

Curvatures and nonmetricities in the nonrelativistic limit of bosonic supergravity

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We construct a metriclike formulation of the nonrelativistic (NR) limit of bosonic supergravity at the Lagrangian level. This formulation is particularly useful for decomposing relativistic tensors, such as powers of the Riemann tensor, in a manifest covariant form with respect to infinitesimal diffeomorphisms. The construction is purely geometrical and is based on a torsionless connection, mimicking the construction of the relativistic theory. The formulation contains nonvanishing nonmetricities, which are associated with the gravitational fields of the theory ($\tau_{\mu\nu}$, $h_{\mu\nu}$, $\tau^{\mu\nu}$, $h^{\mu\nu}$). The nonmetricities are fixed by requiring compatibility with the relativistic metric, before taking the NR expansion. In this sense, they can be interpreted as intrinsic nonmetricities. We provide a fully covariant decomposition of the relativistic Riemann tensor, Ricci tensor, and scalar curvature. Our results establish an equivalence between the vielbein approach of string Newton-Cartan geometry at the level of the Lagrangian and the proposed construction. We also discuss potential applications, including a pure metric rewriting of the two-derivative finite bosonic supergravity Lagrangian under the NR limit, a powerful simplification in deriving NR-bosonic α' corrections and extensions to more general $f(R, Q)$ Newton-Cartan geometries.

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I. INTRODUCTION

In recent years, nonrelativistic (NR) limits of string theory [1–3] and supergravity [4–17] have attracted considerable attention, motivated by both conceptual developments and potential phenomenological applications. In particular, the systematic study of the supergravity limit of NR string theory has led to the emergence of Newton-Cartan and string Newton-Cartan geometries as the natural geometric frameworks underlying these limits (for reviews and complementary introductions, see [18–20] and references therein).

While the NR limit of Neveu-Schwarz (NS)-NS gravity has been deeply studied in [13], where the authors have addressed the construction in the vielbein formalism of NR theory, in this work we will reformulate this same setup, but using a purely metric approach, where the full Lorentz symmetry is manifest from the starting point. In other words, we will address NS-NS gravity in its metric formulation directly from the relativistic theory, and then we will study extensions which keep this structure. This way of avoiding the vielbein formulation will be extremely useful to address higher-derivative contributions in its NR

limit, which can be written entirely with curvatures, like powers of the Riemann tensor. In this sense, one can avoid the use of the spin connection, and work solely with an affine connection. Our canonical example here is the Metsaev and Tseytlin formulation of the four-derivative corrections of bosonic string theory [21].

Let us begin by considering the universal bosonic NS-NS supergravity, whose relativistic field content consists of a spacetime metric $\hat{g}_{\mu\nu}$, a Kalb-Ramond two-form $\hat{B}_{\mu\nu}$, and a dilaton $\hat{\phi}$. A controlled NR limit can be obtained [13] by introducing an explicit expansion in powers of c ,

$$\hat{g}_{\mu\nu} = c^2 \tau_{\mu\nu} + h_{\mu\nu}, \quad (1)$$

$$\hat{g}^{\mu\nu} = \frac{1}{c^2} \tau^{\mu\nu} + h^{\mu\nu}, \quad (2)$$

$$\hat{B}_{\mu\nu} = -c^2 c_{\mu\nu} + b_{\mu\nu}, \quad (3)$$

$$\hat{\phi} = \ln(c) + \varphi, \quad (4)$$

where $\mu, \nu = 0, \dots, 25$. The Newton-Cartan variables satisfy the defining relations

$$h^{\mu\nu} \tau_{\nu\rho} = h^{\mu\nu} c_{\nu\rho} = 0, \quad (5)$$

$$h_{\mu\nu} \tau^{\nu\rho} = 0, \quad (6)$$

$$h_{\mu\nu} h^{\nu\rho} = \delta_\mu^\rho - \tau_{\mu\nu} \tau^{\nu\rho}. \quad (7)$$

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Although individual geometric quantities constructed from the relativistic Levi-Civita connection diverge in the limit $c \rightarrow \infty$, the bosonic NS-NS action

$$S = \int d^6x \sqrt{-\hat{g}} e^{-2\hat{\phi}} \left(\hat{R} + 4\partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{1}{12} \hat{H}_{\mu\nu\rho} \hat{H}^{\mu\nu\rho} \right), \quad (8)$$

with $\hat{H}_{\mu\nu\rho} = 3\partial_{[\mu} \hat{B}_{\nu\rho]}$, remains finite due to a nontrivial cancellation between divergent contributions arising from the Ricci scalar and the Kalb-Ramond sector (for a review of the cancellation mechanism in metric formalism see [22], and in vielbein formalism see the original construction [13]).

In vielbein formalism the longitudinal part can be conveniently parametrized in terms of the vector τ^μ_a and the 1-form τ_μ^a , with $a, b = 0, 1$ the transversal flat directions, i.e.,

$$\tau_{\mu\nu} = \tau_\mu^a \eta_{ab} \tau_\nu^b, \quad (9)$$

$$\tau^{\mu\nu} = \tau^\mu_a \eta^{ab} \tau^\nu_b, \quad (10)$$

while $c_{\mu\nu} = \tau_\mu^a \tau_\nu^b \epsilon_{ab}$. We will use this composition for inspecting the divergences in the four-derivative theory.

The two-derivative cancellation mechanism underlies the construction of various nonrelativistic string supergravity theories, including heterotic formulations [22–24], supersymmetric extensions [15,16], and double field theory constructions [25–28], as well as their formulation within non-Riemannian double geometries [29–37] and their NR limits [38]. Motivated by these developments, increasing attention has recently been devoted to understanding higher-derivative corrections in the NR limit [39–42]. At the relativistic level, many of these corrections are naturally written in terms of covariant tensors constructed from the Levi-Civita connection. In order to study their NR limit while preserving covariance under spacetime diffeomorphisms, it is useful to express these objects in terms of curvatures and covariant derivatives adapted to the NR geometric variables appearing in the expansion (1).

Previous formulations of the NR limit of NS-NS gravity provide a consistent description of the dynamics in terms of Newton-Cartan geometry and connections that naturally accommodate the local boost, $SO(1,1)$, and transverse rotation symmetries of the theory [13]. While this framework is well suited for describing the two-derivative dynamics of NR supergravity, certain applications, in particular the systematic decomposition of relativistic curvature tensors appearing in higher-derivative corrections, benefit from a pure metric formulation. In such a formulation, the relativistic curvature invariants could be reorganized in a covariant manner in terms of their different contributions in powers of c without the need of using the vielbein decomposition or spin connections (see also [43] for a metric formalism in p-brane Galilean geometries).

The main goal of this work is to provide such a formulation. We construct a torsionless affine connection adapted to the NR metric variables $\tau_{\mu\nu}$ and $h_{\mu\nu}$ and use it to rewrite relativistic geometric quantities in a manifestly covariant form under spacetime diffeomorphisms. Importantly, the resulting connection is not metric compatible with respect to the fundamental NR fields. Instead, the covariant derivatives of the fundamental fields are controlled by a set of nonmetricity tensors that arise naturally from expanding the relativistic metric compatibility condition. These nonmetricities are uniquely determined by requiring consistency with the relativistic Levi-Civita structure (compatibility with the relativistic metric).

This construction allows relativistic curvature tensors to be systematically reorganized into NR geometric invariants. In particular, we show that the full bosonic supergravity action can be rewritten in terms of covariant NR quantities, with all contributions remaining finite and well defined in the limit $c \rightarrow \infty$.

The purpose of this paper is threefold. First, we present an explicit construction of a torsionless affine connection and the associated nonmetricity tensors within the Newton-Cartan framework. Second, we compute the corresponding curvature tensors and demonstrate how relativistic curvature invariants decompose into NR geometric objects in a covariant manner. Third, we illustrate several applications of the formalism. These include the derivation of the complete NR two-derivative bosonic supergravity Lagrangian in a manifestly covariant form, the analysis of the finite form of the Metsaev-Tseytlin α' contributions, and the construction of more general Newton-Cartan-inspired theories based on alternative choices of nonmetricity that extend beyond the specific structure inherited from bosonic supergravity.

Overall, our results provide a geometric framework for the NR limit of bosonic supergravity that parallels the role played by the Levi-Civita connection in relativistic theories, while naturally incorporating nonmetricity effects. We expect this formalism to provide a useful starting point for the systematic study of higher-derivative corrections and duality-covariant formulations in the nonrelativistic limit of string theory.

II. PURE METRIC FORMALISM AND THE AFFINE CONNECTION

We start by recalling the transformation rule of the fundamental fields with respect to infinitesimal diffeomorphisms,

$$\begin{aligned} \delta_\xi \tau_{\mu\nu} &= \xi^\rho \partial_\rho \tau_{\mu\nu} + 2\partial_{(\mu} \xi^{\rho} \tau_{\rho|\nu)}, \\ \delta_\xi \tau^{\mu\nu} &= \xi^\rho \partial_\rho \tau^{\mu\nu} - 2\partial_\rho \xi^{(\mu} \tau^{\rho\nu)}, \\ \delta_\xi h_{\mu\nu} &= \xi^\rho \partial_\rho h_{\mu\nu} + 2\partial_{(\mu} \xi^{\rho} h_{\rho|\nu)}, \\ \delta_\xi h^{\mu\nu} &= \xi^\rho \partial_\rho h^{\mu\nu} - 2\partial_\rho \xi^{(\mu} h^{\rho\nu)}. \end{aligned} \quad (11)$$

The torsionless connection is given by

$$\Gamma_{\mu\nu}^\rho(\tau, h) = \frac{1}{2}h^{\rho\sigma}(2\partial_{(\mu}h_{\nu)\sigma} - \partial_\sigma h_{\mu\nu}) + \frac{1}{2}\tau^{\rho\sigma}(2\partial_{(\mu}\tau_{\nu)\sigma} - \partial_\sigma\tau_{\mu\nu}), \quad (12)$$

which is the c^0 contribution coming from the relativistic connection,

$$\hat{\Gamma}_{\mu\nu}^\rho(\hat{g}) = c^2\hat{\Gamma}_{\mu\nu}^{(2)\rho}(\tau, h) + \Gamma_{\mu\nu}^\rho(\tau, h) + \frac{1}{c^2}\hat{\Gamma}_{\mu\nu}^{(-2)\rho}(\tau, h), \quad (13)$$

preserving the nontensorial transformation rule. In other words, $\hat{\Gamma}_{\mu\nu}^{(2)\rho}(\tau, h)$ and $\hat{\Gamma}_{\mu\nu}^{(-2)\rho}(\tau, h)$ transform as tensors, so we can shift away their contributions and consider only $\Gamma_{\mu\nu}^\rho(\tau, h)$ as the nonrelativistic connection. This result only requires the notion of the longitudinal part of the metric, $\tau_{\mu\nu}$, and the transversal one, $h_{\mu\nu}$ (and similarly for the inverse metric). The choice of the connection is not unique, and it can be (re-)shifted by covariant contributions.

The covariant derivatives are therefore defined as

$$\nabla_\mu\tau_{\nu\sigma} = \partial_\mu\tau_{\nu\sigma} - 2\Gamma_{\mu(\nu}^\rho\tau_{\rho|\sigma)}, \quad (14)$$

$$\nabla_\mu\tau^{\nu\rho} = \partial_\mu\tau^{\nu\rho} + 2\Gamma_{\mu\sigma}^{(\nu}\tau^{\sigma|\rho)}, \quad (15)$$

$$\nabla_\mu h_{\nu\rho} = \partial_\mu h_{\nu\rho} - 2\Gamma_{\mu(\nu}^\sigma h_{\rho)\sigma}, \quad (16)$$

$$\nabla_\mu h^{\nu\rho} = \partial_\mu h^{\nu\rho} + 2\Gamma_{\mu\sigma}^{(\nu}h^{\sigma)\rho}. \quad (17)$$

The Riemann tensor can be easily computed from the commutator of the covariant derivatives acting on an arbitrary vector v^μ ,

$$[\nabla_\mu, \nabla_\nu]v^\rho = R^\rho_{\epsilon\mu\nu}v^\epsilon, \quad (18)$$

giving the usual expression

$$R^\rho_{\epsilon\mu\nu}(\tau, h) = 2\partial_{[\mu}\Gamma_{\nu]}^\rho + 2\Gamma_{[\mu}^\rho\Gamma_{\nu]}^\alpha. \quad (19)$$

III. INTRINSIC NONMETRICITIES

The nonmetricities play a fundamental role in the theory, since they guarantee that the NR limit can be taken from the relativistic theory,

$$\nabla_\mu\tau_{\nu\rho} = Q_{\mu\nu\rho}^{(\tau)}, \quad (20)$$

$$\nabla_\mu\tau^{\nu\rho} = Q_\mu^{(\tau^{-1})\nu\rho}, \quad (21)$$

$$\nabla_\mu h_{\nu\rho} = Q_{\mu\nu\rho}^{(h)}, \quad (22)$$

$$\nabla_\mu h^{\nu\rho} = Q_\mu^{(h^{-1})\nu\rho}. \quad (23)$$

The fixed/intrinsic values of the nonmetricities come from the metric conditions,

$$\hat{\nabla}_\mu\hat{g}_{\nu\rho} = 0, \quad (24)$$

$$\hat{\nabla}_\mu\hat{g}^{\nu\rho} = 0, \quad (25)$$

where $\hat{\nabla}$ is the usual covariant derivative with respect to the relativistic Levi-Civita connection. Therefore, after performing the NR expansion, one needs the following non-metricity combinations:

$$Q_{\mu\nu\rho}^{(\tau)} = 2\Gamma^{(2)\sigma}_{\mu(\nu}h_{\sigma|\rho)}, \quad (26)$$

$$Q_\rho^{(\tau^{-1})\mu\nu} = -\tau^{(\mu\alpha}(2\partial_{(\rho}h_{\sigma)\alpha} - \partial_\alpha h_{\rho\sigma})h^{\sigma\nu}), \quad (27)$$

$$Q_{\mu\nu\rho}^{(h)} = 2\Gamma^{(-2)\sigma}_{\mu(\nu}h_{\sigma|\rho)}, \quad (28)$$

$$Q_\rho^{(h^{-1})\mu\nu} = -h^{(\mu\alpha}(2\partial_{(\rho}\tau_{\sigma)\alpha} - \partial_\alpha\tau_{\rho\sigma})\tau^{\sigma\nu}), \quad (29)$$

where

$$\Gamma_{\mu\nu}^{(2)\sigma} = \frac{1}{2}h^{\alpha\sigma}(\nabla_\mu\tau_{\nu\alpha} + \nabla_\nu\tau_{\mu\alpha} - \nabla_\alpha\tau_{\mu\nu}), \quad (30)$$

$$\Gamma_{\mu\nu}^{(-2)\sigma} = \frac{1}{2}\tau^{\alpha\sigma}(\nabla_\mu h_{\nu\alpha} + \nabla_\nu h_{\mu\alpha} - \nabla_\alpha h_{\mu\nu}). \quad (31)$$

The NR nonmetricities are not boost invariants. We recall that the boost transformations are given by [13]

$$\delta_\lambda h^{\mu\nu} = 0, \quad (32)$$

$$\delta_\lambda h_{\mu\nu} = -2\lambda_{aa'}e_{(\mu}^{a'}\tau_{\nu)}^a, \quad (33)$$

$$\delta_\lambda\tau^{\mu\nu} = 2\lambda_a^{a'}e^{(\mu}_{a'}\tau^{\nu)a}, \quad (34)$$

$$\delta_\lambda\tau_{\mu\nu} = 0, \quad (35)$$

$$\delta_\lambda b_{\mu\nu} = -2\epsilon_{ab}\lambda^a_{a'}\tau_{[\mu}^be_{\nu]}^{a'}, \quad (36)$$

$$\delta_\lambda c_{\mu\nu} = 0, \quad (37)$$

where $\lambda_{aa'}$ is an arbitrary boost parameter. Here we are momentarily including the $SO(1, 1)$ index, as discussed in the previous section, to explicitly show that in this geometry, $Q = 0$ is not possible for all the metricities. In other words, some of the nonmetricities cannot be set to zero without compromising the boost symmetry. We will return to this point in the discussion section, since one might be interested in exploring Newton-Cartan geometries with more arbitrary nonmetricities, beyond bosonic supergravity.

IV. DECOMPOSITION OF THE RELATIVISTIC CURVATURES

A. The Riemann tensor

The relativistic Riemann tensor is given by

$$\hat{R}^\rho_{\epsilon\mu\nu} = c^4 \hat{R}^{(4)\rho}_{\epsilon\mu\nu} + c^2 \hat{R}^{(2)\rho}_{\epsilon\mu\nu} + \hat{R}^{(0)\rho}_{\epsilon\mu\nu} + c^{-2} \hat{R}^{(-2)\rho}_{\epsilon\mu\nu} + c^{-4} \hat{R}^{(-4)\rho}_{\epsilon\mu\nu}. \quad (38)$$

So far we have just identified part of the covariant contribution inside $\hat{R}^{(0)\rho}_{\epsilon\mu\nu}$, given by $R^\rho_{\epsilon\mu\nu}$, but other covariant contributions are encoded in $\hat{R}^{(0)\rho}_{\epsilon\mu\nu}$ due to c^2 and c^{-2} contributions in the relativistic Levi-Civita connection. All the contributions in the relativistic Riemann tensor can be written in covariant form, and this is what we are going to show now. Let us start with the higher-order contribution,

$$\hat{R}^{(4)\rho}_{\epsilon\mu\nu} = \frac{1}{2} h^{\rho\sigma} h^{\alpha\beta} (\partial_\alpha \tau_{[\mu\sigma} - \partial_\sigma \tau_{[\mu\alpha}) \times (\partial_{\nu]} \tau_{\epsilon\beta} + \partial_\epsilon \tau_{\nu]\beta} - \partial_\beta \tau_{\nu]\epsilon}). \quad (39)$$

The previous quantity transforms as a tensor, and it can be written in a manifest covariant form using the covariant derivatives,

$$\hat{R}^{(4)\rho}_{\epsilon\mu\nu} = \frac{1}{2} h^{\rho\sigma} h^{\alpha\beta} (\nabla_\alpha \tau_{[\mu\sigma} - \nabla_\sigma \tau_{[\mu\alpha}) \times (\nabla_{\nu]} \tau_{\epsilon\beta} + \nabla_\epsilon \tau_{\nu]\beta} - \nabla_\beta \tau_{\nu]\epsilon}). \quad (40)$$

The same technique can be implemented for the lowest contribution of the Riemann tensor,

$$\hat{R}^{(-4)\rho}_{\epsilon\mu\nu} = \frac{1}{2} \tau^{\rho\sigma} \tau^{\alpha\beta} (\partial_\alpha h_{[\mu\sigma} - \partial_\sigma h_{[\mu\alpha}) \times (\partial_{\nu]} h_{\epsilon\beta} + \partial_\epsilon h_{\nu]\beta} - \partial_\beta h_{\nu]\epsilon}), \quad (41)$$

which therefore can be written as

$$\hat{R}^{(-4)\rho}_{\epsilon\mu\nu} = \frac{1}{2} \tau^{\rho\sigma} \tau^{\alpha\beta} (\nabla_\alpha h_{[\mu\sigma} - \nabla_\sigma h_{[\mu\alpha}) \times (\nabla_{\nu]} h_{\epsilon\beta} + \nabla_\epsilon h_{\nu]\beta} - \nabla_\beta h_{\nu]\epsilon}). \quad (42)$$

In these cases we have identified that the structure of $R^{(-4)\rho}_{\epsilon\mu\nu}$ is just an interchange of $\tau_{\mu\nu} \leftrightarrow h_{\mu\nu}$ and $h^{\mu\nu} \leftrightarrow \tau^{\mu\nu}$ with respect to the highest-order contribution, $R^{(4)\rho}_{\epsilon\mu\nu}$.

The c^2 contributions of the relativistic Riemann tensor take an important role, since it is part of bosonic supergravity action once one takes the NR limit. These contributions can be written in terms of the covariant derivatives

$$\hat{R}^{(2)\rho}_{\epsilon\mu\nu} = 2\nabla_{[\mu} [h^{\rho\alpha} (\nabla_{\nu]} \tau_{\epsilon\alpha} + \nabla_\epsilon \tau_{\nu]\alpha} - \nabla_\alpha \tau_{\nu]\epsilon}). \quad (43)$$

Again, we can interchange $\tau_{\mu\nu} \leftrightarrow h_{\mu\nu}$ and $h^{\mu\nu} \leftrightarrow \tau^{\mu\nu}$ to construct the $c^{(-2)}$ contributions,

$$\hat{R}^{(-2)\rho}_{\epsilon\mu\nu} = 2\nabla_{[\mu} [\tau^{\rho\alpha} (\nabla_{\nu]} h_{\epsilon\alpha} + \nabla_\epsilon h_{\nu]\alpha} - \nabla_\alpha h_{\nu]\epsilon}). \quad (44)$$

Finally, we proceed to construct the c^0 contributions to the relativistic Riemann tensor. This quantity is given by

$$\begin{aligned} \hat{R}^{(0)\rho}_{\epsilon\mu\nu} &= R^\rho_{\epsilon\mu\nu} + \frac{1}{2} h^{\rho\sigma} \tau^{\alpha\beta} (\nabla_{[\mu} \tau_{\alpha\sigma} + \nabla_\alpha \tau_{[\mu\sigma} - \nabla_\sigma \tau_{[\mu\alpha}) (\nabla_{\nu]} h_{\epsilon\beta} + \nabla_\epsilon h_{\nu]\beta} - \nabla_\beta h_{\nu]\epsilon}) \\ &\quad + \frac{1}{2} \tau^{\rho\sigma} h^{\alpha\beta} (\nabla_{[\mu} h_{\alpha\sigma} + \nabla_\alpha h_{[\mu\sigma} - \nabla_\sigma h_{[\mu\alpha}) (\nabla_{\nu]} \tau_{\epsilon\beta} + \nabla_\epsilon \tau_{\nu]\beta} - \nabla_\beta \tau_{\nu]\epsilon}). \end{aligned} \quad (45)$$

In the next section we will study the decomposition of the (relativistic) Ricci tensor in terms of the Newton-Cartan fields, and in covariant form with respect to infinitesimal diffeomorphisms.

B. The Ricci tensor

The relativistic Ricci tensor is given by

$$\hat{R}_{\epsilon\nu} = c^4 \hat{R}^{(4)}_{\epsilon\nu} + c^2 \hat{R}^{(2)}_{\epsilon\nu} + \hat{R}^{(0)}_{\epsilon\nu} + c^{-2} \hat{R}^{(-2)}_{\epsilon\nu} + c^{-4} \hat{R}^{(-4)}_{\epsilon\nu}. \quad (46)$$

Using the results of the previous subsection, we can directly write all the contributions in covariant form,

$$\hat{R}^{(4)}_{\epsilon\nu} = \frac{1}{2} h^{\mu\sigma} h^{\alpha\beta} (\nabla_\alpha \tau_{[\mu\sigma} - \nabla_\sigma \tau_{[\mu\alpha}) (\nabla_{\nu]} \tau_{\epsilon\beta} + \nabla_\epsilon \tau_{\nu]\beta} - \nabla_\beta \tau_{\nu]\epsilon}), \quad (47)$$

$$\hat{R}^{(2)}_{\epsilon\nu} = 2\nabla_{[\mu} [h^{\mu\alpha} (\nabla_{\nu]} \tau_{\epsilon\alpha} + \nabla_\epsilon \tau_{\nu]\alpha} - \nabla_\alpha \tau_{\nu]\epsilon}), \quad (48)$$

$$\hat{R}^{(0)}_{\epsilon\nu} = R_{\epsilon\nu}(\tau, h) + \frac{1}{2} h^{\mu\sigma} \tau^{\alpha\beta} \nabla_\alpha \tau_{[\mu\sigma} (\nabla_{\nu]} h_{\epsilon\beta} + \nabla_\epsilon h_{\nu]\beta} - \nabla_\beta h_{\nu]\epsilon}) + \frac{1}{2} \tau^{\mu\sigma} h^{\alpha\beta} \nabla_\alpha h_{[\mu\sigma} (\nabla_{\nu]} \tau_{\epsilon\beta} + \nabla_\epsilon \tau_{\nu]\beta} - \nabla_\beta \tau_{\nu]\epsilon}), \quad (49)$$

$$\hat{R}_{\epsilon\nu}^{(-2)} = 2\nabla_{[\mu}[\tau^{\mu\alpha}(\nabla_{\nu]}h_{\epsilon\alpha} + \nabla_{\epsilon}h_{\nu]\alpha} - \nabla_{\alpha}h_{\nu]\epsilon}], \quad (50)$$

$$\hat{R}^{(-4)}_{\epsilon\nu} = \frac{1}{2}\tau^{\mu\sigma}\tau^{\alpha\beta}(\nabla_{\alpha}h_{[\mu\sigma} - \nabla_{\sigma}h_{\mu\alpha})(\nabla_{\nu]}h_{\epsilon\beta} + \nabla_{\epsilon}h_{\nu]\beta} - \nabla_{\beta}h_{\nu]\epsilon}). \quad (51)$$

C. The Ricci scalar

The relativistic Ricci scalar is decomposed as

$$\hat{R} = c^2\hat{R}^{(2)} + \hat{R}^{(0)} + c^{-2}\hat{R}^{(-2)} + c^{-4}\hat{R}^{(-4)}, \quad (52)$$

where the covariant form of each contribution is given by

$$\hat{R}^{(2)} = \hat{R}_{\mu\nu}^{(2)}h^{\mu\nu} + \hat{R}_{\mu\nu}^{(4)}\tau^{\mu\nu}, \quad (53)$$

$$\hat{R}^{(0)} = \hat{R}_{\mu\nu}^{(0)}h^{\mu\nu} + \hat{R}_{\mu\nu}^{(2)}\tau^{\mu\nu}, \quad (54)$$

$$\hat{R}^{(-2)} = \hat{R}_{\mu\nu}^{(-2)}h^{\mu\nu} + \hat{R}_{\mu\nu}^{(0)}\tau^{\mu\nu}, \quad (55)$$

$$\hat{R}^{(-4)} = \hat{R}_{\mu\nu}^{(-4)}h^{\mu\nu}. \quad (56)$$

Using the results of the previous subsection we can easily see that the right-hand side of the previous contributions can be written in a fully covariant way. We will explore the explicit form of $\hat{R}^{(0)}$ in the next section.

V. APPLICATIONS

In this section, we explore three direct applications of the proposed formalism. First, we derive the finite nonrelativistic limit of the two-derivative bosonic supergravity Lagrangian. We then discuss the convenience of this formulation for extracting covariant contributions at any desired order in the expansion in powers of c . As an illustrative example, we compute all finite contributions arising from the Metsaev-Tseytlin Lagrangian, which constitutes part of the four-derivative corrections to the bosonic Lagrangian in the NR limit.¹ Finally, we discuss the possibility of employing more general nonmetricities, focusing in particular on the case in which all such tensors vanish. This choice explicitly breaks the boost symmetry of the NR limit of bosonic supergravity.

A. NR-bosonic supergravity Lagrangian: Metriclike (re-)formulation

The finite NR-bosonic supergravity Lagrangian contains contributions coming from the Ricci scalar, the dilaton term, and the $\hat{H}$ -terms, and using the results of the previous section it can be written in the following way:

$$\begin{aligned} e^{-1}e^{2\varphi}L^{(0)}|_{NR} = & R_{\mu\nu}(\tau, h)h^{\mu\nu} + h^{\mu\sigma}\tau^{\alpha\beta}\nabla_{\alpha}\tau_{[\mu\sigma}\left(\nabla_{\nu]}h_{\epsilon\beta} - \frac{1}{2}\nabla_{\beta}h_{\nu]\epsilon}\right)h^{\epsilon\nu} + \tau^{\mu\sigma}h^{\alpha\beta}\nabla_{\alpha}h_{[\mu\sigma}\left(\nabla_{\epsilon}\tau_{\nu]\beta} - \frac{1}{2}\nabla_{\beta}\tau_{\nu]\epsilon}\right)h^{\epsilon\nu} \\ & + 4\nabla_{[\beta}\left[h^{\beta\alpha}(\nabla_{\mu}\tau_{\nu]\alpha} - \frac{1}{2}\nabla_{\alpha}\tau_{\nu]\mu}\right]\tau^{\mu\nu} + 4h^{\mu\nu}\nabla_{\mu}\varphi\nabla_{\nu}\varphi - \frac{1}{12}h_{\mu\nu\rho}h_{\sigma\gamma\epsilon}h^{\mu\sigma}h^{\nu\gamma}h^{\rho\epsilon} - \frac{1}{2}h_{\mu\nu\rho}C_{\sigma\gamma\epsilon}\tau^{\mu\sigma}\tau^{\nu\gamma}h^{\rho\epsilon} \\ & - \frac{1}{4}C_{\mu\nu\rho}C_{\sigma\gamma\epsilon}\tau^{\mu\sigma}\tau^{\nu\gamma}h^{\rho\epsilon}, \end{aligned} \quad (57)$$

where

$$h_{\mu\nu\rho} = 3\partial_{[\mu}b_{\nu\rho]}, \quad C_{\mu\nu\rho} = -3\nabla_{[\mu}c_{\nu\rho]}.$$

Now we can see that the finite bosonic Lagrangian after taking the NR limit contains a $R_{\mu\nu}h^{\mu\nu}$ contribution, plus other terms related to the nonmetricities.

As we mentioned before, the construction is purely written in metric formalism and is equivalent to the torsionful formalism of string Newton-Cartan, in the sense that no geometric constraints are imposed at the Lagrangian level. The nonmetricities are just a convenient way of rewriting diffeomorphism-covariant quantities. Moreover, one cannot consider them as true nonmetricities, in the sense that the fields τ and h are not true metrics in the

Newton-Cartan geometry. It is worth mentioning that the compatibility explored in this paper matches at the level of the action. The compatibility at the equations of motion is not guaranteed, since one should find a metric constraint equivalent to the torsion constraint found in [13].

B. Nonrelativistic bosonic α' corrections

The four-derivative corrections to the bosonic string supergravity were historically computed considering three- and four-point scattering amplitudes for the massless states [44,45] (see [46] for a pedagogical review of this topic).

¹Additional contributions are expected to arise in order to cancel the higher-derivative divergences of the action.

The method is based on the study of the different types of string interactions through the S-matrix to construct an effective Lagrangian, originally computed by Metsaev and Tseytlin [21],

$$S_{MT} = \int d^{26}x \sqrt{-g} e^{-2\phi} (L^{(0)} + L_{MT}), \quad (58)$$

where

$$L_{MT} = -\frac{\alpha'}{4} \left[\hat{R}_{\mu\nu\rho\sigma} \hat{R}^{\mu\nu\rho\sigma} - \frac{1}{2} \hat{H}^{\mu\nu\rho} \hat{H}_{\mu\sigma\lambda} \hat{R}_{\nu\rho}{}^{\sigma\lambda} + \frac{1}{24} \hat{H}^4 - \frac{1}{8} \hat{H}_{\mu\nu}^2 \hat{H}^{2\mu\nu} \right], \quad (59)$$

and

$$\hat{H}_{\mu\nu}^2 = \hat{H}_{\mu}{}^{\rho\sigma} \hat{H}_{\nu\rho\sigma}, \quad \hat{H}^2 = \hat{H}_{\mu\nu\rho} \hat{H}^{\mu\nu\rho}. \quad (60)$$

The formalism developed in this work allows one to easily inspect the NR contributions of the previous quantities without losing covariance.

1. Divergences of the four-derivative bosonic action

Recalling that $\alpha' \rightarrow \frac{\alpha'_{NR}}{c^2}$ [41,42], the divergent contributions coming from the Metsaev and Tseytlin Lagrangian are given by

$$L_{Div}^{(1)} = \frac{3c^2}{2} \alpha'_{NR} \eta_{ab} \eta_{cd} \partial_\mu \tau_\nu{}^a \partial_\rho \tau_\sigma{}^b \partial_\gamma \tau_\epsilon{}^c \partial_\delta \tau_\lambda{}^d (h^{\mu\epsilon} h^{\nu\gamma} h^{\rho\lambda} h^{\sigma\delta} + h^{\mu\gamma} h^{\nu\epsilon} h^{\rho\delta} h^{\sigma\lambda} - 2h^{\mu\gamma} h^{\nu\epsilon} h^{\rho\lambda} h^{\sigma\delta} + h^{\mu\rho} h^{\nu\sigma} h^{\gamma\delta} h^{\epsilon\lambda} - 2h^{\mu\rho} h^{\nu\sigma} h^{\gamma\lambda} h^{\epsilon\delta} + h^{\mu\sigma} h^{\nu\rho} h^{\gamma\lambda} h^{\epsilon\delta}), \quad (61)$$

where we have used the vielbein decomposition $c_{\mu\nu} = \tau_\mu{}^a \tau_\nu{}^b \epsilon_{ab}$. The previous divergent terms survive the non-relativistic expansion, showing that on the Metsaev and Tseytlin action the limit $c \rightarrow \infty$ leads to divergences. However, it is expected that they could be absorbed in field redefinitions (see the next section) and work in progress is being made toward this proof [47]. Since every partial derivative can be turned into a covariant one (by including the affine connection), another possibility is to impose the constraint

$$\tau_\nu{}^a \nabla_\mu h^{\nu\gamma} = 0 \quad (62)$$

(or simply $\nabla_\mu h^{\nu\gamma} = 0$), which cancels the divergences. The price to pay is that the finite action might need higher-derivative boost transformations, to restore its boost invariance due to the extra constraint.

2. Finite contributions to the four-derivative action

The finite contributions of the four-derivative action can also be easily computed with the formalism here developed. The finite Riemann-squared contributions are given by

$$\begin{aligned} & 2\hat{R}^{(4)\mu}{}_{\nu\rho\sigma} \hat{R}^{(-4)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} h^{\nu\epsilon} h^{\rho\delta} h^{\sigma\lambda} + 2\hat{R}^{(2)\mu}{}_{\nu\rho\sigma} \hat{R}^{(-2)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} h^{\nu\epsilon} h^{\rho\delta} h^{\sigma\lambda} + 2\hat{R}^{(4)\mu}{}_{\nu\rho\sigma} \hat{R}^{(-2)\gamma}{}_{\epsilon\delta\lambda} h_{\mu\gamma} h^{\nu\epsilon} h^{\rho\delta} h^{\sigma\lambda} \\ & + 4\hat{R}^{(4)\mu}{}_{\nu\rho\sigma} \hat{R}^{(-2)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} \tau^{\rho\delta} h^{\nu\epsilon} h^{\sigma\lambda} + 2\hat{R}^{(4)\mu}{}_{\nu\rho\sigma} \hat{R}^{(-2)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} \tau^{\nu\epsilon} h^{\rho\delta} h^{\sigma\lambda} + \hat{R}^{(0)\mu}{}_{\nu\rho\sigma} \hat{R}^{(0)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} h^{\nu\epsilon} h^{\rho\delta} h^{\sigma\lambda} \\ & + 2\hat{R}^{(0)\mu}{}_{\nu\rho\sigma} \hat{R}^{(2)\gamma}{}_{\epsilon\delta\lambda} h_{\mu\gamma} h^{\nu\epsilon} h^{\rho\delta} h^{\sigma\lambda} + 4\hat{R}^{(0)\mu}{}_{\nu\rho\sigma} \hat{R}^{(2)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} \tau^{\rho\delta} h^{\nu\epsilon} h^{\sigma\lambda} + 2\hat{R}^{(0)\mu}{}_{\nu\rho\sigma} \hat{R}^{(2)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} \tau^{\nu\epsilon} h^{\rho\delta} h^{\sigma\lambda} \\ & + 4\hat{R}^{(0)\mu}{}_{\nu\rho\sigma} \hat{R}^{(4)\gamma}{}_{\epsilon\delta\lambda} \tau^{\rho\delta} h_{\mu\gamma} h^{\nu\epsilon} h^{\sigma\lambda} + 2\hat{R}^{(0)\mu}{}_{\nu\rho\sigma} \hat{R}^{(4)\gamma}{}_{\epsilon\delta\lambda} \tau^{\nu\epsilon} h_{\mu\gamma} h^{\rho\delta} h^{\sigma\lambda} + 2\hat{R}^{(0)\mu}{}_{\nu\rho\sigma} \hat{R}^{(4)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} \tau^{\rho\delta} \tau^{\sigma\lambda} h^{\nu\epsilon} \\ & + 4\hat{R}^{(0)\mu}{}_{\nu\rho\sigma} \hat{R}^{(4)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} \tau^{\nu\epsilon} \tau^{\rho\delta} h^{\sigma\lambda} + 2\hat{R}^{(2)\mu}{}_{\nu\rho\sigma} \hat{R}^{(2)\gamma}{}_{\epsilon\delta\lambda} \tau^{\rho\delta} h_{\mu\gamma} h^{\nu\epsilon} h^{\sigma\lambda} + \hat{R}^{(2)\mu}{}_{\nu\rho\sigma} \hat{R}^{(2)\gamma}{}_{\epsilon\delta\lambda} \tau^{\nu\epsilon} h_{\mu\gamma} h^{\rho\delta} h^{\sigma\lambda} \\ & + \hat{R}^{(2)\mu}{}_{\nu\rho\sigma} \hat{R}^{(2)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} \tau^{\rho\delta} \tau^{\sigma\lambda} h^{\nu\epsilon} + 2\hat{R}^{(2)\mu}{}_{\nu\rho\sigma} \hat{R}^{(2)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} \tau^{\nu\epsilon} \tau^{\rho\delta} h^{\sigma\lambda} + 2\hat{R}^{(2)\mu}{}_{\nu\rho\sigma} \hat{R}^{(4)\gamma}{}_{\epsilon\delta\lambda} \tau^{\rho\delta} \tau^{\sigma\lambda} h_{\mu\gamma} h^{\nu\epsilon} \\ & + 4\hat{R}^{(2)\mu}{}_{\nu\rho\sigma} \hat{R}^{(4)\gamma}{}_{\epsilon\delta\lambda} \tau^{\nu\epsilon} \tau^{\rho\delta} h_{\mu\gamma} h^{\sigma\lambda} + 2\hat{R}^{(2)\mu}{}_{\nu\rho\sigma} \hat{R}^{(4)\gamma}{}_{\epsilon\delta\lambda} \tau_{\mu\gamma} \tau^{\nu\epsilon} \tau^{\rho\delta} \tau^{\sigma\lambda} + \hat{R}^{(4)\mu}{}_{\nu\rho\sigma} \hat{R}^{(4)\gamma}{}_{\epsilon\delta\lambda} \tau^{\nu\epsilon} \tau^{\rho\delta} \tau^{\sigma\lambda} h_{\mu\gamma}. \end{aligned} \quad (63)$$

Due to the length of the expressions, we provide the remaining H-dependent terms in the Appendix. We clarify that we are not imposing (62) or any other constraint to simplify the action.

The framework constructed in this manuscript now allows one to explore field redefinition for the different Newton-Cartan fields. For example, since $h^{\mu\nu}$ is boost invariant one can propose

$$\tilde{h}^{\mu\nu} = h^{\mu\nu} + ac^4 h^{\mu\rho} R_{\rho\sigma}^{(4)} h^{\sigma\nu}, \quad (64)$$

which is a possible boost-invariant field redefinition for the $h^{\mu\nu}$ field since $\delta_\lambda R_{\mu\nu}^{(4)} = 0$.

The idea of the previous field redefinition is to induce divergences through the two-derivative Lagrangian by fixing the arbitrary coefficients a , which might cancel the divergences of the Metsaev and Tseytlin Lagrangian. Similar field redefinitions can be done for the rest of the fields, i.e., $h_{\mu\nu}$, $\tau_{\mu\nu}$, $\tau^{\mu\nu}$, $b_{\mu\nu}$, and ϕ , consistently with the NR symmetries (one might expect deformations in the leading symmetries, coming from noncovariant field redefinitions). While the Metsaev and Tseytlin action can be written purely in terms of metric degrees of freedom, the boost invariance of the resulting action is not guaranteed due to the absence of the Stueckelberg field m_μ in the current formulation. Therefore, at four-derivative level one might expect new deformations to compensate the boost transformations [13].

A similar feature is expected to happen with the dilatation symmetry, since the missed Poisson equation is expected to have α' corrections. In this regard, we expect corrections to the dilation symmetry coming from field redefinitions.

C. General $f(\mathbf{R}, \mathbf{Q})$ nonrelativistic theories

While in this work we have shown that the nonmetricities (26)–(29) need to be fixed in a very particular way to provide the relativistic metric conditions (24)–(25), one can construct more general NR theories with arbitrary nonmetricities after taking the nonrelativistic limit. One interesting example is to demand the compatibility condition on the Newton-Cartan fields,

$$\nabla_\mu \tau_{\nu\rho} = 0, \quad (65)$$

$$\nabla_\mu \tau^{\nu\rho} = 0, \quad (66)$$

$$\nabla_\mu h_{\nu\rho} = 0, \quad (67)$$

$$\nabla_\mu h^{\nu\rho} = 0, \quad (68)$$

which automatically implies that both the relativistic Riemann and Ricci tensors are now finite,

$$\hat{R}^\rho{}_{\epsilon\mu\nu} = R^\rho{}_{\epsilon\mu\nu}, \quad \hat{R}_{\epsilon\nu} = R_{\epsilon\nu}, \quad (69)$$

and the more general (gravitational) Lagrangian in this setup is given by

$$L_{NR}|_{Q=0} = a_1 R_{\mu\nu} h^{\mu\nu} + a_2 R_{\mu\nu} \tau^{\mu\nu}, \quad (70)$$

with a_1 and a_2 being two arbitrary coefficients. In this case, the $SO(8)$ and $SO(1,1)$ symmetry from the vielbein formalism can be preserved, but the boost symmetry is broken. Further theories with these properties can be explored by constructing generic $L(\mathbf{R}, \mathbf{Q})$ Lagrangians in terms of τ and h , where a particular combination will be associated to the low-energy limit of the bosonic supergravity. For example, it is well known that the Lagrangian

$$L(\hat{g}, Q) = \sqrt{-g} \left(c_1 Q_\lambda Q^\lambda + c_2 Q_\lambda P^\lambda + c_3 P_\lambda P^\lambda + c_4 Q^\mu{}_{\nu\lambda} Q^\nu{}_{\mu}{}^{\lambda} + c_5 Q^\mu{}_{\nu\lambda} Q^\lambda{}_{\mu}{}^{\nu} \right), \quad (71)$$

with $Q^\mu = Q^\mu{}_{\rho\sigma} \hat{g}^{\rho\sigma}$ and $P_\nu = Q^\mu{}_{\nu\mu}$, recovers the dynamics of General Relativity (GR) when $c_1 = -\frac{1}{4}$, $c_2 = \frac{1}{2}$, $c_3 = 0$, $c_4 = \frac{1}{4}$, and $c_5 = -\frac{1}{2}$, while more arbitrary combinations allow new diffeomorphism-invariant Lagrangians. A similar setup is now possible to explore in the Newton-Cartan geometry.

VI. DISCUSSION

The present construction provides a geometrical formulation of the nonrelativistic limit of bosonic supergravity in terms of a Newton-Cartan geometry with nonmetricities. Within this framework, an appropriate choice of connection allows a metric formulation, mimicking the construction of the relativistic cases. As a result, the full two-derivative bosonic supergravity in its NR limit can be written in a covariant form with respect to infinitesimal diffeomorphisms, thereby complementing other approaches based on subgroups of the Lorentz symmetry. Moreover, we find compatibility of the approaches at the level of the action, while the study of the equations of motion deserves further study to understand if it is a necessary new constraint, equivalent to the torsion constraint [13].

A key advantage of the construction is that it allows for a straightforward decomposition of bosonic higher-derivative contributions, such as α' corrections. These corrections are naturally formulated in the metric formalism, where the Lorentz symmetry is manifest and no vielbein (or spin connection) is required. From a practical perspective, this provides a systematic method for analyzing higher-derivative terms by expressing relativistic curvature invariants directly in terms of nonrelativistic geometric data. In particular, the use of fixed nonmetricities enables a transparent identification of divergent and finite contributions, a task that is typically challenging.

Although we do not yet have full control over the divergences that appear at the four-derivative (61), the formalism developed in this work already proves useful for isolating and organizing several finite contributions to the four-derivative bosonic Lagrangian in the NR limit. In particular, we have explicitly demonstrated how to compute the divergent and finite contributions arising from the

Metsaev-Tseytlin Lagrangian. We expect that the covariant rewriting presented here will provide a valuable starting point for future investigations aimed at resolving these divergences in the bosonic α' corrections, as well as for extending the analysis to heterotic supergravity and more general nonrelativistic gravity models.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the author.

APPENDIX: FULL FINITE METSAEV AND TSEYTLIN LAGRANGIAN H-DEPENDENT TERMS

The terms only containing $\hat{H}$ contributions can also be easily written in covariant form. For example, the $\hat{H}^4$ term contributes with the following finite terms:

$$\begin{aligned}
& 4\hat{H}_{\mu\nu\rho}^{(2)} h^{\mu\sigma} h^{\nu\gamma} h^{\rho\epsilon} h^{\delta\lambda} h^{\alpha\beta} h^{\chi\psi} h_{\sigma\gamma\epsilon} h_{\delta\alpha\chi} h_{\lambda\beta\psi} + 6\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\mu\sigma} h^{\nu\gamma} h^{\rho\epsilon} h^{\delta\lambda} h^{\alpha\beta} h^{\chi\psi} h_{\delta\alpha\chi} h_{\lambda\beta\psi} \\
& + 6\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\delta\lambda} h^{\mu\sigma} h^{\nu\gamma} h^{\rho\epsilon} h^{\alpha\beta} h^{\chi\psi} h_{\delta\alpha\chi} h_{\lambda\beta\psi} + 24\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\mu\delta} h^{\nu\lambda} h^{\rho\alpha} h^{\sigma\beta} h^{\gamma\chi} h^{\epsilon\psi} h_{\delta\lambda\alpha} h_{\beta\chi\psi} \\
& + 12\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\sigma} \tau^{\nu\gamma} h^{\rho\epsilon} h^{\delta\beta} h^{\lambda\chi} h^{\alpha\psi} h_{\beta\chi\psi} + 36\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\sigma} \tau^{\delta\beta} h^{\nu\gamma} h^{\rho\epsilon} h^{\lambda\chi} h^{\alpha\psi} h_{\beta\chi\psi} \\
& + 12\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\beta} \tau^{\nu\chi} h^{\rho\psi} h^{\sigma\delta} h^{\gamma\lambda} h^{\epsilon\alpha} h_{\beta\chi\psi} + 2\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \hat{H}_{\beta\chi\psi}^{(2)} \tau^{\mu\sigma} \tau^{\nu\gamma} \tau^{\rho\epsilon} h^{\delta\beta} h^{\lambda\chi} h^{\alpha\psi} \\
& + 18\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \hat{H}_{\beta\chi\psi}^{(2)} \tau^{\mu\sigma} \tau^{\nu\gamma} \tau^{\delta\beta} h^{\rho\epsilon} h^{\lambda\chi} h^{\alpha\psi}, \tag{A1}
\end{aligned}$$

while $H_{\mu\nu}^2 H^{2\mu\nu}$ contributes with

$$\begin{aligned}
& 4\hat{H}_{\mu\nu\rho}^{(2)} h^{\mu\sigma} h^{\nu\gamma} h^{\rho\epsilon} h^{\delta\lambda} h^{\alpha\beta} h^{\chi\psi} h_{\sigma\gamma\delta} h_{\epsilon\alpha\chi} h_{\lambda\beta\psi} + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\mu\sigma} h^{\nu\gamma} h^{\rho\delta} h^{\epsilon\lambda} h^{\alpha\beta} h^{\chi\psi} h_{\delta\alpha\chi} h_{\lambda\beta\psi} \\
& + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\mu\delta} h^{\nu\sigma} h^{\rho\gamma} h^{\epsilon\lambda} h^{\alpha\beta} h^{\chi\psi} h_{\delta\alpha\chi} h_{\lambda\beta\psi} + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\delta\lambda} h^{\mu\sigma} h^{\nu\gamma} h^{\rho\alpha} h^{\epsilon\beta} h^{\chi\psi} h_{\delta\alpha\chi} h_{\lambda\beta\psi} \\
& - 8\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\mu\delta} h^{\nu\sigma} h^{\rho\lambda} h^{\gamma\alpha} h^{\epsilon\beta} h^{\chi\psi} h_{\delta\lambda\chi} h_{\alpha\beta\psi} + 2\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\mu\sigma} h^{\nu\delta} h^{\rho\lambda} h^{\gamma\alpha} h^{\epsilon\beta} h^{\chi\psi} h_{\delta\lambda\chi} h_{\alpha\beta\psi} \\
& + 2\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\delta\lambda} h^{\mu\sigma} h^{\nu\alpha} h^{\rho\beta} h^{\gamma\chi} h^{\epsilon\psi} h_{\delta\alpha\beta} h_{\lambda\chi\psi} + 8\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\mu\delta} h^{\nu\lambda} h^{\rho\alpha} h^{\sigma\beta} h^{\gamma\chi} h^{\epsilon\psi} h_{\delta\lambda\beta} h_{\alpha\chi\psi} \\
& + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \tau^{\mu\delta} h^{\nu\lambda} h^{\rho\alpha} h^{\sigma\beta} h^{\gamma\chi} h^{\epsilon\psi} h_{\delta\beta\chi} h_{\lambda\alpha\psi} + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\sigma} \tau^{\nu\gamma} h^{\rho\delta} h^{\epsilon\beta} h^{\lambda\chi} h^{\alpha\psi} h_{\beta\chi\psi} \\
& - 8\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\sigma} \tau^{\nu\delta} h^{\rho\gamma} h^{\epsilon\beta} h^{\lambda\chi} h^{\alpha\psi} h_{\beta\chi\psi} + 16\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\sigma} \tau^{\delta\beta} h^{\nu\gamma} h^{\rho\lambda} h^{\epsilon\chi} h^{\alpha\psi} h_{\beta\chi\psi} \\
& + 8\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\sigma} \tau^{\nu\beta} h^{\rho\gamma} h^{\epsilon\delta} h^{\lambda\chi} h^{\alpha\psi} h_{\beta\chi\psi} + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\beta} \tau^{\nu\chi} h^{\rho\sigma} h^{\gamma\delta} h^{\epsilon\lambda} h^{\alpha\psi} h_{\beta\chi\psi} \\
& - 8\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\sigma} \tau^{\nu\beta} h^{\rho\gamma} h^{\epsilon\delta} h^{\lambda\chi} h^{\alpha\psi} h_{\beta\chi\psi} + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\sigma} \tau^{\delta\beta} h^{\nu\lambda} h^{\rho\alpha} h^{\gamma\chi} h^{\epsilon\psi} h_{\beta\chi\psi} \\
& - 8\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \tau^{\mu\beta} \tau^{\sigma\chi} h^{\nu\delta} h^{\rho\lambda} h^{\gamma\alpha} h^{\epsilon\psi} h_{\beta\chi\psi} + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \hat{H}_{\beta\chi\psi}^{(2)} \tau^{\mu\sigma} \tau^{\nu\gamma} \tau^{\rho\delta} h^{\epsilon\beta} h^{\lambda\chi} h^{\alpha\psi} \\
& + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \hat{H}_{\beta\chi\psi}^{(2)} \tau^{\mu\sigma} \tau^{\nu\gamma} \tau^{\delta\beta} h^{\rho\lambda} h^{\epsilon\chi} h^{\alpha\psi} + 8\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \hat{H}_{\beta\chi\psi}^{(2)} \tau^{\mu\sigma} \tau^{\nu\delta} \tau^{\gamma\beta} h^{\rho\lambda} h^{\epsilon\chi} h^{\alpha\psi} \\
& + 4\hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{H}_{\delta\lambda\alpha}^{(2)} \hat{H}_{\beta\chi\psi}^{(2)} \tau^{\mu\sigma} \tau^{\nu\delta} \tau^{\gamma\beta} h^{\rho\epsilon} h^{\lambda\chi} h^{\alpha\psi}. \tag{A2}
\end{aligned}$$

For $\hat{H}^{\mu\nu\rho} \hat{H}_{\mu\sigma\lambda} \hat{R}_{\nu\rho}^{\sigma\lambda}$ the finite contributions are given by

$$\begin{aligned}
& \hat{R}^{(2)\mu}_{\nu\rho\sigma} h^{\nu\gamma} h^{\rho\epsilon} h^{\sigma\delta} h^{\lambda\alpha} h_{\mu\gamma\lambda} h_{\epsilon\delta\alpha} - \hat{R}^{(4)\mu}_{\nu\rho\sigma} \tau^{\nu\gamma} h^{\nu\delta} h^{\rho\lambda} h^{\sigma\alpha} h_{\mu\gamma\delta} h_{\epsilon\lambda\alpha} + 2\hat{R}^{(4)\mu}_{\nu\rho\sigma} \tau^{\rho\gamma} h^{\nu\epsilon} h^{\sigma\delta} h^{\lambda\alpha} h_{\mu\epsilon\lambda} h_{\gamma\delta\alpha} \\
& + \hat{R}^{(4)\mu}_{\nu\rho\sigma} \tau^{\nu\gamma} h^{\rho\epsilon} h^{\sigma\delta} h^{\lambda\alpha} h_{\mu\gamma\lambda} h_{\epsilon\delta\alpha} - \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(0)\sigma}_{\gamma\epsilon\delta} h^{\mu\epsilon} h^{\nu\delta} h^{\rho\lambda} h^{\gamma\alpha} h_{\sigma\lambda\alpha} - \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(2)\sigma}_{\gamma\epsilon\delta} \tau^{\mu\lambda} h^{\nu\epsilon} h^{\rho\delta} h^{\gamma\alpha} h_{\sigma\lambda\alpha} \\
& - 2\hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(2)\sigma}_{\gamma\epsilon\delta} \tau^{\mu\epsilon} h^{\nu\delta} h^{\rho\lambda} h^{\gamma\alpha} h_{\sigma\lambda\alpha} + \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(2)\sigma}_{\gamma\epsilon\delta} \tau^{\gamma\lambda} h^{\mu\epsilon} h^{\nu\delta} h^{\rho\alpha} h_{\sigma\lambda\alpha} + 2\hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(4)\sigma}_{\gamma\epsilon\delta} \tau^{\mu\epsilon} \tau^{\nu\lambda} h^{\rho\delta} h^{\gamma\alpha} h_{\sigma\lambda\alpha} \\
& - \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(4)\sigma}_{\gamma\epsilon\delta} \tau^{\mu\epsilon} \tau^{\nu\delta} h^{\rho\lambda} h^{\gamma\alpha} h_{\sigma\lambda\alpha} - \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(4)\sigma}_{\gamma\epsilon\delta} \tau^{\mu\lambda} \tau^{\gamma\alpha} h^{\nu\epsilon} h^{\rho\delta} h_{\sigma\lambda\alpha} + 2\hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(4)\sigma}_{\gamma\epsilon\delta} \tau^{\mu\epsilon} \tau^{\gamma\lambda} h^{\nu\delta} h^{\rho\alpha} h_{\sigma\lambda\alpha} \\
& + \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(0)\mu}_{\sigma\gamma\epsilon} h^{\nu\sigma} h^{\rho\delta} h^{\gamma\lambda} h^{\epsilon\alpha} h_{\delta\lambda\alpha} - \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(2)\mu}_{\sigma\gamma\epsilon} \tau^{\nu\delta} h^{\rho\sigma} h^{\gamma\lambda} h^{\epsilon\alpha} h_{\delta\lambda\alpha} - 2\hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(2)\mu}_{\sigma\gamma\epsilon} \tau^{\gamma\delta} h^{\nu\sigma} h^{\rho\lambda} h^{\epsilon\alpha} h_{\delta\lambda\alpha}
\end{aligned}$$

$$\begin{aligned}
& + \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(2)\mu}_{\sigma\gamma\epsilon} \tau^{\nu\sigma} h^{\rho\delta} h^{\gamma\lambda} h^{\epsilon\alpha} h_{\delta\lambda\alpha} - 2 \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(4)\mu}_{\sigma\gamma\epsilon} \tau^{\nu\delta} \tau^{\gamma\lambda} h^{\rho\sigma} h^{\epsilon\alpha} h_{\delta\lambda\alpha} + \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(4)\mu}_{\sigma\gamma\epsilon} \tau^{\gamma\delta} \tau^{\epsilon\lambda} h^{\nu\sigma} h^{\rho\alpha} h_{\delta\lambda\alpha} \\
& + \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(4)\mu}_{\sigma\gamma\epsilon} \tau^{\nu\sigma} \tau^{\rho\delta} h^{\gamma\lambda} h^{\epsilon\alpha} h_{\delta\lambda\alpha} - 2 \hat{H}_{\mu\nu\rho}^{(2)} \hat{R}^{(4)\mu}_{\sigma\gamma\epsilon} \tau^{\nu\sigma} \tau^{\gamma\delta} h^{\rho\lambda} h^{\epsilon\alpha} h_{\delta\lambda\alpha} - \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(-2)\mu}_{\delta\lambda\alpha} h^{\nu\sigma} h^{\rho\delta} h^{\gamma\lambda} h^{\epsilon\alpha} \\
& - \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(0)\mu}_{\delta\lambda\alpha} \tau^{\nu\sigma} h^{\rho\delta} h^{\gamma\lambda} h^{\epsilon\alpha} + 2 \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(0)\mu}_{\delta\lambda\alpha} \tau^{\sigma\lambda} h^{\nu\gamma} h^{\rho\delta} h^{\epsilon\alpha} + \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(0)\mu}_{\delta\lambda\alpha} \tau^{\nu\delta} h^{\rho\sigma} h^{\gamma\lambda} h^{\epsilon\alpha} \\
& - 2 \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(2)\mu}_{\delta\lambda\alpha} \tau^{\nu\sigma} \tau^{\gamma\lambda} h^{\rho\delta} h^{\epsilon\alpha} - \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(2)\mu}_{\delta\lambda\alpha} \tau^{\sigma\lambda} \tau^{\gamma\alpha} h^{\nu\epsilon} h^{\rho\delta} - \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(2)\mu}_{\delta\lambda\alpha} \tau^{\nu\sigma} \tau^{\rho\delta} h^{\gamma\lambda} h^{\epsilon\alpha} \\
& - 2 \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(2)\mu}_{\delta\lambda\alpha} \tau^{\nu\delta} \tau^{\sigma\lambda} h^{\rho\gamma} h^{\epsilon\alpha} - \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(4)\mu}_{\delta\lambda\alpha} \tau^{\nu\sigma} \tau^{\gamma\lambda} \tau^{\epsilon\alpha} h^{\rho\delta} - 2 \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(4)\mu}_{\delta\lambda\alpha} \tau^{\nu\sigma} \tau^{\rho\delta} \tau^{\gamma\lambda} h^{\epsilon\alpha} \\
& + \hat{H}_{\mu\nu\rho}^{(2)} \hat{H}_{\sigma\gamma\epsilon}^{(2)} \hat{R}^{(4)\mu}_{\delta\lambda\alpha} \tau^{\nu\delta} \tau^{\sigma\lambda} \tau^{\gamma\alpha} h^{\rho\epsilon}.
\end{aligned} \tag{A3}$$

At present, we cannot control all the divergences of the full Metsaev-Tseytlin Lagrangian [terms given in (61)] without imposing further constraints [such as (62)]. For that reason we are not giving an exhaustive analysis of the four-derivative contributions of this Lagrangian in its NR limit. However, the covariant framework developed in this work could be very useful for attacking this open problem and part of the finite action is given by the previous contributions.

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