correlators (a) baryon-charge χ_{11}^{BQ} , (b) baryon-strangeness $-\chi_{11}^{BS}$, (c) charge-strangeness χ_{11}^{QS} vs temperature at $\mu_B = 100, 300$, and 500 MeV. Each μ_B slice has its own color, and thicker lines mark lower μ_B . Sign convention as in Fig. 5. Solid (dashed) lines correspond to PCQM (with) the fermion vacuum term. Forest green right-edge arrows mark the SB limits. In panels (b) and (c) the SB limit is μ_B -dependent at finite μ_B , so each slice carries its own arrow with the arrowhead filled by slice color (Appendix).

chiefly in the onset temperature and steepness of the rise. The $\text{vac} = 1$ curves rise more steeply through the crossover (Sec. III A), and both variants converge toward the SB limit at high T .

C. Fourth-order off-diagonal correlators at finite μ_B

Figure 12 presents the twelve independent fourth-order off-diagonal correlators at $\mu_B = 200$ and 400 MeV. The panel ordering and sign convention match Fig. 8. Each μ_B slice carries a distinct color, and line thickness decreases as μ_B grows. SB limits appear as forest-green arrows on the right edge. The same twelve correlators have been computed at $\mu_B = 0$ MeV in $(2+1)$ -flavor PQM with the vacuum term [43], and in lattice QCD [9]. A nuclear-matter calculation [47] reports the BQ subset at finite μ_B and finds

$\chi_{31}^{BQ} > \chi_{22}^{BQ} > \chi_{13}^{BQ}$ along the chemical freeze-out line. No prior calculation has reported the full twelve-component fourth-order off-diagonal tensor at finite μ_B .

Many $\text{vac} = 1$ panels carry a double-peak fine structure in the chiral region at finite μ_B that resolves the chiral-deconfinement splitting. In panel (a), the $\text{vac} = 1$ curve develops twin chiral peaks at both μ_B values. At $\mu_B = 200$ MeV, the peaks lie near $T = 135$ and 154 MeV. At $\mu_B = 400$ MeV, they sharpen and shift to $T = 121$ and 136 MeV, and the dip between them deepens. The $\text{vac} = 0$ curve carries the same structure at a lower amplitude. The same double-peak structure appears in $\text{vac} = 1$ at $\mu_B = 200$ MeV in panels (b), (c), (d), (f), (g), (j), and (k). It dominates the curve in panels (b), (c), and (g), and lies alongside a strange-melting peak in panels (d), (f),

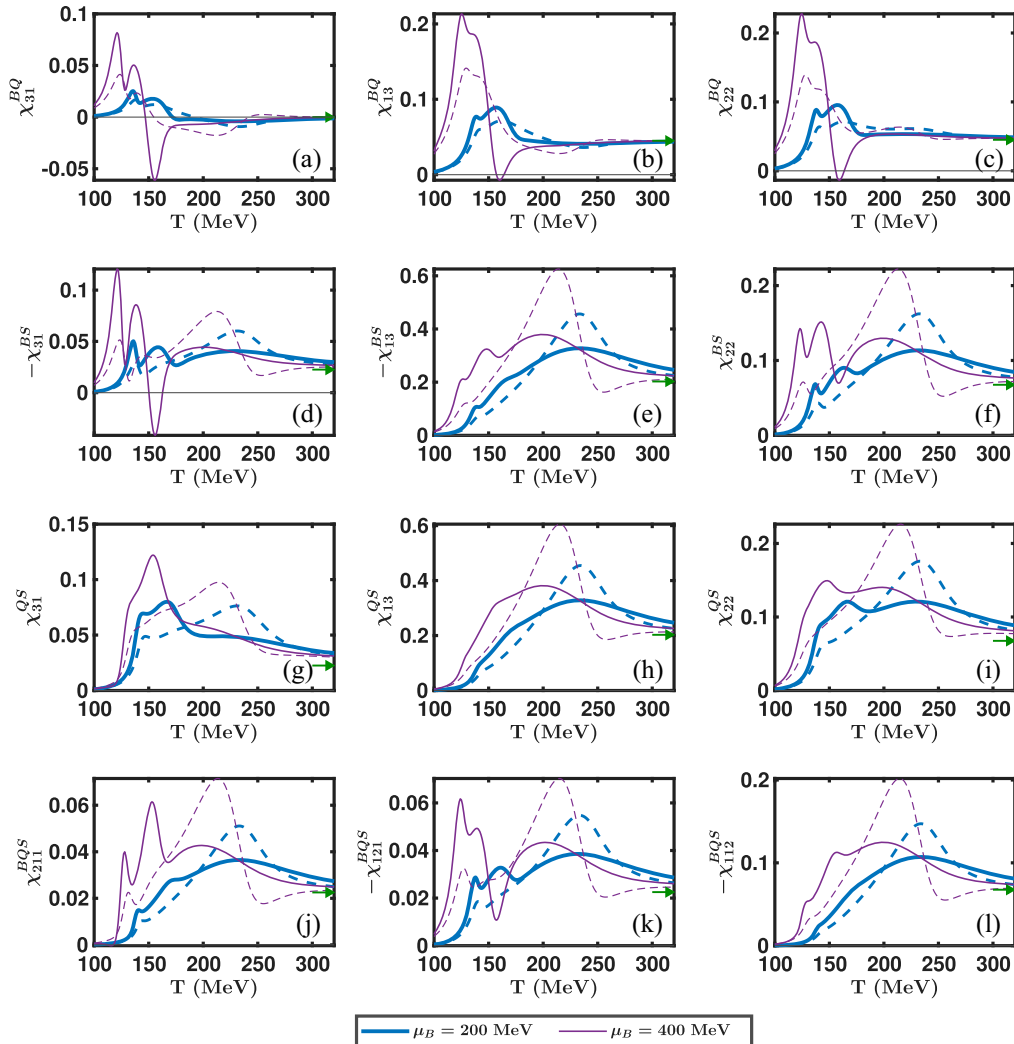


FIG. 12. Complete fourth-order off-diagonal BQS tensor vs temperature at $\mu_B = 200$ and 400 MeV. Rows 1–3 show the two-charge correlators $\chi_{ij}^{BQ,BS,QS}$ with $i + j = 4$, columns ordered as $\chi_{31}, \chi_{13}, \chi_{22}$. Row 4 shows the mixed correlators χ_{ijk}^{BQS} with $i + j + k = 4$, columns ordered as $\chi_{211}, -\chi_{121}, -\chi_{112}$. Panels (d), (e), (k), (l) carry an overall minus sign to match the convention of Abuali *et al.* [9]. Each μ_B slice has its own color, and thicker lines mark lower μ_B . Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Forest green right-edge arrows mark the SB limits (Appendix).

(j), and (k). At $\mu_B = 400$ MeV, the twin sharpens further in panels (a), (d), (f), (j), and (k). In panels (b), (c), and (g), the twin instead merges into a single tall, sharp peak. T_{pc} drops faster than T_{dec} with increasing μ_B , and the chiral peak grows in amplitude. In panels (b), (c), and (g), the growing chiral peak absorbs the deconfinement peak. The $\text{vac} = 0$ variant carries the same splitting in panels (a), (c), and (d) at $\mu_B = 200$ MeV with smaller amplitudes throughout. Larger chiral-peak amplitudes in $\text{vac} = 1$ make the splitting more visible than in $\text{vac} = 0$.

At $\mu_B = 400$ MeV, the $\text{vac} = 1$ curve dips below zero after the chiral peaks in panels (a), (b), (c), and (d). In panel (a), the $\text{vac} = 1$ curve crosses zero after the chiral peaks and reaches a deep minimum at $T = 156$ MeV. The $\text{vac} = 1$ curve in panel (d) shows the same deep dip at $T = 156$ MeV alongside the strange-melting peak. Panels (b) and (c) carry shallower troughs near $T \approx 160$ MeV after the tall chiral peak. The $\text{vac} = 0$ curve dips below zero only in panel (a) and stays positive in panels (b), (c), and (d). The taller chiral peaks in $\text{vac} = 1$ drive the curve below zero between the peaks. Beyond the chiral region, the curves carry additional structure. In most BS, QS, and BQS panels, a strange-melting peak dominates the curve at higher temperatures. The $\text{vac} = 0$ strange peaks exceed the $\text{vac} = 1$ strange peaks, the reverse of the chiral-region hierarchy. At $\mu_B = 400$ MeV in $\text{vac} = 1$, the growing chiral peak overtakes the strange-melting peak in panels (d), (f), (j), and (k). The $\text{vac} = 1$ chiral peaks exceed those of $\text{vac} = 0$,

while the $\text{vac} = 0$ strange-melting peaks exceed those of $\text{vac} = 1$.

D. Diagonal baryon susceptibilities at finite μ_B

Figure 13 presents the diagonal baryon susceptibilities χ_n^B at $\mu_B = 200$ and 400 MeV. The top row shows the odd-order $\chi_1^B, \chi_3^B, \chi_5^B$ and the bottom row the even-order $\chi_4^B, \chi_6^B, \chi_8^B$. The odd-order χ_n^B vanish at $\mu_B = 0$ MeV by charge-conjugation symmetry and are accessible only at finite μ_B . Each μ_B slice carries a distinct color, and line thickness decreases as μ_B grows. SB limits appear as right-edge arrows. Panels (a) and (b) carry two arrows each because $\chi_1^B|_{SB}$ and $\chi_3^B|_{SB}$ depend on μ_B/T and take distinct values at each slice. The remaining panels carry a single μ_B -independent SB arrow.

The odd-order χ_n^B develops richer chiral-region structure as n grows. In Fig. 13(a), χ_1^B rises monotonically through the crossover and saturates at high T at both μ_B values. The $\text{vac} = 1$ curve lies above $\text{vac} = 0$ in the chiral crossover region, and the two merge at high T . In Fig. 13(b), χ_3^B remains positive across the full T range and carries twin peaks in $\text{vac} = 1$ at both μ_B values. The peaks shift to lower T and sharpen as μ_B grows. The $\text{vac} = 0$ curve shows only a single broad maximum at $\mu_B = 200$ MeV and develops the same twin structure at $\mu_B = 400$ MeV. A third broad bump develops at higher T in both variants at $\mu_B = 400$ MeV. The twin peaks resolve the chiral-deconfinement splitting

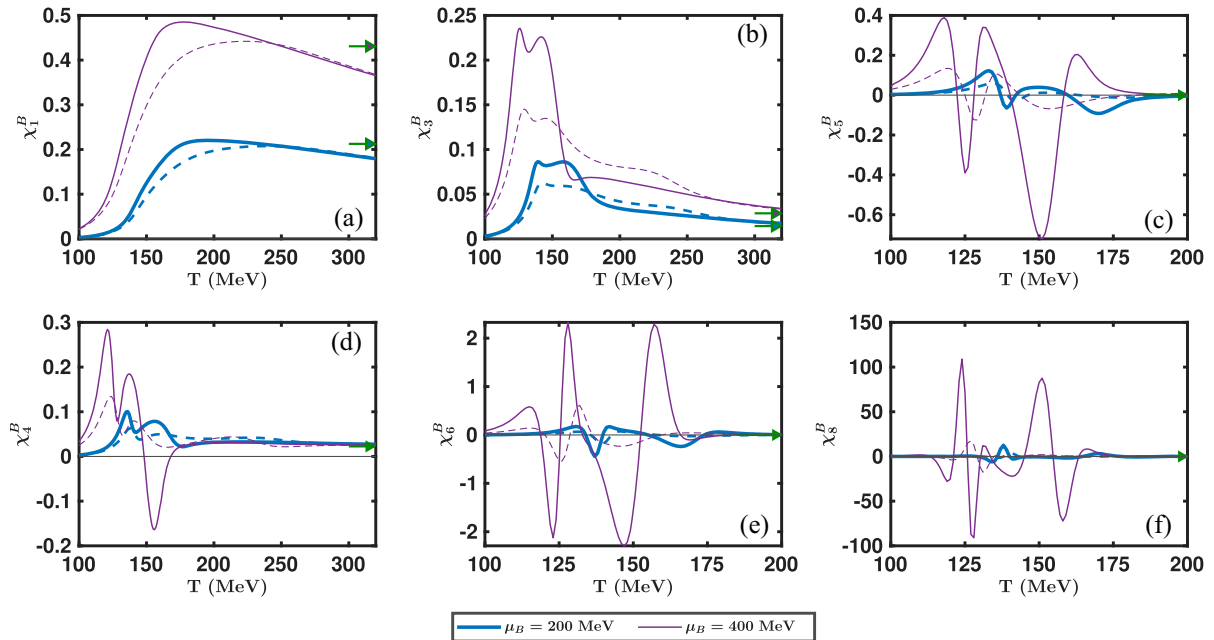


FIG. 13. Diagonal baryon susceptibilities at $\mu_B = 200$ and 400 MeV. The top row shows odd-order (a) χ_1^B , (b) χ_3^B , and (c) χ_5^B . The bottom row shows even-order (d) χ_4^B , (e) χ_6^B , and (f) χ_8^B . Each μ_B slice has its own color, and thicker lines mark lower μ_B . Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Forest-green right-edge arrows mark the SB limits. For the slice-dependent SB values in panels (a) and (b), the arrowhead is filled by slice color (Appendix).

through the glue-to-Yang-Mills mapping [75]. In Fig. 13(c), χ_5^B oscillates strongly across the chiral region in both variants. In $\text{vac} = 1$ at $\mu_B = 400$ MeV, the curve develops a large negative minimum. The $\text{vac} = 0$ extrema occur at smaller amplitudes and at higher temperatures than $\text{vac} = 1$ at both μ_B values.

The even-order χ_n^B develops sign reversals and large oscillation amplitudes across the chiral region. In Fig. 13(d), χ_4^B repeats the double peak and sign reversal of Fig. 10(d) at these slices. The sign reversal in χ_4^B and the off-diagonal sign reversals in Fig. 12 share the same mechanism. PNJL studies of fourth-order net-baryon cumulants near a critical endpoint report sign-changing χ_4^B along freeze-out lines [38,39]. In those studies, proximity to the CEP drives the effect. In Fig. 13(e), χ_6^B oscillates strongly with both signs across the chiral region. The $\text{vac} = 1$ curve carries more visible extrema than $\text{vac} = 0$ at both μ_B values. Amplitudes grow substantially from $\mu_B = 200$ to $\mu_B = 400$ MeV. In Fig. 13(f), χ_8^B shows the most violent oscillation of the figure. At $\mu_B = 400$ MeV, the $\text{vac} = 1$ curve oscillates with multiple sign changes across the narrow T range. The $\text{vac} = 1$ amplitudes far exceed those of $\text{vac} = 0$. The fRG calculation of hyperorder net-baryon fluctuations through tenth order at finite μ_B [55] reports a similar growth of chiral-region oscillations with μ_B . PQM with and without the vacuum

term reports the corresponding R_{n2}^B ratios through twelfth order at finite μ_B [42] but does not isolate individual high-order χ_n^B .

E. Cumulant ratios along the pseudocritical line

Figure 14 presents the cumulant ratios $R_{12}^B \equiv \chi_1^B/\chi_2^B$, $R_{42}^B \equiv \chi_4^B/\chi_2^B$, $R_{51}^B \equiv \chi_5^B/\chi_1^B$, and $R_{62}^B \equiv \chi_6^B/\chi_2^B$ along the pseudocritical line $T_{pc}(\mu_B)$ as functions of $\hat{\mu}_B = \mu_B/T_{pc}$. The HRG baseline $\tanh \hat{\mu}_B$ is shown in panel (a) and unity in panels (b), (c), and (d) [29,50]. STAR BES-II net-proton data are overlaid in panels (b), (c), and (d) [13,14]. The STAR points lie on the chemical freeze-out trajectory under the heavy-ion constraints of strangeness neutrality, $\langle n_S \rangle = 0$, and fixed electric-to-baryon ratio, $n_Q/n_B = 0.4$. Each panel carries two sets of model curves. The thick curves impose both constraints. At each (T, μ_B) , the two conditions are solved together with the gap equations, and the solution determines $\mu_Q(T, \mu_B)$ and $\mu_S(T, \mu_B)$. The susceptibilities are the partial derivatives at fixed μ_Q and μ_S , evaluated at the constrained point. This is what the net-proton cumulants measure. Lattice studies impose the constraints in the same way [45,83]. The ratios are then evaluated on the pseudocritical line of the constrained system. The thin curves set $\mu_Q = \mu_S = 0$. Both μ_Q and μ_S vanish at $\mu_B = 0$ by

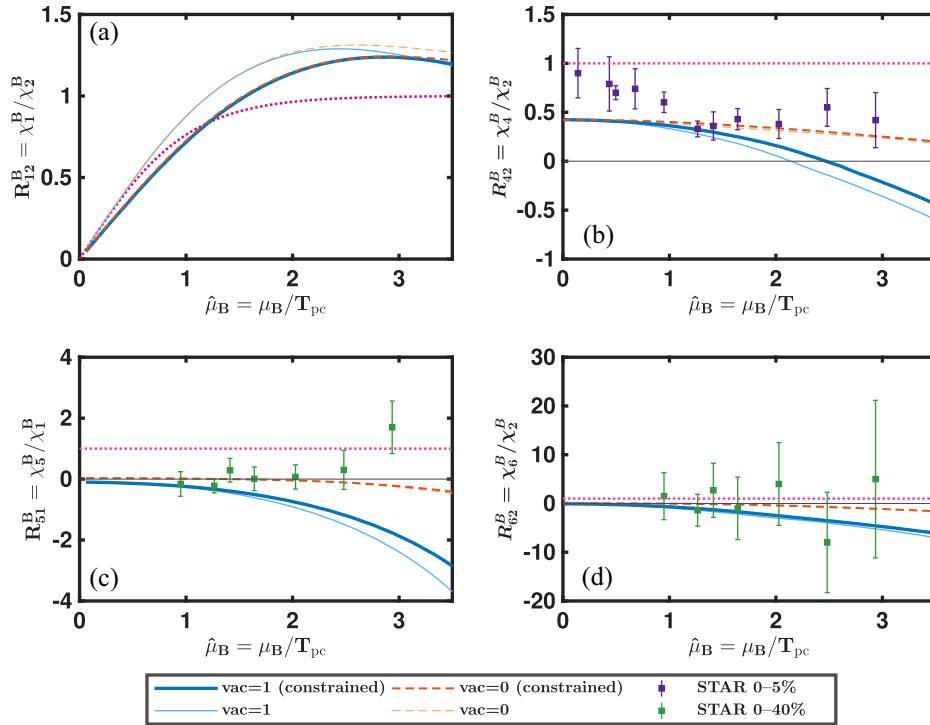


FIG. 14. Cumulant ratios along the pseudocritical line $T_{pc}(\mu_B)$ as functions of $\hat{\mu}_B = \mu_B/T_{pc}$. (a) $R_{12}^B = \chi_1^B/\chi_2^B$, (b) $R_{42}^B = \chi_4^B/\chi_2^B$, (c) $R_{51}^B = \chi_5^B/\chi_1^B$, and (d) $R_{62}^B = \chi_6^B/\chi_2^B$. Thick curves impose strangeness neutrality, $\langle n_S \rangle = 0$, and a fixed electric-to-baryon ratio, $n_Q/n_B = 0.4$, and lie on the pseudocritical line of the constrained system. Thin curves set $\mu_Q = \mu_S = 0$. Solid (dashed) lines correspond to PCQM with (without) the fermion vacuum term. STAR BES-II net-proton data are 0–5% centrality [13] (purple squares) in panel (b) and 0–40% centrality [14] (green squares) in panels (c) and (d). Magenta dotted lines mark the HRG baselines, $\tanh \hat{\mu}_B$ in panel (a) and unity in panels (b)–(d).

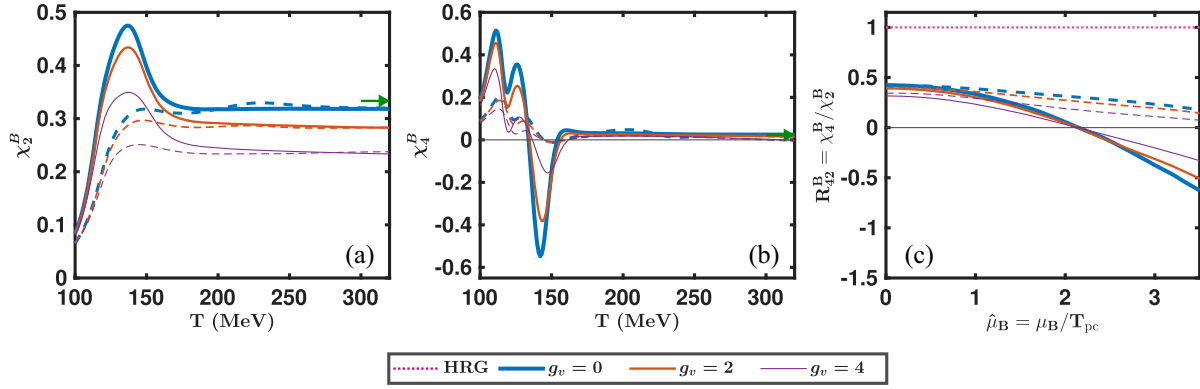


FIG. 15. Diagonal baryon susceptibilities and their fourth-to-second cumulant ratio for $g_v = 0, 2$, and 4 . (a) χ_2^B and (b) χ_4^B vs temperature at $\mu_B = 500$ MeV. (c) $R_{42}^B = \chi_4^B/\chi_2^B$ along the pseudocritical line $T_{pc}(\mu_B)$ as a function of $\hat{\mu}_B = \mu_B/T_{pc}$. Each g_v value has its own color, and thicker lines mark lower g_v . Solid (dashed) lines correspond to PCQMF with (without) the fermion vacuum term. Forest green right-edge arrows in panels (a) and (b) mark the SB limits (Appendix). Magenta dotted horizontal line in panel (c) marks the HRG baseline $R_{42}^B = 1$.

charge-conjugation symmetry, so the two sets of curves meet at $\hat{\mu}_B = 0$ and every $\mu_B = 0$ result of Sec. III is unchanged.

The ratios R_{12}^B and R_{42}^B depart from the HRG baselines as $\hat{\mu}_B$ grows, and $\text{vac} = 1$ develops a sign reversal at fourth order. In Fig. 14(a), R_{12}^B rises from zero in every curve and turns over after a maximum. The constrained curves lie about 20% below the unconstrained ones at low $\hat{\mu}_B$, and the gap narrows as $\hat{\mu}_B$ grows. The unconstrained curves lie above the HRG reference $\tanh \hat{\mu}_B$ over the whole range, and the constrained curves only above $\hat{\mu}_B \approx 1.2$. The HRG limit no longer applies because mean-field quark degrees of freedom dominate at $T_{pc}(\mu_B)$. In Fig. 14(b), both variants start at $R_{42}^B \approx 0.42$ at $\hat{\mu}_B = 0$. The $\text{vac} = 1$ curve decreases monotonically and crosses zero at $\hat{\mu}_B \approx 2.15$ without the constraints and at ≈ 2.47 with them. The curve continues to large negative values at higher $\hat{\mu}_B$. The $\text{vac} = 0$ curve also decreases but stays positive with and without the constraints over the plotted range. In $\text{vac} = 1$, χ_4^B develops the sign reversal seen in Fig. 13(d), and R_{42}^B inherits the sign change because χ_2^B is positive throughout. The constraints suppress χ_2^B on the line, and the suppression grows with $\hat{\mu}_B$. STAR central values lie above both model curves at low $\hat{\mu}_B$ and remain positive throughout.

In $\text{vac} = 1$, R_{51}^B and R_{62}^B start negative and grow more negative as $\hat{\mu}_B$ increases. In $\text{vac} = 0$, both ratios start marginally positive and become negative only at higher $\hat{\mu}_B$. In Fig. 14(c), the $\text{vac} = 1$ curves fall within the STAR error bars at the two lowest- $\hat{\mu}_B$ points and below them at higher $\hat{\mu}_B$, while the $\text{vac} = 0$ curves stay within the bars over most of the range. In Fig. 14(d), both variants fall within the STAR error bars throughout, since the experimental uncertainties at sixth order are large. The constraints reduce the $\text{vac} = 1$ amplitudes at large $\hat{\mu}_B$ without changing their signs. The $\text{vac} = 1$ amplitudes far exceed $\text{vac} = 0$ in both panels and grow faster with $\hat{\mu}_B$ at sixth order. The $\text{vac} = 1$ amplification follows the same pattern

that drives the high-order χ_n^B amplification in Figs. 13(e) and 13(f).

F. Sensitivity to the vector coupling

The vector coupling is held at $g_v = 0$ in the main results. Its effect on the finite- μ_B baryon sector is assessed here. Figure 15 presents χ_2^B and χ_4^B versus temperature at $\mu_B = 500$ MeV in panels (a) and (b), and the ratio $R_{42}^B = \chi_4^B/\chi_2^B$ along the pseudocritical line $T_{pc}(\mu_B)$ as a function of $\hat{\mu}_B = \mu_B/T_{pc}$ in panel (c). A nonzero g_v generates repulsive vector mean fields that lower the effective quark chemical potentials and weaken the baryon-number response. In panel (a), χ_2^B at $\mu_B = 500$ MeV declines monotonically with g_v , and the crossover peak drops as g_v rises from 0 to 4. The fourth-order response in panel (b) retains the two maxima below the crossover and the negative dip of Figs. 10(d) and 13(d), but its amplitude falls with g_v . The $\text{vac} = 0$ curves stay far flatter than the $\text{vac} = 1$ curves throughout. The fourth-order oscillation is therefore a vacuum-term effect that g_v only dampens.

The features that distinguish the two model variants survive at every g_v . Panel (c) sets $\mu_Q = \mu_S = 0$. R_{42}^B starts below the HRG baseline at $\mu_B = 0$ and decreases with $\hat{\mu}_B$. The $\text{vac} = 1$ curves cross zero and turn negative, while the $\text{vac} = 0$ curves remain positive over $\mu_B \leq 500$ MeV, as in Fig. 14(b). The zero crossing does not move with g_v . A larger g_v suppresses the susceptibilities without altering their sign structure. The choice $g_v = 0$ in the main results, therefore, leaves every qualitative conclusion of this work unchanged.

V. DISCUSSION

The fermion vacuum term of Eq. (20) enters the PCQMF effective potential and modifies the gap-equation solutions. In $\text{vac} = 1$, Fig. 1(a) shows that the scalar field σ drops

more steeply with temperature through the chiral crossover than in $\text{vac} = 0$. This sharpening is opposite to the $(2 + 1)$ -flavor PQM result, where the vacuum term smooths the crossover [42,43]. The light constituent quark masses follow, since they are set by the scalar fields through Eq. (8). In Fig. 3(b), $-d\Delta_{l,s}/dT$ develops an inflection near T_{dec} in $\text{vac} = 1$ and rises monotonically into the chiral peak in $\text{vac} = 0$. The deconfinement-driven inflection in $\Delta_{l,s}(T)$ near T_{dec} becomes visible in $\text{vac} = 1$ through the coupled gap equations. The chiral-deconfinement splitting is generated by the glue-to-Yang-Mills mapping of Eq. (17) [75] and is present in both variants. At higher derivative orders, Fig. 6 shows the chiral-deconfinement splitting resolved as a double-peak structure in χ_4^B and as multiple zero crossings in χ_6^B and χ_8^B in both variants. The $\text{vac} = 1$ peak-to-trough contrast exceeds that of $\text{vac} = 0$.

The $\text{vac} = 1/\text{vac} = 0$ difference in the fourth-order off-diagonal tensor (Fig. 8) depends on the temperature region of each peak. Chiral-crossover peaks are larger in $\text{vac} = 1$, while strange-melting peaks are larger in $\text{vac} = 0$. The BQ row sits in the first category, since light quarks dominate its correlators at the chiral crossover. The steeper $\sigma(T)$ drop in $\text{vac} = 1$ drives the $\text{vac} = 1$ dominance throughout the chiral region of the tensor, where the chiral-deconfinement double-peak resolves at low T . The dominant peaks of the BS, QS, and BQS rows sit in the second category, where $\text{vac} = 0$ produces the higher amplitudes despite the steeper $\zeta(T)$ drop in $\text{vac} = 1$ [Fig. 1(b)]. Both variants reproduce the same vacuum observables (Sec. II C), so the strange-melting reversal lies in the high- T response of the two parameter sets rather than in the chiral-field response at low T alone.

The qualitative differences between $\text{vac} = 1$ and $\text{vac} = 0$ persist at finite μ_B . The chiral-region $\text{vac} = 1$ dominance and the strange-region $\text{vac} = 0$ dominance both persist into the finite-density region (Sec. IV A–IV C). Along $T_{\text{pc}}(\mu_B)$, R_{42}^B changes sign in $\text{vac} = 1$ but stays positive in $\text{vac} = 0$, and R_{51}^B and R_{62}^B are persistently negative in $\text{vac} = 1$ (Sec. IV E).

PCQMF treats the scalar fields (σ, ζ, δ) and the Polyakov loop ($\Phi, \bar{\Phi}$) at the mean-field level, and fluctuations of these fields are absent. All three diagonal χ_2 underestimate the lattice (Sec. III B), and the deficit is largest for χ_2^Q . Pions are the lightest charged carriers and dominate χ_2^Q in the hadronic phase, but in PCQMF they freeze at classical mean-field values and contribute no fluctuations. At low temperatures, PCQMF does not approach the HRG limit because hadrons are not realized as quasiparticle states. Beyond-mean-field methods such as the fRG incorporate both bosonic and fermionic fluctuations [55]. The mean-field treatment also places $T_{\text{pc}}(0)$ at 170.5 MeV, 12.5 MeV above the continuum lattice value 158.0 ± 0.6 MeV [5]. The chiral-sector results of Secs. III and IV inherit that shift. The pseudocritical line of Fig. 3(a) carries it to finite μ_B , and the cumulant ratios of Sec. IV E are evaluated on that line. The chiral-deconfinement splitting of 26.1 MeV is

about five times the separation obtained in functional QCD, where the chiral and confinement-deconfinement cross-overs lie at 157 and 152 MeV [84]. The lattice resolves the same ordering, and its separation is also a few MeV [4]. The model reproduces the ordering and overestimates the size. The larger splitting produces the χ_4^B double peak and the extra χ_6^B sign changes. The frozen pions also shift the χ_4^Q peak. It lies 37 MeV above the lattice one, against a few MeV for the correlators that carry a baryon or strangeness derivative.

VI. SUMMARY AND OUTLOOK

This work reports the first calculation of generalized susceptibilities of conserved charges in the PCQMF model with the fermion vacuum term. At $\mu_B = 0$ MeV, the calculation covers all diagonal $\chi_n^{B,Q,S}$ through eighth order and all twelve independent fourth-order off-diagonal correlators. An independently refitted no-sea variant ($\text{vac} = 0$) sets the comparison baseline. The chiral pseudocritical temperature is $T_{\text{pc}} = 170.5$ MeV ($\text{vac} = 1$) and 166.4 MeV ($\text{vac} = 0$), and the deconfinement counterpart is $T_{\text{dec}} = 144.4$ MeV and 146.6 MeV. The chiral-deconfinement splitting reaches 26.1 MeV in $\text{vac} = 1$ and 19.8 MeV in $\text{vac} = 0$. The inflection in $-d\Delta_{l,s}/dT$ near T_{dec} appears in $\text{vac} = 1$ only. Higher derivative orders resolve the chiral-deconfinement splitting as twin maxima in χ_4^B and χ_6^Q , as twin minima in χ_8^B and χ_8^Q , and as multiple zero crossings in χ_6^B . Among the fourth-order off-diagonal correlators, the BQ components carry $\text{vac} = 1$ dominance across the chiral crossover. The dominant peaks of the BS, QS, and mixed BQS components occur at higher strange-melting temperatures, where $\text{vac} = 0$ dominates. The model reproduces the qualitative crossover behavior seen on the lattice. Channel-by-channel magnitude differences are detailed in Sec. III.

The calculation extends to finite baryon chemical potential up to $\mu_B = 500$ MeV, both at $\mu_Q = \mu_S = 0$ and under the heavy-ion constraints of strangeness neutrality, $\langle n_S \rangle = 0$, and $n_Q/n_B = 0.4$. Along the pseudocritical line $T_{\text{pc}}(\mu_B)$, R_{42}^B starts at ≈ 0.42 in both variants and decreases with $\hat{\mu}_B = \mu_B/T_{\text{pc}}$. The $\text{vac} = 1$ curve crosses zero at $\hat{\mu}_B \approx 2.15$ and continues to large negative values, while $\text{vac} = 0$ stays positive over the plotted range. The constraints move the $\text{vac} = 1$ crossing to $\hat{\mu}_B \approx 2.47$. The $\text{vac} = 0$ curve stays positive in both cases. R_{51}^B and R_{62}^B start negative in $\text{vac} = 1$ and grow more negative as $\hat{\mu}_B$ increases. In $\text{vac} = 0$, they start marginally positive and turn negative at higher $\hat{\mu}_B$. The $\text{vac} = 1$ amplitudes far exceed $\text{vac} = 0$ in both. The chiral and strange-melting dominance pattern observed at $\mu_B = 0$ persists into the finite-density region. The $\text{vac} = 1$ curves dominate the chiral region, and the $\text{vac} = 0$ curves dominate the strange-melting region.

The STAR BES-II data lie on the heavy-ion freeze-out trajectory of strangeness neutrality and $n_Q/n_B = 0.4$. Section IV E imposes both constraints. The two conditions

are solved together with the gap equations for $\mu_Q(T, \mu_B)$ and $\mu_S(T, \mu_B)$, and the cumulant ratios are evaluated on the pseudocritical line of the constrained system. Isentropes of constant entropy per net baryon density s/n_B trace the hydrodynamic cooling path at fixed $\sqrt{s_{NN}}$ and connect model predictions to specific BES collision energies [85,86]. Critical end point localization on the extended (T, μ_B) plane in both variants quantifies the $\text{vac} = 1/\text{vac} = 0$ shift. The earlier PCQMF endpoint estimate [67] used neither the vacuum term nor the glue-to-Yang-Mills mapping. No previous PCQMF calculation gives the full BQS tensor with the fermion vacuum term. The twelve fourth-order off-diagonal correlators exist in PQM with the vacuum term at $\mu_B = 0$ only [43], and the earlier PCQMF susceptibility work omitted the vacuum term [68]. The fRG reaches tenth order [55] and holographic QCD twelfth [60], and neither reports the off-diagonal tensor. The R_{62}^B peak overshoots the lattice peak, and large amplitudes for higher-order baryon ratios also appear in PNJL [34] and NJL [54]. The fRG resums the mesonic and fermionic fluctuations that mean-field models omit and recovers the lattice R_{42}^B and R_{62}^B quantitatively [55]. The splitting decides whether the χ_4^B double peak appears. It is present in every set of Table II with a splitting of 19.8 MeV or more and absent in every set at 16 MeV or less. PQM with the vacuum term computes the same quantity and gives a single peak, with a polynomial Polyakov potential and no glue-to-Yang-Mills mapping [43]. The underestimation of χ_2^Q below T_{pc} is not specific to PCQMF. It appears in PNJL with eight-quark interactions [35] and in PQM with the fermion vacuum term [43]. Both describe χ_2^B and χ_2^S well against lattice data and fall short only in the charge channel. Adding the Polyakov loop to the eight-quark model brings χ_2^B, χ_2^S and the BS and QS correlators into fair agreement with the lattice, and leaves χ_2^Q unimproved for either Polyakov potential [35].

ACKNOWLEDGMENTS

D. S. sincerely acknowledges the support for this work from the Ministry of Education (MoE), Government of India, through an Institute fellowship at the National Institute of Technology Jalandhar. A. K. sincerely acknowledges the Anusandhan National Research Foundation (ANRF), Government of India, for funding the research project under the Science and Engineering Research Board-Core Research Grant (SERB-CRG) scheme (File No. CRG/2023/000557). A. K. sincerely acknowledges the DST-SERB, Government of India, for funding the research Project No. CRG/2019/000096. The authors gratefully acknowledge the High Performance Computing (HPC) facility provided by Dr. B.R. Ambedkar National Institute of Technology Jalandhar (NIT Jalandhar) for supporting the computational requirements of this research work.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: STEFAN-BOLTZMANN LIMITS

In the high-temperature limit, the constituent quark masses approach zero, and the gluonic sector decouples, so the pressure reduces to that of a free three-flavor massless quark gas. The single-flavor pressure for one massless Dirac quark with N_c colors at $\hat{\mu}_f \equiv \mu_f/T$ reads [87]

$$\frac{p_f^{\text{SB}}(T, \mu_f)}{T^4} = N_c \left[\frac{7\pi^2}{180} + \frac{\hat{\mu}_f^2}{6} + \frac{\hat{\mu}_f^4}{12\pi^2} \right], \quad (\text{A1})$$

which is a quartic polynomial in $\hat{\mu}_f$. Differentiating with respect to $\hat{\mu}_f$ at $N_c = 3$ gives the only nonzero single-flavor susceptibilities,

$$\begin{aligned} \chi_1^{ff}|_{\text{SB}} &= \hat{\mu}_f + \frac{\hat{\mu}_f^3}{\pi^2}, & \chi_2^{ff}|_{\text{SB}} &= 1 + \frac{3\hat{\mu}_f^2}{\pi^2}, \\ \chi_3^{ff}|_{\text{SB}} &= \frac{6\hat{\mu}_f}{\pi^2}, & \chi_4^{ff}|_{\text{SB}} &= \frac{6}{\pi^2}, \end{aligned} \quad (\text{A2})$$

with $\chi_n^{ff}|_{\text{SB}} = 0$ for $n \geq 5$. Generalized BQS susceptibilities follow from the linear charge-matrix relation between $\hat{\mu}_f$ and the conserved-charge potentials of Eq. (10),

$$\chi_{ijk}^{\text{BQS}}|_{\text{SB}} = \sum_{f=u,d,s} B_f^i Q_f^j S_f^k \chi_n^{ff}|_{\text{SB}}, \quad n = i + j + k, \quad (\text{A3})$$

with $B_f = 1/3$ for all flavors, $(Q_u, Q_d, Q_s) = (2/3, -1/3, -1/3)$, and $(S_u, S_d, S_s) = (0, 0, -1)$. The values at $\hat{\mu}_B = \hat{\mu}_Q = \hat{\mu}_S = 0$ for every quantity computed in this work are collected in Table III. All sixth- and eighth-order diagonal susceptibilities and all odd-order baryon susceptibilities with $n \geq 5$ vanish at SB, since p_f^{SB}/T^4 is exactly quartic in $\hat{\mu}_f$.

Setting $\mu_Q = \mu_S = 0$ in Eq. (A3) gives the $\hat{\mu}_B$ -dependent SB limits

$$\chi_1^B|_{\text{SB}} = \frac{\hat{\mu}_B}{3} + \frac{\hat{\mu}_B^3}{27\pi^2}, \quad (\text{A4})$$

$$\chi_2^B|_{\text{SB}} = \frac{1}{3} + \frac{\hat{\mu}_B^2}{9\pi^2}, \quad (\text{A5})$$

$$\chi_3^B|_{\text{SB}} = \frac{2\hat{\mu}_B}{9\pi^2}. \quad (\text{A6})$$

TABLE III. Stefan-Boltzmann limits at $\hat{\mu}_B = \hat{\mu}_Q = \hat{\mu}_S = 0$ for the free three-flavor massless quark gas with $N_c = 3$, covering every quantity computed in this work. Slice-dependent values at finite μ_B for $\chi_1^B, \chi_2^B, \chi_3^B, \chi_2^Q$, and χ_2^S are given in Eqs. (A4)–(A8).

Observable	Expression	Observable	Expression
χ_2^B	1/3	χ_{11}^{BQ}	0
χ_2^Q	2/3	χ_{11}^{BS}	-1/3
χ_2^S	1	χ_{11}^{QS}	1/3
χ_4^B	2/(9 π^2)	χ_{31}^{BQ}	0
χ_4^Q	4/(3 π^2)	χ_{22}^{BQ}	4/(9 π^2)
χ_4^S	6/ π^2	χ_{13}^{BQ}	4/(9 π^2)
χ_{31}^{BS}	-2/(9 π^2)	χ_{31}^{QS}	2/(9 π^2)
χ_{22}^{BS}	2/(3 π^2)	χ_{22}^{QS}	2/(3 π^2)
χ_{13}^{BS}	-2/ π^2	χ_{13}^{QS}	2/ π^2
χ_{211}^{BQS}	2/(9 π^2)	C_{BS}	1
χ_{121}^{BQS}	-2/(9 π^2)	χ_{11}^{BQ}/χ_2^B	0
χ_{112}^{BQS}	-2/(3 π^2)	χ_{11}^{QS}/χ_2^S	1/3
R_{42}^B	2/(3 π^2)	χ_2^B/χ_2^Q	1/2
R_{42}^Q	2/ π^2	$\chi_5^B, \chi_6^B, \chi_6^Q, \chi_6^S, \chi_8^B, \chi_8^Q, \chi_8^S$	0
R_{42}^S	6/ π^2	$R_{62}^B, R_{62}^Q, R_{62}^S, R_{82}^B$	0

The charge and strangeness second-order limits are

$$\chi_2^Q|_{\text{SB}} = \frac{2}{3} + \frac{2\hat{\mu}_B^2}{9\pi^2}, \quad (\text{A7})$$

$$\chi_2^S|_{\text{SB}} = 1 + \frac{\hat{\mu}_B^2}{3\pi^2}. \quad (\text{A8})$$

$\chi_4^B|_{\text{SB}} = 2/(9\pi^2)$ becomes independent of $\hat{\mu}_B$, and $\chi_n^B|_{\text{SB}} = 0$ for $n \geq 5$. The slice-dependent SB arrows in Figs. 13(a) and 13(b) evaluate Eqs. (A4) and (A6) at the arrow-head temperature $T = 315$ MeV. At $\mu_B = 200$ MeV this gives $\chi_1^B|_{\text{SB}} = 0.2126$ and $\chi_3^B|_{\text{SB}} = 0.01430$. At $\mu_B = 400$ MeV, $\chi_1^B|_{\text{SB}} = 0.4310$ and $\chi_3^B|_{\text{SB}} = 0.02860$. The slice-dependent SB arrows in the top row of Fig. 10 likewise evaluate Eqs. (A5), (A7), and (A8) at $T = 315$ MeV. At $\mu_B = 500$ MeV this gives $\chi_2^B|_{\text{SB}} = 0.362$, $\chi_2^Q|_{\text{SB}} = 0.723$, and $\chi_2^S|_{\text{SB}} = 1.085$. The $\mu_B = 100$ and 300 MeV arrows sit close to the $\hat{\mu}_B = 0$ values 1/3, 2/3, and 1. Panels (b) and (c) of Fig. 11 use the same slice-dependent form, $-\chi_{11}^{BS}|_{\text{SB}} = \chi_{11}^{QS}|_{\text{SB}} = 1/3 + \hat{\mu}_B^2/(9\pi^2)$, while panel (a) keeps $\chi_{11}^{BQ}|_{\text{SB}} = 0$.

- [1] Y. Aoki, G. Endrődi, Z. Fodor, S. D. Katz, and K. K. Szabó, The order of the quantum chromodynamics transition predicted by the standard model of particle physics, *Nature (London)* **443**, 675 (2006).
- [2] A. Bazavov *et al.* (HotQCD Collaboration), Chiral cross-over in QCD at zero and non-zero chemical potentials, *Phys. Lett. B* **795**, 15 (2019).
- [3] S. Borsanyi *et al.* (Wuppertal-Budapest Collaboration), Is there still any T_c mystery in lattice QCD? Results with physical masses in the continuum limit III, *J. High Energy Phys.* **09** (2010) 073.
- [4] S. Borsanyi, Z. Fodor, J. N. Guenther, R. Kara, P. Parotto, A. Pasztor, L. Pirelli, and C. H. Wong, Chiral versus deconfinement properties of the QCD crossover, *Phys. Rev. D* **111**, 014506 (2025).
- [5] S. Borsanyi, Z. Fodor, J. N. Guenther, R. Kara, S. D. Katz, P. Parotto, A. Pasztor, C. Ratti, and K. K. Szabó, QCD crossover at finite chemical potential from lattice simulations, *Phys. Rev. Lett.* **125**, 052001 (2020).
- [6] A. M. Halasz, A. D. Jackson, R. E. Shrock, M. A. Stephanov, and J. J. M. Verbaarschot, Phase diagram of QCD, *Phys. Rev. D* **58**, 096007 (1998).
- [7] M. A. Stephanov, K. Rajagopal, and E. V. Shuryak, Signatures of the tricritical point in QCD, *Phys. Rev. Lett.* **81**, 4816 (1998).
- [8] M. A. Stephanov, QCD phase diagram: An overview, *Proc. Sci. LAT2006* (2006) 024.
- [9] A. Abuali, S. Borsanyi, Z. Fodor, J. N. Guenther, R. Kara, A. Pasztor, C. Ratti, and K. K. Szabó, New four-dimensional lattice QCD equation of state: Extended density coverage from a generalized T' -expansion, *Phys. Rev. D* **112**, 054502 (2025).
- [10] D. A. Clarke, F. Karsch, A. Lahiri, M. Neumann, and C. Schmidt, Searching for the QCD critical endpoint using multipoint Padé approximants, *Phys. Rev. D* **112**, L091504 (2025).
- [11] G. Başar, QCD critical point, Lee–Yang edge singularities, and Padé resummations, *Phys. Rev. C* **110**, 015203 (2024).
- [12] J. Adam *et al.* (STAR Collaboration), Nonmonotonic energy dependence of net-proton number fluctuations, *Phys. Rev. Lett.* **126**, 092301 (2021).
- [13] B. E. Aboona *et al.* (STAR Collaboration), Precision measurement of net-proton number fluctuations in Au + Au collisions at RHIC, *Phys. Rev. Lett.* **135**, 142301 (2025).
- [14] B. E. Aboona *et al.* (STAR Collaboration), Measurement of fifth- and sixth-order fluctuations of (net-)proton number in Au + Au collisions from phase II of the beam energy scan program at RHIC, *Phys. Rev. C* **113**, L051901 (2026).
- [15] B. E. Aboona *et al.* (STAR Collaboration), Measurements of proton high order cumulants in $\sqrt{s_{NN}} = 3$ GeV Au + Au

- collisions and implications for the QCD critical point, *Phys. Rev. Lett.* **128**, 202303 (2022).
- [16] A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov, and N. Xu, Mapping the phases of quantum chromodynamics with beam energy scan, *Phys. Rep.* **853**, 1 (2020).
- [17] A. Pandav, D. Mallick, and B. Mohanty, Search for the QCD critical point in high energy nuclear collisions, *Prog. Part. Nucl. Phys.* **125**, 103960 (2022).
- [18] X. Luo and N. Xu, Search for the QCD critical point with fluctuations of conserved quantities in relativistic heavy-ion collisions at RHIC: An overview, *Nucl. Sci. Tech.* **28**, 112 (2017).
- [19] V. Koch and V. Vovchenko, Exploring the QCD phase diagram through correlations and fluctuations, *Eur. Phys. J. Special Topics* (2026).
- [20] V. Koch, Hadronic fluctuations and correlations, in *Relativistic Heavy Ion Physics*, edited by R. Stock (Springer, New York, 2010), pp. 626–652.
- [21] M. Kitazawa and M. Asakawa, Relation between baryon number fluctuations and experimentally observed proton number fluctuations in relativistic heavy ion collisions, *Phys. Rev. C* **86**, 024904 (2012).
- [22] V. Koch, A. Majumder, and J. Randrup, Baryon-strangeness correlations: A diagnostic of strongly interacting matter, *Phys. Rev. Lett.* **95**, 182301 (2005).
- [23] S. Ejiri, F. Karsch, and K. Redlich, Hadronic fluctuations at the QCD phase transition, *Phys. Lett. B* **633**, 275 (2006).
- [24] Y. Hatta and T. Ikeda, Universality, the QCD critical and tricritical point, and the quark number susceptibility, *Phys. Rev. D* **67**, 014028 (2003).
- [25] A. Bazavov *et al.* (HotQCD Collaboration), Fluctuations and correlations of net baryon number, electric charge, and strangeness: A comparison of lattice QCD results with the hadron resonance gas model, *Phys. Rev. D* **86**, 034509 (2012).
- [26] S. Borsanyi, Z. Fodor, S. D. Katz, S. Krieg, C. Ratti, and K. K. Szabó, Fluctuations of conserved charges at finite temperature from lattice QCD, *J. High Energy Phys.* **01** (2012) 138.
- [27] R. Bellwied, S. Borsanyi, Z. Fodor, S. D. Katz, A. Pasztor, C. Ratti, and K. K. Szabó, Fluctuations and correlations in high temperature QCD, *Phys. Rev. D* **92**, 114505 (2015).
- [28] D. Bollweg, J. Goswami, O. Kaczmarek, F. Karsch, S. Mukherjee, P. Petreczky, C. Schmidt, and P. Scior (HotQCD Collaboration), Second order cumulants of conserved charge fluctuations revisited: I. Vanishing chemical potentials, *Phys. Rev. D* **104**, 074512 (2021).
- [29] F. Karsch, B.-J. Schaefer, M. Wagner, and J. Wambach, Towards finite density QCD with Taylor expansions, *Phys. Lett. B* **698**, 256 (2011).
- [30] A. Bazavov *et al.* (HotQCD Collaboration), Skewness, kurtosis, and the fifth and sixth order cumulants of net baryon-number distributions from lattice QCD confront high-statistics STAR data, *Phys. Rev. D* **101**, 074502 (2020).
- [31] K. Fukushima, Chiral effective model with the Polyakov loop, *Phys. Lett. B* **591**, 277 (2004).
- [32] C. Ratti, M. A. Thaler, and W. Weise, Phases of QCD: Lattice thermodynamics and a field theoretical model, *Phys. Rev. D* **73**, 014019 (2006).
- [33] S. K. Ghosh, T. K. Mukherjee, M. G. Mustafa, and R. Ray, Susceptibilities and speed of sound from the Polyakov–Nambu–Jona-Lasinio model, *Phys. Rev. D* **73**, 114007 (2006).
- [34] A. Bhattacharyya, P. Deb, A. Lahiri, and R. Ray, Susceptibilities with multiquark interactions in the PNJL model, *Phys. Rev. D* **82**, 114028 (2010).
- [35] J. Moreira, J. Morais, B. Hiller, A. A. Osipov, and A. H. Blin, Thermodynamical properties of strongly interacting matter in a model with explicit chiral symmetry breaking interactions, *Phys. Rev. D* **98**, 074010 (2018).
- [36] A. Sarkar, P. Deb, B. Mandal, and R. Varma, Study of net-baryon higher moments in PNJL model and their expectation for net-proton using the subensemble acceptance method for the search of QCD critical point, *Phys. Rev. D* **110**, 074023 (2024).
- [37] A. Sarkar, P. Deb, and R. Varma, Net strangeness fluctuations and their experimental implications in the SU(3) PNJL model using the subensemble acceptance method for the search of QCD critical point, *Eur. Phys. J. C* **86**, 107 (2026).
- [38] G.-y. Shao, Z.-d. Tang, X.-y. Gao, and W.-b. He, Baryon number fluctuations and the phase structure in the PNJL model, *Eur. Phys. J. C* **78**, 138 (2018).
- [39] M. Ferreira, P. Costa, and C. Providência, Presence of a critical endpoint in the QCD phase diagram from the net-baryon number fluctuations, *Phys. Rev. D* **98**, 034006 (2018).
- [40] B.-J. Schaefer, J. M. Pawłowski, and J. Wambach, Phase structure of the Polyakov-quark-meson model, *Phys. Rev. D* **76**, 074023 (2007).
- [41] B.-J. Schaefer, M. Wagner, and J. Wambach, Thermodynamics of $(2 + 1)$ -flavor QCD: Confronting models with lattice studies, *Phys. Rev. D* **81**, 074013 (2010).
- [42] B.-J. Schaefer and M. Wagner, QCD critical region and higher moments for three-flavor models, *Phys. Rev. D* **85**, 034027 (2012).
- [43] S. Chatterjee and K. A. Mohan, Fluctuations and correlations of conserved charges in the $(2 + 1)$ Polyakov–quark-meson model, *Phys. Rev. D* **86**, 114021 (2012).
- [44] S. Pal, G. Kadam, H. Mishra, and A. Bhattacharyya, Effects of hadronic repulsive interactions on the fluctuations of conserved charges, *Phys. Rev. D* **103**, 054015 (2021).
- [45] D. Bollweg *et al.*, Strangeness-correlations on the pseudocritical line in $(2 + 1)$ -flavor QCD, *Phys. Rev. D* **110**, 054519 (2024).
- [46] J. Goswami and F. Karsch, Higher order net-baryon number cumulants and baryon-strangeness correlations: Comparing QCD results on the pseudo-critical line with RHIC-BES II results on the freeze-out line, *EPJ Web Conf.* **364**, 15013 (2026).
- [47] X.-R. Yang, G.-Y. Shao, C.-L. Xie, and Z.-P. Li, Correlations of the net baryon number and electric charge in nuclear matter, *Nucl. Sci. Tech.* **36**, 138 (2025).
- [48] M. A. Stephanov, Non-Gaussian fluctuations near the QCD critical point, *Phys. Rev. Lett.* **102**, 032301 (2009).
- [49] M. A. Stephanov, Sign of kurtosis near the QCD critical point, *Phys. Rev. Lett.* **107**, 052301 (2011).
- [50] B. Friman, F. Karsch, K. Redlich, and V. Skokov, Fluctuations as probe of the QCD phase transition and freeze-out in heavy ion collisions at LHC and RHIC, *Eur. Phys. J. C* **71**, 1694 (2011).

- [51] S. Borsanyi, Z. Fodor, J. N. Guenther, S. D. Katz, P. Parotto, A. Pasztor, D. Pesznyak, K. K. Szabó, and C. H. Wong, Continuum-extrapolated high-order baryon fluctuations, *Phys. Rev. D* **110**, L011501 (2024).
- [52] A. Adam, S. Borsanyi, Z. Fodor, J. N. Guenther, P. Kumar, P. Parotto, A. Pasztor, and C. H. Wong, High-precision baryon number cumulants from lattice QCD in a finite box: Cumulant ratios, Lee-Yang zeros, and critical endpoint predictions, *Phys. Rev. D* **113**, 074525 (2026).
- [53] W. Fan, X. Luo, and H.-S. Zong, Mapping the QCD phase diagram with susceptibilities of conserved charges within Nambu–Jona-Lasinio model, *Int. J. Mod. Phys. A* **32**, 1750061 (2017).
- [54] W. Fan, X. Luo, and H.-S. Zong, Probing the QCD phase structure with higher-order baryon number susceptibilities within the NJL model, *Chin. Phys. C* **43**, 033103 (2019).
- [55] W.-j. Fu, X. Luo, J. M. Pawłowski, F. Rennecke, R. Wen, and S. Yin, Hyper-order baryon number fluctuations at finite temperature and density, *Phys. Rev. D* **104**, 094047 (2021).
- [56] W.-j. Fu, J. M. Pawłowski, and F. Rennecke, Strangeness neutrality and QCD thermodynamics, *SciPost Phys. Core* **2**, 002 (2020).
- [57] W.-j. Fu, X. Luo, J. M. Pawłowski, F. Rennecke, and S. Yin, Ripples of the QCD critical point, *Phys. Rev. D* **111**, L031502 (2025).
- [58] W.-j. Fu, J. M. Pawłowski, and F. Rennecke, QCD phase structure at finite temperature and density, *Phys. Rev. D* **101**, 054032 (2020).
- [59] W.-j. Fu, C. Huang, J. M. Pawłowski, F. Rennecke, A. E. Wen, and S. Yin, Strangeness neutrality and the QCD phase diagram, [arXiv:2603.13455](https://arxiv.org/abs/2603.13455).
- [60] Z. Li, J. Liang, S. He, and L. Li, Holographic study of higher-order baryon number susceptibilities at finite temperature and density, *Phys. Rev. D* **108**, 046008 (2023).
- [61] V. Skokov, B. Friman, E. Nakano, K. Redlich, and B.-J. Schaefer, Vacuum fluctuations and the thermodynamics of chiral models, *Phys. Rev. D* **82**, 034029 (2010).
- [62] S. Chatterjee and K. A. Mohan, Including the fermion vacuum fluctuations in the $(2+1)$ flavor Polyakov quark-meson model, *Phys. Rev. D* **85**, 074018 (2012).
- [63] U. S. Gupta and V. K. Tiwari, Revisiting the phase structure of the Polyakov-quark-meson model in the presence of vacuum fermion fluctuation, *Phys. Rev. D* **85**, 014010 (2012).
- [64] S. K. Rai and V. K. Tiwari, Phase structure of the on-shell parametrized $2+1$ flavor Polyakov quark-meson model, *Phys. Rev. D* **109**, 034025 (2024).
- [65] D. Singh and A. Kumar, Finite-size effects on the transport coefficients of strongly interacting QCD matter, *Phys. Rev. D* **111**, 074017 (2025).
- [66] D. Singh and A. Kumar, Impact of anisotropy on the QCD phase structure and transport coefficients of quark matter, *Phys. Rev. D* **113**, 054005 (2026).
- [67] M. Kumari and A. Kumar, Quark matter within Polyakov chiral SU(3) quark mean field model at finite temperature, *Eur. Phys. J. Plus* **136**, 19 (2021).
- [68] N. Chahal, S. Dutt, and A. Kumar, Quark matter properties and fluctuations of conserved charges in a $(2+1)$ -flavored quark model, *Chin. Phys. C* **46**, 063104 (2022).
- [69] N. Chahal, S. Dutt, and A. Kumar, Effects of finite volume and magnetic fields on thermodynamic properties of quark matter and fluctuations of conserved charges, *Phys. Rev. C* **107**, 045203 (2023).
- [70] P. Papazoglou, S. Schramm, J. Schaffner-Bielich, H. Stöcker, and W. Greiner, Chiral Lagrangian for strange hadronic matter, *Phys. Rev. C* **57**, 2576 (1998).
- [71] P. Papazoglou, D. Zschesche, S. Schramm, J. Schaffner-Bielich, H. Stöcker, and W. Greiner, Nuclei in a chiral SU(3) model, *Phys. Rev. C* **59**, 411 (1999).
- [72] P. Wang, Z. Y. Zhang, Y. W. Yu, R. K. Su, and H. Q. Song, Strange hadronic matter in a chiral SU(3) quark mean field model, *Nucl. Phys. A* **688**, 791 (2001).
- [73] P. Wang, V. E. Lyubovitskij, T. Gutsche, and A. Faessler, Strange quark matter in a chiral SU(3) quark mean field model, *Phys. Rev. C* **67**, 015210 (2003).
- [74] S. Rössner, C. Ratti, and W. Weise, Polyakov loop, diquarks, and the two-flavor phase diagram, *Phys. Rev. D* **75**, 034007 (2007).
- [75] L. M. Haas, R. Stiele, J. Braun, J. M. Pawłowski, and J. Schaffner-Bielich, Improved Polyakov-loop potential for effective models from functional calculations, *Phys. Rev. D* **87**, 076004 (2013).
- [76] J. Bradbury, R. Frostig, P. Hawkins, M. J. Johnson, Y. Katariya, C. Leary, D. Maclaurin, G. Nacula, A. Paszke, J. VanderPlas, S. Wanderman-Milne, and Q. Zhang, JAX: Composable transformations of Python + NumPy programs, <https://github.com/jax-ml/jax> (2018), version 0.4.35.
- [77] A. Bazavov *et al.* (HotQCD Collaboration), Equation of state and QCD transition at finite temperature, *Phys. Rev. D* **80**, 014504 (2009).
- [78] A. Bazavov *et al.* (HotQCD Collaboration), Equation of state in $(2+1)$ -flavor QCD, *Phys. Rev. D* **90**, 094503 (2014).
- [79] S. Borsanyi *et al.* (Wuppertal-Budapest Collaboration), Lattice QCD constraints on the critical point from an improved precision equation of state, *Phys. Rev. D* **112**, L111505 (2025).
- [80] A. Andronic, P. Braun-Munzinger, K. Redlich, and J. Stachel, Decoding the phase structure of QCD via particle production at high energy, *Nature (London)* **561**, 321 (2018).
- [81] L. Adamczyk *et al.* (STAR Collaboration), Bulk properties of the medium produced in relativistic heavy-ion collisions from the beam energy scan program, *Phys. Rev. C* **96**, 044904 (2017).
- [82] T. K. Herbst, J. M. Pawłowski, and B.-J. Schaefer, The phase structure of the Polyakov–quark-meson model beyond mean field, *Phys. Lett. B* **696**, 58 (2011).
- [83] S. Borsanyi, Z. Fodor, J. N. Guenther, S. D. Katz, A. Pasztor, I. Portillo, C. Ratti, and K. K. Szabó, Higher order fluctuations and correlations of conserved charges from lattice QCD, *J. High Energy Phys.* **10** (2018) 205.
- [84] Y. Lu, F. Gao, Y.-X. Liu, and J. M. Pawłowski, Finite density signatures of confining and chiral dynamics in QCD thermodynamics and fluctuations of conserved charges, *Phys. Rev. D* **113**, 054019 (2026).
- [85] D. Bollweg, D. A. Clarke, J. Goswami, O. Kaczmarek, F. Karsch, S. Mukherjee, P. Petreczky, C. Schmidt, and S. Sharma (HotQCD Collaboration), Equation of state and

- speed of sound of $(2 + 1)$ -flavor QCD in strangeness-neutral matter at nonvanishing net baryon-number density, [Phys. Rev. D **108**, 014510 \(2023\)](#).
- [86] M. Motta, R. Stiele, W.M. Alberico, and A. Beraudo, Isentropic evolution of the matter in heavy-ion collisions and the search for the critical endpoint, [Eur. Phys. J. C **80**, 770 \(2020\)](#).
- [87] J. I. Kapusta and C. Gale, *Finite-Temperature Field Theory: Principles and Applications*, 2nd ed. (Cambridge University Press, Cambridge, England, 2006).