

Summing large Pomeron loops in the saturation region: Nucleus-nucleus collision

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In this paper we found the nucleus-nucleus scattering amplitude at high energies by summing large Pomeron loops. It turns out that the energy dependence of this amplitude is the same for dipole-dipole scattering. This happens since the evolution of one dipole cascade contributes to this scattering.

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I. INTRODUCTION

In this paper we continue [1,2] to develop our approach for summing large BFKL Pomeron loops [3].¹ Our goal is to sum these loops for nucleus-nucleus scattering at high energies. This process gives a clear example of the dense-dense parton system scattering for which we do not have a nonlinear evolution equation. Actually, the only nonlinear evolution that we know is able to describe the evolution of dilute-dense parton system (dipole-nucleus scattering) [4–22]. This nonlinear BK [8,9]² equation stems from summing the “fan” Pomeron diagrams in the BFKL Pomeron calculus [4,11]. In our previous paper [2] we demonstrated that large Pomeron loops changed drastically the scattering dipole-nucleus amplitude making it asymptotically at ultrahigh energy the same as

dipole-dipole scattering, which is an example of dilute-dilute parton system scattering.

In Fig. 1 the first Pomeron diagrams are shown and one can note that they have a much more complicated structure than the “fan” diagrams. It is well known that in spite of intensive work [4–58], the problem of summation of the Pomeron loops has not been solved. In our recent papers [1,2] we have suggested the way to sum the large Pomeron loops for dipole-dipole and dipole-nucleus scattering deeply in the saturation region. Our summation was based on the t -channel unitarity, which has been rewritten in the convenient form for the dipole approach to color glass condensate in Refs. [7,22,23,26,30,44,57,59,60] (see Fig. 2). The analytic expression takes the form for the imaginary part of the scattering amplitude (N) [22,30,57],

$$N(Y, r, R; \mathbf{b}) = \sum_{n=1}^{\infty} (-1)^{n+1} n! \int \prod_i^n d^2 r_i d^2 r'_i d^2 b'_i \int d^2 \delta b_i \gamma^{BA}(r_1, r'_1, \mathbf{b}_i - \mathbf{b}'_i \equiv \delta \mathbf{b}_i) \rho_n^P(Y - Y_0, \{\mathbf{r}_i, \mathbf{b}_i\}) \rho_n^T(Y_0, \{\mathbf{r}'_i, \mathbf{b}'_i\}). \quad (1)$$

γ^{BA} is the scattering amplitude of two dipoles in the Born approximation of perturbative QCD. The parton densities $\rho_i^P(Y; \{\mathbf{r}_i, \mathbf{b}_i\})$ for projectile and $\rho_i^T(Y; \{\mathbf{r}_i, \mathbf{b}_i\})$ for target have been introduced in Ref. [22] as follows:

$$\rho_n(Y - Y_0; r_1, b_1, \dots, r_n, b_n) = \frac{1}{n!} \prod_{i=1}^n \frac{\delta}{\delta u_i} Z(Y - Y_0; [u])|_{u=1} \quad (2)$$

where the generating functional Z is

$$Z(Y, \mathbf{r}, \mathbf{b}; [u_i]) = \sum_{n=1}^{\infty} \int P_n(Y, \mathbf{r}, \mathbf{b}; \{\mathbf{r}_i, \mathbf{b}_i\}) \prod_{i=1}^n u(\mathbf{r}_i, \mathbf{b}_i) d^2 r_i d^2 b_i \quad (3)$$

where $u(\mathbf{r}_i, \mathbf{b}_i) \equiv u_i$ is an arbitrary function and P_n is the probability to have n dipoles with the given kinematics. The initial and boundary conditions for the BFKL cascade stem from one dipole have the following form for the functional Z :

$$Z(Y = 0, \mathbf{r}, \mathbf{b}; [u_i]) = u(\mathbf{r}, \mathbf{b}); \quad Z(Y, r, [u_i = 1]) = 1; \quad (4a)$$

$$\begin{aligned} \rho_1(Y = 0; r, b, r_1, b_1) &= \delta^{(2)}(\mathbf{r} - \mathbf{r}_1) \delta^{(2)}(\mathbf{b} - \mathbf{b}_1); \\ \rho_n(Y = 0; \mathbf{r}, \mathbf{b}; [r_i, b_i]) &= 0 \quad \text{at } n \geq 2. \end{aligned} \quad (4b)$$

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¹BFKL stands for Balitsky, Fadin, Kuraev, and Lipatov.

²BK stands for Balitsky and Kovchegov.

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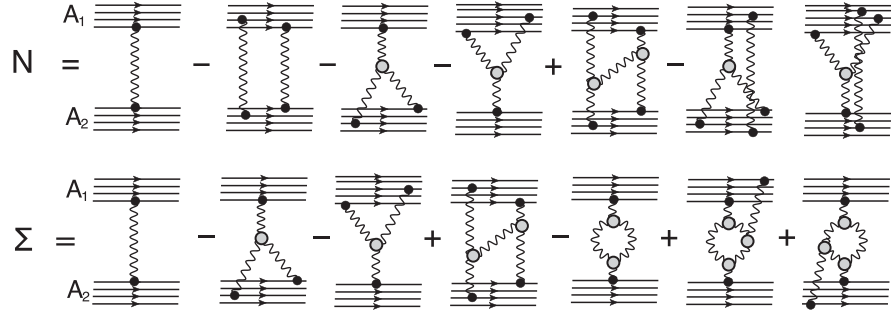


FIG. 1. First BFKL Pomeron diagrams for nucleus-nucleus scattering. The imaginary part of the scattering amplitude (N) is equal to $N = 1 - \exp(-\Sigma)$. Σ is the set of the two nuclei irreducible Pomeron diagrams. Wavy lines denote the BFKL Pomerons. The black circles stand for Pomeron-dipole interaction vertex. The gray circles denote the triple Pomeron vertices.

The parton densities for fast dipole and fast nucleus have been found in Ref. [1,2]. For the completeness of presentation we will briefly discuss them in the next section. However we wish to stress here that they satisfy both the evolution equations and the recurrence relations derived in QCD (see Refs. [22,28,57]) and the analytic solution to the nonlinear BK equation of Ref. [27]. Based on parton densities, found in Sec. II, we show in Sec. III that the scattering amplitude for nucleus-nucleus collisions is determined by the dipole (nucleon)-dipole(nucleon) interaction at ultrahigh energies. In the conclusions we summarize our results.

II. DIPOLE DENSITIES $\rho_n(r, b, \{r_i, b_i\})$

A. The BFKL Pomeron in the saturation region

The Green's function for the BFKL Pomeron exchange satisfies the linear equation [3], which has the form,

$$\frac{\partial G_{IP}(Y, r; b)}{\partial Y} = \bar{\alpha}_S \int \frac{d^2 r'}{2\pi} \frac{r^2}{r'^2 (r - r')^2} \left\{ G_{IP}\left(Y, r'; b - \frac{1}{2}(r - r')\right) + G_{IP}\left(Y, r - r'; b - \frac{1}{2}r'\right) - G_{IP}(Y, r; b) \right\}. \quad (5)$$

The solution to this equation is the sum over the eigenfunction $\phi_\gamma(r, r_1, b)$ with the eigenvalues $\bar{\alpha}_S \chi(\gamma)$,

$$G_P(Y, r, r_1; b) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} e^{\bar{\alpha}_S \chi(\gamma) Y} \phi_\gamma(r, r_1, b) \phi_{in}(r_1). \quad (6)$$

The eigenfunction $\phi_\gamma(r, r_1, b)$ (the scattering amplitude of two dipoles with sizes r and r_1) is equal to [29]

$$\phi_\gamma(r, r_1, b) = \left(\frac{r^2 r_1^2}{(b + \frac{1}{2}(r - r_1))^2 (b - \frac{1}{2}(r - r_1))^2} \right)^\gamma = e^{\gamma \xi_{r, r_1}} \quad \text{with} \quad 0 < \text{Re} \gamma < 1. \quad (7)$$

$\phi_{in}(r_1)$ can be found from the initial conditions: at $Y = 0$ G_P has to describe the scattering amplitude in the Born approximation of perturbative QCD (the exchange of two

gluons between dipoles with sizes r and r_1 , see insertion in Fig. 2). The eigenvalue $\omega(\bar{\alpha}_S, \gamma) = \bar{\alpha}_S \chi(\gamma)$ is given by

$$\omega(\bar{\alpha}_S, \gamma) = \bar{\alpha}_S \chi(\gamma) = \bar{\alpha}_S (2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)). \quad (8)$$

$\psi(z)$ is the Euler ψ function [see Ref. [61] formula (8.36)].

At high energies (in the saturation region) the exchange of the BFKL Pomeron shows two features. The new scale, saturation momentum, appears in the solution to the BK equation [4], which has the following Y dependence [4,39,40]:

$$Q_s^2(Y, b) = Q_s^2(Y = 0, b) e^{\bar{\alpha}_S \kappa Y - \frac{3}{\gamma_{cr}} \ln Y} \quad (9)$$

where $Y = 0$ is the initial value of rapidity and κ and γ_{cr} are determined by the following equations:

$$\kappa \equiv \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} = -\frac{d\chi(\gamma_{cr})}{d\gamma_{cr}}. \quad (10)$$

In the saturation region the exchange of the BFKL Pomeron shows the geometric scaling behavior [62], being a function of one argument,

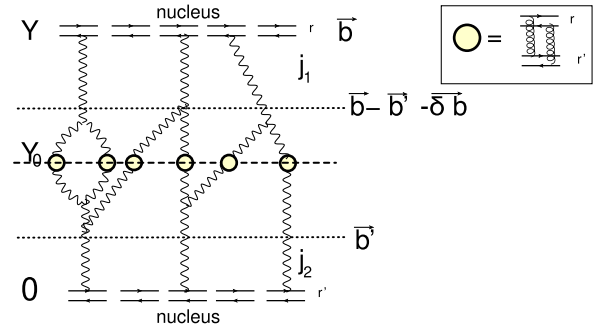


FIG. 2. Summing large Pomeron loops for nucleus-nucleus scattering: t -channel unitarity for Σ . The wavy lines denote the BFKL Pomeron exchanges. The circles denote the amplitude γ in the Born approximation of perturbative QCD. The nuclei are viewed as the bags of dipoles with the size r and r' .

$$z = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} (Y - Y_0) + \xi_{r,r_1} \quad (11)$$

where ξ_{r,r_1} is given by Eq. (7).

It turns out that the Green's function for the exchange of one BFKL Pomeron for scattering of dipoles with sizes r and r_1 can be written in the following way [4,39]:

$$G_P(z) = N_0 e^{\bar{\gamma} z} \quad (12)$$

where $\bar{\gamma} = 1 - \gamma_{cr}$, N_0 is a constant, z in Eq. (12) is defined in Eq. (11).

The Pomeron Green's function satisfies the t -channel unitarity [3] analytically continued to the s channel and can be rewritten as the integration over two Reggeized gluons at fixed momentum Q_T that carries each Pomeron [4,7],

$$G_P(Y, Q_T, r, R) = \int \frac{d^2 k_T}{(2\pi)^2} G_P(Y - y', Q_T, r, k_T) G_P(y', Q_T, R, k_T) \quad (13)$$

where

$$G_{IP}(Y - y', Q_T, r, r') = r'^2 \int \frac{d^2 k_T}{(2\pi)^2} e^{ik_T \cdot r'} G_{IP}(Y - y', Q_T, r, k_T). \quad (14)$$

Equation (13) can be rewritten through $G_P(Y - y', r, r', Q_T)$ in the form

$$G_P(Y, r, R; \mathbf{b}) = \int d^2 r_i d^2 b_i d^2 r'_i d^2 b'_i \rho_1(Y - Y_0; r, b - b_i, r_i) \gamma^{BA}(r_i, r'_i, \delta b) \rho_1(Y_0; r', b'_i, r'_i) \quad (15a)$$

$$= \frac{1}{4\pi^2} \int \frac{d^2 r_i}{r_i^4} d^2 b_i G_P(Y - Y_0, r, r_i; \mathbf{b} - \mathbf{b}_i) G_P(Y_0, r_i; \mathbf{b}_i) \quad (15b)$$

where $\gamma^{BA}(r_i, r'_i, \delta b)$ is the scattering amplitude of two dipoles in the Born approximation of perturbative QCD. The first equation is derived in Ref. [7] (see also Refs. [26,44]) while the second is in Ref. [63].

B. $\rho_n^d(r, b, \{r_i, b_i\})$ for dipole

In Ref. [1] we found the parton (dipole) densities for the projectile (fast dipole with size r). They are equal to

$$\rho_n^d(Y - Y_0; r, b, \{r_i, b_i\}) = C \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\omega}{2\pi i} e^{\frac{\bar{\gamma}_\kappa \omega^2}{2}} \frac{1}{n!} \frac{\Gamma(\omega + n)}{\Gamma(\omega)} \prod_{i=1}^n \frac{G_P(z_i)}{N_0} \quad (16)$$

where

$$z_i = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} (Y - Y_0) + \xi_{r,r_i} \quad (17)$$

with ξ_{r,r_i} from Eq. (7) in which r_1 is replaced by r_i .

It has been shown that Eq. (16) leads to the solution of the BK nonlinear equation deep in the saturation region [27]. This solution takes the form,

$$N^{\text{DIS}}(z = r^2 Q_s^2(Y, b)) = 1 - C(z) \exp\left(-\frac{z^2}{2\kappa}\right) \quad (18)$$

where $z = \ln(r^2 Q_s^2(Y))$. One can see that this solution shows the geometric scaling behavior [62] being a function of one variable. Function $C(z)$ is a smooth function that can be considered as a constant in our approach.

Actually Eq. (16) stems from our main idea that this solution to the BK equation has to sum the “fan” Pomeron diagrams in the BFKL Pomeron calculus. In Ref. [1] we present Eq. (18) as a sum of many Pomerons exchanges and in doing so we find the parton densities ρ_n .

C. $\rho_n^A(r, b, \{r_i, b_i\})$ for nucleus

We consider a nucleus as a bag of dipoles with size r that do not interact with each other at $Y_0 = 0$. It means that [see Eq. (4b)]

$$\rho_n^A(Y_0=0; r', b', \{r'_i, b'_i\}) = \frac{S_A^n(b')}{n!} \prod_{i=1}^n \delta^{(2)}(r' - r_i) \delta^{(2)}(b' - b_i). \quad (19)$$

Factor $S_A^n(b')$ describes the probability to find n nucleons (dipoles) in a nucleus at the impact parameter b' .

Looking in Fig. 2 one can see that parton cascades with rapidities less than Y_0 include only annihilation of two dipoles to one ($2IP \rightarrow IP$ vertices). Hence each nucleon of the target at $Y_0 = 0$ creates the dipole cascade that produces k_i dipoles. The dipole densities for a nucleus can be written as

$$\begin{aligned} \rho_n^A(Y_0; r, b, \{r_i, b_i\}) &= \sum_{j=1}^n \underbrace{\frac{1}{j!} S_A^j(b)}_{\text{probability to find } j\text{-nucleons}} \underbrace{\sum_{k_1=1, \dots, k_j=1}^{k_1=n-j+1, \dots, k_j=n-j+1} \delta_{\sum_{i=1}^j k_i=n}}_{\text{summing all } k_i} \\ &\rho_{k_1}^d(Y_0; r', b', r_1, \dots, r_{k_1}) \times \dots \times \underbrace{\rho_{k_i}^d(Y_0; r', b', r_{k_{i-1}+1}, \dots, r_{k_i})}_{\text{dipole density produced by } i\text{-th nucleon}} \times \dots \times \rho_{k_j}^d(Y_0; r', b', r_{k_{j-1}+1}, \dots, r_{k_j}). \end{aligned} \quad (20)$$

This formula describes the simple fact that each i -th nucleon produces k_i dipoles and $\sum_{i=1}^j k_i = n$, where n is the total number of dipoles at $Y = Y_0$. Each k_i runs from $k_i = 1$ to $k_i = n - j + 1$ with the additional restriction that $\sum_{i=1}^j k_i = n$ as it is written in the sum of the second factor of Eq. (20).

Equation (19) can be easily derived from the McLerran-Venugopalan formula for the scattering amplitude [13] in which

$$\gamma(r_i, r'; b) = S_A(b) \int d\delta b \gamma^{BA}(r_i, r'; \delta b_i) \quad (21)$$

where $S(b)$ is the nucleus profile function, which is equal to

$$S_A(b) = \int_{-\infty}^{\infty} dz \rho_A(\sqrt{b^2 + z^2}) \quad (22)$$

with ρ_A the density of nucleons in the nucleus A.

In Eq. (20) it is included that for interaction with nuclei all impact parameters are large (all $b' \rightarrow b$) for large nuclei (see Fig. 2). Summing in Eq. (20) over k_i is restricted by $\sum_{i=1}^j k_i = n$. Using Eq. (16) we can obtain that

$$\rho_n^A(Y_0; r, b, \{r_i, b_i\}) = \prod_{i=1}^n \frac{G_P(z_i)}{N_0} \sum_{j=1}^n \frac{1}{j!} S_A^j(b) \sum_{k_1=1, \dots, k_j=1}^{k_1=n-j+1, \dots, k_j=n-j+1} \delta_{\sum_{i=1}^j k_i=n} \prod_{i=1}^j C \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_i}{2\pi i} e^{\frac{\gamma}{2}\omega_i^2} \frac{1}{k_i!} \frac{\Gamma(\omega_i + k_i)}{\Gamma(\omega_i)}. \quad (23)$$

In this paper we use Eq. (23) for $\rho_n^T(r, b, \{r_i, b_i\})$ replacing $\Gamma(\omega_i + k_i)$ by the integrals over t_i , viz.,

$$\Gamma(\omega_i + k_i) = \int_0^{\infty} dt_i t_i^{\omega_i + k_i - 1} e^{-t_i}. \quad (24)$$

In doing so we obtain

$$\begin{aligned} &\rho_n^A(Y_0; r, b, \{r_i, b_i\}) \\ &= \prod_{i=1}^n \frac{G_P(z_i)}{N_0} \sum_{j=1}^n \frac{1}{j!} S_A^j(b) \sum_{k_1=1, \dots, k_j=1}^{k_1=n-j+1, \dots, k_j=n-j+1} \delta_{\sum_{i=1}^j k_i=n} S^j(b) \prod_{i=1}^j C \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_i}{2\pi i} e^{\frac{\gamma}{2}\omega_i^2} \frac{1}{k_i!} \frac{1}{\Gamma(\omega_i)} \int_0^{\infty} dt_i t_i^{\omega_i + k_i - 1} e^{-t_i}. \end{aligned} \quad (25)$$

In Eq. (25) we put all $b'_i = b$ neglecting the size of the dipole cascade in the impact parameter space. z_i is defined in Eq. (17) for rapidity Y_0 .

III. SCATTERING AMPLITUDE

A. Single BFKL Pomeron contribution

Let us start with the simplest interaction: the exchange of one BFKL Pomeron. The impact parameter picture of this interaction is shown in Fig. 3(a). In this figure $\mathbf{b}$ is the impact parameter between the centers of two nuclei. $\mathbf{b} - \mathbf{b}' - \delta\mathbf{b}$ is the position in the impact parameters of the dipole r from the nucleus A_1 , while $\mathbf{b}'$ is the position of the dipole r' . $\delta\mathbf{b}$ is the position of dipole r' with respect to dipole r in impact parameters. The typical value of $\delta\mathbf{b}$ is the size of the interacting dipoles, which is much smaller than the sizes of nuclei: R_{A_1} or R_{A_2} . Hence we can replace $\mathbf{b} - \mathbf{b}' - \delta\mathbf{b}$ by $\mathbf{b} - \mathbf{b}'$ neglecting $\delta\mathbf{b}$. In Eq. (1) we integrate over $\delta\mathbf{b}$. Therefore, we can write the contribution of the single Pomeron exchange as follows:

$$N(Y, r, r'; \mathbf{b}) = \int d^2 b' S_{A_1}(\mathbf{b} - \mathbf{b}') S_{A_2}(\mathbf{b}') \times \int d^2 \delta\mathbf{b} G_{\mathbf{P}}(Y, r, r'; \delta\mathbf{b}). \quad (26)$$

$\int d^2 \delta\mathbf{b} G_{\mathbf{P}}(Y, r, r'; \delta\mathbf{b})$ is the Green's function of the Pomeron at a fixed momentum $Q_T = 0$

$$G_{\mathbf{P}}(Y, Q_T = 0; r, r') = \int d^2 \delta\mathbf{b} G_{\mathbf{P}}(Y, r, r'; \delta\mathbf{b}). \quad (27)$$

Q_T is the momentum that carries each Pomeron. Extracting factor $2\pi r r'$ from this Green's function, $G_{\mathbf{P}}(Y, Q_T; r, r') = 2\pi r r' \tilde{G}(Y, Q_T; r, r')$ we can rewrite the t -channel unitarity of Eq. (15) in the following form for $\tilde{G}_{\mathbf{P}}$:

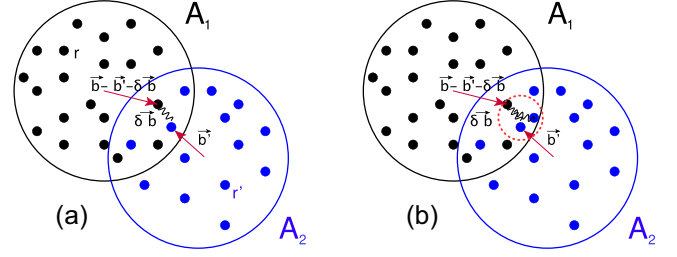


FIG. 3. Pomeron interaction: the impact parameter view. (a) The exchange of one BFKL Pomeron. (b) The Pomeron interaction of one nucleon from the nucleus A_1 with two nucleons in the nucleus A_2 . The wavy lines denote the BFKL Pomeron exchanges. The circles denote the dipoles with size r or r' . $\mathbf{b} - \mathbf{b}' - \delta\mathbf{b}(\mathbf{b}')$ is the position of the nucleon in the nucleus A_1 (A_2). $\delta\mathbf{b}$ is the impact parameter size of the BFKL Pomeron or several BFKL Pomerons that carry the interaction between nucleons.

$$\tilde{G}(Y, Q_T; r, r') = \frac{1}{2\pi} \int \frac{d^2 r_i}{r_i^2} \tilde{G}_{\mathbf{P}}(Y - Y_0, Q_T; r, r_i) \times \tilde{G}_{\mathbf{P}}(Y_0, Q_T; r_i, r'). \quad (28)$$

In particular, for $Q_T = 0$ we have

$$\tilde{G}(Y, 0; r, r') = \frac{1}{2\pi} \int \frac{d^2 r_i}{r_i^2} \tilde{G}_{\mathbf{P}}(Y - Y_0, 0; r, r_i) \times \tilde{G}_{\mathbf{P}}(Y_0, 0; r_i, r'). \quad (29)$$

It is worthwhile mentioning that the Green's function $\tilde{G}(Y, Q_T = 0; r, r')$ has the form of Eq. (12) in the saturation region where N_0 is the dimensionless constant and z is equal to

$$z = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} Y + \xi_{r, r'}. \quad (30)$$

$\xi_{r, r'}$ is equal to $\ln(\frac{r^2}{r'^2})$.

Plugging Eq. (29) into Eq. (26) we obtain,

$$\begin{aligned} N(Y, r, r'; \mathbf{b}) &= \int d^2 b' S_{A_1}(\mathbf{b} - \mathbf{b}') S_{A_2}(\mathbf{b}') \int d^2 \delta\mathbf{b} G_{\mathbf{P}}(Y, r, r'; \delta\mathbf{b}) \\ &= \int d^2 b' S_{A_1}(\mathbf{b} - \mathbf{b}') S_{A_2}(\mathbf{b}') \frac{1}{2\pi} \int \frac{d^2 r_i}{r_i^2} \tilde{G}_{\mathbf{P}}(Y - Y_0, 0; r, r_i) \tilde{G}_{\mathbf{P}}(Y_0, 0; r_i, r') \\ &= \int d^2 b' \frac{1}{2\pi} \int \frac{d^2 r_i}{r_i^2} \rho^{A_1}(r, \mathbf{b} - \mathbf{b}', r_i, \mathbf{b}_i = \mathbf{b} - \mathbf{b}') \rho^{A_2}(r', \mathbf{b}', r_i, \mathbf{b}'_i = \mathbf{b}'). \end{aligned} \quad (31)$$

One can see that for large nuclei we simplify the general formulas of Eq. (1) integrating over the impact parameters of the dipoles that originate from the cascade of one nucleon (dipole) in a nucleus.

B. Impact parameter dependence

In this section we are going to integrate over $\delta\mathbf{b}_i(\delta\mathbf{b}'_i)$. Our approach is based on the results, which have been discussed in the previous section, and we are going to use two features of the impact parameter dependence for

nucleus-nucleus collisions. First is the general property of Eq. (1) shown in Fig. 2 that for $Y > Y_0$ all Pomerons interact with vertex $IP \rightarrow 2IP$ while for $Y < Y_0$ only vertex $2IP \rightarrow IP$ contribute to their interactions. Therefore, each nucleon (dipole) produces its own dipole cascade and these dipole cascades interact at $Y = Y_0$. Bearing this in mind we can rewrite $\rho_n^A(r, b, \{r_i, b_i\})$ through the dipole cascades $\rho^d(Y, r, \{r_i, \delta b_i\})$, which are generated by j nucleons,

$$\rho_n^A(Y; r, b, \{r_i, b_i\}) = \sum_{j=1}^A \frac{1}{j!} \prod_{j'=1}^j S_A(b_{j'}) \rho_{n_{j'}}^d(Y, r, \{r_i, \delta b_i\}) \delta\left(\sum_{j'=1}^j n_{j'} = n\right) \quad (32)$$

where δb_i is the values of the impact parameters of dipoles with respect to the nucleons. The second feature is that the dipole develops a dipole cascade, which has $\delta b \ll b_j$. Indeed, each nucleon (dipole) is distributed over the impact parameters in a nucleus with $S_A(b_j)$ [see Eq. (22)]. Recall that $S_A(b_j)$ is the probability to have a nucleon with a large impact parameter $b_j \sim R_A$. The dipole cascades, described by the Pomeron exchanges, have small typical δb_i [see Fig. 3(b) and Eq. (26)] of the order of the dipole size: $r \geq \delta b_i \geq 1/Q_s(Y)$. Hence we can integrate over them in Eq. (1) without affecting the value of b_j and replace Eq. (1) by

$$\begin{aligned} N(Y, r, R; \mathbf{b}) &= \sum_{n=1}^{\infty} (-1)^{n+1} n! \sum_{j_1=1}^{A_1} \sum_{j_2=1}^{A_2} \int \prod d^2 b'_{j'_1} \frac{1}{j_1! j_2!} \prod_{j'_1=1}^{j_1} \prod_{j'_2=1}^{j_2} S_{A_1}(\mathbf{b} - \mathbf{b}'_{j'_1}) S_{A_2}(\mathbf{b}'_{j'_2}) \\ &\times \prod_i^n d^2 r_i d^2 r'_i d^2 \delta b_i d^2 \delta b'_i \gamma^{BA}(r_1, r'_i, \delta \mathbf{b} - \delta \mathbf{b}'_i) \\ &\times \rho_{n_{j'_1}}^d(Y - Y_0; r, \{r_i, \delta b_i\}) \delta\left(\sum_{j'_1=1}^{j_1} n_{j'_1} = n\right) \rho_{n_{j'_2}}^d(Y_0; r', \{r'_i, \delta b'_i\}) \delta\left(\sum_{j'_2=1}^{j_2} n_{j'_2} = n\right). \end{aligned} \quad (33)$$

Equation (33) is written for $j_1 \leq j_2$. $\mathbf{b}'_{j'_2}$ for $j'_2 > j_1$ coincide with some of $\mathbf{b}'_{j'_1}$. Figure 3(b) gives the example of one such situation. It demonstrates the interaction for which two dipoles in the nucleus A_2 have almost the same impact parameters.

Using Eq. (15b) we can rewrite Eq. (33) in a more convenient form,

$$\begin{aligned} N(Y, r, R; \mathbf{b}) &= \sum_{n=1}^{\infty} (-1)^{n+1} n! \sum_{j_1=1}^{A_1} \sum_{j_2=1}^{A_2} \int \prod d^2 b'_{j'_1} \frac{1}{j_1! j_2!} \prod_{j'_1=1}^{j_1} \prod_{j'_2=1}^{j_2} S_{A_1}(\mathbf{b} - \mathbf{b}'_{j'_1}) S_{A_2}(\mathbf{b}'_{j'_2}) \\ &\times \prod_i^n \int \frac{d^2 r_i d^2 \delta b_i}{(2\pi)^2 r_i^A} \rho_{n_{j'_1}}^d(Y - Y_0; r, \{r_i, \delta b_i\}) \delta\left(\sum_{j'_1=1}^{j_1} n_{j'_1} = n\right) \rho_{n_{j'_2}}^d(Y_0; r', \{r_i, \delta b_i\}) \delta\left(\sum_{j'_2=1}^{j_2} n_{j'_2} = n\right). \end{aligned} \quad (34)$$

This equation can be simplified by taking into account Eqs. (29) and (31). Indeed, as we have discussed in the previous section, in the case of nucleus interaction the parton densities in the parton cascade, generated by one nucleon (dipole) can be described by the integral over δb_i of dipole densities, viz.,

$$\tilde{\rho}_n^A(r, b, \{r_i\}) = \int \prod_i d^2 \delta b_i \rho_n^A(r, b, \{r_i, b_i\}). \quad (35)$$

They can be written as follows:

$$\tilde{\rho}_n^A(r, b, \{r_i\}) = \prod_{i=1}^n \frac{G_{\mathbf{P}}(z_i)}{N_0} \sum_{j=1}^n \frac{1}{j!} (2\pi r^2 S_A(b))^j \sum_{k_1=1, \dots, k_j=1}^{k_1=n-j-1, \dots, k_j=n-j-1} \delta_{\sum_{i=1}^j k_i=n} C \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_i}{2\pi i} e^{\frac{\gamma}{2}\omega_i^2} \frac{1}{k_i!} \frac{\Gamma(\omega_i + k_i)}{\Gamma(\omega_i)} \quad (36)$$

with

$$z_i = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} Y + \ln\left(\frac{r^2}{r_i^2}\right). \quad (37)$$

For further simplification of the impact parameter dependence we consider the set of Pomeron diagrams Σ that are irreducible with respect two nuclei states (see Fig. 1 and compare the diagrams for the scattering amplitude and Σ). The scattering amplitude is equal to

$$N(Y, r, r'; \mathbf{b}) = 1 - \exp(-\Sigma(Y, r, r'; \mathbf{b})). \quad (38)$$

For Σ we can simplify Eq. (33) introducing

$$\Sigma(Y, r, r'; \mathbf{b}) = \sum_{j_1, j_2} \int \frac{d^2 \mathbf{b}'}{2\pi r r'} (\pi r^2 S_{A_1}(\mathbf{b} - \mathbf{b}'))^{j_1} (\pi r'^2 S_{A_2}(\mathbf{b}'))^{j_2} \frac{1}{j_1! j_2!} \Sigma_{j_2}^{j_1}(Y, r, r'). \quad (39)$$

For $\Sigma_{j_2}^{j_1}(Y, r, r')$ we have the simple form of Eqs. (1) and (33),

$$\begin{aligned} \Sigma_{j_2}^{j_1}(Y, r, r') &= \sum_{n=1}^{\infty} (-1)^{n+1} n! \prod_{i=1}^n \int dz_i \frac{G_P(z_i)}{N_0} \sum_{k_1=1, \dots, k_j=1}^{k_1=n-j-1, \dots, k_j=n-j_1-1} \delta_{\sum_{i=1}^{j_1} k_i=n} C \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_i}{2\pi i} e^{\frac{\gamma_K}{2}\omega_i^2} \frac{1}{k_i!} \frac{\Gamma(\omega_i + k_i)}{\Gamma(\omega_i)} \\ &\times \frac{G_P(z'_i)}{N_0} \sum_{l_1=1, \dots, l_j=1}^{l_1=n-j_2-1, \dots, l_j=n-j-1} \delta_{\sum_{i=1}^{j_2} l_i=n} C \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_i}{2\pi i} e^{\frac{\gamma_K}{2}\omega_i^2} \frac{1}{l_i!} \frac{\Gamma(\omega_i + l_i)}{\Gamma(\omega_i)} \end{aligned} \quad (40)$$

where z_i is given by Eq. (37) and z'_i is equal to

$$z'_i = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} (Y - Y_0) + \ln\left(\frac{r^2}{r_i'^2}\right). \quad (41)$$

Using Eq. (29) we can take the integral over z_i and obtain,

$$\begin{aligned} \Sigma_{j_2}^{j_1}(Y, r, r') &= \sum_{n=1}^{\infty} (-1)^{n+1} n! \left(\frac{G_P(z')}{N_0^2}\right)^n \prod_{i=1}^n \sum_{k_1=1, \dots, k_j=1}^{k_1=n-j-1, \dots, k_j=n-j_1-1} \delta_{\sum_{i=1}^{j_1} k_i=n} C \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_i}{2\pi i} e^{\frac{\gamma_K}{2}\omega_i^2} \frac{1}{k_i!} \frac{\Gamma(\omega_i + k_i)}{\Gamma(\omega_i)} \\ &\times \sum_{l_1=1, \dots, l_j=1}^{l_1=n-j_2-1, \dots, l_j=n-j-1} \delta_{\sum_{i=1}^{j_2} l_i=n} C \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_i}{2\pi i} e^{\frac{\gamma_K}{2}\omega_i'^2} \frac{1}{l_i!} \frac{\Gamma(\omega'_i + l_i)}{\Gamma(\omega'_i)} \end{aligned} \quad (42)$$

where z' is equal to

$$z' = \bar{\alpha}_S \frac{\chi(\bar{\gamma})}{\bar{\gamma}} Y_0 + \ln\left(\frac{r^2}{r'^2}\right). \quad (43)$$

C. Contribution of two nucleons (dipoles) scattering

In this section we consider the instructive example: the interaction of pairs of nucleons (dipoles) from different nuclei. In this case

$$\Sigma_1^1(Y, r, r'; \mathbf{b}) = \int \frac{d^2 \mathbf{b}'}{2\pi} S_{A_1}(\mathbf{b} - \mathbf{b}') S_{A_2}(\mathbf{b}') \int \frac{d^2 \delta \mathbf{b}}{2\pi} N_{\text{dipole-dipole}}(Y, r, r'; \delta \mathbf{b}). \quad (44)$$

$N_{\text{dipole-dipole}}(Y, r, r'; \delta \mathbf{b})$ has been found in Ref. [1]:

$$N_{\text{dipole-dipole}}(Y, r, r'; \delta \mathbf{b}) = 1 - S(z') = 1 - C'^2 \exp\left(-\frac{z'^2}{4\kappa}\right) \quad (45)$$

where z' is given by Eq. (30). $C(z')$ is a smooth function of z' .

From Eq. (45) we have

$$\int d^2 \delta b N_{\text{dipole-2dipoles}}(Y, r, r'; \delta \mathbf{b}) = \sigma_0 \left(1 - C' \exp\left(-\frac{z'^2}{4\kappa}\right)\right) \quad (46)$$

where $\sigma_0 = 2\pi r r'$ but the exact numerical factor we cannot find in this general consideration.

Plugging Eq. (46) into Eq. (44) we obtain

$$\begin{aligned} \Sigma_1^1(Y, r, r'; \mathbf{b}) &= \sigma_0 \int \frac{d^2 b'}{2\pi} S_{A_1}(\mathbf{b} - \mathbf{b}') S_{A_2}(\mathbf{b}') \\ &\times \left(1 - C \exp\left(-\frac{z'^2}{4\kappa}\right)\right). \end{aligned} \quad (47)$$

Hence the scattering amplitude is equal to

$$\begin{aligned} N(Y, r, r'; \mathbf{b}) &= 1 - \exp(-\Sigma_1^1(Y, r, r'; \mathbf{b})) \\ &= 1 - \exp\left(-\sigma_0 \int \frac{d^2 b'}{2\pi} S_{A_1}(\mathbf{b} - \mathbf{b}') S_{A_2}(\mathbf{b}')\right) \\ &\quad - \sigma_0 \int \frac{d^2 b'}{2\pi} S_{A_1}(\mathbf{b} - \mathbf{b}') S_{A_2}(\mathbf{b}') C \exp\left(-\frac{z'^2}{4\kappa}\right). \end{aligned} \quad (48)$$

$$\begin{aligned} N(Y, r, r'; \mathbf{b}) &= 1 - \exp(-\Sigma_2^1(Y, r, r'; \mathbf{b})) \\ &= 1 - \exp\left(-\frac{2\pi r'^2}{2!} \sigma_0 \int d^2 b' S_{A_1}(\mathbf{b} - \mathbf{b}') S_{A_2}^2(\mathbf{b}')\right) - \sigma_0 2\pi r'^2 \int d^2 b' S_{A_1}(\mathbf{b} - \mathbf{b}') S_{A_2}^2(\mathbf{b}') C' \exp\left(-\frac{z'^2}{4\kappa}\right). \end{aligned} \quad (53)$$

E. $j_1 = j_2 = 2$

In this section we consider the first case of Fig. 4(b), which crucially differs from dipole-dipole and dipole-nucleus scattering. Using Eqs. (39) and (42) with ρ_n from Eq. (20) we obtain for Σ_2^2

$$\begin{aligned} \tilde{\Sigma}_2^2(Y, r, r') &= C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_2}{2\pi i} e^{\frac{z'^2}{2}(\omega_1^2 + \omega_2^2 + \omega'^2_1 + \omega'^2_2)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int dz_i \prod_{i=1}^n \frac{G_{\mathbf{P}}(z_i)}{N_0} \frac{G_{\mathbf{P}}(z'_i)}{N_0} \\ &\times \sum_{l=1}^{n-1} \frac{n!}{l!(n-l)!} \frac{\Gamma(\omega_1 + l)}{\Gamma(\omega_1)} \frac{\Gamma(\omega_2 + n - l)}{\Gamma(\omega_2)} \sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} \frac{\Gamma(\omega'_1 + k)}{\Gamma(\omega'_1)} \frac{\Gamma(\omega'_2 + n - k)}{\Gamma(\omega'_2)}. \end{aligned} \quad (54)$$

Using Eqs. (24) and (15a) we can sum over l and k and make the integration over r_i and r'_i reducing Eq. (54) to the form,

D. $j_1 = 1, j_2 = 2 (j_1 = 2, j_2 = 1)$

In this section we consider the example of interaction: the case when the fast dipole from one nucleus interacts with two dipoles of another nucleus [see Fig. 4(a)]. Σ for this case has the following form using Eq. (38):

$$\begin{aligned} \Sigma_2^1(Y, r, r'; \mathbf{b}) &= \frac{1}{2!} \int \frac{d^2 b'}{2\pi} S_{A_1}(\mathbf{b} - \mathbf{b}') 2\pi r'^2 S_{A_2}^2(\mathbf{b}') \\ &\times \int d\delta b N_{\text{dipole-2dipoles}(A_2)}(Y, r, r'; \delta \mathbf{b}). \end{aligned} \quad (49)$$

The scattering amplitude $N_{\text{dipole-2dipoles}(A_2)}(Y, r, r'; \delta \mathbf{b})$ has been found in Ref. [2] and it has the form,

$$N_{\text{dipole-2dipoles}}(Y, r, r'; \delta \mathbf{b}) = 1 - 2C' \exp\left(-\frac{z'^2}{4\kappa}\right) \quad (50)$$

where z' is given by Eq. (30) and C' is the smooth function of z' . Using this equation we obtain

$$\int d^2 \delta b N_{\text{dipole-2dipoles}}(Y, r, r'; \delta \mathbf{b}) = \sigma_0 \left(1 - 2C' \exp\left(-\frac{z'^2}{4\kappa}\right)\right) \quad (51)$$

with σ_0 , which is of the order of $2\pi r'^2$.

Plugging Eq. (51) into Eq. (49) we have,

$$\begin{aligned} \Sigma_2^1(Y, r, r'; \mathbf{b}) &= \frac{1}{2!} \sigma_0 \int d^2 b' S_{A_1}(\mathbf{b} - \mathbf{b}') 2\pi r'^2 S_{A_2}^2(\mathbf{b}') \\ &\times \left(1 - 2C \exp\left(-\frac{z'^2}{4\kappa}\right)\right). \end{aligned} \quad (52)$$

Equation (52) leads to the following scattering amplitude:

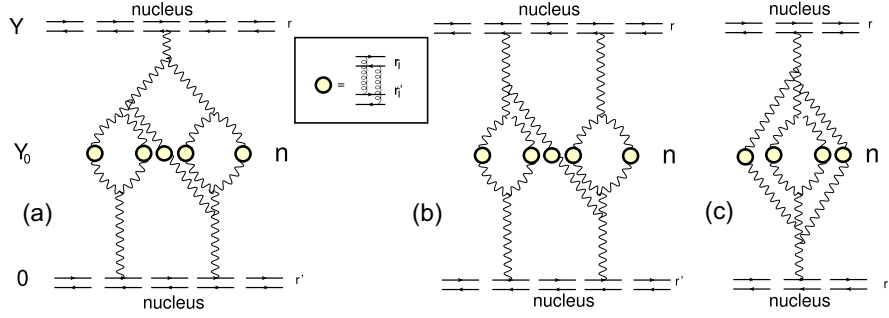


FIG. 4. Summing large Pomeron loops for nucleus-nucleus scattering. (a) The parton cascade of the fast dipole from one nucleus ($j_1 = 1$) interacts with the parton cascades generated by two nucleons in the second nucleus ($j_2 = 2$). (b) The parton cascades of two fast dipoles from one nucleus ($j_1 = 2$) interacts with the parton cascades generated by two nucleons in the second nucleus ($j_2 = 2$). (c) The parton cascade of the fast dipole from one nucleus interacts with the parton cascades generated by one nucleon in the second nucleus, which gives the main contribution to nucleus-nucleus interaction at high energy. The circles denote the amplitude γ in the Born approximation of perturbative QCD. A nucleus is viewed as the bag of dipoles with the size r' .

$$\begin{aligned} \tilde{\Sigma}_2^2(Y, r, r') = & C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_2}{2\pi i} e^{\frac{\gamma^2}{2}(\omega_1^2 + \omega_2^2 + \omega'^2_1 + \omega'^2_2)} \\ & \times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\infty dt_1 e^{-t_1} t_1^{\omega_1-1} \int_0^\infty dt_2 e^{-t_2} t_2^{\omega_2-1} \int_0^\infty dt'_1 e^{-t'_1} t'^{\omega'_1-1}_1 \int_0^\infty dt'_2 e^{-t'_2} t'^{\omega'_2-1}_2 \left(\frac{G_P(z'_i)}{N_0^2} \right)^n \\ & \times \frac{1}{\Gamma(\omega_1)\Gamma(\omega_2)\Gamma(\omega'_1)\Gamma(\omega'_2)} ((t_1 + t_2)^n - t_1^n - t_2^n)((t'_1 + t'_2)^n - t'^n_1 - t'^n_2). \end{aligned} \quad (55)$$

In this equation we introduce a different function $\tilde{\Sigma}_2^2(Y, r, r'; \mathbf{b})$. As one sees from Eq. (54) we started to sum with $n = 0$, while in Σ summation should go from $n = 1$ since Σ is the contribution of two nuclei irreducible Pomeron diagrams to the scattering amplitude. Therefore, $\tilde{\Sigma}$ is the contribution of the irreducible diagrams to the S matrix.

Using Eq. (24) we can integrate over t_1 and t_2 and rewrite Eq. (55) in the form

$$\begin{aligned} \tilde{\Sigma}_2^2(Y, r, r') = & C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_2}{2\pi i} e^{\frac{\gamma^2}{2}(\omega_1^2 + \omega_2^2 + \omega'^2_1 + \omega'^2_2)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\Gamma(\omega_1 + \omega_2 + n)}{\Gamma(\omega_1 + \omega_2)} - \frac{\Gamma(\omega_1 + n)}{\Gamma(\omega_1)} - \frac{\Gamma(\omega_2 + n)}{\Gamma(\omega_2)} \right) \\ & \times \left(\frac{G_{IP}(z'_i)}{N_0^2} \right)^n \int_0^\infty dt'_1 e^{-t'_1} t'^{\omega'_1-1}_1 \int_0^\infty dt'_2 e^{-t'_2} t'^{\omega'_2-1}_2 \frac{1}{\Gamma(\omega'_1)\Gamma(\omega'_2)} ((t'_1 + t'_2)^n - t'^n_1 - t'^n_2). \end{aligned} \quad (56)$$

We can perform summing over n using that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(\omega + n)}{\Gamma(\omega)} \left(\frac{G_P(z'_i)}{N_0^2} \right)^n = \left(1 + \frac{G_P(z'_i)}{N_0^2} \right)^{-\omega} \xrightarrow{\frac{G_P}{N_0^2} \gg 1} \left(\frac{G_P(z'_i)}{N_0^2} \right)^{-\omega}. \quad (57)$$

Equation (56) takes the form,

$$\begin{aligned} \tilde{\Sigma}_2^2(Y, r, r') = & C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_2}{2\pi i} e^{\frac{\gamma^2}{2}(\omega_1^2 + \omega_2^2 + \omega'^2_1 + \omega'^2_2)} \left\{ \left(1 + (t'_1 + t'_2) \frac{G_{IP}(z'_i)}{N_0^2} \right)^{-\omega_1 - \omega_2} - \left(1 + t'_1 \frac{G_{IP}(z'_i)}{N_0^2} \right)^{-\omega_1 - \omega_2} \right. \\ & - \left(1 + t'_2 \frac{G_{IP}(z'_i)}{N_0^2} \right)^{-\omega_1 - \omega_2} - \left(1 + (t'_1 + t'_2) \frac{G_{IP}(z'_i)}{N_0^2} \right)^{-\omega_1} + \left(1 + t'_1 \frac{G_{IP}(z'_i)}{N_0^2} \right)^{-\omega_1} + \left(1 + t'_2 \frac{G_{IP}(z'_i)}{N_0^2} \right)^{-\omega_1} \\ & \left. - \left(1 + (t'_1 + t'_2) \frac{G_{IP}(z'_i)}{N_0^2} \right)^{-\omega_2} + \left(1 + t'_1 \frac{G_{IP}(z'_i)}{N_0^2} \right)^{-\omega_2} + \left(1 + t'_2 \frac{G_{IP}(z'_i)}{N_0^2} \right)^{-\omega_2} \right\} \frac{1}{\Gamma(\omega'_1)\Gamma(\omega'_2)}. \end{aligned} \quad (58)$$

For $\frac{G_P(z'_i)}{N_0^2} \gg 1$ we can take integrals over $t'1$ and t'_2 . The result takes the form,

$$\begin{aligned} \tilde{\Sigma}_2^2(Y, r, r') = & C^2 \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_1}{2\pi i} \int_{\epsilon_i\infty}^{\epsilon+i\infty} \frac{d\omega_2}{2\pi i} \int_{\epsilon_i\infty}^{\epsilon+i\infty} \frac{d\omega'_1}{2\pi i} \int_{\epsilon_i\infty}^{\epsilon+i\infty} \frac{d\omega'_2}{2\pi i} e^{\frac{\bar{\gamma}^2 \kappa}{2}(\omega_1^2 + \omega_2^2 + \omega_1'^2 + \omega_2'^2)} \\ & \times \left\{ \left[\frac{\Gamma(\omega'_1 + \omega'_2 - \omega_1 - \omega_2)}{\Gamma(\omega'_1 + \omega'_2)} - \frac{\Gamma(\omega'_1 - \omega_1 - \omega_2)}{\Gamma(\omega'_1)} - \frac{\Gamma(\omega'_2 - \omega_1 - \omega_2)}{\Gamma(\omega'_2)} \right] \exp \left(-(\omega_1 + \omega_2) \ln \left(\frac{G_P(z')}{N_0} \right) \right) \right. \\ & \times \left[-\frac{\Gamma(\omega'_1 + \omega'_2 - \omega_1)}{\Gamma(\omega'_1 + \omega'_2)} + \frac{\Gamma(\omega'_1 - \omega_1)}{\Gamma(\omega'_1)} + \frac{\Gamma(\omega'_2 - \omega_1)}{\Gamma(\omega'_2)} \right] \exp \left(-\omega_1 \ln \left(\frac{G_P(z')}{N_0} \right) \right) \\ & \left. \times \left[-\frac{\Gamma(\omega'_1 + \omega'_2 - \omega_2)}{\Gamma(\omega'_1 + \omega'_2)} + \frac{\Gamma(\omega'_1 - \omega_2)}{\Gamma(\omega'_1)} + \frac{\Gamma(\omega'_2 - \omega_2)}{\Gamma(\omega'_2)} \right] \exp \left(-\omega_2 \ln \left(\frac{G_P(z')}{N_0} \right) \right) \right\}. \end{aligned} \quad (59)$$

Closing contour of integrations over ω_i and ω'_i on the first pole of the Γ function we reduce Eq. (59) to the sum of integrals that can be taken using the method of steepest descent at large z' . Finally we obtain,

$$\tilde{\Sigma}_2^2(Y, r, r') = C_1^2 \exp \left(-\frac{z'^2}{2\kappa} \right) - 4C_2^2 C_1^2 \exp \left(-\frac{z'^2}{23\kappa} \right) + 4C_3^2 C_1^2 \exp \left(-\frac{z'^2}{4\kappa} \right). \quad (60)$$

From this equation the contribution to $\Sigma(Y, r, r'; \mathbf{b})$ with $j_1 = j_2 = 2$ is equal to

$$\begin{aligned} \tilde{\Sigma}_2^{j_1}(Y, r, r'; \mathbf{b}) = & \frac{1}{2!} \frac{1}{2!} \int \frac{d^2 b'}{2\pi r r'} (2\pi r^2)^2 S_{A_1}^2(\mathbf{b} - \mathbf{b}') (2\pi r'^2)^2 S_{A_2}^2(\mathbf{b}') \\ & \times \left(1 - C_1^2 \exp \left(-\frac{z'^2}{2\kappa} \right) + 4C_2^2 C_1^2 \exp \left(-\frac{z'^2}{23\kappa} \right) - 4C_3^2 C_1^2 \exp \left(-\frac{z'^2}{4\kappa} \right) \right). \end{aligned}$$

F. General case: Arbitrary j_1 and j_2

After thorough derivation of the $j_1 = j_2 = 2$ case we can write the general formula,

$$\begin{aligned} \tilde{\Sigma}_{j_2}^{j_1}(Y, r, r';) = & C^2 \prod_{i=1}^{j_1} \prod_{l=1}^{j_2} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_i}{2\pi i} \int_{\epsilon_i\infty}^{\epsilon+i\infty} \frac{d\omega'_l}{2\pi i} e^{\frac{\bar{\gamma}^2 \kappa}{2}(\omega_i^2 + \omega_l'^2)} \sum_{j'_1=1}^{j_1} \frac{j_1!}{j'_1!(j_1 - j'_1)!} \sum_{j'_2=1}^{j_2} \frac{j_2!}{j'_2!(j_2 - j'_2)!} (-1)^{j'_1+j'_2} \\ & \times \exp \left(-\left(\sum_{i=1}^{j'_1} \omega_i \right) \ln \left(\frac{G_P(z')}{N_0} \right) \right) \exp \left(-\left(\sum_{l=1}^{j'_2} \omega_l \right) \ln \left(\frac{G_P(z')}{N_0} \right) \right) \frac{\Gamma(\sum_{i=1}^{j'_1} \omega_i - \sum_{l=1}^{j'_2} \omega'_l)}{\Gamma(\sum_{l=1}^{j'_2} \omega'_l)}. \end{aligned}$$

Closing the contour of integration over $\sum_{i=1}^{j'_1} \omega_i \equiv \omega$ on the pole $\omega = \sum_{l=1}^{j'_2} \omega'_l$ we have the following integrals over ω_i and ω'_l :

$$\begin{aligned} \tilde{\Sigma}_{j_2}^{j_1}(Y, r, r') = & \sum_{j'_1=1}^{j_1} \frac{j_1!}{j'_1!(j_1 - j'_1)!} \sum_{j'_2=1}^{j_2} \frac{j_2!}{j'_2!(j_2 - j'_2)!} (-1)^{j'_1+j'_2} C^2 \prod_{i=2}^{j_1} \prod_{l=2}^{j_2} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \exp \left(-\omega \ln \left(\frac{G_P(z')}{N_0^2} \right) \right) \\ & \times \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_i}{2\pi i} \int_{\epsilon_i\infty}^{\epsilon+i\infty} \frac{d\omega'_l}{2\pi i} \exp \left(\frac{\bar{\gamma}^2 \kappa}{2} \left(\left(\omega - \sum_{l=2}^{j'_2} \omega_l \right)^2 + \sum_{i=2}^{j_1} \omega_i^2 + \left(\omega - \sum_{l=2}^{j'_2} \omega'_l \right)^2 + \sum_{l=2}^{j'_2} \omega_l'^2 \right) \right). \end{aligned} \quad (61)$$

Expecting that $\omega \propto \bar{\gamma} z$ is large we can take the integrals over ω 's using the method of steepest descent. The equations for the saddle points are the following:

$$(1) \bar{\gamma}^2 \kappa \left(2\omega^{\text{SP}} - \sum_{i=2}^{j'_1} \omega_i^{\text{SP}} - \sum_{l=2}^{j'_2} \omega_l^{\text{SP}} \right) - \bar{\gamma} z = 0; \quad (i) \omega_i^{\text{SP}} + \sum_{i=2}^{j'_1} \omega_i^{\text{SP}} - \omega^{\text{SP}} = 0; \quad (l) \omega_l^{\text{SP}} + \sum_{l=2}^{j'_2} \omega_l^{\text{SP}} - \omega^{\text{SP}} = 0. \quad (62)$$

Solutions to these equations are

$$\omega_i^{\text{SP}} = \frac{1}{j'_1} \omega^{\text{SP}}; \quad \omega_l^{\text{SP}} = \frac{1}{j'_2} \omega^{\text{SP}}; \quad \omega^{\text{SP}} = \frac{z}{\tilde{\gamma} \kappa j'_1 + j'_2}. \quad (63)$$

Plugging Eq. (63) into Eq. (61) we obtain that each term in the sum of this equation behaves as

$$\exp\left(-\frac{z'^2}{2\kappa} \frac{j'_1 j'_2}{j'_1 + j'_2}\right). \quad (64)$$

Therefore, we can conclude that the main contribution stems from $j'_1 = j'_2 = 1$ and

$$\tilde{\Sigma}_{j_2}^{j_1}(Y, r, r') = j_1 j_2 \tilde{\Sigma}_1^1(Y, r, r') \propto \exp\left(-\frac{z'^2}{4\kappa}\right). \quad (65)$$

Therefore, one can see that the only interaction between two nucleons (dipoles) from different nuclei are essential at high energy. This interaction leads to the nucleus-nucleus scattering amplitude deep in the saturation region, which has the same z' behavior as the dipole-dipole scattering. Indeed, Eq. (61) for $j'_1 = 1$ and $j'_2 = 1$ reduces to the following expression:

$$\begin{aligned} \tilde{\Sigma}_{j_2}^{j_1}(Y, r, r') &= j_1 j_2 C^2 \prod_{i=2}^{j_1} \prod_{l=2}^{j_2} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} \exp\left(-\omega \ln\left(\frac{G_P(z')}{N_0^2}\right)\right) \\ &\times \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega_i}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega'_l}{2\pi i} \exp\left(\frac{\tilde{\gamma}^2 \kappa}{2} \left(2\omega^2 + \sum_{i=2}^{j_1} \omega_i^2 + \sum_{l=2}^{j_2} \omega'_l{}^2\right)\right). \end{aligned} \quad (66)$$

One can see that $\omega \sim \ln\left(\frac{G_P(z')}{N_0^2}\right)$, while all other ω do not depend on z' , being of the order of $\omega_i \sim \omega'_l \sim 1/\sqrt{\tilde{\gamma}^2 \kappa}$.

Hence the scattering amplitude for the nucleus-nucleus interaction can be reduced to the contribution of the interaction of two nucleons from different nuclei that has been discussed in Sec. III C [see Eq. (48)]. Integration over ω_i and ω'_l with $i, l \geq 2$ are absorbed in the smooth function C in this equation.

IV. CONCLUSIONS

In this paper we summed the large BFKL Pomeron loops in the framework of the BFKL Pomeron calculus for the nucleus-nucleus scattering. The main result is that this scattering amplitude has the same energy dependence as the dipole-dipole amplitude. Therefore, all three types of scattering: dilute-dilute dipoles systems (dipole-dipoles scattering [1]), dilute-dense dipole system (dipole-nucleus scattering [2]), and dense-dense dipole system (this paper), show the same energy dependence due to contributions of large BFKL Pomeron loops. In other words, it turns out that the evolution of one dipole cascade contributes to all three cases [see Fig. 4(c)]. From Fig. 4(c) one can see that the interaction of two dipoles can be interpreted as the exchange of the “dressed” Pomeron with the Green’s function given by Eq. (48).

We believe that this observation could be valuable for summing the Pomeron loops of arbitrary sizes. We wish to recall that we have summed only Pomeron loops with the sizes larger than $\min\{Y - Y_0, Y_0\}$. The smaller loops are still out of our control and we do not know how important their contributions could be.

We could not provide reliable estimates of what value of energy of the calculated contributions will be dominant. Reference [64] states that the Pomeron loops started to dominate over single Pomeron exchange at $Y > 10$. We gave in Ref. [2] that for proton-nucleus scattering we expect that our contribution will prevail at $Y > 18$. For nucleus-nucleus scattering the border would be larger.

We thank our referee who noticed that the Pomeron loops that we have calculated are included in the estimates by the Monte Carlo generators, like ARIADNE [65] and DIPSY [66] from the Lund group. Indeed, these generators are based on the dipole approach [7,26] and their dipole cascades take into account our loop diagrams. Therefore, we believe that the real estimate for the values of rapidity when our contribution will dominate, can be obtained from collaboration with these groups.

We hope that this paper will contribute to the further study of Pomeron calculus in QCD.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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