

Type-II spacetimes and the double copy for fluids metrics

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(Received 25 April 2025; accepted 31 July 2025; published 25 August 2025)

In our previous paper [Keeler *et al.*, From Navier-Stokes to Maxwell via Einstein, *J. High Energy Phys.* **08** (2020) 147.] we discussed type-D and type-N fluid-dual spacetimes and provided their associated single copies in the context of the Weyl double copy. In this work we extend our analysis to more general fluids thereby requiring the application of the double-copy picture to type-II spacetimes. By combining our previous type-D and type-N fluids via their associated stream functions we construct an example of a viable type-II double copy. We show that the gravity duals of these fluids perturbatively satisfy Einstein's equations. We use a near-horizon expansion to identify the type-II double copy for the fluid-dual spacetimes. We show a Maxwell spinor ansatz containing a heterogeneous bispinor component is necessary to provide a viable type-II double copy at the lowest order.

DOI: 10.1103/6h4v-3l3r

I. INTRODUCTION

The Parke-Taylor formula of [1] succinctly demonstrates how n -gluon tree amplitudes, if they are maximal helicity violating, simplify significantly. At the same time, a broader understanding was obtained of how closed string amplitudes can be written as squares of open string amplitudes [2–6]. A culmination of these insights results in the double-copy paradigm [7–10]. Fundamentally, the double-copy picture posits that tree-level gravitational amplitudes can be “double-copies” of Yang-Mills amplitudes.

Recent formulations of the classical double copy make more explicit this classical/tree-level mapping between spin-2 and spin-1 fields. Work by the authors of [11] demonstrates that spacetimes which admit a Kerr-Schild-like expansion, i.e. when written as

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu \tilde{k}_\nu, \quad (1.1)$$

allow a mapping between the null vector fields k_μ and gauge fields A_μ^a . This allows us to identify Yang-Mills fields as

$$A_\mu^a \sim k_\mu c^a. \quad (1.2)$$

The Weyl double copy [12] provides a second formulation, using spinor notation to identify the Weyl spinor with the field strength spinors via

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$$C_{ABCD} = \frac{1}{5} f_{(AB} f_{CD)}. \quad (1.3)$$

Since this initial relation applies to type-D and type-N spacetimes, it was important to extend the Weyl double copy to more general Petrov types [13] by utilizing a twistor space representation (see [14–17] for more). More recently [18] construct examples of the Weyl double copy in general dimensions and provide novel examples in five dimensions, while [19–21] considered sources and symmetries in the Weyl context. See [22,23] for recent reviews.

We proceed from our previous work in [24] where we demonstrate how the Weyl double copy of [25] with the fluid-gravity duality of [26] can be utilized to map solutions of the Navier-Stokes equations to Maxwell's equations. We obtained a simple mapping between components of the fluid and their associated single- and double-copy fields.

On the gravity side, we found that irrotational potential fluid flows are associated with type-N spacetimes, while constant vorticity fluids are associated with type-D spacetimes. With respect to their associated single-copy gauge fields, the type-D, constant vorticity fluids allowed for a simple mapping between the fluid velocity onto gauge fields, or equivalently allowed the vorticity of the fluid to be proportional to the magnetic field of the single-copy gauge field [24]. The scalar potential associated with the potential flow of the type-N spacetime was mapped onto the zeroth-copy scalar field.

However, in [24] to map the single-copy fields onto fluids it was necessary to reduce the type-II fluid metric onto either type-N or type-D metrics. In this paper we extend our previous analysis and demonstrate that type-II

fluid metrics are amenable to the double-copy procedure. This stream function allows us to specify all two-dimensional incompressible fluids.

Although we work with a specific metric, which we write as a near-horizon expansion to Rindler space, throughout this paper we have attempted to write expressions in terms of generic geometric quantities—specifically Weyl scalars Ψ_2 and Ψ_4 . We do this as the restrictions we find in solving the Maxwell and scalar equations should be translatable to other spacetimes that can similarly be expressed as expansions near Rindler horizons.

The structure of this paper is as follows: in Sec. II we review the near-horizon expansion in the context of the fluid-gravity duality, highlighting our previous work [24] and restating its results in terms of a fluids stream function. In Sec. III we provide an example of a simple type-II fluid-dual metric by combining our previous type-D and type-N metrics. In Sec. IV we consider a more general form of single- and zeroth-copy fields for type-II fluid-dual spacetimes. In Appendix A we review the necessary Newman-Penrose formulation and set our notation, and

in Appendix B we relate the fluid vorticity equation and stream function to the Weyl scalars.

II. THE FLUIDS METRIC AND THE STREAM FUNCTION

In this section we summarize the methodology used. We first describe the near horizon limit of the fluid-dual metric as stated in [26] and used by us previously in [24], then we identify the Weyl scalars thus obtained and map them onto a scalar function for the fluid and the stream function, and finally we discuss the conditions necessary on these functions so as to solve all the necessary equations of motion.

A. The fluids metric in the near horizon limit and steps to a type-II double copy

In [26], perturbations on a timelike cutoff surface are mapped to a fluid-dual, referred to as the near-horizon expansion, and the near-null limit provides the following fluids dual spacetime metric:

$$\begin{aligned} d\hat{s}^2 = & -\frac{r}{\lambda} d\tau^2 \\ & + [2d\tau dr + dx_i dx^i - 2(1-r)v_i dx^i d\tau + (1-r)(v^2 + 2P)d\tau^2] \\ & + \lambda[(1-r)v_i v_j dx^i dx^j - 2v_i dx^i dr + (v^2 + 2P)d\tau dr \\ & + (r-1)[-(r+1)\partial^2 v_i + (v^2 + 2P)2v_i + 4\partial_i P]dx^i d\tau] + \mathcal{O}(\lambda^2). \end{aligned} \quad (2.1)$$

Here, λ is the near-horizon expansion parameter ($r = 0$ is the Rindler horizon). The fluid's velocity v_i in the transverse direction x_i is, in general, a function of all transverse directions x_j as well as of the time coordinate τ . The fluid's pressure P is also a function of the same coordinates. r , the radial direction of the gravitational dual solution, is explicitly listed wherever it appears.

Einstein's equations for this metric correspond to the Navier-Stokes equations for an incompressible fluid; specifically, we see

$$\begin{aligned} G_{00} = 0 & \Rightarrow \partial_i v_i = 0, \\ G_{0i} = 0 & \Rightarrow \partial_\tau v_i - \eta \partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0, \end{aligned} \quad (2.2)$$

and the Weyl scalars for this metric are

$$\begin{aligned} \Psi_2 &= \frac{-i}{4}(\partial_x v_y - \partial_y v_x) + \mathcal{O}(\lambda), \\ \Psi_4 &= \frac{-i}{2}(\partial_x v_y + \partial_y v_x - i(\partial_x v_x - \partial_y v_y)) + \mathcal{O}(\lambda). \end{aligned} \quad (2.3)$$

For fluids in two dimensions, we can characterize flows by a single scalar whose curl provides the velocity of the

fluid. This scalar, χ , is referred to as the stream function. That is, we have

$$v_i = \epsilon_{ij} \partial_j \chi. \quad (2.4)$$

Incompressibility follows trivially from the above relation since the divergence of a curl vanishes. Taking a curl of the above expression, we can see that the Laplacian of the stream function is related to the vorticity of the fluid,

$$\partial^2 \chi = -\omega(x, y). \quad (2.5)$$

Using the above, the Weyl scalars can be expressed directly in terms of this stream function¹

$$\Psi_2 = i\partial_z \partial_{\bar{z}} \chi, \quad \Psi_4 = 2i\partial_{\bar{z}}^2 \chi. \quad (2.6)$$

This identification of the Weyl scalars in terms of the stream function will have utility as we probe the

¹Note that these quantities are more clearly expressed in terms of holomorphic and antiholomorphic functions, and thus for the remainder of this paper we will be writing the stream function in terms of complex coordinates where $z = x + iy$.

existence of type-II fluid-dual metrics that satisfy the double copy.

1. Revisiting type-D and type-N double copies

We begin by revisiting the type-D and type-N double copies in [24] written in terms of the stream function χ . We require the following conditions to hold for type-D and type-N spacetimes, respectively:

- (1) *For type-D spacetimes* the Weyl scalar Ψ_4 must vanish (see [27] for a comprehensive overview); that is, from Eq. (2.6) we require that the stream function χ_D be such that

$$\psi_4 = 2i\partial_{\bar{z}}^2\chi_D = 0. \quad (2.7)$$

Along with the above, requiring the type-D fluid velocity to be real gives us the following form for the type-D stream function as

$$\chi_D = A(\tau)z + \bar{A}(\tau)\bar{z} + B(\tau)z\bar{z}. \quad (2.8)$$

Such a stream function is associated with a spatially constant Weyl scalar Ψ_2 , and more precisely from Eqs. (2.5) and (2.6) we find

$$\Psi_2 = -i\frac{\omega}{2} = iB(\tau). \quad (2.9)$$

- (2) *For type-N spacetimes* we can identify the stream function such that the Weyl scalar Ψ_2 vanishes, giving us a type-N fluid. Therefore, we find

$$\Psi_2 = i\partial_z\partial_{\bar{z}}\chi_N = 0. \quad (2.10)$$

A stream function χ_N that is a sum of holomorphic and antiholomorphic pieces easily satisfies this constraint. Such a stream function can be identified with a fluid potential associated with a vanishing Laplacian. Such fluid flows are commonly referred to as potential flows. The associated stream function can be written as

$$\chi_N = -if(z, \tau) + i\bar{f}(\bar{z}, \tau). \quad (2.11)$$

In the next section we first describe a simple example of a type-II double copy obtained by simply combining the type-D and type-N fluid stream functions above in Eqs. (2.9) and (2.11). Subsequently, we examine more general type-II double-copy fields and discuss the constraints associated with satisfying the associated Maxwell and scalar field equations of motion.

III. A SIMPLE TYPE-II FLUIDS DOUBLE COPY

The Weyl spinor associated with a type-II spacetime can be written as²

$$C_{ABCD} = 6\Psi_2 o_{(A} l_B o_C l_{D)} + \Psi_4 o_A o_B o_C o_D. \quad (3.1)$$

In the context of the double copy this Weyl spinor is mapped to a zeroth-copy scalar S and a mixed single copy, with Maxwell spinors $f_{AB}^{(1)}$ and $f_{AB}^{(2)}$. In contrast to the type-D case, the associated Maxwell spinors need not be identical,

$$\frac{1}{S}f_{(AB}^{(1)}f_{CD)}^{(2)} = C_{ABCD}. \quad (3.2)$$

The simplest type-II double copy can be constructed by combining stream functions for type-D and type-N fluids via

$$\chi_{II} = \chi_D + \chi_N. \quad (3.3)$$

A simple stream function that provides us with a type-II double copy of the form as written in Eq. (3.2) will correspond to a velocity at most linear in coordinates. The stream function for such a fluid is

$$\chi(\tau, z, \bar{z}) = A(\tau)z + \bar{A}(\tau)\bar{z} + B(\tau)z\bar{z} + i\bar{C}(\tau)\bar{z}^2 - iC(\tau)z^2. \quad (3.4)$$

Flows between parallel plates, i.e. Couette flows, are a simple example of fluids with such stream functions. Utilizing the mapping between the stream function and Weyl scalars as provided in Eq. (2.6), we find the following Weyl scalars for this type-II spacetime:

$$\Psi_2 = iB(\tau), \quad \Psi_4 = -4\bar{C}(\tau). \quad (3.5)$$

The form of our Maxwell spinors that follows from Eq. (3.2) is

$$f_{AB}^{(i)} = P^{(i)} o_{(A} l_{B)} + Q^{(i)}(\tau) o_{(A} o_{B)}. \quad (3.6)$$

Here, to satisfy equations of motion, $P^{(1)}$ and $P^{(2)}$ must be constant, while $Q^{(1)}$ and $Q^{(2)}$ can only depend on time.

The double-copy relation Eq. (3.2) consists of five separate equations, one for each Weyl spinor component; in this case, only three of the equations are nontrivial. Solving these three equations yields the Maxwell field

²Note, however, a Weyl spinor with other nonvanishing components may also be type-II; however, it can be rotated to the provided form via a tetrad rotation.

$$\begin{aligned}
f_{AB}^{(1)} &= P^{(1)} o_{(A} l_{B)} + 2P^{(1)} \sqrt{\frac{\bar{C}}{iB}} o_A o_B, \\
f_{AB}^{(2)} &= P^{(2)} o_{(A} l_{B)} - 2iP^{(2)} \sqrt{\frac{\bar{C}}{iB}} o_A o_B,
\end{aligned} \quad (3.7)$$

while the associated zeroth copy that solves the scalar wave equation is

$$S_{II} = -i \frac{P^{(1)} P^{(2)}}{B(\tau)} \Rightarrow \square S_{II} = 0. \quad (3.8)$$

Here, $P^{(1)}$ and $P^{(2)}$ are arbitrary constants, but $B(\tau)$ and $\bar{C}(\tau)$ are set by the specific stream function of the fluid whose gravity dual is under discussion.

The Maxwell and scalar wave equations we solve live on a Rindler-like background metric. This metric can be obtained from the full metric in Eq. (2.1) by setting the fluid velocity and pressures to vanish:

$$d\hat{s}^2 = -\frac{r}{\lambda} d\tau^2 + 2d\tau dr + dx_i dx^i. \quad (3.9)$$

Taken together Eqs. (3.4)–(3.8) provide us with a viable type-II double copy of the form Eq. (3.2) which satisfies the scalar wave and Maxwell equations on the background metric above in Eq. (3.9). These Maxwell spinors may not provide the most general single-copy fields. However, the single-copy fields do satisfy their respective spin-1 Maxwell and spin-0 wave equation on the Rindler-like background.

As mentioned earlier, the stream function Eq. (3.4) includes Couette flow. In the simplest (planar) case of Couette flow, $A = 0$, B is a real constant, and $C = B/2i$. This stream function describes a fluid with $v_y = 0$ and $v_x = -4By$, as expected for the fluid flow between two parallel flat plates separated in the y direction, when the bottom plate is held steady while the top plate moves in the x direction. Since this is a steady state solution, the scalar and Maxwell spinors are all constants. Interestingly, the two Maxwell fields point in different directions; both have an F^{rr} component, but $f^{(1)}$ has nonzero F^{rx} and $F_{\tau x}$ whereas $f^{(2)}$ instead has nonzero F^{ry} and $F_{\tau y}$. It is the specific combination of components between the two fields which conspire to give the fluid with only v_x nonzero, and linearly dependent on y .

In the next section we consider more general type-II solutions and examine the constraints that arise while trying to satisfy the equations of motion for the associated single and zeroth copies.

IV. GENERALIZING THE TYPE-II DOUBLE COPY

In the previous section we examined how a type-II fluid double copy was achievable, however for a relatively

simple fluid velocity profile, one with a velocity at most linear in its spatial coordinates and a vorticity that is spatially invariant.

In this section we attempt a more general double copy and identify the constraints that present themselves while extending the double-copy picture for the fluids metric more complex fluid flows. To do so, in the next two subsections we list out the general functional forms of these Maxwell spinors as well as the scalars that satisfy the Maxwell and scalar wave equations on the background.

A. General solution for the scalar equation on the fluids background

For the scalar function S , we perturbatively expand $O(\lambda)$,

$$S \equiv S^{(0)} + \lambda S^{(1)} + \lambda^2 S^{(2)} + O(\lambda^3). \quad (4.1)$$

We are interested in identifying a general functional form of this function such that the scalar equation $\square S = 0$ on the background metric as stated in Eq. (3.9) above are satisfied. Plugging the above ansatz into $\square S = 0$ we find

$$O(\lambda^{-1}): \partial_r(r\partial_r S^{(0)}) = 0, \quad (4.2)$$

$$O(\lambda^0): \partial_r(r\partial_r S^{(1)}) + 2\partial_\tau \partial_r S^{(0)} + \partial_x^2 S^{(0)} + \partial_y^2 S^{(0)} = 0, \quad (4.3)$$

$$O(\lambda^1): \partial_r(r\partial_r S^{(2)}) + 2\partial_\tau \partial_r S^{(1)} + \partial_x^2 S^{(1)} + \partial_y^2 S^{(1)} = 0. \quad (4.4)$$

At higher orders a similar pattern as the one noted in Eq. (4.4) continues; i.e. we have

$$\begin{aligned}
O(\lambda^i): \partial_r(r\partial_r S^{(i+1)}) + 2\partial_\tau \partial_r S^{(i)} \\
+ \partial_x^2 S^{(i)} + \partial_y^2 S^{(i)} = 0 \quad \forall i > 0.
\end{aligned} \quad (4.5)$$

At the lowest order, $O(\lambda^{-1})$, from Eq. (4.2) we have

$$\begin{aligned}
\partial_r(r\partial_r S^{(0)}(\tau, r, x, y)) &= 0 \Rightarrow S^{(0)} \\
&= S_a^{(0)}(\tau, x, y) + \log(r) S_b^{(0)}(\tau, x, y).
\end{aligned} \quad (4.6)$$

Plugging in this lowest order solution for the Maxwell scalar we have at the next higher order at $O(\lambda)$

$$\begin{aligned}
\partial^2(S_a^{(0)}(\tau, x, y) + \log(r) S_b^{(0)}(\tau, x, y)) \\
+ \frac{2}{r} \partial_\tau S_b^{(0)}(\tau, x, y) + \partial_r(r\partial_r S^{(1)}(\tau, r, x, y)) = 0.
\end{aligned} \quad (4.7)$$

The above expression can be integrated in r to identify the r -dependence for the next higher order piece in the scalar function $S^{(1)}$. We have

$$\begin{aligned}
& S^{(1)}(\tau, r, x, y) \\
& = \log(r) S_a^{(1)}(\tau, x, y) + S_b^{(1)}(\tau, x, y) - r \partial^2 S_a^{(0)}(\tau, x, y) \\
& \quad - r(\log(r) - 2) \partial^2 S_b^{(0)}(\tau, x, y) - \log(r)^2 \partial_\tau S_b^{(0)}(\tau, x, y). \quad (4.8)
\end{aligned}$$

We can continue to iterate so to obtain components of the scalar that satisfy the wave equation at higher orders. The r -dependence of the next higher order piece is informed by the lower order piece before it.

The forms of Eqs. (4.6) and (4.8) will be useful as we proceed in this paper to understand where the type-II double copy works and where it may not.

B. General solution for the wave equation on the fluids background

We begin with a look at a generic Maxwell spinor, f_{AB} , and examine the conditions necessary for it to satisfy Maxwell's equations. Note that in our conventions (see Appendix A) our principal null spinors are

$$o_A = \frac{1}{\sqrt{2}} \{1, 1\} \quad \text{and} \quad i_A = \frac{1}{\sqrt{2}} \{1, -1\}. \quad (4.9)$$

A general Maxwell spinor may thus have the form

$$\begin{aligned}
f_{AB} &= P(\tau, r, z, \bar{z}) o_{(A} i_{B)} + Q(\tau, r, z, \bar{z}) o_{(A} o_{B)} \\
&\quad + R(\tau, r, z, \bar{z}) i_{(A} i_{B)}, \quad (4.10)
\end{aligned}$$

or equivalently (suppressing the functional dependence),

$$f_{AB} = \frac{1}{2} \begin{pmatrix} P + Q + R & Q - R \\ Q - R & -P + Q + R \end{pmatrix}, \quad (4.11)$$

where

$$\bar{f}_{AB} = (f_{AB})^* = (P \rightarrow \bar{P}, Q \rightarrow \bar{Q}). \quad (4.12)$$

The corresponding Maxwell tensor for this spinor has the form

$$F_{ab} = \frac{1}{4} \begin{pmatrix} 0 & -(P + \bar{P}) & -i(Q - \bar{Q} + R - \bar{R}) & (Q + \bar{Q} - R - \bar{R}) \\ (P + \bar{P}) & 0 & -i(Q - \bar{Q} - R + \bar{R}) & (Q + \bar{Q} + R + \bar{R}) \\ i(Q - \bar{Q} + R - \bar{R}) & i(Q - \bar{Q} - R + \bar{R}) & 0 & i(P - \bar{P}) \\ -(Q + \bar{Q} - R - \bar{R}) & -(Q + \bar{Q} + R + \bar{R}) & -i(P - \bar{P}) & 0 \end{pmatrix}. \quad (4.13)$$

Using the vierbiens and noting the frame tensor we can construct the Maxwell tensor as $F_{\mu\nu} \equiv F_{ab} e_\mu^a e_\nu^b$, and then write the exterior derivatives dF and d^*F as

$$\begin{aligned}
dF_{\tau x} &= -\partial_x(P + \bar{P}) + \frac{i}{\sqrt{\lambda}} \partial_r[r(R - \bar{R}) + (Q - \bar{Q})] \\
&\quad + 2i\sqrt{\lambda} \partial_\tau[R - \bar{R}], \quad (4.14)
\end{aligned}$$

$$\begin{aligned}
dF_{\tau y} &= -\partial_y(P + \bar{P}) + \frac{1}{\sqrt{\lambda}} \partial_r[r(R + \bar{R}) - (Q + \bar{Q})] \\
&\quad + 2\sqrt{\lambda} \partial_\tau[R + \bar{R}], \quad (4.15)
\end{aligned}$$

$$\begin{aligned}
dF_{\tau y} &= i\partial_\tau(P - \bar{P}) + \frac{1}{\sqrt{\lambda}} \partial_x[r(R + \bar{R}) - (Q + \bar{Q})] \\
&\quad - \frac{i}{\sqrt{\lambda}} \partial_y[r(R - \bar{R}) + (Q - \bar{Q})], \quad (4.16)
\end{aligned}$$

$$\begin{aligned}
dF_{\tau y} &= i\partial_r(P - \bar{P}) - 2\sqrt{\lambda} \partial_x(R + \bar{R}) + 2i\sqrt{\lambda} \partial_y(R - \bar{R}), \quad (4.17)
\end{aligned}$$

$$\begin{aligned}
d^*F_{\tau x} &= i\partial_x(P - \bar{P}) + \frac{1}{\sqrt{\lambda}} \partial_r[r(R + \bar{R}) + (Q + \bar{Q})] \\
&\quad + 2\sqrt{\lambda} \partial_\tau[R + \bar{R}], \quad (4.18)
\end{aligned}$$

$$\begin{aligned}
d^*F_{\tau y} &= i\partial_y(P - \bar{P}) - \frac{i}{\sqrt{\lambda}} \partial_r[r(R - \bar{R}) - (Q - \bar{Q})] \\
&\quad - 2i\sqrt{\lambda} \partial_\tau[R - \bar{R}], \quad (4.19)
\end{aligned}$$

$$d^*F_{\tau y} = \partial_r(P + \bar{P}) + 2\sqrt{\lambda} \partial_y(R + \bar{R}) + 2i\sqrt{\lambda} \partial_x(R - \bar{R}), \quad (4.20)$$

$$\begin{aligned}
d^*F_{\tau y} &= \partial_\tau(P + \bar{P}) - \frac{i}{\sqrt{\lambda}} \partial_x[r(R - \bar{R}) - (Q - \bar{Q})] \\
&\quad - \frac{1}{\sqrt{\lambda}} \partial_y[r(R + \bar{R}) + (Q + \bar{Q})]. \quad (4.21)
\end{aligned}$$

To correctly identify order by order solutions to the Maxwell's equations as written above we first identify expansions in functions P (similarly for Q and R) in λ ,

$$P \equiv P^0 \lambda^0 + P^{1/2} \lambda^{1/2} + P^1 \lambda^1 + O(\lambda^{3/2}). \quad (4.22)$$

By utilizing the expansion above we see that the lowest order contribution for Maxwell's equations is singularly provided by the $O(\lambda^{-1/2})$ coefficients in Eq. (4.14) to Eqs. (4.21),

$$dF_{\tau x} = \partial_r[r(R^0 - \bar{R}^0) + (Q^0 - \bar{Q}^0)] = 0, \quad (4.23)$$

$$dF_{\tau\gamma} = -\partial_\tau[r(R^0 + \bar{R}^0) - (Q^0 + \bar{Q}^0)] = 0, \quad (4.24)$$

$$dF_{\text{rx}\gamma} = 0, \quad (4.25)$$

$$dF_{\text{rx}\gamma} = \partial_x[r(R^0 + \bar{R}^0) - (Q^0 + \bar{Q}^0)] - i\partial_y[r(R^0 - \bar{R}^0) + (Q^0 - \bar{Q}^0)], \quad (4.26)$$

$$d^*F_{\tau\text{rx}} = \partial_\tau[r(R^0 + \bar{R}^0) + (Q^0 + \bar{Q}^0)] = 0, \quad (4.27)$$

$$d^*F_{\tau\gamma} = -\partial_\tau[r(R^0 - \bar{R}^0) - (Q^0 - \bar{Q}^0)] = 0, \quad (4.28)$$

$$d^*F_{\text{rx}\gamma} = 0, \quad (4.29)$$

$$d^*F_{\text{rx}\gamma} = \partial_x[r(R^0 - \bar{R}^0) - (Q^0 - \bar{Q}^0)] - i\partial_y[r(R^0 + \bar{R}^0) + (Q^0 + \bar{Q}^0)]. \quad (4.30)$$

In the next subsection we will see how an identification of the Weyl scalars with these functions P, Q, and R at different orders in λ corresponds to solving Maxwell's equations.

C. Mapping the Weyl scalars to P, Q, and R and conditions on solving Maxwell's equations

A more correct approach to solving Maxwell's equations for the case at hand is to consider solutions order by order in the expansion parameter for the Maxwell spinor and map them to the Weyl tensor. Table 1 gives the nonvanishing orders of our Maxwell spinor.

Using the above, we map an order by order expansion in the Maxwell spinor functions P, Q, and R onto the Weyl scalars Ψ_i via the double copy [12], i.e. by utilizing Eq. (1.3) above,

$$\frac{1}{S} f_{(AB}^{(1)} f_{CD)}^{(2)} = \Psi_0 l_A l_B l_C l_D + 4\Psi_1 l_{(A} l_B l_C o_{D)} + 6\Psi_2 o_{(A} l_B o_C l_{D)} + 4\Psi_3 l_{(A} o_B o_C o_{D)} + \Psi_4 o_A o_B o_C o_D. \quad (4.31)$$

Recall that we have identified what a general form of the Maxwell spinor may look like in terms of functions P, Q, and R; thus, for each of the spinors $f^{(1)}$ and $f^{(2)}$ we have

TABLE I. Lowest order nonvanishing components of Weyl scalars associated with the near-horizon fluids metric in Eq. (2.1). We also state at which orders perturbative limits of the metric in Eq. (2.1) translate to errors in aforementioned Weyl scalars.

	Lowest order nonvanishing component	Error at $O(\lambda^k)$
Ψ_0	First nonzero term is an error term	$O(\lambda)$
Ψ_1	$O(\lambda^{1/2})$	$O(\lambda^{3/2})$
Ψ_2	$O(\lambda^0)$	$O(\lambda^1)$
Ψ_3	$O(\lambda^{1/2})$	$O(\lambda^{3/2})$
Ψ_4	$O(\lambda^0)$	$O(\lambda^1)$

$$f_{AB}^{(i)} = P(\tau, r, z, \bar{z})^{(i)} o_{(A} l_{B)} + Q(\tau, r, z, \bar{z})^{(i)} o_{(A} o_{B)} + R(\tau, r, z, \bar{z})^{(i)} l_{(A} l_{B)}. \quad (4.32)$$

Utilizing the above expansion for both Maxwell spinors along with the double copy,

$$C_{ABCD} = \frac{1}{S_{II}} f_{(AB}^{(1)} f_{CD)}^{(2)}, \quad (4.33)$$

we now have

$$C_{ABCD} = \frac{1}{S_{II}} \{ [P^{(1)} P^{(2)} + (Q^{(1)} R^{(2)} + Q^{(2)} R^{(1)})] o_{(A} l_B o_C l_{D)} + [Q^{(1)} Q^{(2)}] o_A o_B o_C o_D + [R^{(1)} R^{(2)}] l_A l_B l_C l_D + (P^{(1)} R^{(2)} + P^{(2)} R^{(1)}) l_{(A} l_B l_C o_{D)} + [(Q^{(1)} P^{(2)} + P^{(2)} Q^{(1)})] l_{(A} o_B o_C o_{D)} \}. \quad (4.34)$$

Matching the above Weyl spinor to the Weyl scalars via the double-copy Eq. (4.33) we have

$$\Psi_0 = \frac{1}{S_{II}} R^{(1)} R^{(2)} = 0 + O(\lambda), \quad (4.35)$$

$$\begin{aligned} \Psi_1 &= \frac{1}{S_{II}} (P^{(1)} R^{(2)} + P^{(2)} R^{(1)}) \\ &= \Psi_1^{1/2} \lambda^{1/2} + O(\lambda^{3/2}), \end{aligned} \quad (4.36)$$

$$\begin{aligned} \Psi_2 &= \frac{1}{S_{II}} [P^{(1)} P^{(2)} + (Q^{(1)} R^{(2)} + Q^{(2)} R^{(1)})] \\ &= \Psi_2^0 \lambda^0 + O(\lambda^1), \end{aligned} \quad (4.37)$$

$$\begin{aligned} \Psi_3 &= \frac{1}{S_{II}} (Q^{(1)} P^{(2)} + P^{(2)} Q^{(1)}) \\ &= \Psi_3^{1/2} \lambda^{1/2} + O(\lambda^{3/2}), \end{aligned} \quad (4.38)$$

$$\Psi_4 = \frac{1}{S_{II}} Q^{(1)} Q^{(2)} = \Psi_4^0 \lambda^0 + O(\lambda), \quad (4.39)$$

where the superscripts not in parentheses on the Weyl scalars indicate the nonzero orders of the Weyl scalars. Following (4.22), we write expansions for functions P (and similarly for Q and R), for the two Maxwell spinors,

$$P^{(i)} \equiv P^{(i),0} \lambda^0 + P^{(i),1/2} \lambda^{1/2} + P^{(i),1} \lambda^1 + O(\lambda^{3/2}), \quad (4.40)$$

the first superscript index $^{(i)}$ indicates which Maxwell spinor the functions P, Q, or R belong to while the second superscript index (not in parentheses) follows the λ order in the expansion. We now have the components necessary to map the expansions in functions P, Q, and R order by order to the nonzero terms in various Weyl scalars as stated in

Eqs. (4.35) to (4.39). At the lowest informative order in this map, i.e. at $O(\lambda^0)$, we have

$$R^{(1),0}R^{(2),0} = 0, \quad (4.41)$$

$$P^{(1),0}R^{(2),0} + P^{(2),0}R^{(1),0} = 0, \quad (4.42)$$

$$P^{(1),0}P^{(2),0} + (Q^{(1),0}R^{(2),0} + Q^{(2),0}R^{(1),0}) = \frac{\Psi_2^0}{S_{II}^0}, \quad (4.43)$$

$$Q^{(1),0}P^{(2),0} + Q^{(2),0}P^{(1),0} = 0, \quad (4.44)$$

$$Q^{(1),0}Q^{(2),0} = \frac{\Psi_4^0}{S_{II}^0}. \quad (4.45)$$

The above expressions result in two kinds of meaningfully distinct solutions,

- (1) the first with no P in either $f^{(1)}$ or $f^{(2)}$ at lowest order,

$$\begin{aligned} P^{(1),0} = P^{(2),0} = 0, \quad Q^{(2),0} &= \frac{\Psi_4^0}{Q^{(1),0}} S_{II}^0, \\ R^{(1),0} = 0, \quad R^{(2),0} &= \frac{\Psi_2^0}{Q^{(1),0}} S_{II}^0; \end{aligned} \quad (4.46)$$

- (2) the second with no R in either $f^{(1)}$ or $f^{(2)}$ at lowest order,

$$\begin{aligned} R^{(1),0} = R^{(2),0} = 0, \quad P^{(1),0} &= iQ^{(1),0} \sqrt{\frac{\Psi_2^0}{\Psi_4^0}}, \\ P^{(2),0} &= -i \frac{\sqrt{\Psi_2^0 \Psi_4^0}}{Q^{(1),0}} S_{II}^0, \quad Q^{(2),0} = \frac{\Psi_4^0}{Q^{(1),0}} S_{II}^0. \end{aligned} \quad (4.47)$$

In the subsequent subsections we examine Maxwell and scalar equations associated with the two solutions above in more detail. We will see that only the second of these solutions, Eq. (4.47), results in a viable single copy; i.e. we satisfy both Maxwell's and scalar equations of motion up to $O(\lambda^{1/2})$, while the first solution Eq. (4.46) does not provide a nonvanishing scalar.

1. Maxwell's equations with vanishing function P in both spinors

Functions P, Q, and R as identified by the first solution in Eq. (4.46) result in the following pair of Maxwell spinors:

$$\begin{aligned} f_{AB}^{(1)} &= Q^{(1),0} o_{(A} o_{B)} + O(\lambda^{1/2}), \\ f_{AB}^{(2)} &= \frac{\Psi_4^0}{Q^{(1),0}} S_{II}^0 o_{(A} o_{B)} + \frac{\Psi_2^0}{Q^{(1),0}} S_{II}^0 l_{(A} l_{B)} + O(\lambda^{1/2}). \end{aligned} \quad (4.48)$$

Since Maxwell spinor $f_{AB}^{(1),0}$ (at its lowest relevant order) above only depends on $Q^{(1),0}$, requiring Maxwell's equations $dF^{(1)} = 0$ and $d^*F^{(1)} = 0$ to hold at their lowest corresponding order $O(\lambda^{-1/2})$ results in the following conditions:

$$\partial_i(Q^{(1),0} - \bar{Q}^{(1),0}) = 0 \quad \text{and} \quad \partial_i(Q^{(1),0} + \bar{Q}^{(1),0}) = 0$$

where $i \in \{r, x, y\}$. (4.49)

The above expressions imply that the function $Q^{(1)}$ is independent of r, x, y ; i.e. we have

$$Q^{(1),0} = Q^{(1),0}(\tau). \quad (4.50)$$

Similarly for the second Maxwell spinor $f_{AB}^{(2)}$ we find that the following conditions must hold in order for Maxwell's equations $dF^{(2)} = 0$ and $d^*F^{(2)} = 0$ to hold:

$$\begin{aligned} \partial_r[r(R^{(2),0} - \bar{R}^{(2),0}) - (Q^{(2),0} - \bar{Q}^{(2),0})] &= 0, \\ \partial_r[r(R^{(2),0} + \bar{R}^{(2),0}) - (Q^{(2),0} + \bar{Q}^{(2),0})] &= 0, \\ \partial_x[r(R^{(2),0} + \bar{R}^{(2),0}) - (Q^{(2),0} + \bar{Q}^{(2),0})] &= 0, \\ -i\partial_y[r(R^{(2),0} - \bar{R}^{(2),0}) + (Q^{(2),0} - \bar{Q}^{(2),0})] &= 0, \\ \partial_x[r(R^{(2),0} - \bar{R}^{(2),0}) - (Q^{(2),0} - \bar{Q}^{(2),0})] &= 0, \\ -i\partial_y[r(R^{(2),0} + \bar{R}^{(2),0}) + (Q^{(2),0} + \bar{Q}^{(2),0})] &= 0. \end{aligned} \quad (4.51)$$

The first two expressions above can be rewritten as

$$\partial_r(rR^{(2),0} - Q^{(2),0}) = 0 \quad \text{and} \quad \text{c.c.} \quad (4.52)$$

Substituting for functions $R^{(2),0}$ and $Q^{(2),0}$ from Eq. (4.46) above, we have

$$\partial_r \left(r \frac{\Psi_2^0}{Q^{(1),0}} S_{II}^0 - \frac{\Psi_4^0}{Q^{(1),0}} S_{II}^0 \right) = 0. \quad (4.53)$$

Using the fact that $Q^{(1),0}$ is r -independent from Eq. (4.50), that the Weyl scalars have no r -dependence from Eq. (2.6), and that the r -dependence in the zeroth-copy scalar is logarithmic in r from Eq. (4.6) we have

$$\begin{aligned} \frac{\Psi_2^0}{Q^{(1),0}} \partial_r(r(S_a^{(0)}(\tau, x, y) + \log(r)S_b^{(0)}(\tau, x, y))) \\ - \frac{\Psi_4^0}{Q^{(1),0}} \partial_r(S_a^{(0)}(\tau, x, y) + \log(r)S_b^{(0)}(\tau, x, y)) = 0. \end{aligned} \quad (4.54)$$

Since functions S_a and S_b are r -independent, the above expression must be solved for coefficients of $r, \log(r)$, and the only solution that is permissible is if the scalar vanishes.

2. Maxwell's equations with vanishing function R in both spinors

Given that the first solution in Eq. (4.46) did not provide a viable double copy, we now proceed to consider the second set of solutions as provided in Eq. (4.47), where at lowest order the function R vanishes; i.e. the coefficient of $\iota_{(A}\iota_{B)}$ is vanishing in both $f^{(1)}$ and $f^{(2)}$, and this provides the following pair of Maxwell spinors:

$$\begin{aligned} f_{AB}^{(1)} &= iQ^{(1),0} \sqrt{\frac{\Psi_2^0}{\Psi_4^0}} o_{(A} \iota_{B)} + Q^{(1),0} o_{(A} o_{B)} + O(\lambda^{1/2}), \\ f_{AB}^{(2)} &= -i \frac{\sqrt{\Psi_2^0 \Psi_4^0}}{Q^{(1),0}} S_{II}^0 o_{(A} \iota_{B)} + \frac{\Psi_4^0}{Q^{(1),0}} S_{II}^0 o_{(A} o_{B)} + O(\lambda^{1/2}). \end{aligned} \quad (4.55)$$

The lowest order, $O(\lambda^{-1/2})$, Maxwell's equations when solved provide solutions to the $O(\lambda^0)$ components for functions, i.e. $Q^{(1),0}$ and $Q^{(2),0}$; specifically we find the following conditions:

$$\partial_k(Q^{(i),0} - \bar{Q}^{(i),0}) = 0 \quad \text{and} \quad \partial_k(Q^{(i),0} + \bar{Q}^{(i),0}) = 0, \quad (4.56)$$

where $k \in \{r, x, y\}$ and $i \in \{1, 2\}$.

These differential equations imply that $Q^{(1),0}$ and $Q^{(2),0}$ are functionally independent of r, x , and y and may only carry a dependence on Rindler time coordinate τ , that is, we have

$$Q^{(i),0}(\tau, r, x, y) = Q^{(i),0}(\tau). \quad (4.57)$$

Recall from Eq. (4.47) we have for $Q^{(2),0}$ [i.e. coefficient of $o_{(A} o_{B)}$ in $f_{AB}^{(2)}$, Eq. (4.55)]

$$Q^{(2),0} = \frac{\Psi_4^0}{Q^{(1),0}} S_{II}^0. \quad (4.58)$$

Consistency between Eqs. (4.57) and (4.58), i.e. that both $Q^{(i),0}$ are only time dependent implies

$$\partial_k(\Psi_4^0 S_{II}^0) = 0, \quad \text{where } k \in \{r, x, y\}. \quad (4.59)$$

The r -independence of Weyl scalars Eq. (2.6) will now require that the zeroth-copy scalar S_{II}^0 must be r -independent, and further its x, y dependence is inversely proportional to the Weyl scalar Ψ_4 . That is, we have

$$\partial_r S_{II}^0 = 0 \quad \text{and} \quad S_{II}^0(x, y) \sim \frac{1}{\Psi_4^0}. \quad (4.60)$$

The above type-II scalar satisfies lowest order scalar equation of motion Eq. (4.2). We thus have a lowest order type-II zeroth and single copies that satisfy the associated lowest order Maxwell and scalar equations of motion.

We can continue our iterative procedure; following Eq. (4.3) we see that the next higher order component of the type-II scalar will be sourced via the Laplacian of $S_{II}^0(\tau, x, y)$.

Similarly, examining Maxwell's equations at the next higher order allows us to put constraints on the functions $P^{(i),0}$. From dF_{rxy} in Eq. (4.17) and d^*F_{rxy} in Eq. (4.20), the $O(\lambda^0)$ components give us the following condition:

$$\partial_r P^{(i),0}(\tau, r, x, y) = 0. \quad (4.61)$$

The functions $P^{(i)}$ in terms of Weyl scalars in Eq. (4.55) depend on the functions $Q^{(i)}$, S_{II} , and the Weyl scalars as

$$P^{(1)} = iQ^{(1),0} \sqrt{\frac{\Psi_2^0}{\Psi_4^0}}, \quad P^{(2)} = -i \frac{\sqrt{\Psi_2^0 \Psi_4^0}}{Q^{(1),0}} S_{II}^0. \quad (4.62)$$

The r -independence of the functions on the right-hand sides of the above expressions allows us to satisfy Eq. (4.61).

However, to fully solve the next order Maxwell equation, we further need to solve $dF_{\tau r x}$, $dF_{\tau r y}$, and $dF_{\tau x y}$ in Eqs. (4.14) to (4.16) and the associated Hodge-duals in Eqs. (4.18) to (4.20). From these expressions, we note that τ, x and y derivatives for $P^{(i),0}$ depend on $O(\lambda^{1/2})$ components of Q and R . We now need to map these $O(\lambda^{1/2})$ components of functions Q and R on to Weyl scalars in (4.35) to (4.39). Using Eqs. (4.31), (4.33), and (4.34) we thus identify at $O(\lambda^{1/2})$,

$$Q^{(2),1/2} = -\frac{Q^{(1),1/2} Q^{(2),0}}{\Psi_4^0}, \quad (4.63)$$

$$P^{(1),1/2} = -\psi_2^0 \psi_3^{1/2} + \psi_4^0 \psi_1^{1/2}, \quad (4.64)$$

$$P^{(2),1/2} = \frac{Q^{(2),0} \psi_1^{1/2}}{\psi_2^0}, \quad (4.65)$$

$$R^{(2),1/2} = 0, \quad (4.66)$$

$$R^{(1),1/2} = \frac{Q^{(1),1/2} \psi_2^0}{\psi_4^0}. \quad (4.67)$$

As we see from Eqs. (4.63), (4.66), and (4.67), we are able to identify the remaining functional dependencies for the function P in terms of $O(\lambda^0)$ components of Weyl scalars Ψ_2 and Ψ_4 as well as functions $Q^{(i),0}$ and $Q^{(i),1/2}$. For our tetrad choice we are thus able to obtain leading order solutions to Maxwell's and scalar wave equations; however, for next higher orders a different tetrad choice is necessary. Beyond $O(\lambda^{1/2})$ we see that further higher-order solutions are restricted by error pieces in the metric.

Consequently we thus observe that for type-II fluid-dual spacetimes, Maxwell spinors of the form Eq. (4.48) do not

provide viable double-copy structures, while Maxwell spinors of the form Eq. (4.55) can be shown to solve background wave and scalar equations at leading order within the double-copy paradigm. Indeed, the simple case considered in Eqs. (3.7) and (3.8) is a special case of Eq. (4.48).

V. CONCLUSIONS

The Weyl double-copy picture has been applied to a large class of metrics. While the initial proposal was specific to type-D spacetimes [12], twistor space formulations [13,14] have shown how the Weyl double copy applies to algebraically general spacetimes, i.e. all the way up to Petrov type-I metrics.

Here we revisit the fluid-gravity duality in the context of the double-copy paradigm [24]. While in [24], we constrained ourselves to type-D and type-N algebraically special spacetimes, and here we extend the double-copy picture to include more general type-II spacetimes.

Our approach in this paper has been guided by the two-dimensional fluid stream function, which we utilize to map geometric Weyl scalars onto specific fluid solutions. Further, the stream function provides a convenient tool that shows how the dynamics of the fluid are related to that of the spacetime. For instance, in Appendix B, we show clearly how the dynamics of Weyl scalar Ψ_2 are very similar to the dynamics of the fluid vorticity.

In line with the above emphasis on the stream function, we see that the simplest type-II fluid that satisfies constraints Eq. (3.1) as well as the scalar and Maxwell equations of motion can be obtained by linearly combining stream functions associated with type-D and type-N fluids metrics.

While considering more general solutions, we find that not all classes of consistent factorization of type-II fluid solutions Eqs. (3.1) provide a nontrivial zeroth copy that satisfies the scalar equation of motion. Perturbatively, using our tetrad in Sec. IV C 2, we see that nontrivial single- and zeroth-copy fields that satisfy algebraic constraints and the equations of motion to lowest order are possible only if the coefficient of the bispinor $\iota_{(A}\iota_{B)}$ vanishes.

Our approach is inherently limited by the perturbative nature of the fluids metric we consider [26]. To alleviate such perturbative limitations one may instead consider a more general fluids picture as proposed in [28] and construct associated single- and zeroth-copy fields. Another interesting extension of our work will be possible if one were to consider relativistic fluids instead [29].

The classical double copy similarly provides an important and insightful tool for studying black hole horizons [30]. Similarly, the fluid-gravity paradigm in its various formulations [26,31–38] allows for a deeper understanding of horizon dynamics and symmetries as fluids. Thus, a broader examination of various fluid-dual geometries can provide physically insightful perspectives toward how

asymptotic symmetries of such geometries map to their associated single-copy factorization and fluid-duals.

ACKNOWLEDGMENTS

N. M. and C. K. thank Tucker Manton for discussions on the topic. C. K. is supported by the U.S. Department of Energy under Grant No. DE-SC0019470 and by the Heising-Simons Foundation “Observational Signatures of Quantum Gravity” collaboration Grant No. 2021-2818.

DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: NEWMAN-PENROSE FORMALISM

Similar to our approach in [24], we once again follow the Newman-Penrose formalism [27]. This formalism allows us to rewrite various geometric and gauge degrees of freedom as outer products of spinors. Central to the formalism is a rewriting of the metric tensor in terms of a null tetrad,

$$g_{\mu\nu} = -l_{(\mu}n_{\nu)} + m_{(\mu}\bar{m}_{\nu)}. \quad (\text{A1})$$

As outlined in [39], one utilizes the above tetrad set $\{l, n, m, \bar{m}\}$ to obtain a set of spin coefficients which are then utilized to write various Weyl scalars, $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \}$,

$$\begin{aligned} \Psi_0 &= D\sigma - \delta\kappa - (\rho + \bar{\rho} + 3\varepsilon + \bar{\varepsilon})\sigma + (\tau - \bar{\pi} + \bar{\alpha} + 3\beta)\kappa, \\ \Psi_1 &= D\beta - \delta\varepsilon - (\alpha + \pi)\sigma - (\bar{\rho} - \bar{\varepsilon})\beta + (\mu + \gamma)\kappa + (\bar{\alpha} - \bar{\pi})\varepsilon, \\ \Psi_2 &= D\mu - \delta\pi + (\varepsilon + \bar{\varepsilon} - \bar{\rho})\mu + (\bar{\alpha} - \beta - \bar{\pi})\pi + \nu\kappa - \sigma\lambda - R/12, \\ \Psi_3 &= \bar{\delta}\gamma - \Delta\alpha + (\rho + \varepsilon)\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma, \\ \Psi_4 &= \bar{\delta}\nu - \Delta\lambda - (\mu + \bar{\mu} + 3\gamma - \bar{\gamma})\lambda + (3\alpha + \bar{\beta} + \pi - \bar{\tau})\nu, \end{aligned} \quad (\text{A2})$$

where the following are directional derivatives:

$$\begin{aligned} D &= l^\mu \nabla_\mu, & \Delta &= n^\mu \nabla_\mu, \\ \delta &= m^\mu \nabla_\mu, & \bar{\delta} &= \bar{m}^\mu \nabla_\mu. \end{aligned} \quad (\text{A3})$$

In particular, the author of [39] allows an easier computation of spin coefficients $\sigma, \kappa, \rho, \dots$, by using partial derivatives. Finally, in terms of these Weyl scalars one can rewrite the Weyl spinor,

$$\begin{aligned} C_{ABCD} &= \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A} \iota_B \iota_C \iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)} \\ &\quad - 4\Psi_3 o_{(A} o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D. \end{aligned} \quad (\text{A4})$$

The spinors o_A, ι_A are related to the frame metric choice one makes. We will make explicit this connection now. The metric written in terms of vierbiens has the form

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \quad \text{where } \eta_{ab} = \text{diag}\{-1, 1, 1, 1\}. \quad (\text{A5})$$

The frame metric η_{ab} can itself be written as outer products of a tetrad set, and this will allow us to make identifications between the vierbiens and the tetrad set:

$$\begin{aligned} \eta_{ab} &= -\hat{l}_{(a} \hat{n}_{b)} + \hat{m}_{(a} \hat{\bar{m}}_{b)} \\ \Rightarrow g_{\mu\nu} &= e_\mu^a e_\nu^b (-\hat{l}_{(a} \hat{n}_{b)} + \hat{m}_{(a} \hat{\bar{m}}_{b)}) \\ \Rightarrow g_{\mu\nu} &= -l_{(\mu} n_{\nu)} + m_{(\mu} \bar{m}_{\nu)}, \end{aligned} \quad (\text{A6})$$

where in the last step we have made the identifications

$$\begin{aligned} e_\mu^a \hat{l}_a &= l_\mu, & e_\mu^a \hat{n}_a &= n_\mu, \\ e_\mu^a \hat{m}_a &= m_\mu, & e_\mu^a \hat{\bar{m}}_a &= \bar{m}_\mu. \end{aligned} \quad (\text{A7})$$

Now the tetrad set that reproduces the Minkowski frame metric is

$$\begin{aligned} \hat{l}_a &= \frac{1}{\sqrt{2}} \{1, -1, 0, 0\} \\ \hat{n}_a &= \frac{1}{\sqrt{2}} \{1, 1, 0, 0\} \\ \hat{m}_a &= \frac{1}{\sqrt{2}} \{0, 0, i, 1\} \\ \hat{\bar{m}}_a &= \frac{1}{\sqrt{2}} \{0, 0, -i, 1\}. \end{aligned} \quad (\text{A8})$$

The expression (A7) can be inverted to go from tetrad to vierbiens via the following:

$$\begin{aligned} e_\mu^0 &= \frac{1}{\sqrt{2}} (l_\mu + n_\mu), & e_\mu^1 &= \frac{1}{\sqrt{2}} (l_\mu - n_\mu), \\ e_\mu^2 &= \frac{i}{\sqrt{2}} (\bar{m}_\mu - m_\mu), & e_\mu^3 &= \frac{1}{\sqrt{2}} (m_\mu + \bar{m}_\mu). \end{aligned} \quad (\text{A9})$$

In order to obtain the spinors $\{o_A, \iota_A\}$ we write these *ISO* (3,1) four vectors in an $SL(2, C)$ representation by contracting them with Pauli vectors, which in our conventions are the set $\sigma_{AA}^a = \{\mathbb{I}, \vec{\sigma}\}$,

$$\begin{aligned} o_A o_{\dot{A}} &\equiv \hat{l}_a \sigma_{A\dot{A}}^a = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \Rightarrow o_A &= \frac{1}{\sqrt{2}} \{1, 1\}, \\ \iota_A \iota_{\dot{A}} &\equiv \hat{n}_a \sigma_{A\dot{A}}^a = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ \Rightarrow \iota_A &= \frac{1}{\sqrt{2}} \{1, -1\}. \end{aligned} \quad (\text{A10})$$

Further noting that one can transform from $SL(2, C)$ left to right by complex conjugation, we use the convention

$$(o_A)^* \equiv o_{\dot{A}}. \quad (\text{A11})$$

This further verifies that the two remaining contractions will hold the following relations correctly (albeit simpler given that our choice of dyads is real):

$$\begin{aligned} \hat{m}_a \sigma_{A\dot{A}}^a &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \iota_A o_{\dot{A}}, \\ \hat{\bar{m}}_a \sigma_{A\dot{A}}^a &= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = o_A \iota_{\dot{A}}. \end{aligned} \quad (\text{A12})$$

In the near horizon (λ) expansion the vierbiens $e_\mu^{(0),a}$, as obtained by setting the velocities and pressures to zero in the fluids metric in Eq. (2.1), are

$$\begin{aligned} e_\mu^{(0),0} &= \left\{ \frac{1+r}{2\sqrt{\lambda}}, -\sqrt{\lambda}, 0, 0 \right\}, \\ e_\mu^{(0),1} &= \left\{ \frac{1-r}{2\sqrt{\lambda}}, \sqrt{\lambda}, 0, 0 \right\}, \\ e_\mu^{(0),2} &= \{0, 0, 1, 0\}, \\ e_\mu^{(0),3} &= \{0, 0, 0, 1\}, \end{aligned} \quad (\text{A13})$$

which reproduce the background metric

$$\eta_{ab} e_\mu^{(0),a} e_\nu^{(0),b} = g_{\mu\nu}^0 = \begin{pmatrix} -\frac{r}{\lambda} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A14})$$

Similarly, we can construct the Maxwell tensor together, i.e. $F_{\mu\nu} = F_{ab} e_\mu^{(0),a} e_\nu^{(0),b}$.

The algebraic speciality of the spacetime is identified by the computation of various covariant combinations of the Weyl scalars,

$$\begin{aligned} I &\equiv \Psi_0 \Psi_4 - 4\Psi_1 \Psi_3 + 3\Psi_2^2, \\ J &\equiv \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}, \\ K &\equiv \Psi_1 \Psi_4^2 - 3\Psi_4 \Psi_3 \Psi_2 + 2\Psi_3^3, \\ L &\equiv \Psi_2 \Psi_4 - \Psi_3^2, \\ N &\equiv 12L^2 - \Psi_4^2 I. \end{aligned} \quad (\text{A15})$$

Of interest to us are spacetimes referred to as being algebraically special. For the spacetime to be algebraically special, it must minimally satisfy [27]

$$I^3 - 27J^2 = 0. \quad (\text{A16})$$

Referred to as Petrov types, when algebraically special, such spacetimes may be most generically Petrov type-II spacetimes. With additional constraints they may further be subclassified into type-D or type-N spacetimes, as was the case in our previous paper [24], or be further simplified to be type-III or type-O spacetimes. For a type-II spacetime, it is always possible to choose a frame such that both Ψ_0 and Ψ_1 vanish. For type-III, Ψ_0, Ψ_1, Ψ_2 can be set to vanish, while for type-N it is possible to choose only Ψ_4 to be nonzero. In the type-D case, the frame can be chosen so $\Psi_0, \Psi_1, \Psi_3, \Psi_4$ all vanish. Note that this vanishing is frame dependent; only a frame aligned with the repeated null principal vectors of the spacetime will exhibit vanishing components of the Weyl spinor. One advantage of using the actual invariants I, J, K, L, N is that their relations are independent of frame choice.

APPENDIX B: A NOTE ON THE VORTICITY EQUATION

Using the Navier-Stokes equation it is possible to identify an expression for the vorticity of the fluid. Recall the Navier-Stokes equation reads

$$\partial_\tau v_i - \eta \partial^2 v_i + \partial_i P + v_j \partial_j v_i = 0 \quad \forall i \in \{1, 2\}. \quad (\text{B1})$$

Taking the curl of the Navier-Stokes equation we get

$$\partial_\tau \epsilon_{ij} \partial_i v_j - \eta \partial^2 \epsilon_{ij} \partial_i v_j + v_k \partial_k \epsilon_{ij} \partial_i v_j = 0, \quad (\text{B2})$$

where the gradient of the pressure vanishes following the curl. In terms of the vorticity, $\omega = \epsilon_{ij} v_i v_j$, the above expression has the form

$$\partial_\tau \omega - \eta \partial^2 \omega + v_k \partial_k \omega = 0. \quad (\text{B3})$$

For our purposes we would like to write this equation in terms of the stream function χ which provides the velocity of the fields via

$$v_i = \epsilon_{ij} \partial_j \chi. \quad (\text{B4})$$

As stated in Eq. (2.5), the vorticity of the fluid can thus be identified in terms of the stream function:

$$\partial^2 \chi = -\omega(x, y). \quad (\text{B5})$$

The above equation allows us to rewrite Eq. (B3) in terms of the stream function. We find

$$i \partial_\tau \partial_z \partial_{\bar{z}} \chi - 4 \eta \partial_z^2 \partial_{\bar{z}}^2 \chi + 2 \partial_z \chi \partial_z (\partial_z \partial_{\bar{z}} \chi) - 2 \partial_z \chi \partial_{\bar{z}} (\partial_z \partial_{\bar{z}} \chi) = 0. \quad (\text{B6})$$

Recall from (2.6) that we were able to rewrite Weyl scalars in terms of the stream function:

$$\Psi_2 = i \partial_z \partial_{\bar{z}} \chi, \quad \Psi_4 = 2 i \partial_z^2 \chi. \quad (\text{B7})$$

We can now identify the Weyl scalar Ψ_2 (or Ψ_4) in Eq. (B6), finally allowing us to rewrite the fluid vorticity equation in Eq. (B3) as

$$\partial_\tau \Psi_2 - 4 \eta \partial_z \partial_{\bar{z}} \Psi_2 - 2 i (\partial_z \chi \partial_z \Psi_2 - 2 \partial_z \chi \partial_{\bar{z}} \Psi_2) = 0. \quad (\text{B8})$$

Rearranging terms in Eq. (B8) and utilizing Eq. (B4), we are able to express dynamics of Weyl scalar Ψ_2 in a very similar form to that of the vorticity of the fluid in Eq. (B3):

$$\partial_\tau \Psi_2 - 4 \eta \partial_z \partial_{\bar{z}} \Psi_2 + v_z \partial_z \Psi_2 + v_{\bar{z}} \partial_{\bar{z}} \Psi_2 = 0. \quad (\text{B9})$$

The above expression can be further canonicalized using a convective derivative, $D_\tau \equiv \partial_\tau + v_j \partial_j$, where $j \in \{z, \bar{z}\}$, becoming

$$4 \eta \partial_z \partial_{\bar{z}} \Psi_2 = D_\tau \Psi_2, \quad D_\tau \Psi_2 \equiv \frac{d\Psi_2(\tau, x, y)}{d\tau} = \partial_\tau \Psi_2 + v_j \partial_j \Psi_2. \quad (\text{B10})$$

Note that the Navier-Stokes equation can also be written more compactly in terms of the same convective derivative D_τ :

$$\eta \partial^2 \omega = D_\tau \omega, \quad D_\tau \omega \equiv \frac{d\omega(\tau, x, y)}{d\tau} = \partial_\tau \omega + v_j \partial_j \omega. \quad (\text{B11})$$

There are two simple classes of solutions that emerge to involve studying the vorticity equation in the inviscid regime (with the viscosity turned off) or by turning off the directional derivative, which reduces to the heat equation. Succinctly, we have two regimes:

- (1) Turning the viscosity off. This regime gives $D_\tau \omega = 0$. These solutions are often referred to as point vortex solutions. Following a similar procedure while searching for a suitable Weyl scalar we have

$$D_\tau \Psi_2 = 0. \quad (\text{B12})$$

- (2) Fluid solutions where $v_j \partial_j \omega = 0$. These solutions reduce the vorticity equation to the heat equation, and such solutions are referred to as Lamb-Oseen vortices. In terms of the Weyl scalar this corresponds to

$$4 \eta \partial_z \partial_{\bar{z}} \Psi_2 = \partial_\tau \Psi_2. \quad (\text{B13})$$

The above equations Eqs. (B9) to (B13) can thus be utilized to construct additional fluid solutions as necessary.

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