

# Convergent sum of EFT corrections to the Schwarzschild metric requires UV locality

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Corrections to vacuum black hole solutions of general relativity (GR) are considered in an effective field theory (EFT) framework, perturbatively in EFT coefficients, focusing on the Schwarzschild solution of GR. We find dominant corrections to the Schwarzschild metric in all orders in the derivative expansion far away from the horizon. These corrections can be summed up in a closed form through EFT coefficients up to all orders in derivatives and to the second order in curvature. It occurs that such a summation is convergent only for localizable theories, making a direct connection between the graviton scattering amplitudes properties and the applicability of a perturbative treatment of an EFT of gravity. We further apply our results to logarithmic form factors which appear in the one-loop effective action for GR in four dimensions. We find out that the corresponding corrections to the Schwarzschild metric are stronger than those from the tree-level EFT operators. The developed framework can be extended to account for the corrections to the other black hole solutions in GR, such as the Kerr metric.

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## I. INTRODUCTION

Corrections to General Relativity (GR) are inevitable and appear in many contexts. Construction of the one-loop effective action [1–4] to GR revealed the appearance of higher curvature and higher derivative and even nonlocal operators. String corrections suggest that higher curvature corrections naturally arise [5]. An effective field theory (EFT) approach [6] to GR [7] generates an infinite tower of operators of all positive mass dimensions arranged in Hilbert series [8]. The more recent idea of asymptotic safety in gravity [9–13] brings out a construction based on the RG flow considerations and the existence of an asymptotically safe fixed point. This construction consists of various terms in addition to an initial GR action arranged

with the help of operator functions of the d'Alembertian, the so-called form factors. Moreover, an idea of higher and infinite derivative gravity theories was developed by many authors in an attempt to pinpoint a unitary and renormalizable gravity theory [14–20].

Presence of new terms in an action nearly always implies that known solutions should be modified. In particular, known black hole (BH) solutions of GR are expected to change upon inclusion of GR corrections. Despite several known results (see [21] for a BH solution with a Gauss-Bonnet term in five dimensions and a short review [22] regarding exact BH solutions in Lanczos-Lovelock models of gravity [23]), solving gravitational field equations in generalized theories of gravity is known to be a problem of immense difficulty. Very interesting results were obtained in quadratic gravity [14] combining analytic and perturbative methods [24,25]. However, in a case when corrections to GR are treated perturbatively, one can also implement perturbative schemes to find corrections to known GR solutions. In this case coefficients of the GR corrections play the role of small perturbation parameters. Perturbative corrections to BH solutions in such an approach in an EFT of GR were intensively studied. Remarkable results concerning the corrections to the metric of vacuum BH solutions are presented in [26–34]. Recently, such

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corrections for the case of near extremal Kerr metric received a lot of attention due to the significant modifications in the observables [35]. Finding and analyzing the dominant perturbative corrections to all orders in an EFT of gravity expansion far away from the horizon to BH solutions in GR using Schwarzschild as the primary departure point comprises the context of the present paper.

It is clear that, in the case of EFT expansion, a set of operators determined by the Hilbert series represents a derivative expansion. It means that the perturbations are, in fact, higher derivative terms compared to those already present in the model. This can give more solutions to equations of motion, including unstable ones. In the EFT context, one should be careful not to take such solutions into account, as they usually cannot be properly described by EFT, since the EFT description will break. The very existence of such unstable solutions depends on the UV completion, which is unknown in the case of gravity. Moreover, even for small corrections to GR solutions, it is not guaranteed that the perturbative construction of the solution is a convergent procedure. There are two reasons for this.

The first reason is as follows. Consider the EFT corrected action truncated to a certain order of mass dimension and derivatives. As the equations are nonlinear, the next order in EFT couplings of the truncated action may be larger than the previous one for certain values of the spacetime coordinates (presumably, in the near-horizon limit of a BH). This can well be the case of near-extremal BH-s, where linear order in EFT couplings naively predicts a near-horizon EFT breakdown [36]. However, it seems to be more accurate to expect that, in reality, this behavior corresponds to the breaking of the linear approximation of the perturbative scheme of constructing a corrected solution [35–40].

The other reason is connected to the truncation of the infinite sequence of the EFT operators. The question is whether it is correct to assume that only a finite subset of operators is responsible for the dynamics at low energies, even below the cutoff scale. It is important to note at this point that consistent EFTs having unitary, causal, and local UV completion must have an infinite subset of operators leading to the  $s^n$  terms in the forward limit ( $t \rightarrow 0_-$ ) of the  $2 \rightarrow 2$  scattering amplitudes [41] ( $s, t, u$  are standard Mandelstam variables here). These terms are constrained by positivity bounds and are required to be present without a large hierarchy among them [42]. In general, corrections to BH solutions generated by EFT operators should form a power series in  $1/r$ . One would naively expect that for a large enough BH, only the first corrections are important. But the situation can be more subtle because under a certain choice of the sequence of EFT couplings, these series can have too small (not allowing to continue the solution from spatial infinity to the horizon) or even zero radius of convergence.

In our work, we focus on studying whether and when a perturbative approximation scheme for finding a vacuum BH solution in the EFT of gravity will be well defined and result in a physically meaningful outcome. We mainly consider an EFT parametrization capturing, among any given mass dimension, all the operators with the maximal number of derivatives acting on a metric field. Such operators are naturally expected to provide dominant corrections to BH solutions among those of the same mass dimension, and these operators have the form

$$\mathcal{O}_n = C_{\mu\nu\lambda\rho} \frac{\square^n}{\Lambda^{2n}} C^{\mu\nu\lambda\rho} \quad (1)$$

for  $n \geq 0$  where  $C_{\alpha\beta\gamma\delta}$  is the Weyl tensor,  $\square$  is the covariant d'Alembert operator, and  $\Lambda$  is the mass parameter associated with the EFT cutoff scale. Although by a sequence of transformations (including the use of the Bianchi identity and field redefinitions), it is possible to rewrite it in a familiar form like in EFT of GR [8] (where it is also explained why only the powers of the Weyl tensor and covariant derivatives are relevant), we will be using the just written form in order to capture the dominant corrections. Such operators allow us to make a link to the properties of the nonperturbative graviton propagator, as they do change the perturbative propagator, while operators with higher degree in the Weyl tensor obviously do not [43]. Remarkably, we will find a tight relation between the very possibility of constructing a BH solution perturbatively and the property of UV locality of the graviton propagator and amplitudes. Additionally, it is possible to consider a series of such corrections

$$\sum_{n \geq 0} f_{C_n} C_{\mu\nu\lambda\rho} \frac{\square^n}{\Lambda^{2n}} C^{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} \mathcal{F}(\square/\Lambda^2) C^{\mu\nu\lambda\rho} \quad (2)$$

and perform a resummation of the contributions of such operators for a BH solution. Here  $f_{C_n}$  are dimensionless coefficients which are EFT coefficients and are to be treated perturbatively.

On the other hand, a similar kinds of operator comprises corrections arising in the one-loop effective action for GR. The latter situation is more subtle as it requires an inclusion of an operator with a logarithmic form factor in four dimensions,

$$\mathcal{O}_{\log} = \gamma(\mu) C_{\mu\nu\lambda\rho} \log(\square/\mu^2) C^{\mu\nu\lambda\rho} \quad (3)$$

where  $\mu$  is the renormalization scale. Being a nonanalytic function at the origin, it requires a special treatment. To start with, even deriving equations of motion is already a problematic task as the latter relies on the Taylor expansion of a form factor and a convergence of the corresponding series. We will find that meaningful results can be obtained by analytically continuing results for monomial operators

containing  $\square^n/\Lambda^{2n}$  and will show that the one-loop contribution with a logarithmic form factor yields the dominant correction to BH solutions compared to those resulting from power law form factors.

The paper is organized as follows. In Sec. II, we set up the model and present the equations of motion to be used in the analysis. In Sec. III, we study dominant corrections to Schwarzschild solution in four dimensions coming from operators of kinds (1) and (2). An extension to an arbitrary dimension  $D > 4$  is done in the Appendix A. In Sec. IV, we study the convergence of various series obtained in Sec. III and in the Appendix A, as well as the physical implications of different sets of EFT parameters. In Sec. V, we attack the situation of operators with a logarithmic form factor (3) arising in the one-loop effective action in GR in four dimensions. We then summarize our findings in the Conclusion section and outline questions for future study. Some more technical expressions and series are presented in extra Appendixes.

## II. MODEL SETUP

The model under consideration is

$$S = \int d^D x \sqrt{-g} \left( \frac{M_P^{D-2}}{2} R + \frac{\lambda M_P^{D-4}}{2} C_{\mu\nu\alpha\beta} \mathcal{F}_C(\square/\Lambda^2) C^{\mu\nu\alpha\beta} \right). \quad (4)$$

Here  $M_P$  is the Planck mass.  $\mathcal{F}_C(\square/\Lambda^2)$  is an operator function of the d'Alembert operator, a form factor. It is assumed to be compatible with the IR limit of GR. This does not necessarily imply to be analytic in the origin as, for example, a log-type form factor does also give a correct IR limit.<sup>1</sup> Here  $\lambda$  is a dimensionless coupling which will be used as the perturbation parameter and thus will play an essential role in our analysis.  $\Lambda$  is the scale of higher derivative modifications, which can have different interpretations, as explained in the Introduction. Note that the perturbative unitarity demands the presence of another term  $\frac{\lambda M_P^{D-4}}{2} R \mathcal{F}_R(\square/\Lambda^2) R$  and, moreover, the form factors should be related as  $\mathcal{F}_R(\square) + \frac{D-3}{D-1} \mathcal{F}_C(\square) = 0$  (more details in case of  $D = 4$  can be found in [45]). However, this extra term is irrelevant to our analysis, as it will be demonstrated in the next paragraph.

Our goal is to find perturbative corrections in  $\lambda$  to existing BH solutions of GR. Terms quadratic and of a higher order in  $R$  (or  $R_{\mu\nu}$ ) can be neglected as they will vanish in the perturbative scheme in the linear order in  $\lambda$ . In this paper, we will concentrate on the linear order in perturbations and, therefore, can neglect such operators.

<sup>1</sup>There is a recent analysis with fraction-type form factors also targeting a similar task of constructing perturbatively corrected solutions for BHs [44].

Moreover, we will build the analysis looking for corrections to the Schwarzschild metric, leaving other important solutions, including the Kerr metric for a rotating BH, for future projects. One naturally expects the new solutions to be BHs, but we should exercise some caution since we have to demonstrate that the perturbative approach provides solutions that can be extended to the horizon. It is clear that the suggested approach is in line with the paradigm of an EFT and is very much justified physically, since we are discussing corrections to IR results considering UV model corrections.

Equations of motion for action (4) in the vacuum have the form

$$\begin{aligned} E_\nu^\mu &= -M_P^{D-2} G_\nu^\mu + \lambda M_P^{D-4} \Delta_\nu^\mu \\ &= -M_P^{D-2} G_\nu^\mu \\ &\quad + \lambda M_P^{D-4} \left[ \frac{1}{2} \delta_\nu^\mu C_{\gamma\delta\alpha\beta} \mathcal{F}_C(\square/\Lambda^2) C^{\gamma\delta\alpha\beta} \right. \\ &\quad - 2 C_{\delta\alpha\beta}^\mu \mathcal{F}_C(\square/\Lambda^2) C_\nu^{\delta\alpha\beta} \\ &\quad + 2 \left( \frac{2}{D-2} R_{\alpha\beta} + 2 \nabla_\alpha \nabla_\beta \right) \mathcal{F}_C(\square/\Lambda^2) C_\nu^{\alpha\beta\mu} \\ &\quad \left. + \mathcal{L}_{C\nu}^\mu - \frac{1}{2} \delta_\nu^\mu (\mathcal{L}_{C\sigma}^\sigma + \tilde{\mathcal{L}}_C) + 4 \mathcal{K}_{C\nu}^\mu \right] = 0 \end{aligned} \quad (5)$$

where

$$\begin{aligned} \mathcal{L}_{C\nu}^\mu &= \sum_{n=1}^{\infty} \frac{f_{Cn}}{\Lambda^{2n}} \sum_{l=0}^{n-1} (\nabla^\mu \square^l C_{\beta\gamma\delta}^\alpha) (\nabla_\nu \square^{n-l-1} C_\alpha^{\beta\gamma\delta}), \\ \tilde{\mathcal{L}}_C &= \sum_{n=1}^{\infty} \frac{f_{Cn}}{\Lambda^{2n}} \sum_{l=0}^{n-1} (\square^l C_{\beta\gamma\delta}^\alpha) (\square^{n-l-1} C_\alpha^{\beta\gamma\delta}), \\ \mathcal{K}_{C\nu}^\mu &= \sum_{n=1}^{\infty} \frac{f_{Cn}}{\Lambda^{2n}} \sum_{l=0}^{n-1} \nabla_\beta [(\square^l C^{\beta\rho}_{\gamma\zeta}) (\nabla^\mu \square^{n-l-1} C_{\nu\rho}^{\gamma\zeta}) \\ &\quad - (\nabla^\mu \square^l C^{\beta\rho}_{\gamma\zeta}) (\square^{n-l-1} C_{\nu\rho}^{\gamma\zeta})] \end{aligned}$$

and everywhere a symmetrization with respect to free indices is assumed. The quantity  $\Delta_\nu^\mu$  is self-explanatory. Note that the above equations are obtained under an essential assumption that the form factor is Taylor expandable at the origin and coefficients  $f_{Cn}$  are introduced as follows:

$$\mathcal{F}_C(\square/\Lambda^2) = \sum_{n \geq 0} f_{Cn} \square^n / \Lambda^{2n}. \quad (6)$$

While this is a sufficient condition to restore the IR limit of GR (not necessary though), one may want to consider a wider class of functions. The most notable and already mentioned is  $\mathcal{F}(\square/\Lambda^2) \sim \log(\square/\Lambda^2)$  appearing in the one-loop correction to GR. To accommodate such form factors,

one can write them using some integral representation or limits which allows one to do the Taylor expansion first and perform the corresponding representation integration later. We will specifically discuss the case of the logarithm in Sec. V.

Moreover, we have neglected potentially infinitely many boundary terms. We can do this in our analysis as long as we are interested in modifications to GR solutions far away from an event horizon. Such terms may become important, however, for consideration of the thermodynamics of the obtained solutions—an important topic which we defer to a future study.

### III. CORRECTIONS TO SCHWARZSCHILD SOLUTION IN FOUR DIMENSIONS FAR AWAY FROM THE HORIZON

We proceed by computing correction to the Schwarzschild metric in the four-dimensional case far away from the horizon. This consideration can be straightforwardly extended to any dimension  $D > 4$  and the corresponding results are collected in Appendix A. Here we proceed with  $D = 4$  for clarity.

The Schwarzschild metric in four dimensions can be written in Schwarzschild coordinates as

$$ds_0^2 = -\left(1 - \frac{r_4}{r}\right) dt^2 + \left(1 - \frac{r_4}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \quad (7)$$

where  $r_4$  is the Schwarzschild BH radius in four dimensions for which  $r_4 M_P^2 \sim M$ , where  $M$  is the BH mass [46,47]. Now we are going to construct corrections to this metric to find a solution to Eq. (5) perturbatively in  $\lambda$ . The following parametrization will be used

$$ds_\lambda^2 = -\left(1 - \frac{r_4}{r} + \lambda \frac{P_4(r)}{r}\right) (1 + \lambda F_4(r)) dt^2 + \left(1 - \frac{r_4}{r} + \lambda \frac{P_4(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \quad (8)$$

Index 4 signifies that these are functions relevant in four dimensions. In a general dimension, we will use index  $D$ . Substituting the latter metric in (5), one obviously restores the Schwarzschild solution in GR in the order  $\lambda^0$ . In the order  $\lambda^1$ , we have to solve the following equation:

$$-M_P^2 \frac{d(G_\nu^\mu|_{ds_\lambda^2})}{d\lambda} \Big|_{\lambda=0} + \Delta_\nu^\mu|_{ds_0^2} = 0, \quad (9)$$

which has a universal form in any dimension. Upon computing the first term in the latter equation one sees the convenience of the specific parametrization (8). The term  $G_0^0|_{ds_\lambda^2}$  is always exactly only linear in  $\lambda$  and is, moreover, monomial in  $P'_4(r)$  only. Hereafter, prime is a

derivative with respect to the radius  $r$  and detailed full formulas can be found in Appendix B.

#### A. The $\binom{0}{0}$ component of equations

The  $\binom{0}{0}$  component of Eq. (9) reads

$$M_P^2 \frac{P'_4(r)}{r^2} = \frac{r_4^2}{r^6} \sum_{n=0}^{\infty} \frac{c_n(4) f_{Cn}}{r^{2n} \Lambda^{2n}} \left(1 + O\left(\frac{r_4}{r}\right)\right) = \frac{r_4^2}{r^6} \Phi(4, r) \left(1 + O\left(\frac{r_4}{r}\right)\right). \quad (10)$$

Here we use one more approximation, assuming that we are far away from the horizon and  $\Phi(4, r)$  is a shorthand for the sum in the right-hand side (rhs) of the equation. We recall that coefficients  $f_{Cn}$  are the theory parameters as they determine the form factor  $\mathcal{F}_C(\square/\Lambda^2)$  through expansion (6). However,  $c_n$  are numeric coefficients coming out as a result of the evaluation of the rhs of Eq. (9) on the four-dimensional Schwarzschild metric. While here we do computations in four dimensions, they, in general, depend on the dimension  $D$ . Note that the zero term in the sum on the rhs of Eq. (10) does not have  $O(r_4/r)$  corrections. The explicit form of coefficients  $c_n$  is presented in Eqs. (B18) and (B20) in Appendix B. Here we write down them for completeness,

$$c_{n \geq 2}(4) = 2\alpha(4) 4^n \Gamma\left(n + \frac{5}{2}\right) \Gamma(n+3), \quad (11)$$

where  $\alpha(4) = 2/\sqrt{\pi}$  and also  $c_0(4) = 0$ ,  $c_1(4) = 90$ .

Finding out the function  $P_4(r)$  far away from the horizon implies a straightforward integration of the series on the rhs of (10). It is, however, possible to sum up this series and express it through the form factor  $\mathcal{F}_C(\square/\Lambda^2)$ . To do so we first split  $\Phi(4, r)$  as follows:

$$\Phi(4, r) = \frac{90 f_{C1}}{r^2 \Lambda^2} + \sum_{n=2}^{\infty} \frac{c_n(4) f_{Cn}}{r^{2n} \Lambda^{2n}} = \frac{90 f_{C1}}{r^2 \Lambda^2} + \Phi_2(4, r). \quad (12)$$

This splitting is coming naturally from the fact that coefficients  $c_n$  starting from  $n = 2$  obey a closed-form matching through  $\Gamma$ -functions. Introducing  $z = (r\Lambda)^2/4$  one can rewrite  $\Phi_2(4, r)$  as

$$\Phi_2(4, z) = \sum_{n=2}^{\infty} 2 \frac{2}{\sqrt{\pi}} \Gamma\left(n + \frac{5}{2}\right) \Gamma(n+3) f_{Cn} z^{-n}. \quad (13)$$

The key observation at this stage is that  $\Gamma$  factors in the latter series can be written using an integral involving the modified Bessel function  $K_\nu(x)$  (C1),

$$\Phi_2(4, z) = 4 \frac{2}{\sqrt{\pi}} z^{\frac{11}{4}} \sum_{n=2}^{\infty} \int_0^{\infty} d\tau f_{Cn} \tau^{n+\frac{7}{4}} K_{\frac{1}{2}}(2\sqrt{z}\tau). \quad (14)$$

While the Bessel function is convenient for comparison with more general answers in Appendix A, in this case, one can use its explicit form

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}}.$$

Now, one immediately notices that the summation over  $n$  can be written as

$$\sum_{n=2}^{\infty} f_{Cn} \tau^n = \mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau. \quad (15)$$

Combining all together we have transformed Eq. (10) to the following form:

$$P'_4(r) = \frac{1}{M_P^2} \frac{r_4^2}{r^4} \cdot \left( \frac{90f_{C1}}{r^2\Lambda^2} + \Phi_2(4, r) \right), \quad (16)$$

where  $\Phi_2(4, r)$  has the following integral representation through  $\mathcal{F}_C(\tau)$ :

$$\begin{aligned} \Phi_2(4, r) &= 4 \frac{2}{\sqrt{\pi}} \left( \frac{r\Lambda}{2} \right)^{\frac{11}{2}} \int_0^{\infty} d\tau (\mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau) \tau^{\frac{7}{4}} \\ &\times K_{\frac{1}{2}}(r\Lambda\sqrt{\tau}), \end{aligned} \quad (17)$$

where we have returned back to  $r$ . Finally, integrating over  $r$  one yields

$$P_4(r) = -\frac{1}{M_P^2} \frac{r_4^2}{r^4} \cdot \left( \frac{90f_{C1}}{5r\Lambda^2} + \tilde{\Phi}_2(4, r) \right), \quad (18)$$

where  $\tilde{\Phi}_2(4, r)$  comes as the result of an integration of  $\Phi_2(4, r)/r^4$  with respect to  $r$  with subsequent factoring of  $-1/r^4$  out. The corresponding expression is

$$\begin{aligned} \tilde{\Phi}_2(4, r) &= \frac{4}{\Lambda} \frac{2}{\sqrt{\pi}} \left( \frac{r\Lambda}{2} \right)^{\frac{11}{2}} \int_0^{\infty} d\tau (\mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau) \tau^{\frac{5}{4}} \\ &\times K_{\frac{3}{2}}(r\Lambda\sqrt{\tau}). \end{aligned} \quad (19)$$

An integration constant must vanish in order not to modify the Newtonian potential at infinity. Here again one can express Bessel function as follows:

$$K_{\frac{3}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \left( 1 + \frac{1}{x} \right).$$

The reason why this integration went smoothly is that we explicitly hit the property of Bessel functions (C2).

## B. The $(\frac{1}{2})$ component of equations

The  $(\frac{1}{2})$  component of Eq. (9) reads

$$\begin{aligned} M_P^2 \left[ \frac{F'_4(r)}{r} \left( 1 + O\left(\frac{r_4}{r}\right) \right) + \frac{P'_4(r)}{r^2} \right] \\ = \frac{r_4^2}{r^6} \sum_{n=0}^{\infty} \frac{d_n(4) f_{Cn}}{r^{2n} \Lambda^{2n}} \left( 1 + O\left(\frac{r_4}{r}\right) \right) \\ = \frac{r_4^2}{r^6} \Psi(4, r) \left( 1 + O\left(\frac{r_4}{r}\right) \right). \end{aligned} \quad (20)$$

Here again, we use an approximation that we are far away from the horizon and  $\Psi(4, r)$  is a shorthand for the sum on the rhs of the equation. Note that the zero term in the sum on the rhs of Eq. (10) does not have  $O(r_4/r)$  corrections. The explicit form of coefficients  $d_n$  is presented in Appendix B, Eqs. (B19) and (B20). Here we write them down for completeness,

$$d_{n \geq 2}(4) = \alpha(4) 4^n \Gamma\left(n + \frac{3}{2}\right) \Gamma(n+3) = \frac{c_n(4)}{2n+3} \quad (21)$$

and also  $d_0(4) = 0$  and  $d_1(4) = 126$ .

Working out the series  $\Psi(4, r)$  in the same way as  $\Phi(4, r)$ , namely, splitting  $n \geq 2$  terms into a series  $\Psi_2(4, r)$  and repeating the next steps in a similar fashion, one yields

$$F'_4(r) = -\frac{P'_4(r)}{r^3} + \frac{1}{M_P^2} \frac{r_4^2}{r^7} \left( \frac{126f_{C1}}{r^2\Lambda^2} + \Psi_2(4, r) \right), \quad (22)$$

where  $\Psi_2(4, r)$  has the following integral representation through  $\mathcal{F}_C(\tau)$ :

$$\begin{aligned} \Psi_2(4, r) &= 2 \frac{2}{\sqrt{\pi}} \left( \frac{r\Lambda}{2} \right)^{\frac{9}{2}} \int_0^{\infty} d\tau (\mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau) \tau^{\frac{5}{4}} \\ &\times K_{\frac{3}{2}}(r\Lambda\sqrt{\tau}). \end{aligned} \quad (23)$$

Given that both  $P'_4(r)$  and  $P_4(r)$  have just been found explicitly above, finding  $F_4(r)$  is a straightforward task. Using Eq. (16) to simplify (22), we see that prefactors in front of parentheses exactly match, and one can write

$$F'_4(r) = \frac{1}{M_P^2} \frac{r_4^2}{r^7} \left( \frac{36f_{C1}}{r^2\Lambda^2} + \Psi_2(4, r) - \Phi_2(4, r) \right). \quad (24)$$

To get  $F_4(r)$ , we have to perform just one integration over  $r$ . Before an actual integration, however, it is instructive to note the following relation:

$$\begin{aligned}\Delta_2(4, r) &= \Psi_2(4, r) - \Phi_2(4, r) \\ &= 2 \frac{2}{\sqrt{\pi}} \left( \frac{r\Lambda}{2} \right)^{\frac{9}{2}} \int_0^\infty d\tau (\mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau) \tau^{\frac{5}{2}} \\ &\quad \times \left[ K_{\frac{3}{2}}(r\Lambda\sqrt{\tau}) - r\Lambda\sqrt{\tau} K_{\frac{1}{2}}(r\Lambda\sqrt{\tau}) \right].\end{aligned}$$

The expression in brackets can be rewritten as follows:

$$K_{\frac{3}{2}}(x) - xK_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \left( 1 + \frac{1}{x} - x \right).$$

Finally, one gets for  $F_4(r)$ ,

$$F_4(r) = -\frac{1}{M_p^2} \frac{r_4^2}{r^7} \left( \frac{36f_{C1}}{8r\Lambda^2} + \tilde{\Delta}_2(4, r) \right), \quad (25)$$

where  $\tilde{\Delta}_2(4, r)$  comes as the result of an integration of  $\Delta_2(4, r)/r^7$  with respect to  $r$  with subsequent factoring of  $-1/r^7$  out. An answer in four dimensions reads

$$\begin{aligned}\tilde{\Delta}_2(4, r) &= -\frac{1}{24} (r\Lambda)^7 \int d\tau \left[ (\mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau) \tau^{5/2} \right. \\ &\quad \left. \times \left( 4\text{Ei}(-x) + e^{-x} \frac{4x^2 - x - 1}{x^3} \right) \right] \\ \text{with } x &= r\Lambda\sqrt{\tau}.\end{aligned} \quad (26)$$

An integration constant must vanish in order not to modify the Newtonian potential at infinity.

### C. General comments to the obtained perturbative solutions

To conclude this section, we underline that we have constructed explicitly and analytically corrections to the Schwarzschild metric in first order in the coupling  $\lambda$  far away from the horizon in four dimensions for a selected set of EFT terms (see Appendix A for a general dimension  $D > 4$ ). Such terms are motivated by the study of asymptotically safe gravity [11–13] and by analyzing the effect of loop corrections. These terms represent operators that result in the most possible derivatives acting on a metric in equations of motion. This translates into taking into account the strongest corrections to the asymptotic behavior of the metric as long as the operators contain powers of the d'Alembertian. In the next section, we will see that a logarithm may provide an even stronger correction.

The notable fact is that thanks to a nice matching of computed coefficients by closed-form series and a subsequent integral rewriting of the sums, we can express corrections through the higher derivative form factor in the model Lagrangian. Computations can be manifestly extended to all orders in  $r_4/r$  (or in general  $r_D/r$  in  $D$  dimensions). On the way to deducing the presented equations, all such terms were explicitly computed; they

are all obviously inverse powers of  $r$  and thus can be readily integrated. This will give subleading contributions to the obtained results, but the actual result will most likely lack any nice closed form with no explicit summations. Moreover, other operators of the same mass dimension will give similar contributions, and in order to be consistent, we have to take into account other operators as well.

To illustrate this, consider a typical example term  $f_{C2} C_{\mu\nu\alpha\beta} (\Box^2/\Lambda^4) C_{\mu\nu\alpha\beta}$  in the action in four dimensions, which results in the following  $\Delta_0^0$  component in the order  $\lambda$ :

$$\Delta_0^0 = 18f_{C2} \frac{r_4^2}{r^{10}} \left( 560 - 1488 \frac{r_4}{r} + 939 \frac{r_4^2}{r^2} \right).$$

Similar computation for operators with higher powers of the d'Alembertian results in more terms of different degrees in parentheses. Our analysis considers only the first term in the parentheses, the constant 560. It is the most significant correction in the asymptotic infinity, and it can originate only upon accounting for the suggested higher derivative terms in the action. Other operators of the same mass dimension, for example,  $C_{\mu\nu\alpha\beta} C^{\mu\nu\sigma\rho} C_{\sigma\rho\gamma\delta} C^{\gamma\delta\alpha\beta}$  will result in the following correction:

$$C_{\mu\nu\alpha\beta} C^{\mu\nu\sigma\rho} C_{\sigma\rho\gamma\delta} C^{\gamma\delta\alpha\beta} \rightsquigarrow \frac{r_4^3}{r^{11}} \left( 64 - 67 \frac{r_4}{r} \right).$$

It does not have the  $\sim 1/r^{10}$  term but will impact the  $\sim 1/r^{11}$  and higher order corrections and thus would have to be added to the model for consistency if subleading asymptotics are to be computed. We, however, do not expect any new effects upon accounting for these corrections. This situation is the same for all the EFT operators of the mass dimension 8 or higher. One, however, would have similar contributions from all the operators of the mass dimension 6 in four dimensions, i.e.,  $C_{\mu\nu\alpha\beta} \Box C^{\mu\nu\alpha\beta}$ ,  $C_{\mu\nu\alpha\beta} C^{\mu\nu\sigma\rho} C_{\sigma\rho}{}^{\alpha\beta}$  and others, which is in line with consideration of amplitudes in an EFT of gravity [8,48].

On the technical side, we note that square in curvature operators without d'Alembertians do not contribute in four dimensions due to the topological nature of the Gauss-Bonnet term. Moreover, in any dimension, these operators give no contribution to the function  $F_D(r)$ . All other operators, however, do change  $F_D(r)$  nontrivially making manifestly metric components  $g_{00}$  and  $g_{11}^{-1}$  different.

The next question to be explored in the subsequent analysis is the convergence of our results for metric corrections for different sets of  $f_{Cn}$  and compatibility of form factors with the perturbation scheme. Moreover, we will generalize our analysis to form factors that do not have a Taylor expansion at zero. In particular, the logarithmic form-factor characteristic to one-loop corrections will be studied by utilizing an integral representation for the logarithm.

#### IV. RESUMMATION OF THE SERIES AND CONVERGENCE ANALYSIS

Here we aim at studying the convergence of the series  $\Phi_2$  in (10) and in general dimension in (A3) emerging as the perturbative correction to the Schwarzschild solution. We proceed with a general dimension parameter  $D$  as its impact on the analysis is minimal. The series in question can be written as

$$\Phi_2(\zeta) = \sum_{n=2}^{\infty} c_n(D) f_{Cn} \zeta^n, \quad (27)$$

where we have defined  $\zeta = \frac{1}{r^2 \Lambda^2}$ . Given that the coefficients  $c_n$  contain  $\Gamma$  functions in the numerator, the series clearly has worse convergence properties compared to the expansion for form factor (6),

$$\mathcal{F}_C(\tau) = \sum_{n=0}^{\infty} f_{Cn} \tau^n. \quad (28)$$

According to the Cauchy-Hadamard theorem, the power series is convergent for all values of  $|\zeta| < \zeta_0$  with the convergence radius  $\zeta_0$  given by

$$\zeta_0 = \lim_{n \rightarrow \infty} |c_n(D) f_{Cn}|^{\frac{1}{n}}. \quad (29)$$

The behavior of the coefficients  $f_{Cn}$  for large  $n$  is related to the asymptotic limit of the function  $\mathcal{F}_C(\tau)$  at  $\tau \rightarrow \infty$  in the complex plane of  $\tau$ . Any entire function  $\phi(z)$  near the complex infinity can be characterized by an order

$$\rho = \max \left( \lim_{|z| \rightarrow \infty} \frac{\log(\log |\phi(z)|)}{\log |z|} \right) = \lim_{n \rightarrow \infty} \frac{n \log n}{n \log a_n}, \quad (30)$$

and a type

$$s = \max \left( \lim_{|z| \rightarrow \infty} \frac{\log |\phi(z)|}{|z|^\rho} \right) = \lim_{n \rightarrow \infty} \frac{\log \left( \frac{n}{a_n} \right)}{|a_n|^{1/n}}. \quad (31)$$

Here  $a_n$  are the series coefficients, such that the entire function  $\phi(\zeta)$  can be presented as

$$\phi(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n \rightarrow e^{s \zeta^\rho} \quad \text{when } |\zeta| \rightarrow \infty. \quad (32)$$

In this way, the large  $n$  limit of the series coefficients is related to the high-energy limit of the form factor  $\mathcal{F}_C(\square)$  on the complex plane of the energy. It is interesting to note that this property is related to the locality of the underlying theory in the UV limit and, in particular, to the polynomial boundedness of the scattering amplitudes. Indeed, substituting our result

$$c_{n \geq 2}(D) = 2\alpha(D) 4^n \Gamma\left(n + \frac{D+1}{2}\right) \Gamma(n+D-1) \quad (33)$$

in Eq. (29), we find after expanding  $\Gamma$  functions for large  $n$ ,

$$\zeta_0 \sim (2\pi n^{3(D-1)/2} e^{2n(\log(2n)-1)} f_{Cn})^{-1/n} \sim \frac{1}{n^2} f_{Cn}^{-1/n}. \quad (34)$$

Here we take only the leading multiplier at  $n \rightarrow \infty$ . It is apparent that we have a nonzero convergence radius for the series only if

$$f_{Cn}^{-1/n} \gtrsim n^2, \quad f_{Cn} \lesssim n^{-2n} \sim \frac{1}{(n!)^2}. \quad (35)$$

It means, in particular, that for an exponential type form factor  $\mathcal{F}(\square) = e^{s\square}$  with coefficients  $f_{Cn} = 1/(n!) > 1/(n!)^2$  the series of corrections to the Schwarzschild solution is divergent everywhere.

The maximal growth rate of a form factor on the complex energy plane which still results in a convergent series for corrections to a BH solution is of the form  $e^{s\square^{1/2}}$  which corresponds to order  $\rho = 1/2$  as defined in Eq. (30), while the type  $s$  as defined in Eq. (31) can be arbitrary. It is interesting to mention that such a form factor would also contribute to the graviton scattering amplitudes in a way corresponding to the localizable field theories [49–51]. Namely, in such theories, amplitudes are polynomially bounded on the first sheet of the complex plane of total energy at fixed momentum transfer. Thus, we have come to a nontrivial conclusion that only in localizable theories do the perturbative corrections to the Schwarzschild metric make sense. For nonlocal exponential form factors, the perturbative corrections form a series that is divergent everywhere except for the spatial infinity.

#### V. ONE-LOOP LOGARITHM

In this section, we generalize our results to the case of form factors nonanalytic at zero appearing in the quantum effective action as loop corrections. In particular, in  $D = 4$  we expect to have an operator

$$\mathcal{F}(\square) = \gamma(\mu) \log \frac{\square}{\mu^2}. \quad (36)$$

Here  $\mu$  can be interpreted as a renormalization scale, and  $\gamma(\mu)$  is a coupling defined at this scale.<sup>2</sup>

We can understand the effect caused by this operator by proposing the analytic continuation of our result obtained for operators of the form  $\square^n$ . To this end, we can write

<sup>2</sup>Study of leading corrections in this case in the near-horizon limit was performed in [52].

$$\log \frac{\square}{\mu^2} = \lim_{n \rightarrow 0} \frac{(\square/\mu^2)^n - 1}{n}. \quad (37)$$

This expression can be used as a definition of a nonlocal operator emerging at the loop level in the quantum effective action (see also [53,54] for more detailed discussion and representation of such an operator in the coordinate space). We use our result obtained for an infinite set of  $\square^n$  operators (13) at integer  $n \geq 2$ ,

$$\begin{aligned} \Phi(4, r) &= 2 \frac{2}{\sqrt{\pi}} \Gamma\left(n + \frac{5}{2}\right) \Gamma(n+3) \left(\frac{r^2 \mu^2}{4}\right)^{-n}, \\ \mathcal{F}(\square) &= (\square/\mu^2)^n. \end{aligned} \quad (38)$$

Then we can define its analytic continuation in a variable  $n$  to  $n = 0 + \epsilon$ ,  $\epsilon > 0$ . Here, we use such a procedure as a definition of a nonlocal logarithmic operator, leaving its more rigorous justification and relation to the coordinate space representation of effective action for future study.

Using Eq. (38) for the  $\square^n$  operator, we can take the formal limit  $n \rightarrow 0$  (as it is prescribed by the analytic continuation) and find

$$\begin{aligned} \Phi_2(4, r) &= \lim_{n \rightarrow 0} \gamma(\mu) \left( \frac{4\Gamma(n + \frac{5}{2})\Gamma(n+3)4^n(\mu^2 r^2)^{-n}}{n\sqrt{\pi}} - \frac{6}{n} \right) \\ &= -\gamma(\mu)(12 \log \mu r + 12\gamma_E - 25). \end{aligned} \quad (39)$$

Then, from Eq. (16) we get

$$P'_4(r) = -\frac{r_4^2}{M_P^2 r^4} \gamma(\mu)(12 \log \mu r + 12\gamma_E - 25). \quad (40)$$

Performing an integration, we obtain

$$P_4(r) = \gamma(\mu) \frac{r_4^2(4 \log(\mu r) + 4\gamma_E - 7)}{r^3 M_P^2}. \quad (41)$$

Remarkably, the same result can be obtained from Eqs. (18) and (19) if one puts  $f_{C0} = 0$ ,  $f_{C1} = 0$  (although these series coefficients are not defined for the logarithm). Namely, the expression

$$\begin{aligned} P_4(r) &= -\gamma(\mu) \int_0^\infty d\tau \left( \frac{\mu^3 r_4^2 \tau^{5/4} (\mu r)^{3/2} \log(\tau)}{4\sqrt{2\pi} M_P^2} K_{3/2}(\mu r \sqrt{\tau}) \right) \\ &= \gamma(\mu) \frac{r_4^2(4 \log(\mu r) + 4\gamma_E - 7)}{r^3 M_P^2} \end{aligned} \quad (42)$$

provides the same result as Eq. (41). This is consistent with our definition of analytic continuation in  $n$  and works for integer  $n \geq 2$ .

In a similar way, we can find the correction  $F_4(r)$  from Eq. (20),

$$M_P^2 \left( \frac{F'_4(r)}{r} + \frac{P'_4(r)}{r^2} \right) = \gamma(\mu) \frac{r_4^2}{r^6} (-4 \log(\mu r) - 4\gamma_E + 7). \quad (43)$$

Expressing  $F'_4(r)$ , one can obtain

$$F'_4(r) = \frac{2\gamma(\mu)r_4^2}{M_P^2 r^5} (4 \log(\mu r) + 4\gamma_E - 9). \quad (44)$$

After integration, the correction to the metric finally takes the form

$$F_4(r) = -\frac{2\gamma(\mu)r_4^2}{M_P^2 r^4} (\log(\mu r) + \gamma_E - 2). \quad (45)$$

The terms omitted in this computation are suppressed by the extra powers of  $r_4/r$ , which makes them subdominant far away from the BH.

To be assured in the correctness of our result, and especially the analytic continuation used to produce an answer for a logarithmic one-loop correction, we can use a parametrization similar to the Schwinger proper time parametrization and define the logarithm using the following integral:

$$\log(z) = \int_0^\infty \frac{e^{-s} - e^{-zs}}{s} ds.$$

However, this definition may not *a priori* be suitable in our case because it involves an entire function of order 1, while we just have proven in the previous section that we can guarantee the convergence of resummation only for functions of the order  $\rho \leq 1/2$ . To circumvent this problem, we can resort to a more general form of the last expression, that is, to use the Frullani integral

$$\int_0^\infty \frac{f(as) - f(bs)}{s} ds = (f(\infty) - f(0)) \log\left(\frac{a}{b}\right), \quad (46)$$

which is valid as long as  $f(x)$  is defined for all non-negative  $x$  and has a limit when  $x \rightarrow +\infty$ , denoted as  $f(\infty)$  in the latter formula. We can suggest an entire function

$$f(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}}.$$

This is clearly an entire function of order  $1/2$ ,  $f(0) = 1$  and  $f(\infty) = 0$ . Then taking  $a = 1$  and  $b = \square/\mu^2$ , we get exactly  $\log(\square/\mu^2)$  on the rhs of formula (46) while keeping the order of the entire function bounded by  $1/2$ . Thus, we represent the form factor  $\mathcal{F}(\tau)$  in the following form:

$$\mathcal{F}(\tau) = \gamma(\mu) \int_0^\infty \left( \frac{\sin(\sqrt{s})}{\sqrt{s}} - \frac{\sin(\sqrt{\tau s})}{\sqrt{\tau s}} \right) \frac{ds}{s}, \quad (47)$$

where  $\tau = \square/\mu^2$ .

At this stage, we should, starting from (47) which has a suitable order  $1/2$ , first postpone an integration over  $s$ , then do computation of the rhs of equations of motion with a subsequent resummation over  $n$ , and after those steps return to the remaining integration over  $s$ . Rewriting series (12) in combination with the definition (13), we get for representation (47),

$$\begin{aligned} \Phi(4, r) &= 0 \cdot f_{C0} + \frac{90f_{C1}}{r^2\mu^2} \\ &+ \sum_{n=2}^{\infty} 2 \frac{2}{\sqrt{\pi}} \Gamma\left(n + \frac{5}{2}\right) \Gamma(n+3) f_{Cn} \left(\frac{4}{(r\mu)^2}\right)^n \\ &= \frac{90f_{C1}}{r^2\mu^2} + \frac{1}{4} \sum_{n=2}^{\infty} \Gamma(2n+5) \frac{f_{Cn}}{(r\mu)^{2n}} \end{aligned} \quad (48)$$

where  $f_{Cn}$  are coming from the Taylor expansion of  $\frac{\sin(\sqrt{\tau s})}{\sqrt{\tau s}}$  with respect to  $\tau$  at zero. This result lacks coefficient  $f_{C0}$  because it was multiplied on the way by  $c_0(4)$  which is zero [as explicitly written in the first line of equation (48), see (B18) for details]. Coefficients  $f_{Cn}$  can be read from

$$-\frac{\gamma(\mu)}{s} \frac{\sin(\sqrt{\tau s})}{\sqrt{\tau s}} = -\frac{\gamma(\mu)}{s} \sum_{n=0}^{\infty} \frac{(-1)^n s^n}{(2n+1)!} \tau^n.$$

Substituting these coefficients, one gets

$$\begin{aligned} \Phi(4, r) &= \gamma(\mu) \int_0^\infty \left[ \frac{90}{r^2\mu^2} \cdot \frac{1}{6} - \frac{1}{4s} \sum_{n=2}^{\infty} (2n+2)(2n+3) \right. \\ &\quad \left. \times (2n+4) \frac{(-s)^n}{(r\mu)^{2n}} \right] ds. \end{aligned} \quad (49)$$

This series has no factorial growth, which supports our findings from the previous section that the order  $1/2$  is the limiting situation for a possible resummation convergence. However, being at the threshold, we still get a divergent series. To overcome this issue, we can compute the sum in (49) in the sense of the Borel summation (see Appendix D for details),

$$\begin{aligned} \Phi(4, r) &= \gamma(\mu) \int_0^\infty \left[ \frac{15}{r^2\mu^2} + \frac{6}{s} - \frac{30}{r^2\mu^2} \right. \\ &\quad \left. - \frac{6}{s} \cdot \frac{1 - s/(r\mu)^2}{(1 + s/(r\mu)^2)^4} \right] ds. \end{aligned} \quad (50)$$

Here the first term is the one that was present before, second and third terms represent absent and now subtracted zero and first terms in the sum, respectively. The last term is

the sum over  $n$  from zero to infinity. If we formally neglect the divergent contributions, the relevant finite part turns out to be

$$\Phi(4, r) = -\gamma(\mu)(12 \log(r\mu) - 13) + \text{divergent terms}, \quad (51)$$

which exactly reproduces including the coefficient, the logarithmic dependence of the result (39) obtained by the analytic continuation  $n \rightarrow 0$  on a series representation level [note that in formula (39)  $\Phi_2$  is actually  $\Phi$  as the terms by which they differ are effectively zero in that derivation].

Obviously, the divergent contributions in (50) need explanation. The first and third terms in (50) are similar but are not canceling each other. They, however, can be canceled by adding a cubic Weyl tensor term. Indeed, as it was already mentioned at the end of Sec. III, operators of mass dimension 6, such as  $C_{\mu\nu\alpha\beta} \square C^{\mu\nu\alpha\beta}$ ,  $C_{\mu\nu\alpha\beta} C^{\mu\nu\sigma\rho} C_{\sigma\rho}^{\alpha\beta}$  result in similar leading behavior at large distances. The second term (50) is just  $6/s$  and it diverges logarithmically both at zero and at infinity. Its divergence at zero in the meantime helps canceling the logarithm divergence at zero of the third term. But for large  $s$ , the divergence coming from the term  $6/s$  is not compensated. The presence of this term would be compensated by operators of mass dimension 4 but such operators were taken as vanishing [see Eq. (48)] in our computation since they correspond to the Gauss-Bonnet term, which is a topological invariant in four dimensions.

The latter is a subtle point, as corresponding operators may not vanish if one comes to the one-loop answer by using the dimensional regularization in which the Gauss-Bonnet term loses its neat property to be a topological invariant. If we assume that the dimensional regularization changes  $D \rightarrow D - 2\epsilon$  we must restore the  $c_0$  coefficient, which becomes  $c_0 = 6\epsilon$  [see (B18)] and do this already in (48), replacing  $0 \rightarrow 2\epsilon$  in front of  $f_{C0}$ . Next, assuming that an inclusion of the cubic in the Weyl tensor term can solve the problem of the linear divergence in (50) we can shift the coefficient in front of  $f_{C1}$  such that it matches the rest of the series in (48), effectively becoming 180, which is  $\Gamma(7)/4$ . Then we can write instead of (50),

$$\begin{aligned} \Phi(4, r) &= \gamma(\mu) \int_0^\infty \left[ \left( \frac{1}{s} \cdot \frac{\sin(\sqrt{s})}{\sqrt{s}} - \frac{1}{s} \right) 6\epsilon \right. \\ &\quad \left. + \frac{6}{s} - \frac{6}{s} \cdot \frac{1 - s/(r\mu)^2}{(1 + s/(r\mu)^2)^4} \right] ds, \end{aligned} \quad (52)$$

where the first parentheses correspond to  $f_{C0}$  as it comes from representation (47) for a logarithm, the last term is the summation of the series from zero to infinity, and the second term is the still subtracted zero term of this series as we should have summed from the unit (recall that the unit

term is now assumed to be adjusted to match the rest of the series by adding a mass dimension 6 operator as explained above). We now see that an analytic continuation of the dimensional regularization parameter  $\epsilon \rightarrow 1$  cancels a couple of terms. Furthermore, computing the integral with  $\epsilon = 1$  we get

$$\Phi(4, r) = -\gamma(\mu)(12 \log(r\mu) + 12\gamma_E - 25), \quad (53)$$

which matches absolutely our previous answer (39) [recall that in (39)  $\Phi_2$  is actually  $\Phi$  as other terms are supposed to be zero in that derivation]. This demonstrates that both methods of getting the result perfectly match upon proper accounting of divergencies. It is, however, interesting to emphasize that the first method of an analytic continuation  $n \rightarrow 0$  in a series produces the result in a much simpler way without the need for dealing with divergent contributions.

It is important to notice here that the corrections in the metric induced by higher derivative EFT corrections [like  $C_{\mu\nu\alpha\beta}\mathcal{F}(\square/\Lambda^2)C^{\mu\nu\alpha\beta}$ ] are always more suppressed than the ones from the  $C_{\mu\nu\alpha\beta}\log(\square/\mu^2)C^{\mu\nu\alpha\beta}$  operator corresponding to the loop effects in GR. Stating this, we assume that the form factor  $\mathcal{F}(\tau)$  is compatible with the resummation of corrections as discussed in the previous section. This situation is specific for four dimensions because the effects of the tree-level quadratic in curvature terms are vanishing due to the topological nature of the Gauss-Bonnet term. This means that logarithmic loop corrections can result in a more significant deformation of the Schwarzschild solution in four dimensions.

## VI. CONCLUSIONS

In this work, we aimed at computing a deformation of the Schwarzschild solution in the presence of a sequence of higher derivative EFT operators up to an infinite order in derivatives. We particularly focus on operators of the form  $C_{\mu\nu\alpha\beta}(\square^n/\Lambda^{2n})C^{\mu\nu\alpha\beta}$  and study the corresponding deformations of the vacuum BH solution of GR perturbatively in the EFT coefficients. For each power  $n$ , we find an explicit expression for corrections to the metric in an arbitrary spacetime dimension  $D \geq 4$  (see Sec. III and Appendix A). Moreover, for  $n \geq 2$ , based on the xAct [55] computation done for  $n \leq 10$ , we observe a regular pattern of coefficients in the leading (most slowly decaying) order in  $r_D/r$  expansion. This allows us to write a general formula for these corrections. This result further leads to an integral representation for the corrections in the presence of a generic operator  $C_{\mu\nu\alpha\beta}\mathcal{F}(\square/\Lambda^2)C^{\mu\nu\alpha\beta}$  solely through the function  $\mathcal{F}(\tau)$ . We provide analytic expressions for the corrected metric components under the assumption that the function  $\mathcal{F}(\tau)$  can be expanded in a Taylor series around zero. However, we find that the deformation expressed in power series is not always convergent, which puts restrictions on the use of the perturbation theory methods for

searching for a solution with an infinite tower of higher derivative EFT operators. Our main result is the observation that the very possibility of finding corrections to a BH solution in the form of a perturbative deformation of the Schwarzschild solution is related to UV locality of the underlying theory. Namely, the term  $C_{\mu\nu\alpha\beta}\mathcal{F}(\square/\Lambda^2)C^{\mu\nu\alpha\beta}$  would contribute to the scattering amplitude of four gravitons and its UV asymptotic. In nonlocalizable theories, the maximal growth of the amplitudes on the complex energy plane is larger than  $e^{|p|}$ . The corresponding form factors  $\mathcal{F}(\tau)$  would cause a deformation of the metric in the form of series that are divergent everywhere except for the spatial infinity  $r \rightarrow \infty$ .

Even though our results are obtained under the assumption that the function  $\mathcal{F}(\tau)$  can be expanded in a Taylor series around zero, we can also perform an analytic continuation with respect to the power  $n$  of the  $\square^n$  operator. Having a closed-form expression of corrections to the metric from the  $C_{\mu\nu\alpha\beta}(\square^n/\Lambda^{2n})C^{\mu\nu\alpha\beta}$  operator as a function of an integer power  $n \geq 2$ , we can use it for fractional powers and the logarithm in the form factor  $\mathcal{F}(\tau)$ . Such an extension allows for computing corrections to the vacuum solution in GR from graviton loops, summarized in a one-loop effective action as the  $C_{\mu\nu\alpha\beta}\log(\square/\mu^2)C^{\mu\nu\alpha\beta}$  operator in four dimensions. We compute an explicit deformation of the Schwarzschild solution for such an operator and have found that it is dominant at large  $r \gg r_4$  compared to the one obtained from the EFT corrections as long as EFT coefficients lead to a convergent resummation of corrections. This situation is specific to four dimensions because there are no local EFT terms quadratic in curvature, which would correct the solution.

Additionally, we verify our method of deriving corrections corresponding to the one-loop effective action in GR in four dimensions by using an alternative representation for the logarithm of the d'Alembertian. Namely, we use a generalized Schwinger proper time parametrization by utilizing the Frullani integral. This allows us to maintain an order of an entire function used to construct the logarithm equal  $1/2$ , which, in turn, allows us to perform the subsequent resummation in the Borel sense. Interestingly, this alternative approach is sensitive to the use of the dimensional regularization in the derivation of the one-loop effective action itself. At the end, both methods result exactly in one and the same answer for the deformation of the Schwarzschild solution.

Our techniques and results can be straightforwardly used for computing corrections to BH solutions in asymptotically safe quantum gravity. In this context, the leading deformation far away from the BH comes from the part of the action determining the nonperturbative graviton propagator. Our result provides a technique that can be used for such a computation, given the explicit form of the graviton two-point function is known. Looking forward, the presented technique can be extended to other vacuum solutions of GR,

and the first future step will be to expand it to the case of a rotating Kerr BH metric. Other configurations, like a charged Reissner-Nordström BH, can also be considered in this framework. One of the most interesting limiting cases would be extremal configurations and constraints coming from the weak gravity conjecture [29,56].

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### DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

### APPENDIX A: CORRECTIONS TO SCHWARZSCHILD SOLUTION IN $D \geq 4$ DIMENSIONS FAR AWAY FROM THE HORIZON

Here we generalize results obtained in Sec. III to any dimension  $D \geq 4$ . Extensive comments are suppressed as the logic of derivation replicates the one in four dimensions. The Schwarzschild metric in  $D$  dimensions can be written in Schwarzschild coordinates as

$$ds_0^2 = -\left(1 - \left(\frac{r_D}{r}\right)^{D-3}\right) dt^2 + \left(1 - \left(\frac{r_D}{r}\right)^{D-3}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2 \quad (\text{A1})$$

where  $r_D$  is the Schwarzschild BH radius in  $D$  dimensions for which  $r_D^{D-3} M_P^{D-2} \sim M$ , where  $M$  is the BH mass. Corrections to this metric are introduced as follows:

$$ds_\lambda^2 = -\left(1 - \left(\frac{r_D}{r}\right)^{D-3} + \lambda \frac{P_D(r)}{r^{D-3}}\right) (1 + \lambda F_D(r)) dt^2 + \left(1 - \left(\frac{r_D}{r}\right)^{D-3} + \lambda \frac{P_D(r)}{r^{D-3}}\right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2. \quad (\text{A2})$$

The  $(0)$  component of Eq. (9) reads

$$M_P^2 \frac{D-2}{2} \frac{P'_D(r)}{r^{D-2}} = \frac{r_D^{2(D-3)}}{r^{2D-2}} \sum_{n=0}^{\infty} \frac{c_n(D) f_{Cn}}{r^{2n} \Lambda^{2n}} \left(1 + O\left(\frac{r_D}{r}\right)\right) = \frac{r_D^{2(D-3)}}{r^{2D-2}} \Phi(D, r) \left(1 + O\left(\frac{r_D}{r}\right)\right). \quad (\text{A3})$$

We assume that we are far away from the horizon and  $\Phi(D, r)$  is a shorthand for the sum on the rhs of the equation.  $c_n$  are numeric coefficients coming out as a result of the evaluation of the rhs of Eq. (9) on the  $D$ -dimensional Schwarzschild metric. The explicit form of coefficients  $c_n$  is presented in Appendix B, Eqs. (B18) and (B20).

We first split  $\Phi(D, r)$  as follows:

$$\Phi(D, r) = c_0(D) f_{C0} + \frac{c_1(D) f_{C1}}{r^2 \Lambda^2} + \sum_{n=2}^{\infty} \frac{c_n(D) f_{Cn}}{r^{2n} \Lambda^{2n}} = c_0(D) f_{C0} + \frac{c_1(D) f_{C1}}{r^2 \Lambda^2} + \Phi_2(D, r). \quad (\text{A4})$$

Introducing  $z = (r\Lambda)^2/4$ , one can rewrite  $\Phi_2(D, r)$  as

$$\Phi_2(D, z) = \sum_{n=2}^{\infty} 2\alpha(D) \Gamma\left(n + \frac{D+1}{2}\right) \Gamma(n+D-1) f_{Cn} z^{-n}. \quad (\text{A5})$$

The  $\Gamma$  factors in the latter series can be written using an integral involving the modified Bessel function  $K_\nu(x)$  (C1),

$$\Phi_2(D, z) = 4\alpha(D) z^{\frac{3D-1}{4}} \sum_{n=2}^{\infty} \int_0^\infty d\tau f_{Cn} \tau^{n+\frac{3D-5}{4}} K_{\frac{D-3}{2}}(2\sqrt{z\tau}). \quad (\text{A6})$$

Note that for even  $D$  this is clearly some form of the Laplace transform since the index of the  $K$  function is a half-integer and the corresponding modified Bessel function can be reexpressed through powers and an exponent of its argument. The summation over  $n$  can be written as

$$\sum_{n=2}^{\infty} f_{Cn} \tau^n = \mathcal{F}_C(\tau) - f_{C0} - f_{C1} \tau. \quad (\text{A7})$$

Combining all together we get instead of Eq. (A3) the following one:

$$P'_D(r) = \frac{2}{D-2} \frac{1}{M_P^2} \frac{r_D^{2(D-3)}}{r^D} \cdot \left(c_0(D) f_{C0} + \frac{c_1(D) f_{C1}}{r^2 \Lambda^2} + \Phi_2(D, r)\right), \quad (\text{A8})$$

where  $\Phi_2(D, r)$  has the following integral representation through  $\mathcal{F}_C(\tau)$ :

$$\Phi_2(D, r) = 4\alpha(D) \left(\frac{r\Lambda}{2}\right)^{\frac{3D-1}{2}} \times \int_0^\infty d\tau (\mathcal{F}_C(\tau) - f_{C0} - f_{C1} \tau) \tau^{\frac{3D-5}{4}} \times K_{\frac{D-3}{2}}(r\Lambda\sqrt{\tau}), \quad (\text{A9})$$

where, moreover, we have returned back to  $r$ . Finally, integrating over  $r$  one yields

$$P_D(r) = -\frac{2}{D-2} \frac{1}{M_P^2} \frac{r_D^{2(D-3)}}{r^D} \cdot \left( r \frac{c_0(D)f_{C0}}{D-1} + \frac{c_1(D)f_{C1}}{r\Lambda^2(D+1)} + \tilde{\Phi}_2(D, r) \right), \quad (\text{A10})$$

where  $\tilde{\Phi}_2(D, r)$  comes as the result of an integration of  $\Phi_2(D, r)/r^D$  with respect to  $r$  with subsequent factoring of  $-1/r^D$  out. The corresponding expression is

$$\tilde{\Phi}_2(D, r) = \frac{4\alpha(D)}{\Lambda} \left( \frac{r\Lambda}{2} \right)^{\frac{3D-1}{2}} \int_0^\infty d\tau (\mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau) \tau^{\frac{3D-7}{4}} K_{\frac{D-1}{2}}(r\Lambda\sqrt{\tau}). \quad (\text{A11})$$

An integration constant must vanish in order not to modify the Newtonian potential at infinity. This integration could utilize the property of Bessel functions (C2).

The  $(1)$  component of Eq. (9) reads

$$\begin{aligned} M_P^2 \frac{D-2}{2} \left[ \frac{F'_D(r)}{r} \left( 1 + O\left(\frac{r_D}{r}\right) \right) + \frac{P'_D(r)}{r^{D-2}} \right] &= \frac{r_D^{2(D-3)}}{r^{2D-2}} \sum_{n=0}^\infty \frac{d_n(D)f_{Cn}}{r^{2n}\Lambda^{2n}} \left( 1 + O\left(\frac{r_D}{r}\right) \right) \\ &= \frac{r_D^{2(D-3)}}{r^{2D-2}} \Psi(D, r) \left( 1 + O\left(\frac{r_D}{r}\right) \right). \end{aligned} \quad (\text{A12})$$

Here again, we assume that we are far away from the horizon and  $\Psi(D, r)$  is a shorthand for the sum on the rhs of the equation. Note that the zero term in the sum on the rhs of Eq. (A12) does not have  $O(r_D/r)$  corrections in any dimension. The explicit form of coefficients  $d_n$  is presented in Appendix B, Eqs. (B19) and (B20).

Proceeding in full analogy with Sec. III, one yields

$$F'_D(r) = -\frac{P'_D(r)}{r^{D-1}} + \frac{2}{D-2} \frac{1}{M_P^2} \frac{r_D^{2(D-3)}}{r^{2D-1}} \left( d_0(D)f_{C0} + \frac{d_1(D)f_{C1}}{r^2\Lambda^2} + \Psi_2(D, r) \right), \quad (\text{A13})$$

where  $\Psi_2(D, r)$  has the following integral representation through  $\mathcal{F}_C(\tau)$ :

$$\Psi_2(D, r) = 2\alpha(D) \left( \frac{r\Lambda}{2} \right)^{\frac{3D-3}{2}} \int_0^\infty d\tau (\mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau) \tau^{\frac{3D-7}{4}} K_{\frac{D-1}{2}}(r\Lambda\sqrt{\tau}). \quad (\text{A14})$$

Using Eq. (A8) to simplify (A13), one can write

$$F'_D(r) = \frac{2}{D-2} \frac{1}{M_P^2} \frac{r_D^{2(D-3)}}{r^{2D-1}} \left( \frac{(d_1(D) - c_1(D))f_{C1}}{r^2\Lambda^2} + \Psi_2(D, r) - \Phi_2(D, r) \right), \quad (\text{A15})$$

where we have used the property of coefficients that  $c_0(D) = d_0(D)$ . It is instructive to note the following relation:

$$\begin{aligned} \Delta_2(D, r) &= \Psi_2(D, r) - \Phi_2(D, r) = 2\alpha(D) \left( \frac{r\Lambda}{2} \right)^{\frac{3D-3}{2}} \int_0^\infty d\tau (\mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau) \tau^{\frac{3D-7}{4}} \\ &\quad \times \left[ K_{\frac{D-1}{2}}(r\Lambda\sqrt{\tau}) - r\Lambda\sqrt{\tau} K_{\frac{D-3}{2}}(r\Lambda\sqrt{\tau}) \right]. \end{aligned}$$

Finally, one gets for  $F_D(r)$ ,

$$F_D(r) = -\frac{2}{D-2} \frac{1}{M_P^2} \frac{r_D^{2(D-3)}}{r^{2D-1}} \left( \frac{(d_1(D) - c_1(D))f_{C1}}{r\Lambda^2 2D} + \tilde{\Delta}_2(D, r) \right), \quad (\text{A16})$$

where  $\tilde{\Delta}_2(D, r)$  comes as the result of an integration of  $\Delta_2(D, r)/r^{2D-1}$  with respect to  $r$  with subsequent factoring of  $-1/r^{2D-1}$  out. An expression to be evaluated is

$$\tilde{\Delta}_2(D, r) = -2\alpha(D)2^{\frac{D+1}{2}}\left(\frac{r\Lambda}{2}\right)^{2D-1} \int \frac{d\tau}{\tau} \left[ (\mathcal{F}_C(\tau) - f_{C0} - f_{C1}\tau)(\sqrt{\tau})^{2D-1} \int x^\nu (K_{\nu+1}(x) - xK_{\nu+2}(x))dx \right]$$

with  $\nu = -\frac{D}{2} - \frac{1}{2}$  and  $x = r\Lambda\sqrt{\tau}$ ,

where we have to perform an integration over  $x$ . The final answer can be found using integrals listed in Appendix C, especially formula (C3) and alternative representations of the integration result after it.

An integration constant must vanish in order not to modify the Newtonian potential at infinity.

## APPENDIX B: EXPLICIT FORM OF EQ. (9)

### 1. General form of $G_\nu^\mu|_{ds_\lambda^2}$

$$G_0^0|_{ds_\lambda^2} = \lambda \frac{D-2}{2} \frac{P_D'(r)}{r^{D-2}}, \quad (B1)$$

$$G_1^1|_{ds_\lambda^2} = \lambda \frac{D-2}{2} \frac{(r^{D-3} - r_D^{D-3})F_D'(r)}{r^{D-2}} + G_0^0|_{ds_\lambda^2} + O(\lambda^2), \quad (B2)$$

$$G_i^i|_{ds_\lambda^2} = \frac{\lambda}{2} \left[ \frac{(r^{D-3} - r_D^{D-3})F_D''(r) + P_D''(r)}{r^{D-3}} + \frac{D-3}{2} \frac{(2r^{D-3} + r_D^{D-3})F_D'(r)}{r^{D-2}} \right] + O(\lambda^2). \quad (B3)$$

Here, index  $i$  runs from 2 to  $D-1$ , and there is no sum over  $i$ .

### 2. Coefficients $c_n(D)$ and $d_n(D)$

#### a. Case $D=4$

$c_0 = 0$ ,  $c_n$  starting from  $c_0$  are

$$0, 90, 10080, 907200, 119750400, 21794572800, 5230697472000, 1600593426432000, \\ 608225502044160000, 281000181944401920000, 155112100433309859840000, \dots$$

Note that  $c_0 = 0$  holds thanks to the fact that the Gauss-Bonnet term in  $D = 4$  is topological and thus does not contribute to equations as long as the boundary term is not accounted for. Using an analytic matching starting from  $c_2$ , the coefficients can be expressed as

$$c_0 = 0, \quad c_1 = 90, \quad c_{n \geq 2} = \frac{1}{\sqrt{\pi}} 4^{n+1} \Gamma\left(n + \frac{5}{2}\right) \Gamma(n+3) = \frac{1}{4} \Gamma(2n+5). \quad (B4)$$

$d_0 = 0$ ,  $d_n$  starting from  $d_0$  are

$$0, 126, 1440, 100800, 10886400, 1676505600, 348713164800, 94152554496000, \\ 32011868528640000, 13380961044971520000, 6744004366665646080000, \dots$$

Using an analytic matching starting from  $d_2$ , the coefficients can be expressed as

$$d_0 = 0, \quad d_1 = 126, \quad d_{n \geq 2} = 2 \frac{1}{\sqrt{\pi}} \cdot 4^n \Gamma\left(n + \frac{3}{2}\right) \Gamma(n+3). \quad (B5)$$

**b. Case  $D=5$** 

$c_0 = -12$ ,  $c_n$  starting from  $c_0$  are

$$-12, 1440, 184320, 22118400, 3715891200, 832359628800, 239719573094400, 86299046313984000, \\ 37971580378152960000, 20048994439664762880000, 12510572530350812037120000, \dots$$

Using an analytic matching starting from  $c_2$ , the coefficients can be expressed as

$$c_0 = -12, \quad c_1 = 1440, \quad c_{n \geq 2} = 4^{n+1} \Gamma(n+3) \Gamma(n+4). \quad (\text{B6})$$

$d_0 = -12$ ,  $d_n$  starting from  $d_0$  are

$$-12, 1248, 23040, 2211840, 309657600, 59454259200, 14982473318400 \\ 4794391461888000, 1898579018907648000, \\ 911317929075671040000, 521273855431283834880000, \dots$$

Using an analytic matching starting from  $d_2$ , the coefficients can be expressed as

$$d_0 = -12, \quad d_1 = 1248, \quad d_{n \geq 2} = 2 \cdot 4^n \Gamma(n+2) \Gamma(n+4). \quad (\text{B7})$$

**c. Case  $D=6$** 

$c_0 = -60$ ,  $c_n$  starting from  $c_0$  are

$$-60, 8400, 1360800, 209563200, 43589145600, 11769069312000, 4001483566080000, \\ 1672620130621440000, 843000545833205760000, \\ 504114326408257044480000, 352880028485779931136000000.$$

Using an analytic matching starting from  $c_2$ , the coefficients can be expressed as

$$c_0 = -60, \quad c_1 = 8400, \quad c_{n \geq 2} = \frac{1}{\sqrt{\pi}} 4^{n+1} \Gamma\left(n + \frac{7}{2}\right) \Gamma(n+5). \quad (\text{B8})$$

$d_0 = -60$ ,  $d_n$  starting from  $d_0$  are

$$-60, 6000, 151200, 19051200, 3353011200, 784604620800, 235381386240000, \\ 88032638453760000, 40142883134914560000, \\ 21918014191663349760000, 14115201139431197245440000, \dots$$

Using an analytic matching starting from  $d_2$ , the coefficients can be expressed as

$$d_0 = -60, \quad d_1 = 6000, \quad d_{n \geq 2} = 2 \frac{1}{\sqrt{\pi}} 4^n \Gamma\left(n + \frac{5}{2}\right) \Gamma(n+5). \quad (\text{B9})$$

**d. Case  $D=7$** 

$c_0 = -180$ ,  $c_n$  starting from  $c_0$  are

$$\begin{aligned} & -180, 31680, 6451200, 1238630400, 312134860800, 99883155456000, 39553729560576000 \\ & 18985790189076480000, 10859871988151746560000 \\ & 7297833976037973688320000, 5692310501309619476889600000, \dots \end{aligned}$$

Using an analytic matching starting from  $c_2$ , the coefficients can be expressed as

$$c_0 = -180, \quad c_1 = 31680, \quad c_{n \geq 2} = \frac{1}{6} 4^{n+1} \Gamma(n+4) \Gamma(n+6). \quad (\text{B10})$$

$d_0 = -180$ ,  $d_n$  starting from  $d_0$  are

$$\begin{aligned} & -180, 20160, 645120, 103219200, 22295347200, 6242697216000, 2197429420032000, \\ & 949289509453824000, 493630544915988480000, \\ & 304076415668248903680000, 218935019281139210649600000, \dots \end{aligned}$$

Using an analytic matching starting from  $d_2$ , the coefficients can be expressed as

$$d_0 = -180, \quad d_1 = 20160, \quad d_{n \geq 2} = \frac{1}{3} 4^n \Gamma(n+3) \Gamma(n+6). \quad (\text{B11})$$

**e. Case  $D=8$** 

$c_0 = -420$ ,  $c_n$  starting from  $c_0$  are

$$\begin{aligned} & -420, 92610, 23284800, 5448643200, 1634592960000, 611337767040000, 278770021770240000, \\ & 152208431886551040000, \dots \end{aligned}$$

Using an analytic matching starting from  $c_2$ , the coefficients can be expressed as

$$c_0 = -420, \quad c_1 = 92610, \quad c_{n \geq 2} = \frac{1}{18} \frac{1}{\sqrt{\pi}} 4^{n+1} \Gamma\left(n + \frac{9}{2}\right) \Gamma(n+7). \quad (\text{B12})$$

$d_0 = -420$ ,  $d_n$  starting from  $d_0$  are

$$\begin{aligned} & -420, 54390, 2116800, 419126400, 108972864000, 35961045120000, 14672106408960000, \\ & 7248020566026240000, \dots \end{aligned}$$

Using an analytic matching starting from  $d_2$ , the coefficients can be expressed as

$$d_0 = -420, \quad d_1 = 54390, \quad d_{n \geq 2} = \frac{1}{9} \frac{1}{\sqrt{\pi}} 4^n \Gamma\left(n + \frac{7}{2}\right) \Gamma(n+7). \quad (\text{B13})$$

**f. Case  $D=9$** 

$c_0 = -840$ ,  $c_n$  starting from  $c_0$  are

$$-840, 228480, 69672960, 19508428800, 6866966937600, \dots$$

Using an analytic matching starting from  $c_2$ , the coefficients can be expressed as

$$c_0 = -840, \quad c_1 = 228480, \\ c_{n \geq 2} = \frac{1}{240} 4^{n+1} \Gamma(n+5) \Gamma(n+8). \quad (\text{B14})$$

$d_0 = -840$ ,  $d_n$  starting from  $d_0$  are  
 $-840, 126336, 5806080, 1393459200, 429185433600, \dots$

Using an analytic matching starting from  $d_2$ , the coefficients can be expressed as

$$d_0 = -840, \quad d_1 = 126336, \\ d_{n \geq 2} = \frac{1}{120} 4^n \Gamma(n+4) \Gamma(n+8). \quad (\text{B15})$$

### g. Case $D = 10$

$c_0 = -1512$ ,  $c_n$  starting from  $c_0$  are

$$-1512, 498960, 181621440, 59935075200, \\ 24453510681600, \dots$$

Using an analytic matching starting from  $c_2$ , the coefficients can be expressed as

$$c_0 = -1512, \quad c_1 = 498960, \\ c_{n \geq 2} = \frac{1}{1350} \frac{1}{\sqrt{\pi}} 4^{n+1} \Gamma\left(n + \frac{11}{2}\right) \Gamma(n+9). \quad (\text{B16})$$

$d_0 = -1512$ ,  $d_n$  starting from  $d_0$  are

$$-1512, 263088, 13970880, 3995671680, \\ 1438441804800, \dots$$

Using an analytic matching starting from  $d_2$ , the coefficients can be expressed as

$$d_0 = -1512, \quad d_1 = 263088, \\ d_{n \geq 2} = \frac{1}{675} \frac{1}{\sqrt{\pi}} 4^n \Gamma\left(n + \frac{9}{2}\right) \Gamma(n+9). \quad (\text{B17})$$

### h. General pattern

The patterns appearing and looking very plausible are as follows.

For  $c_n$ ,

$$c_0(D) = -\frac{1}{2}(D-4)(D-3)(D-2)(D-1) \\ c_1(D) = 90, 1440, 8400, 31680, 92610, 228480, \\ 498960, \dots \\ c_{n \geq 2}(D) = 2\alpha(D) 4^n \Gamma\left(n + \frac{D+1}{2}\right) \Gamma(n+D-1). \quad (\text{B18})$$

For  $d_n$ ,

$$d_0(D) = c_0(D) = -\frac{1}{2}(D-4)(D-3)(D-2)(D-1) \\ d_1(D) = 126, 1248, 6000, 20160, 54390, \\ 126336, 263088, \dots \\ d_{n \geq 2}(D) = \alpha(D) 4^n \Gamma\left(n + \frac{D-1}{2}\right) \Gamma(n+D-1). \quad (\text{B19})$$

Here  $\alpha(D)$  takes the values

$$\alpha(D) = \frac{2}{\sqrt{\pi}}, \frac{2}{\sqrt{\pi}}, \frac{1}{9\sqrt{\pi}}, \frac{1}{675\sqrt{\pi}} \text{ in } D = 4, 6, 8, 10 \\ \alpha(D) = 2, \frac{1}{3}, \frac{1}{120} \text{ in } D = 5, 7, 9 \quad (\text{B20})$$

and both  $c_1(D)$  and  $d_1(D)$  [as well as  $c_1(D) - d_1(D)$ ] do not obey an easily recognizable pattern. We also notice that for  $n \geq 2$  one has

$$c_n(D) = 2\left(n + \frac{D-1}{2}\right) d_n(D).$$

### APPENDIX C: INTEGRALS INVOLVING MODIFIED BESSEL FUNCTION $K_\nu(x)$

Integrals listed below can be found in [57] or evaluated using Wolfram *Mathematica*. An integral used to transform (13) is as follows:

$$\int_0^\infty x^\mu K_\nu(ax) dx = 2^{\mu-1} a^{-\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right). \quad (\text{C1})$$

A standard integral relevant for our analysis is

$$\int x^\nu K_{\nu-1}(x) dx = -x^\nu K_\nu(x). \quad (\text{C2})$$

Another integral relevant for our analysis is

$$\int x^\nu K_{\nu+1}(x) dx = \pi 2^{-\nu-3} \frac{1}{\sin(\pi\nu)} \left( \frac{4^\nu (8\nu \log(x) - x^2 {}_2F_3(1, 1; 2, 2, 1 - \nu; \frac{x^2}{4}))}{\Gamma(1 - \nu)} + \frac{x^{2\nu+2} {}_1F_2(\nu + 1; \nu + 2, \nu + 2; \frac{x^2}{4})}{(\nu + 1)^2 \Gamma(\nu + 1)} \right). \quad (C3)$$

In this case, we have  $\nu = -D/2 - 1/2$ . Then, for even dimensions, one can find

$$\int (x^\nu K_{\nu+1}(x) - x^{\nu+1} K_{\nu+2}(x)) dx \Big|_{D=4} = \frac{1}{3} \sqrt{\frac{\pi}{2}} \left( 4\text{Ei}(-x) + e^{-x} \frac{4x^2 - x - 1}{x^3} \right) \quad (C4)$$

$$\int (x^\nu K_{\nu+1}(x) - x^{\nu+1} K_{\nu+2}(x)) dx \Big|_{D=6} = \frac{1}{5} \sqrt{\frac{\pi}{2}} \left( -2\text{Ei}(-x) - e^{-x} \frac{2x^4 - 2x^3 - x^2 + 3x + 3}{x^5} \right) \quad (C5)$$

and a similar pattern containing exponent, integral exponents, and fractions repeats for higher even  $D$ . For odd  $D$ , the situation is a bit more complicated, as the first integral (C3) is not very convenient given an apparent singularity for integer  $\nu$  due to the sine in the denominator. However, an alternative form using Meijer  $G$  function can be found. The following is the result for  $D = 5$ :

$$\int (x^\nu K_{\nu+1}(x) - x^{\nu+1} K_{\nu+2}(x)) dx \Big|_{D=5} = \frac{1}{16} \left( 2G_{1,3}^{3,0} \left( \frac{x}{2}, \frac{1}{2} \middle| -1, 0, 0 \right) - G_{1,3}^{3,0} \left( \frac{x}{2}, \frac{1}{2} \middle| -2, 0, 0 \right) \right), \quad (C6)$$

where the generalized Meijer  $G$  function is defined as follows:

$$G_{p,q}^{m,n} \left( z, r \middle| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right) = \frac{r}{2\pi i} \int \frac{\prod_{i=1}^n (-a_i - rs + 1) \prod_{j=1}^m (b_j + rs)}{\prod_{i=n+1}^p (a_i + rs) \prod_{j=1+m}^q (-b_j - rs + 1)} z^{-s} ds.$$

#### APPENDIX D: BOREL SUMMATION FOR FORMULA (49)

For a series

$$S = \sum_{n=0}^{\infty} x^n,$$

which converges for  $|x| < 1$ , we define its Borel series as

$$\tilde{S} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n,$$

which sums to

$$\tilde{S} = e^{tx}.$$

We then define the Borel sum for the initial series  $S$  to be

$$S \rightarrow \int_0^{\infty} e^{-t} e^{tx} dt = \frac{1}{1-x},$$

which is a valid answer for  $\text{Re}(x) < 1$ .

Formally, it is enough in our case since in our case  $x = -s/(r\mu)^2 < 0$ .

We need the following sums:

$$\begin{aligned} S &= \sum_{n=0}^{\infty} x^n \rightarrow \frac{1}{1-x}, & S_1 &= \frac{d}{dx} S = \sum_{n=0}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n \rightarrow \frac{1}{(1-x)^2} \\ S_2 &= \frac{d^2}{dx^2} S = \sum_{n=0}^{\infty} n(n-1) x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) x^n \rightarrow \frac{2}{(1-x)^3} \\ S_3 &= \frac{d^3}{dx^3} S = \sum_{n=0}^{\infty} n(n-1)(n-2) x^{n-3} = \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) x^n \rightarrow \frac{6}{(1-x)^4}. \end{aligned}$$

Then we readily get for an expression of interest

$$\sum_{n=0}^{\infty} (n+1)(n+2) \left(n + \frac{3}{2}\right) x^n = S_3 - \frac{3}{2} S_2 = 3 \frac{1+x}{(1-x)^4}.$$

Expression (49) can be rewritten as

$$\Phi(4, r) = \frac{15}{r^2 \mu^2} - \frac{2}{s} \left( 3 \frac{1 - s/(r\mu)^2}{(1 + s/(r\mu)^2)^4} - 3 + 15 \frac{s}{(r\mu)^2} \right).$$

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