

Tomography of the rho meson in the QCD instanton vacuum: Transverse momentum dependent parton distribution functions

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We analyze the rho meson unpolarized and polarized transverse momentum dependent parton distribution functions (TMDPDFs) in the instanton liquid model (ILM). The corresponding TMDs in ILM are approximated by a constituent quark beam function in the leading Fock state multiplied by a rapidity-dependent factor resulting from the staple-shaped Wilson lines, for fixed longitudinal momentum, transverse separation, and rapidity. At the resolution of the ILM, all of the rho meson TMDs are symmetric in parton x for fixed transverse momentum, and Gaussian-like in the transverse momentum for fixed parton x . The latter is a direct consequence of the profiling of the quark zero modes in the ILM. The evolved TMDs at higher rapidity using the Collins-Soper kernel, and higher resolution using the renormalization group, show substantial skewness towards low parton x .

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I. INTRODUCTION

Fast moving hadrons are composed of quarks and gluons (partons). The partonic structure of hadrons is probed through inclusive and semi-inclusive deep inelastic scattering experiments [1]. Transverse momentum distributions (TMDs) for a given hadron map the probability distribution of a parton in both transverse momentum and longitudinal momentum fraction x [2]. They reduce to the unimodular parton distribution functions (PDFs) after transverse integration [3].

There is continuous effort both experimentally [4–10] and theoretically including the instanton-based nonperturbative model [11,12] and large scale lattice simulations [13–18], to quantify these nonperturbative functions. Their understanding is part of a large experimental effort at COMPASS at CERN, JLAB in Virginia, LHC at CERN, and the future EIC at BNL. Currently, TMDPDFs are empirically extracted from Drell-Yan (DY) processes and semi-inclusive deep inelastic scattering (SIDIS), with small final hadron transverse momentum [19].

In the leading twist approximation, TMDs are defined as correlation functions of bilocal quarks with stapled lightlike

Wilson lines that are lined up on the light cone. Recent lattice developments using the large momentum effective theory have proven useful for their possible Euclidean extraction and matching using quasi-TMDs, where the staple-shaped Wilson line (soft function) can be related to a mesonic form factor [20].

Spin-0 targets are characterized by one T-even TMD (unpolarized function) and one T-odd (Boer-Mulders function). Specifically, the T-even one reduces to the unpolarized parton distribution. Spin- $\frac{1}{2}$ targets are characterized by six T-even TMDs and two T-odd TMDs [2,21]. The T-odd TMDs are known as the Sivers and Boer-Mulders functions, while the T-even TMDs are tied to the standard unpolarized, helicity, and transversity PDFs. Spin-1 targets do not have Sivers and Boer-Mulders functions. Alternatively, they have three additional T-even TMDs related to unpolarized partons in a tensor-polarized target, which amount to an additional PDF upon transverse integration [22]. The purpose of this paper is to evaluate the rho meson unpolarized and polarized TMDs at low resolution, using the instanton liquid model (ILM) description of the QCD vacuum. It is a follow-up on our recent study of the spin-0 TMDs for the pion and kaon in the ILM [11].

The Yang-Mills vacuum at high resolution is dominated by UV gauge fluctuations. When the gradient flow cooling is applied to these configurations [23–30], a smoother landscape in the IR emerges. This landscape is dominated by topologically active configurations known as pseudo-particles (instantons and anti-instantons), with a mean size $\rho \sim 13$ fm, and a mean density (tunneling rate) $n_{I+A} \sim 1/\text{fm}^4$. These features are well captured by the

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ILM [31–41]. For a more comprehensive review, see also [42–44] (and references therein).

The organization of the paper is as follow: In Sec. II we outline the general aspects of the rho TMDs, and develop their analysis in the ILM, at low resolution. We also derive their relationships to the rho PDFs and sum rules. In Sec. III we show how to implement the soft subtraction scheme on the TMD distributions, and carry their double evolution in scale and rapidity. Our conclusions are in Sec. IV.

A number of Appendixes are added to detail and support the arguments in the main text.

II. RHO TMDS

A. General

The leading twist-2 TMDs describe the unpolarized, helicity, and several transversity parton distributions in the transverse plane:

$$\begin{aligned} q_\rho(x, k_\perp) &= \int_{-\infty}^{\infty} \frac{dz^-}{2\pi} \int \frac{d^2 b_\perp}{(2\pi)^2} e^{ixP^+ z^- - ik_\perp \cdot b_\perp} \langle \rho | \bar{\psi}(0) \gamma^+ W^{(\pm)}[0, 0_\perp; z^-, b_\perp] \psi(z^-, b_\perp) | \rho \rangle \\ \Delta q_\rho(x, k_\perp) &= \int_{-\infty}^{\infty} \frac{dz^-}{2\pi} \int \frac{d^2 b_\perp}{(2\pi)^2} e^{ixP^+ z^- - ik_\perp \cdot b_\perp} \langle \rho | \bar{\psi}(0) \gamma^+ \gamma^5 W^{(\pm)}[0, 0_\perp; z^-, b_\perp] \psi(z^-, b_\perp) | \rho \rangle \\ \delta q_\rho^\alpha(x, k_\perp) &= \int_{-\infty}^{\infty} \frac{dz^-}{2\pi} \int \frac{d^2 b_\perp}{(2\pi)^2} e^{ixP^+ z^- - ik_\perp \cdot b_\perp} \langle \rho | \bar{\psi}(0) i\sigma^{\alpha+} \gamma^5 W^{(\pm)}[0, 0_\perp; z^-, b_\perp] \psi(z^-, b_\perp) | \rho \rangle. \end{aligned} \quad (1)$$

Here the hadronic state label includes the quantum numbers carried by the on-shell vector meson of mass m_ρ , i.e., $\rho(p, \lambda)$. The rho kinematics is represented by the longitudinal light front unit vectors $n = (1, 0^-, 0_\perp)$ and $\bar{n} = (0^+, 1, 0_\perp)$ in light-cone signature, and the transverse component. The hadronic momentum with energy $E_{\vec{p}} = \sqrt{m_\rho^2 + \vec{p}^2}$ and three-momentum $\vec{p}$ can be decomposed:

$$p^\mu = p^+ \bar{n}^\mu - \frac{m_\rho^2}{2p^+} n^\mu + p_\perp^\mu. \quad (2)$$

The Wilson lines in the matrix elements (1) enforce gauge invariance. They are different in semi-inclusive deep inelastic scattering (SIDIS) process (spacelike), and in Drell-Yan process (timelike) [45]. The resummation in the SIDIS processes of the intermediate gluons yields the Wilson line defined as $W^{(+)}$. The resummations in the Drell-Yan processes of the intermediate gluons lead the Wilson line defined as $W^{(-)}$ [46–50],

$$\begin{aligned} W^{(\pm)}[0, 0_\perp; z^-, b_\perp] &= W_n[0, 0_\perp; \pm\infty, 0_\perp] \\ &\times W_\perp[\pm\infty, 0_\perp; \pm\infty, b_\perp] \\ &\times W_n[\pm\infty, b_\perp; z^-, b_\perp], \end{aligned} \quad (3)$$

with the generic gauge link defined as

$$W[x; y] = \mathcal{P} \exp \left[ig \int_x^y dz_\mu A^\mu(z) \right]. \quad (4)$$

B. Momentum distribution

The rho TMDs can be decomposed into several functions based on the quark spin direction compared to the polarization of the hadron states. The notations follow the convention in PDFs. The functions f , g , and h denote the quark polarization for being unpolarized (γ^+), longitudinally polarized ($\gamma^+ \gamma^5$), and transversely polarized ($i\sigma^{\alpha+} \gamma^5$), respectfully. The lower index 1 refers to the leading twist, and the subscripts L and T refer to the hadron polarization.

For rho mesons, the Lorentz decomposition makes use of the spin quantization axis $\vec{s} = (s_\perp, s_L)$, and the vector meson polarization $\lambda = 0, \pm 1$ projected onto $\vec{s}$. With this in mind, the TMDs for vector mesons can be decomposed as [51]

$$\begin{aligned} q_\rho(x, k_\perp) &= f_1^{q/\rho}(x, k_\perp) + S_{LL} f_{1LL}^{q/\rho}(x, k_\perp) \\ &+ \frac{S_{LT} \cdot k_\perp}{m_\rho} f_{1LT}^{q/\rho}(x, k_\perp) \\ &+ \frac{S_{TT}^{\alpha\beta} k_\perp^\alpha k_\perp^\beta}{m_\rho^2} f_{1TT}^{q/\rho}(x, k_\perp) \end{aligned} \quad (5)$$

$$\Delta q_\rho(x, k_\perp) = \lambda \left[s_L g_1^{q/\rho}(x, k_\perp) + \frac{k_\perp \cdot s_\perp}{m_\rho} g_{1T}^{q/\rho}(x, k_\perp) \right] \quad (6)$$

$$\begin{aligned} \delta q_\rho^\alpha(x, k_\perp) &= \lambda \left[s_\perp^\alpha h_1^{q/\rho}(x, k_\perp) + s_L \frac{k_\perp^\alpha}{m_\rho} h_{1L}^{q/\rho}(x, k_\perp) \right. \\ &\left. + \frac{1}{2m_\rho^2} (s_\perp^\alpha k_\perp^2 - 2k_\perp \cdot s_\perp k_\perp^\alpha) h_{1T}^{q/\rho}(x, k_\perp) \right] \end{aligned} \quad (7)$$

with the spin polarization tensors

$$S_{LL} = (3\lambda^2 - 2) \left(\frac{1}{6} - \frac{1}{2} s_L^2 \right) \quad (8)$$

$$S_{LT}^\alpha = (3\lambda^2 - 2) s_L s_\perp^\alpha \quad (9)$$

$$S_{TT}^{\alpha\beta} = (3\lambda^2 - 2) \left(s_\perp^\alpha s_\perp^\beta - \frac{1}{2} s_\perp^2 \delta^{\alpha\beta} \right). \quad (10)$$

There are three tensor polarized TMDs f_{1LL} , f_{1LT} , and f_{1TT} that are specific to spin-1 mesons. As we noted earlier, for spin-1 mesons there are nine leading-twist TMDs and four PDFs, if we restrict ourselves to the T -even distributions.

With the Fourier transform of a generic TMD $F^{q/\rho}(x, k_\perp)$ to the conjugated transverse coordinate space defined as

$$\tilde{F}^{q/\rho}(x, b_\perp) = \int \frac{d^2 k_\perp}{(2\pi)^2} e^{ik_\perp \cdot b_\perp} F^{q/\rho}(x, k_\perp), \quad (11)$$

the Lorentz decomposition of TMD distribution in $b_\perp$ space can be written as [52]

$$\begin{aligned} \tilde{q}_\rho(x, b_\perp) &= \tilde{f}_1^{q/\rho}(x, b_\perp) + S_{LL} \tilde{f}_{1LL}^{q/\rho}(x, b_\perp) \\ &\quad + i S_{LT} \cdot b_\perp m_\rho \tilde{f}_{1LT}^{q/\rho}(x, b_\perp) \\ &\quad + S_{TT}^{\alpha\beta} b_\perp^\alpha b_\perp^\beta m_\rho^2 \tilde{f}_{1TT}^{q/\rho}(x, b_\perp) \end{aligned} \quad (12)$$

$$\Delta \tilde{q}_\rho(x, b_\perp) = \lambda \left[s_L \tilde{g}_1^{q/\rho}(x, b_\perp) + i b_\perp \cdot s_\perp m_\rho \tilde{g}_{1T}^{q/\rho}(x, b_\perp) \right] \quad (13)$$

$$\begin{aligned} \delta \tilde{q}_\rho^\alpha(x, b_\perp) &= \lambda \left[s_\perp^\alpha \tilde{h}_1^{q/\rho}(x, b_\perp) + i s_L b_\perp^\alpha m_\rho \tilde{h}_{1L}^{q/\rho}(x, b_\perp) \right. \\ &\quad \left. - \frac{m_\rho^2}{2} (s_\perp^\alpha b_\perp^2 - 2 b_\perp \cdot s_\perp b_\perp^\alpha) \tilde{h}_{1T}^{q/\rho}(x, b_\perp) \right]. \end{aligned} \quad (14)$$

The relations between the $b_\perp$ space TMD and the $k_\perp$ space TMDs are listed in Appendix E.

C. Rho TMD in ILM

The computation of TMD quark distribution in the vector meson can be simply addressed by the light front wave functions (LFWFs). The softly factored contribution of the staple-shaped Wilson line $W^{(\pm)}$ at low resolution will be addressed below. With this in mind, the rho meson TMD reads

$$\begin{aligned} q_\rho(x, k_\perp) &= \frac{1}{(2\pi)^3} \sum_{s_1, s_2} \Phi_\rho^\dagger(x, k_\perp, s_1, s_2) \Phi_\rho(x, k_\perp, s_1, s_2) \\ \Delta q_\rho(x, k_\perp) &= \frac{1}{(2\pi)^3} \sum_{s_1, s'_1, s_2} \Phi_\rho^\dagger(x, k_\perp, s_1, s_2) \\ &\quad \times \sigma_{s_1 s'_1}^3 \Phi_\rho(x, k_\perp, s'_1, s_2) \\ \delta q_\rho^\alpha(x, k_\perp) &= \frac{1}{(2\pi)^3} \sum_{s_1, s'_1, s_2} \Phi_\rho^\dagger(x, k_\perp, s_1, s_2) \\ &\quad \times \sigma_{s_1 s'_1}^\alpha \Phi_\rho(x, k_\perp, s'_1, s_2), \end{aligned} \quad (15)$$

with the vector meson wave functions polarized in the $\vec{s}$ direction. Their general form follows from the derivation given in [32], using an arbitrary spin quantization axis. The result for the longitudinal vector meson LFWF, is

$$\begin{aligned} \Phi_\rho^0(x, k_\perp, s_1, s_2) &= \frac{1}{\sqrt{N_c}} \left[\frac{C_\rho}{\sqrt{x\bar{x}}(m_\rho^2 - \frac{k_\perp^2 + M^2}{x\bar{x}})} \mathcal{F}\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \right] \\ &\quad \times \epsilon_0^\mu(p) \bar{u}_{s_1}(k_1) \gamma_\mu \tau^a v_{s_2}(k_2). \end{aligned} \quad (16)$$

For the transverse vector meson LFWFs, the results are

$$\begin{aligned} \Phi_\rho^\pm(x, k_\perp, s_1, s_2) &= \frac{1}{\sqrt{N_c}} \left[\frac{C_\rho}{\sqrt{x\bar{x}}(m_\rho^2 - \frac{k_\perp^2 + M^2}{x\bar{x}})} \mathcal{F}\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \right] \\ &\quad \times \epsilon_\pm^\mu(p) \bar{u}_{s_1}(k_1) \gamma_\mu \tau^a v_{s_2}(k_2). \end{aligned} \quad (17)$$

Here $u_{s_1}(k_1)$ denotes the light front quark spinor with spin s_1 and internal momentum $k_1^+ = xp^+$, $k_{1\perp} = k_\perp$. Similarly, $v_{s_2}(k_2)$ denotes the light front antiquark spinor with spin s_2 and internal momentum $k_2^+ = \bar{x}p^+$, $k_{2\perp} = -k_\perp$, as detailed in Appendix B. The polarization direction is encoded in the polarization vector $\epsilon^\mu(p)$. The normalization condition for the light front wave functions yields

$$\begin{aligned} C_{\rho_T} &= \left[\frac{1}{4\pi^2} \int_0^1 dx \int_0^\infty dk_\perp^2 \frac{k_\perp^2 + M^2 - 2x\bar{x}k_\perp^2}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \right. \\ &\quad \left. \times \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \right]^{-1/2} \end{aligned} \quad (18)$$

and

$$\begin{aligned} C_{\rho_L} &= \left[\frac{1}{4\pi^2} \int_0^1 dx \int_0^\infty dk_\perp^2 \left(\frac{1}{m_\rho^2} + \frac{4x\bar{x}(k_\perp^2 + M^2)}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \right) \right. \\ &\quad \left. \times \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \right]^{-1/2} \end{aligned} \quad (19)$$

with $\lambda_\rho = 1.260$ fixed by demanding the proper normalization of the rho distribution amplitude (DA) for an on-shell rho of mass m_ρ , as detailed in Appendix C. In other words, $\lambda_\rho = 1.260$ reproduces the rho electroweak decay

constant f_ρ . By setting the pseudoparticle (instanton) size $\rho = 0.313$ fm, and a constituent mass $M = 398.17$ MeV, the rho mass is found to be $m_\rho = 791$ MeV, and the normalization constants in (18) and (19) are

$$C_{\rho_L} = C_{\rho_T} = C_\rho = 1.9888. \quad (20)$$

It is important to note that, to recover full covariance on the light front for the rho meson, terms of order

$$p^- - \frac{k_\perp^2 + M^2}{2x\bar{x}p^+} \sim \mathcal{O}(n_{I+A}^2)$$

need to be retained in the light front analysis [32]. Note that the Ward identity is readily enforced,

$$p_\mu \bar{u}_{s_1}(k_1) \gamma^\mu v_{s_2}(k_2) = 0. \quad (21)$$

Inserting (16) and (17) into (15) and reducing yields the leading twist TMDs in the ILM:

$$\begin{aligned} f_1^{q/\rho}(x, k_\perp) &= \frac{1}{3} \frac{C_\rho^2}{(2\pi)^3} \left[\left(\frac{2}{m_\rho^2} + \frac{8x\bar{x}(k_\perp^2 + M^2)}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \right) \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \right] \\ &+ \frac{2}{3} \frac{C_\rho^2}{(2\pi)^3} \left[\frac{2(k_\perp^2 + M^2 - 2x\bar{x}k_\perp^2)}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \right] \end{aligned} \quad (22)$$

$$\begin{aligned} f_{1LL}^{q/\rho}(x, k_\perp) &= \frac{C_\rho^2}{(2\pi)^3} \left[\left(\frac{2}{m_\rho^2} + \frac{8x\bar{x}(k_\perp^2 + M^2)}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \right) \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \right] \\ &- \frac{C_\rho^2}{(2\pi)^3} \left[\frac{2(k_\perp^2 + M^2 - 2x\bar{x}k_\perp^2)}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \right] \end{aligned} \quad (23)$$

$$\begin{aligned} f_{1LT}^{q/\rho}(x, k_\perp) &= \frac{C_\rho^2}{(2\pi)^3} \left[\frac{2(x - \bar{x})(x\bar{x}m_\rho^2 + k_\perp^2 + M^2)}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \right] \\ &\times \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \end{aligned} \quad (24)$$

$$f_{1TT}^{q/\rho}(x, k_\perp) = \frac{C_\rho^2}{(2\pi)^3} \left[\frac{4x\bar{x}m_\rho^2}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \right] \quad (25)$$

$$g_1^{q/\rho}(x, k_\perp) = \frac{C_\rho^2}{(2\pi)^3} \frac{4((x - \bar{x})k_\perp^2 + M^2)}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \quad (26)$$

$$g_{1T}^{q/\rho}(x, k_\perp) = \frac{C_\rho^2}{(2\pi)^3} \frac{2(x\bar{x}m_\rho^2 + k_\perp^2 + M^2)}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \quad (27)$$

TABLE I. Table of the $k_\perp$ moment of TMDs defined in (32). Dots indicate the corresponding TMDs are vanishing. All units are given in GeV. Our results are compared to the Nambu–Jona-Lasinio (NJL) model [22], the light-front holographic model (LFHM), the light-front quark model (LFQM) [53], and the Bethe-Salpeter equation (BSE) [51].

	$\langle k_\perp \rangle_{\text{NJL}}$	$\langle k_\perp \rangle_{\text{LFHM}}$	$\langle k_\perp \rangle_{\text{LFQM}}$	$\langle k_\perp \rangle_{\text{BSE}}$	$\langle k_\perp \rangle_{\text{ILM}}$
f_1	0.32	0.238	0.328	0.399	0.0810
g_1	0.08	0.204	0.269	0.318	0.0751
g_{1T}	0.34	0.229	0.269	0.358	0.0785
h_1	0.34	0.229	0.307	0.367	0.0785
h_{1L}	0.33	0.204	0.269	0.368	0.0751
h_{1T}	...	...	0.237	0.365	...
f_{1TT}	0.32	0.211	...	0.338	0.0734

$$h_1^{q/\rho}(x, k_\perp) = \frac{C_\rho^2}{(2\pi)^3} \frac{M}{m_\rho} \frac{2(x\bar{x}m_\rho^2 + k_\perp^2 + M^2)}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \quad (28)$$

$$h_{1L}^{q/\rho}(x, k_\perp) = -\frac{C_\rho^2}{(2\pi)^3} \frac{4x\bar{x}Mm_\rho}{(x\bar{x}m_\rho^2 - k_\perp^2 - M^2)^2} \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \quad (29)$$

$$h_{1T}^{q/\rho}(x, k_\perp) = 0. \quad (30)$$

We note the relationship

$$g_{1T}^{q/\rho}(x, k_\perp) = \frac{m_\rho}{M} h_1^{q/\rho}(x, k_\perp) \quad (31)$$

which is in agreement with the relationship noted in [22]. For comparison with lattice results and other works, we define the mean transverse momentum of a specific TMD function $F^{q/\rho}$ as

$$\langle k_\perp \rangle = \frac{\int dx d^2k_\perp |k_\perp| F^{q/\rho}(x, k_\perp)}{\int dx d^2k_\perp F^{q/\rho}(x, k_\perp)}. \quad (32)$$

The result is shown in Table I.

D. TMDs at low resolution

In Fig. 1, we show the rho TMD $f_1^{q/\rho}$ (22) (left column) and the rho TMD $f_{1LT}^{q/\rho}$ (24) (right column) in the two-body Fock space approximation, calculated by the ILM at a renormalization scale, or resolution, $\mu = 1/\rho$ with $\rho = 0.313$ fm. The rho LFWF parameters are $C_\rho = 1.9888$, $m_\rho = 791.0$ MeV, and $M = 398.17$ MeV. Figures 1(a) and 1(b) are the 2D density plots in $x, k_\perp$, and Figs. 1(c) and 1(d) their 3D visualization. The support of the TMDs in longitudinal momentum is dominant at $x \approx \frac{1}{2}$, with a width $\sqrt{\langle k_\perp^2 \rangle} \approx 0.1$ GeV. The profiles of the TMDs (22) and (24),

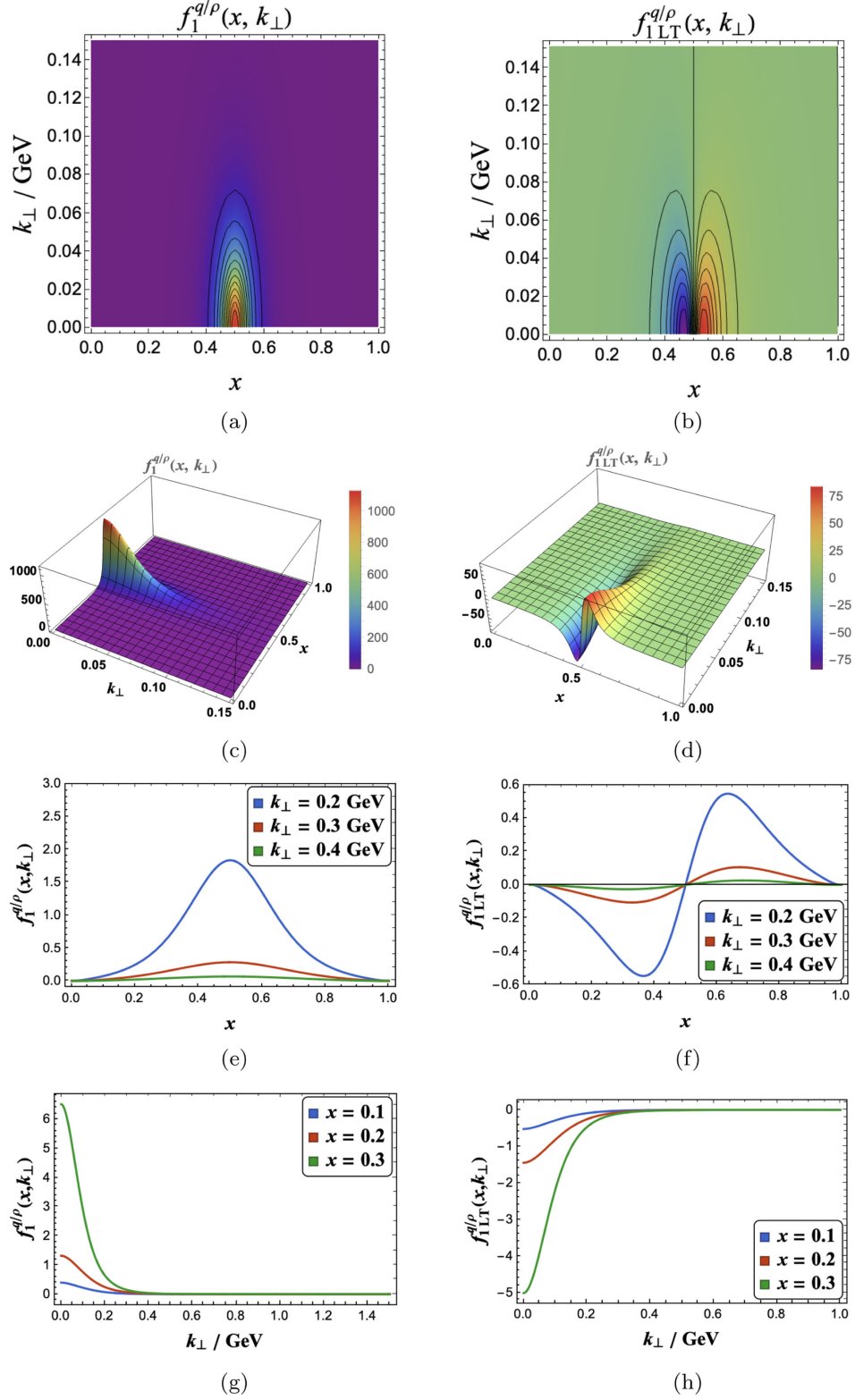


FIG. 1. Rho meson TMD $f_1^{q/\rho}$ (left) (22) and $f_{1LT}^{q/\rho}$ (right) (24) beam functions with only constituent quark contribution at low resolution $\mu = 1/\rho$ with $\rho = 0.313$ fm: (a), (b) are the density plots, (c), (d) the 3D plots, (e), (f) the longitudinal momentum dependence for fixed $k_\perp$, and (g), (h) the transverse momentum dependent plots for fixed x . The rho light front wave function (LFWF) parameters are $C_\rho = 1.9888$, $m_\rho = 791.0$ MeV, and $M = 398.17$ MeV.

in parton x for $k_\perp = 0.2, 0.3, 0.4$ GeV, are shown in Figs. 1(e) and 1(f), and the corresponding profiles in $k_\perp$ for $x = 0.1, 0.2, 0.3$ are shown in 1(g) and 1(h), respectively. For fixed $k_\perp$, the distribution in parton x of $f_1^{q/\rho}$ is symmetric, while that of $f_{1LT}^{q/\rho}$ is antisymmetric. Both distributions in $k_\perp$ for fixed parton x are Gaussian-like. This final feature arises from the profile of the quark zero modes in the ILM.

In Fig. 2 we show the rho TMD $f_{1LL}^{q/\rho}$ (23) (left column) and the rho TMD $f_{1TT}^{q/\rho}$ (25) (right column), in the two-body Fock space approximation, in the ILM at a resolution $\mu = 1/\rho$ with $\rho = 0.313$ fm. The rho LFWF parameters are $C_\rho = 1.9888$, $m_\rho = 791.0$ MeV, and $M = 398.17$ MeV. Figures 2(a) and 2(b) are the 2D density plots in $x, k_\perp$, and 2(c) and 2(d) their 3D visualization. The support of the TMDs in longitudinal momentum is dominant at $x \approx \frac{1}{2}$, with a width $\sqrt{\langle k_\perp^2 \rangle} \approx 0.1$ GeV. The profiles of the TMDs (23) and (25), in parton- x for $k_\perp = 0.2, 0.3, 0.4$ GeV, are shown in Figs. 2(e) and 2(f), and the corresponding profiles in $k_\perp$ for $x = 0.1, 0.2, 0.3$ are shown in 2(g) and 2(h), respectively. For fixed $k_\perp$, both distributions in parton x of $f_{LL}^{q/\rho}$ and $f_{TT}^{q/\rho}$ are symmetric, with the former displaying two zeros, an indication of a possible P-wave admixture. Both distributions in $k_\perp$ for fixed parton x are Gaussian-like, the feature introduced by the profile of the instanton zero modes.

In Fig. 3 we show the rho TMD $g_1^{q/\rho}$ (26) (left column) and the rho TMD $g_{1T}^{q/\rho}$ (27) (right column), in the two-body Fock space approximation, in the ILM at a resolution $\mu = 1/\rho$ with $\rho = 0.313$ fm. The rho LFWF parameters are $C_\rho = 1.9888$, $m_\rho = 791.0$ MeV, and $M = 398.17$ MeV. Figures 3(a) and 3(b) are the 2D density plots in $x, k_\perp$, and 3(c) and 3(d) their 3D visualization. The support of the TMDs in longitudinal momentum is dominant at $x \approx \frac{1}{2}$, with a wider spread in $\sqrt{\langle k_\perp^2 \rangle} \approx 0.1$ GeV. The profiles of the TMDs (26) and (27), in parton x for, are shown in, are shown in 3(e) and 3(f), and the corresponding profiles in $k_\perp$ for $x = 0.1, 0.2, 0.3$ are shown in 3(g) and 3(h), respectively. For fixed $k_\perp$, both distributions in parton x of $g_1^{q/\rho}$ and $g_{1T}^{q/\rho}$ are symmetric. Both distributions in $k_\perp$ for fixed parton x are Gaussian-like, due to the profile of the instanton zero modes.

In Fig. 4 we show the rho TMD $h_1^{q/\rho}$ (28) (left column) and the rho TMD $h_{1L}^{q/\rho}$ (29) (right column), calculated by the two-body Fock space approximation, in the ILM at a resolution $\mu = 1/\rho$ with $\rho = 0.313$ fm. The rho LFWF parameters are $C_\rho = 1.9888$, $m_\rho = 791.0$ MeV, and $M = 398.17$ MeV. Figures 4(a) and 4(b) are the 2D density plots in $x, k_\perp$, and 4(c) and 4(d) their 3D visualization. The support of the TMDs in longitudinal momentum is

dominant at $x \approx \frac{1}{2}$, with a wider spread in $\sqrt{\langle k_\perp^2 \rangle} \approx 0.1$ GeV. The profiles of the TMDs (28) and (29), in parton x for $k_\perp = 0.2, 0.3, 0.4$ GeV, are shown in 4(e) and 4(f), and the corresponding profiles in $k_\perp$ for $x = 0.1, 0.2, 0.3$ are shown in 4(g) and 4(h), respectively. For fixed $k_\perp$, both distributions in parton x of $h_1^{q/\rho}$ and $h_{1L}^{q/\rho}$ are symmetric. Both distributions in $k_\perp$ for fixed parton x are Gaussian-like due to the profile of the zero modes in the ILM.

E. Rho PDFs from TMDs

The integration over the transverse momentum $k_\perp$ dependence of the TMDs yields the PDFs. As we noted in the introduction and for vector mesons, we have four twist-2 PDFs, $f_1^{q/\rho}(x)$, $f_{1LL}^{q/\rho}(x)$, $g_1^{q/\rho}(x)$, and $h_1^{q/\rho}(x)$, given by [22]

$$\int d^2 k_\perp q_\rho(x, k_\perp) = f_1^{q/\rho}(x) + S_{LL} f_{1LL}^{q/\rho}(x) \quad (33)$$

$$\int d^2 k_\perp \Delta q_\rho(x, k_\perp) = \lambda s_L g_1^{q/\rho}(x) \quad (34)$$

$$\int d^2 k_\perp \delta q_\rho^\alpha(x, k_\perp) = \lambda s_\perp^\alpha h_1^{q/\rho}(x) \quad (35)$$

with

$$\begin{aligned} & [f_1^{q/\rho}, f_{1LL}^{q/\rho}, g_1^{q/\rho}, h_1^{q/\rho}](x) \\ &= \int d^2 k_\perp [f_1^{q/\rho}, f_{1LL}^{q/\rho}, g_1^{q/\rho}, h_1^{q/\rho}](x, k_\perp). \end{aligned} \quad (36)$$

For a spin-1 meson, the quark and antiquark distributions are related by charge conjugation symmetry:

$$\begin{aligned} & [f_1^{u/\rho^+}, f_{1LL}^{u/\rho^+}, g_1^{u/\rho^+}, h_1^{u/\rho^+}](x) \\ &= [f_1^{\bar{d}/\rho^+}, f_{1LL}^{\bar{d}/\rho^+}, g_1^{\bar{d}/\rho^+}, h_1^{\bar{d}/\rho^+}](1-x) \end{aligned} \quad (37)$$

and

$$\begin{aligned} & [f_1^{u/\rho^+}, f_{1LL}^{u/\rho^+}, g_1^{u/\rho^+}, h_1^{u/\rho^+}](x) \\ &= [f_1^{\bar{u}/\rho^-}, f_{1LL}^{\bar{u}/\rho^-}, g_1^{\bar{u}/\rho^-}, h_1^{\bar{u}/\rho^-}](x). \end{aligned} \quad (38)$$

For spin-1 mesons there are four PDFs. Compared to the three nucleon PDFs, the new parton distribution $f_{1LL}^{q/\rho}$ was noted originally by Hoodbhoy *et al.* [54]. This new PDF, also referred to as b_1^q structure function, represents the difference of unpolarized quark distributions in a longitudinally polarized spin-1 meson with spin projection $\lambda = 0$ and $\lambda = \pm 1$,

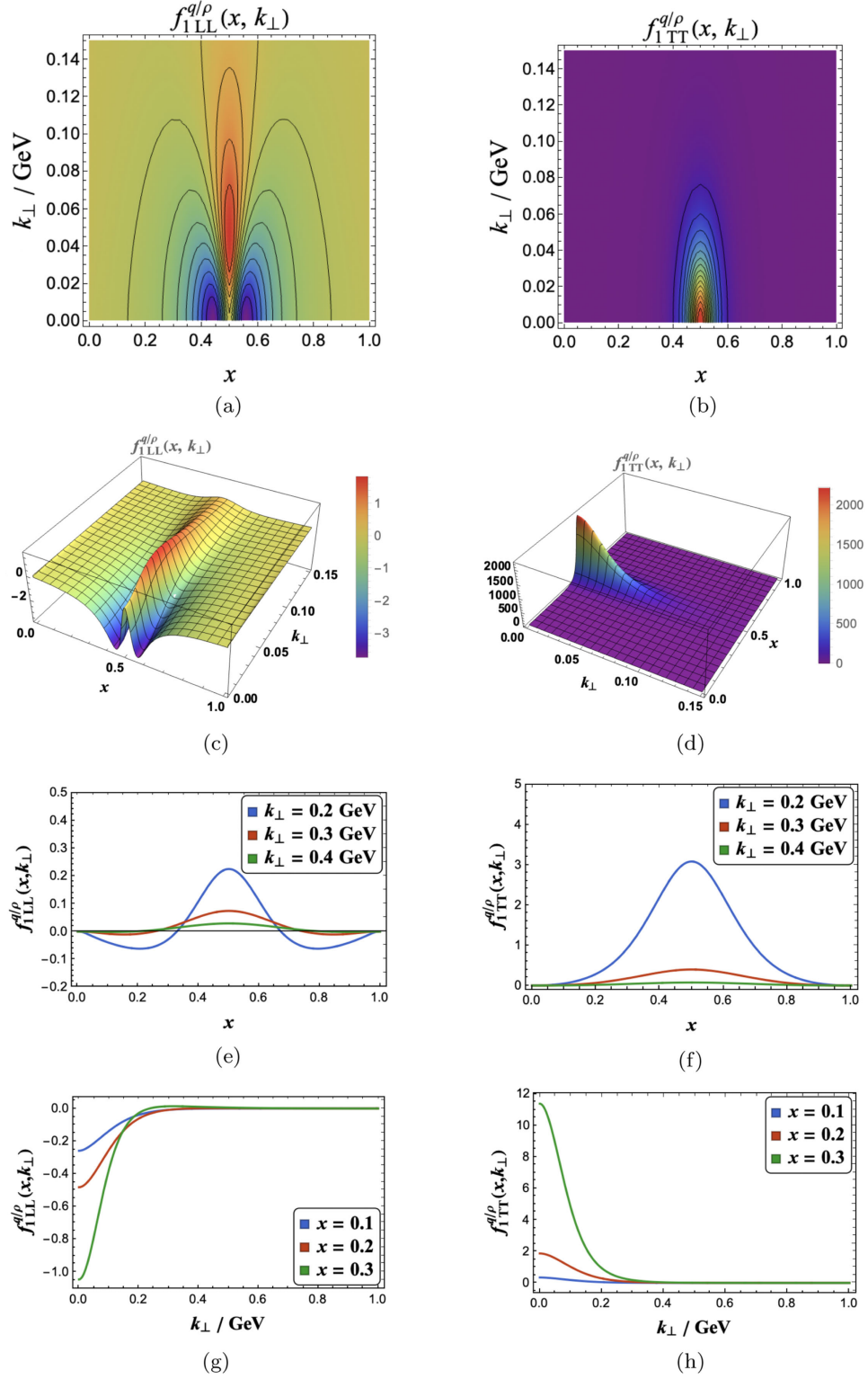


FIG. 2. Rho meson tensor-polarized TMD $f_{1LL}^{q/\rho}$ (left) (23) and $f_{1TT}^{q/\rho}$ (right) (25) beam functions with only constituent quark contribution at low resolution $\mu = 1/\rho$ with $\rho = 0.313$ fm: (a), (b) are the density plots, (c), (d) the 3D plots, (e), (f) the longitudinal momentum dependence for fixed $k_\perp$, and (g), (h) the transverse momentum dependent plots for fixed x . The rho LFWF parameters are $C_\rho = 1.9888$, $m_\rho = 791.0$ MeV, and $M = 398.17$ MeV.

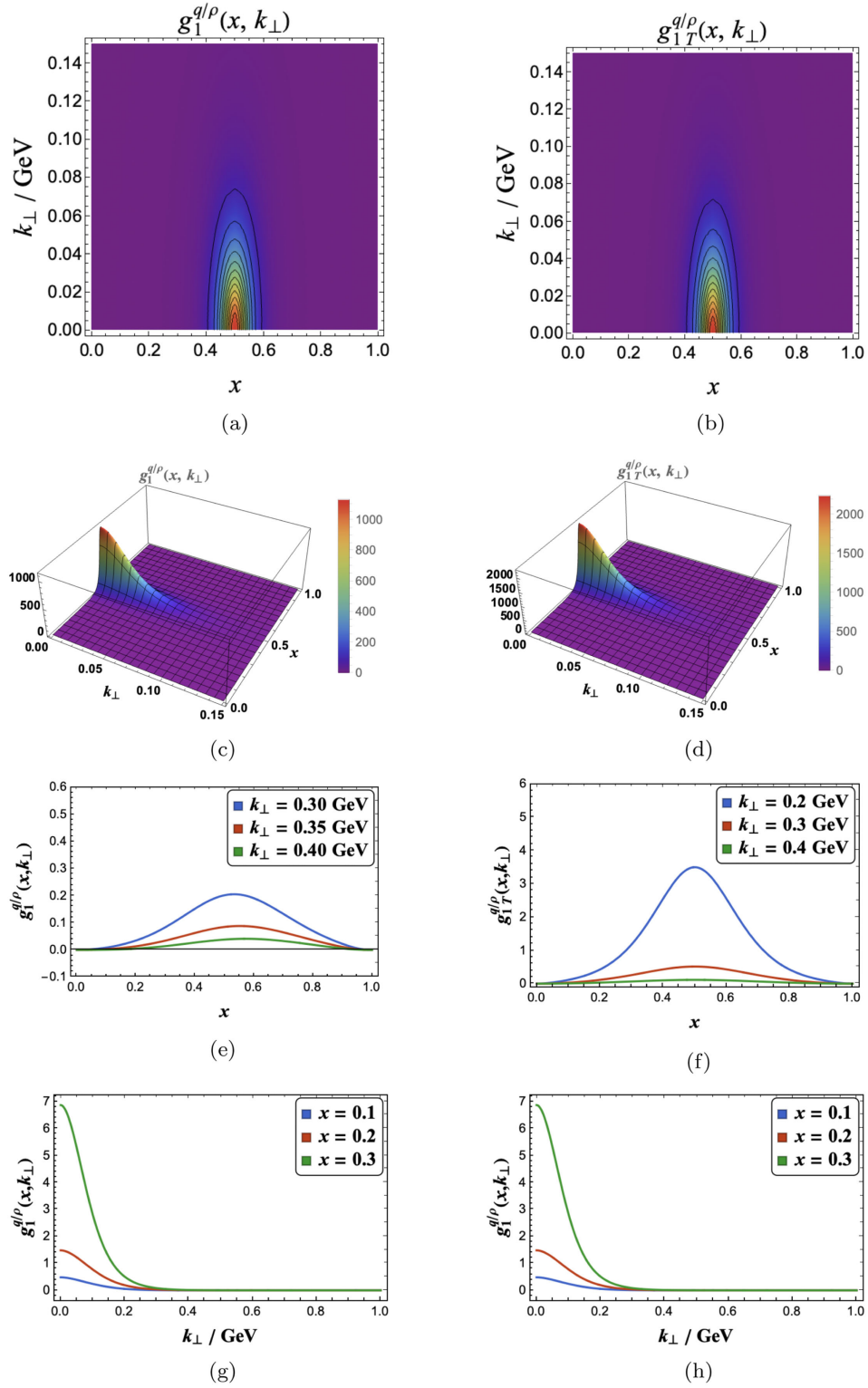


FIG. 3. Rho meson TMD $g_1^{q/\rho}$ (left) (26) and $g_{1T}^{q/\rho}$ (right) (27) beam functions with only constituent quark contribution at low resolution $\mu = 1/\rho$ with $\rho = 0.313$ fm: (a), (b) are the density plots, (c), (d) the 3D plots, (e), (f) the longitudinal momentum dependence for fixed $k_\perp$, and (g), (h) the transverse momentum dependent plots for fixed x . The rho LFWF parameters are $C_\rho = 1.9888$, $m_\rho = 791.0$ MeV, and $M = 398.17$ MeV.

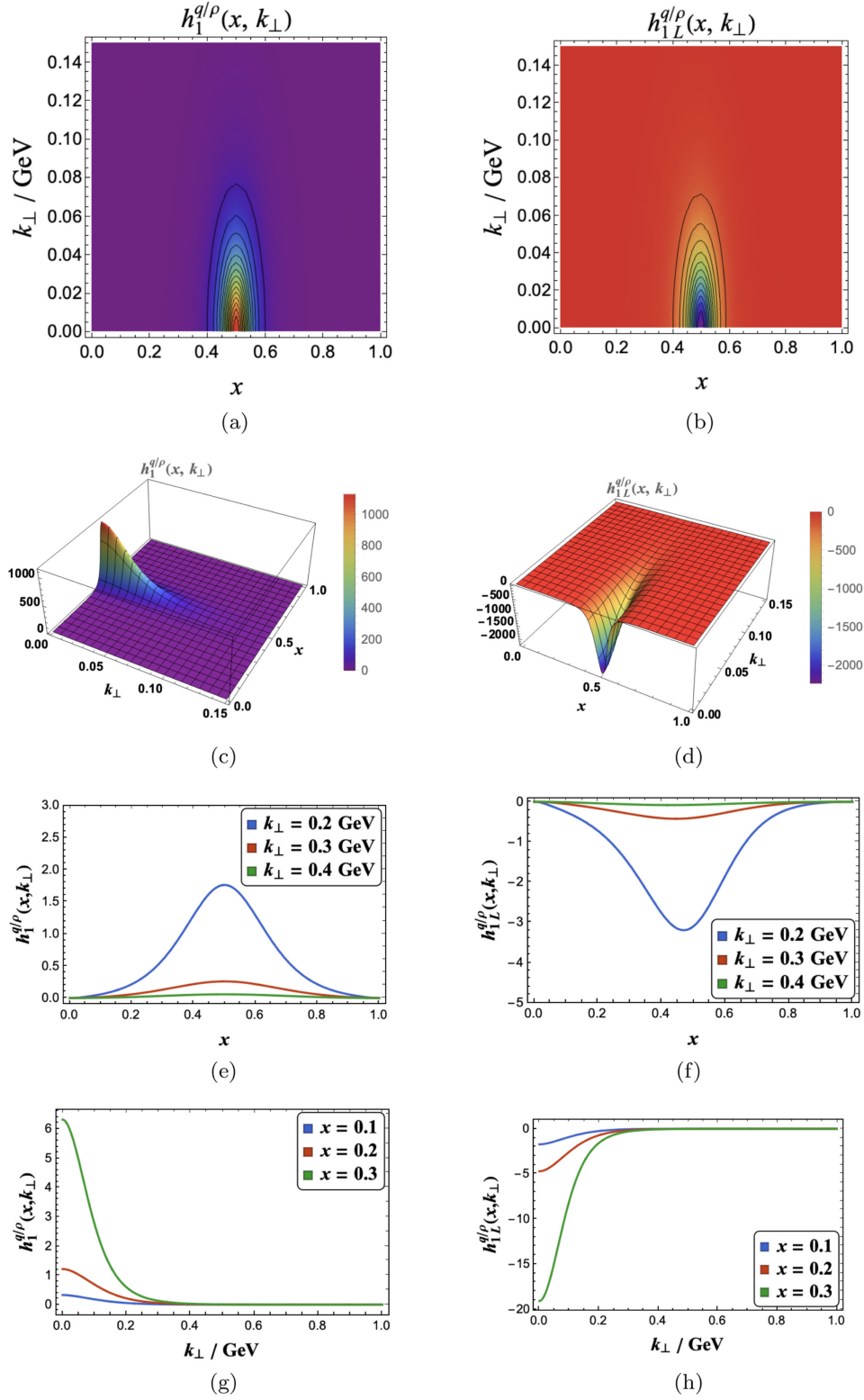


FIG. 4. Rho meson TMD $h_1^{q/\rho}$ (left) (28) and $h_{1L}^{q/\rho}$ (right) (29) beam functions with only constituent quark contribution at low resolution $\mu = 1/\rho$ with $\rho = 0.313$ fm: (a), (b) are the density plots, (c), (d) the 3D plots, (e), (f) the longitudinal momentum dependence for fixed $k_\perp$, and (g), (h) the transverse momentum dependent plots for fixed x . The rho LFWF parameters are $C_\rho = 1.9888$, $m_\rho = 791.0$ MeV, and $M = 398.17$ MeV.

TABLE II. Our instanton estimation on the first moment of unpolarized PDF $f_1^{q/\rho}$ is compared to the NJL model [22], and lattice calculation on the rho meson structure function with renormalization ≈ 2.4 GeV. We also evolve our result to 2.4 GeV by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation.

	NJL [22]	Lattice [57]	ILM ($\mu \sim 1/\rho$)	ILM ($\mu = 2.4$ GeV)
$\int dx x f_1^{q/\rho}(x)$	0.5	0.334(21)	0.5	0.201

$$\begin{aligned} \frac{1}{2} f_{1LL}^{q/\rho}(x) &\equiv b_1^{q/\rho}(x) \\ &= \frac{1}{4} \left[2q_{\rho}^{(\lambda=0)}(x) - q_{\rho}^{(\lambda=1)}(x) - q_{\rho}^{(\lambda=-1)}(x) \right]. \end{aligned} \quad (39)$$

This PDF is sensitive to the orbital motion of the constituents, and may also be carrying information on QCD exotica [55,56].

F. Rho sum rules

The standard parton distribution sum rule guarantees

$$\int_0^1 dx f_1^{q/\rho}(x) = 1. \quad (40)$$

The first moment of the unpolarized parton distribution, $f_1^{q/\rho}$, is about the longitudinal momentum carried by the quarks. In Table II we compare our results with those of other effective models, and selected lattice results.

For spin-1 mesons, there is an additional sum rule [58,59],

$$\int_0^1 dx x f_{1LL}^{q/\rho}(x) = 0, \quad (41)$$

which reflects on the fact that the valence quark number is independent of the hadron spin state. It was shown in [60] that Lorentz covariance is important for the enforcement of this sum rule, even for nonrelativistic constituents. The first moment of $f_{1LL}^{q/\rho}$ is compared to the lattice result from [57] at the resolution ≈ 2.4 GeV in Table III. If the quarks are in a relative S-wave state in the infinite momentum frame, its first moment is zero. The lattice results are evidence for an

TABLE III. Our instanton estimation on the first moment of tensor polarized PDF $f_{1LL}^{q/\rho}$ is compared to the NJL model [22], and lattice calculation on the rho meson structure function with renormalization ≈ 2.4 GeV. We also evolve our result to 2.4 GeV by the DGLAP equation.

	NJL [22]	Lattice [57]	ILM ($\mu \sim 1/\rho$)	ILM ($\mu = 2.4$ GeV)
$\int_0^1 dx x f_{1LL}^{q/\rho}(x)$	0	0.29(23)	0	0

TABLE IV. Table of the quark spin sum defined in (40). Our instanton estimation is compared to the NJL model [22], the Bethe-Salpeter equation (BSE) [51], and lattice calculation on the rho meson structure function with renormalization ≈ 2.4 GeV.

	NJL [22]	BSE [51]	Lattice [57]	ILM
Δq	0.56	0.67	0.570(32)	0.967

orbital contribution to the quark content [57]. The null result for our analysis in the ILM, suggests that a higher P-wave Fock contribution to the rho meson is needed, to account for the missing orbital contribution.

The quark spin sum (axial charge) for each quark in the meson state is defined as

$$\Delta q = \int_0^1 dx g_1^{q/\rho}(x). \quad (42)$$

The comparison of our ILM estimate to other models and selected lattice results is shown in Table IV. The results in the ILM show that the total valence quark and antiquark contribution to the spin of the ρ meson is 96.7%.

The tensor charge for each quark in the rho meson state is defined as

$$\delta q = \int_0^1 dx h_1^{q/\rho}(x). \quad (43)$$

The tensor charge is renormalization-scale dependent [61]. The one-loop renormalization group (RG) evolution reads

$$\delta q(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \right)^{\frac{C_F}{\beta_0}} \delta q(\mu_0). \quad (44)$$

The comparison of our ILM estimation to the other model is shown in Table V. Our instanton estimation shows the tensor charge of the ρ meson is $\delta q = 0.9807$ at original instanton scale $1/\rho \simeq 630$ MeV and $\delta q = 0.704$ at 2 GeV.

In Fig. 5 we show the rho meson PDF $f_{1LL}^{q/\rho}$ at the resolution $\mu \simeq 1/\rho$ in the ILM, and their evolved version at different resolutions $\mu = 1, 2, 3, 4$ GeV in 5(b). These rho PDFs are obtained before the soft subtraction to be discussed below. The rho meson PDFs $f_1^{q/\rho}$ (33), $g_1^{q/\rho}$ (34), and $h_1^{q/\rho}$ (35) are shown in Fig. 6 at the resolution $\mu \simeq 1/\rho$: (a) $f_1^{q/\rho}$, (b) $g_1^{q/\rho}$, and (c) $h_1^{q/\rho}$, compared to the

TABLE V. Table of the tensor charge defined in (42). Our instanton estimation is compared to the NJL model [22] and the Bethe-Salpeter equation (BSE) [51].

	NJL [22]	Lattice [57]	BSE [51]	ILM ($\mu \sim 1/\rho$)	ILM ($\mu = 2$ GeV)
δq	0.94	0.803	0.79	0.981	0.704

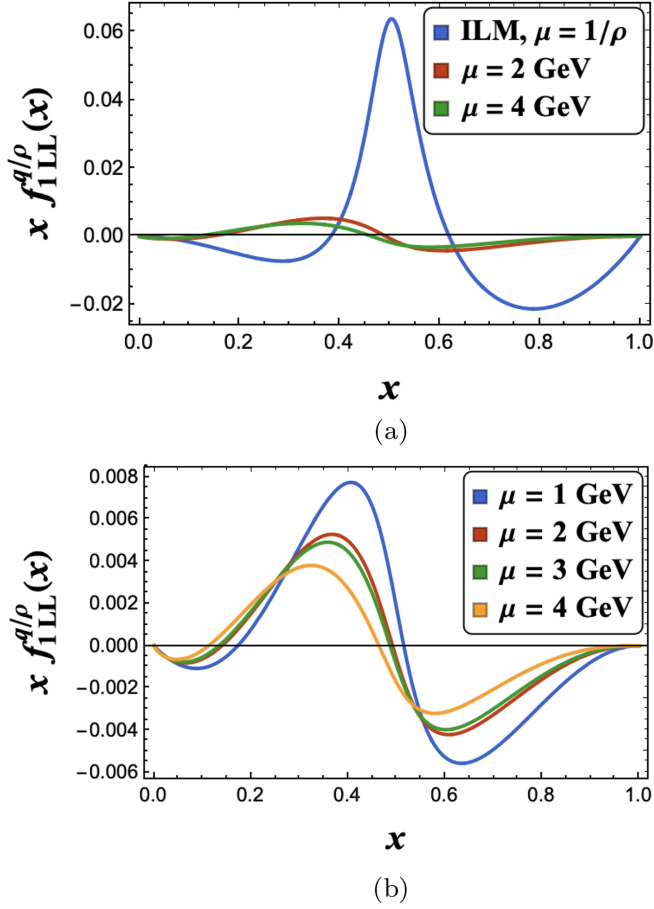


FIG. 5. The rho parton distribution $f_{1LL}^{q/p}$ at $\mu \simeq 1/\rho$ (a), and after DGLAP evolution (b).

results evolved to 2 and 4 GeV by DGLAP. Their other evolved results at different resolutions $\mu = 1, 2, 3, 4$ GeV are shown in (d), (e), and (f), respectively. Overall, there is a substantial shift of the evolved PDFs towards low x .

III. SOFT SUBTRACTION AND TMD EVOLUTION

A. Soft subtracted TMD distributions

The definition of TMD distributions as twist-2 matrix elements in (1) accounts for the transverse momentum dependent structure of a parton inside a hadron in the entire range of kinematics. In the following, we will denote them by $\tilde{F}^{q/p}(x, b_\perp, \mu, y_q - y_n)$, where the rapidity arises from the stapled Wilson lines. However, in TMD factorization, the definition in (1), which is referred to as unsubtracted TMD functions, often leads to an ambiguous overlap between the collinear and soft kinematic region. This leads to light-cone divergences that appear in the lightlike Wilson lines, regulated by y_n , in both the TMD distributions and soft factor. This is because, in the real world, these gluon exchanges are not distinguishable. Therefore, to disentangle their contributions in the factorization, we introduce rapidity regularization by implementing a soft subtraction.

For a generic unsubtracted rho meson TMD, say $\tilde{F}^{q/p}(x, b_\perp, \mu, y_q - y_n)$, defined in (1), the soft subtracted TMDs in $b_\perp$ follow through,

$$\tilde{F}^{q/p}(x, b_\perp; \mu, \zeta) = \frac{\tilde{F}^{q/p}(x, b_\perp, \mu, y_q - y_n)}{S(b_\perp, \mu, y - y_n)}, \quad (45)$$

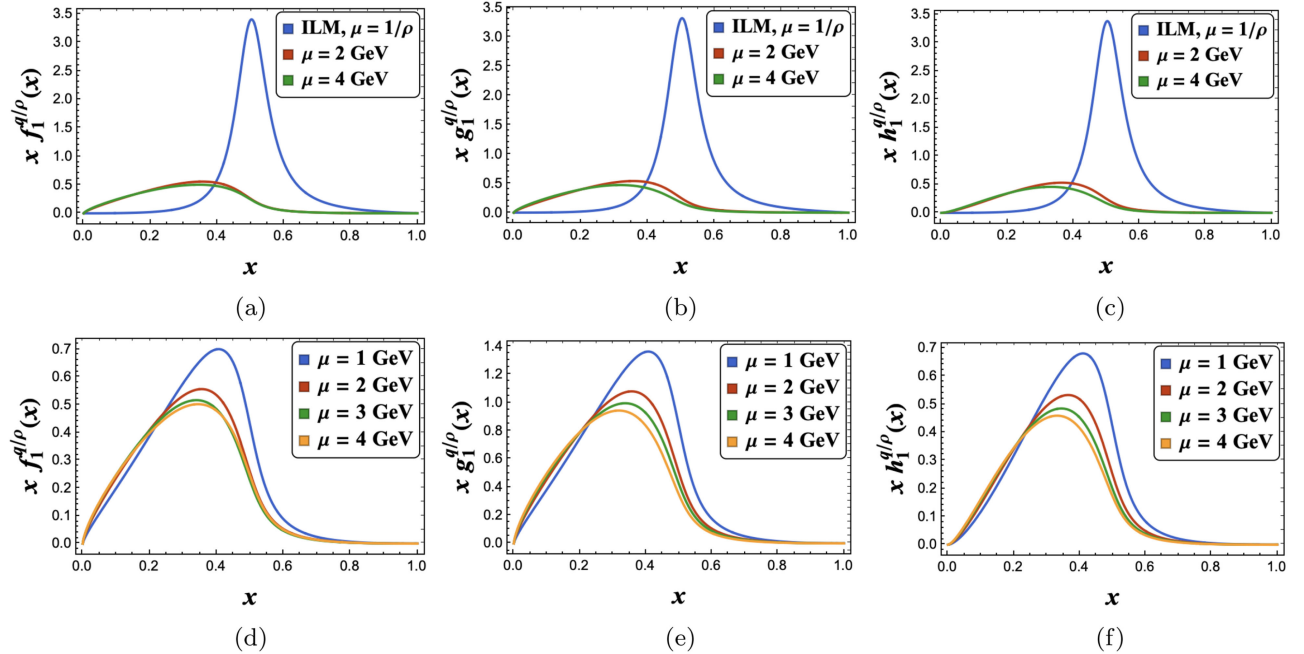


FIG. 6. The rho parton distributions at $\mu \simeq 1/\rho$: (a) $f_1^{q/p}$, (b) $g_1^{q/p}$, and (c) $h_1^{q/p}$ compared to their DGLAP evolved counterparts (d), (e), and (f).

where $y_q = \frac{1}{2} \ln(k^+/k^-)$ is the hadron rapidity and $y_n = \frac{1}{2} \ln(n^+/n^-)$ is the rapidity for the near-light-cone direction n of Wilson line in the hadronic matrix element definition (1). This direction n usually refers to the moving direction of the other hadron in the process. The rapidity scale

$$\zeta = 2(k^+)^2 e^{-y}$$

represents the energy of the hadron with a rapidity regulator y , distinguishing the energy between two involving hadrons in the TMD factorization process, and $\zeta_\rho \zeta_h = Q^4$, where ζ_h is the rapidity scale of the other involved hadron h . The soft factor $S(b_\perp, \mu, y_n - y_{\bar{n}})$ is defined as [12,62,63]

$$S(b_\perp, \mu, y_n - y_{\bar{n}}) = \frac{1}{N_c} \text{Tr} \langle W_{\pm\bar{n}}(0) W_{\pm n}^\dagger(0) W_{\pm n}(b_\perp) W_{\pm\bar{n}}^\dagger(b_\perp) \rangle. \quad (46)$$

B. TMD evolution

For any kind of leading-twist TMD functions with soft subtraction $\tilde{F}^{q/\rho}$ in $b_\perp$ space, the Collins-Soper-Sterman

(CSS) renormalization group equations reads [64,65]

$$\begin{aligned} \frac{d}{d \ln \sqrt{\zeta}} \ln \tilde{F}^{q/\rho}(x, b_\perp; \mu, \zeta) &= K_{\text{CS}}(b_\perp, \mu) \\ \frac{d}{d \ln \mu} \ln \tilde{F}^{q/\rho}(x, b_\perp; \mu, \zeta) &= \gamma_F(\alpha_s(\mu)) \\ &\quad - \Gamma_{\text{cusp}}(\alpha_s(\mu)) \ln \left(\frac{\zeta}{\mu^2} \right) \end{aligned} \quad (47)$$

with the Collin-Sober kernel K_{CS} , also known as the rapidity anomalous dimension that governs the rapidity ζ scale evolution. The anomalous dimension γ_F depends on the quark spin sources $\gamma^+, \gamma^+ \gamma^5, i\sigma^{\alpha+} \gamma^5$. The perturbative expansions of the anomalous dimension functions Γ_{cusp} and γ_F , are given in Appendix G.

The TMDs involve two evolution scales, μ , ζ , and one intrinsic transverse scale, $b_\perp$. Their evolution is driven by both perturbative and nonperturbative kernels [19,66–69]. For simplicity, we can choose a simple evolution scheme, with $\zeta = \mu^2$. In the perturbative region $b_\perp \ll b_{\text{NP}}$, the transverse scale of the TMD at high resolution $\mu \gg 1/\rho$ is completely governed by the perturbative analysis, with the result

$$\begin{aligned} \tilde{F}^{q/\rho}(x, b_\perp; \mu, \mu^2) \big|_{b \ll b_{\text{NP}}} &\simeq \sum_{q'=q, \bar{q}, g} \int_x^1 \frac{dx'}{x'} C_{q/q'}^f(x/x', b_\perp, \mu_b) f^{q'/\rho}(x', \mu_b) \\ &\times \exp \left[- \int_{\mu_b}^\mu \frac{d\mu'}{\mu'} \left[\Gamma_{\text{cusp}}(\alpha_s(\mu'^2)) \ln \left(\frac{\mu^2}{\mu'^2} \right) - \gamma_F(\alpha_s(\mu'^2)) \right] + K_{\text{CS}}^{(\text{pert})}(b_\perp, \mu_b) \ln \left(\frac{\mu}{\mu_b} \right) \right]. \end{aligned} \quad (48)$$

Here $f^{q/\rho}(x, \mu_b) = \int d^2 k_\perp F^{q/\rho}(x, k_\perp)$ is the corresponding leading-twist collinear PDF, namely $f_1^{q/\rho}, f_{1LL}^{q/\rho}, g_1^{q/\rho}, h_1^{q/\rho}$, at μ_b scale defined as

$$\mu_b = 2e^{-\gamma_E}/b_\perp. \quad (49)$$

The convolution is the operator product expansion (OPE), which describes the small- $b_\perp$ behavior of the

TMDs in terms of the collinear PDFs $f^{q/\rho}$, convoluted with perturbative Wilson coefficients $C_{i/j}^f$, where f denotes the type of the collinear PDF. The details are briefly given in Appendix G.

In the nonperturbative region $b_\perp \gg b_{\text{NP}}$, the instanton and anti-instanton fields contribute. The corresponding TMD reads

$$\tilde{F}^{q/\rho}(x, b_\perp; \mu, \mu^2) \big|_{b \gg b_{\text{NP}}} \simeq \tilde{F}^{q/\rho}(x, b_\perp; \mu_0, \mu_0^2) e^{K_{\text{CS}}^{(\text{inst})}(b_\perp/\rho) \ln \sqrt{\frac{\mu^2}{\mu_0^2}}} \exp \left[\int_{\mu_0}^\mu \frac{d\mu'}{\mu'} \left(\gamma_F(\alpha_s(\mu'^2)) - \Gamma_{\text{cusp}}(\alpha_s(\mu'^2)) \ln \left(\frac{\mu_0^2}{\mu'^2} \right) \right) \right], \quad (50)$$

where the low resolution TMD in the ILM is defined by

$$\tilde{F}^{q/\rho}(x, b_\perp; \mu_0, \zeta_0) = \left[\int d^2 k_\perp e^{ik_\perp \cdot b_\perp} F_{(\text{inst})}^{q/\rho}(x, k_\perp) \right] e^{K_{\text{CS}}^{(\text{inst})}(b_\perp/\rho) \ln \sqrt{\frac{\zeta_0}{1/\rho^2}}}. \quad (51)$$

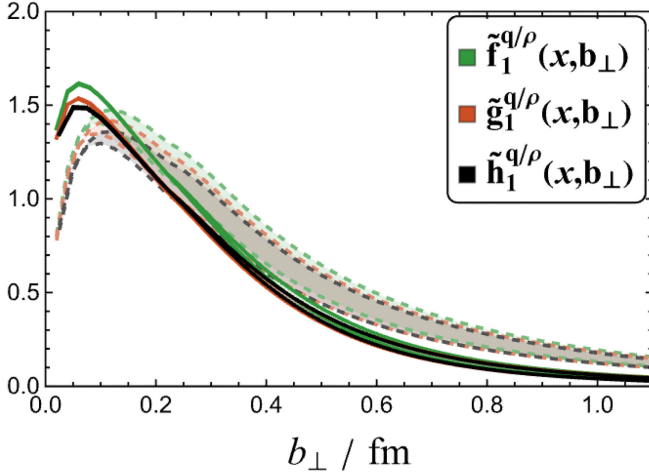


FIG. 7. Soft subtracted unpolarized $\tilde{f}_1^{q/\rho}$ (green), spin-dependent TMD distributions $\tilde{g}_1^{q/\rho}$ (red), and $\tilde{h}_1^{q/\rho}$ (black) at two different high resolutions, $\mu = 2$ GeV (dashed lines) and $\mu = 10$ GeV (solid lines), using (52), with the bands corresponding to the fitting parameter $b_{\text{NP}} = 1.3\text{--}2.0$ GeV $^{-1}$ [12].

The unsubtracted TMD at low resolution is composed of the overlap of the LFWFs $F_{(\text{inst})}^{q/\rho}(x, k_\perp)$ in (15) and a rapidity factor $\exp(K_{\text{CS}}^{(\text{inst})} \ln \sqrt{\xi_0/x^2 m_\rho^2})$ from the stapled Wilson line. Thus, the low energy unsubtracted TMD can be fully calculated in the instanton liquid model.

The evolution now is separated into a perturbative and nonperturbative region, where the cutoff separating the two regions is set as

$$b_{\text{NP}} = \frac{2e^{-\gamma_E}}{\mu_0} = 1.5 \text{ GeV}^{-1},$$

where $\mu_0 \sim 1/\rho$ is the nonperturbative scale fixed in the ILM. The full TMD at high resolution $\mu \gg 1/\rho$ interpolates between the large $b_\perp$ and small $b_\perp$, by optimally matching the linear combination

$$\begin{aligned} \tilde{F}^{q/\rho}(x, b_\perp; \mu, \mu^2) &= \tilde{F}^{q/\rho}(x, b_\perp \ll b_{\text{NP}}; \mu, \mu^2) e^{-b_\perp/b_{\text{NP}}} \\ &+ \tilde{F}^{q/\rho}(x, b_\perp \gg b_{\text{NP}}; \mu, \mu^2) \\ &\times (1 - e^{-b_\perp/b_{\text{NP}}}) \end{aligned} \quad (52)$$

with b_{NP} the optimization parameter.

In Fig. 7 we show the unpolarized $\tilde{f}_1^{q/\rho}$ (green), spin-dependent $\tilde{g}_1^{q/\rho}$ (red), and $\tilde{h}_1^{q/\rho}$ (black) TMDs at the resolution $\mu = 2$ GeV (solid lines) compared to a higher resolution $\mu = 10$ GeV (dashed lines). The results follow from (52) with the matching parameter in the range $b_{\text{NP}} = 1.3\text{--}2.0$ GeV $^{-1}$ [12]. In Fig. 8 we show the separate results for the corresponding TMDs (a) $\tilde{f}_1^{q/\rho}$, (b) $\tilde{g}_1^{q/\rho}$, and (c) $\tilde{h}_1^{q/\rho}$ at $x = 0.3$, with the pQCD results (green), the ILM (red), and combined result (black) with a band corresponding to the matching parameter in the range $b_{\text{NP}} = 1.3\text{--}2.0$ GeV $^{-1}$ at high resolution $\mu = 2$ GeV with $\zeta = \mu^2$. Only the small- $b_\perp$ perturbative contribution inside green band ($b_\perp = 0.56\text{--}1.5$ GeV $^{-1}$) is perturbatively reliable. The smaller region ($b_\perp \leq 1/\mu$) suffers the sensitivity to the UV regularization scheme and uncertainty from the large logarithm. The phenomenological prescriptions to this region have been proposed in [8,9,70,71] (and references therein). On the other hand, the large $b_\perp$ TMD is dominated by ILM. The large $b_\perp$ of perturbative contribution in Fig. 8 is simply modified by b_* prescription [72] such that the saturation occurs to avoid the Landau pole Λ_{QCD} .

IV. CONCLUSIONS

We have developed an analysis of the leading-twist TMDs of the spin-1 rho meson, using the ILM of the QCD vacuum. By making use of the most general spin quantization, we derived all nine T-even TMDs for the rho meson, and their corresponding reductions to PDFs. The analysis uses a leading Fock approximation for the longitudinal and transverse rho meson light front wave functions. $O(3)$ symmetry is enforced through the proper counting rule of instanton density n_{I+A} in the ILM, at low resolution. Lorentz covariance is dynamically restored,

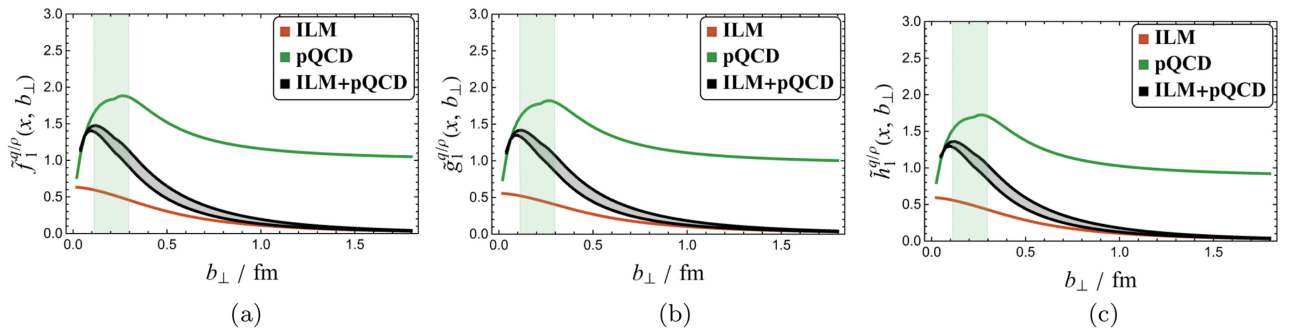


FIG. 8. The unpolarized TMD distributions (a) $\tilde{f}_1^{q/\rho}$, and spin-dependent TMDs (b) $\tilde{g}_1^{q/\rho}$, (c) $\tilde{h}_1^{q/\rho}$ are shown at the resolution $\mu = 2$ GeV for also $x = 0.3$ with black bands corresponding to the interpolating parameter $b_{\text{NP}} = 1.3\text{--}2.0$ GeV $^{-1}$ (52). The green interval corresponds to the perturbatively reliable region $2e^{-\gamma_E}/\mu < b_\perp < 1.5$ GeV $^{-1}$.

with the momentum sum rules for the unpolarized and tensor polarized PDFs satisfied in the ILM.

Before evolution and at the ILM resolution $\mu = 1/\rho$, the $h_{1T}^{q/\rho}$ rho TMD is found to be null in the leading Fock approximation. The remaining and nonvanishing T-even rho TMDs show a Gaussian-like behavior in $k_\perp$ for fixed parton x . This feature follows from a profiling of the zero modes in the ILM, and carries to the rest of the TMDs. The $f_{1TT}^{q/\rho}$ rho TMD is found to be antisymmetric with respect to the $x = \frac{1}{2}$ -line, while the $f_{1LL}^{q/\rho}$ vanishes around two symmetric lines about $x = \frac{1}{2}$, a possible sensitivity to a small P-wave contribution in the spin-1 state, in the two-body Fock space approximation. At low resolution, the rho tensor charge is about 1, in agreement with the constituent quark model, or NJL.

Following our recent analysis of the pion and kaon TMDs [11], we have suggested that the rho TMDs in the ILM at low resolution should be supplemented with the pertinent CS kernels $K_{CS}^{(inst)}$ to allow for evolution at higher resolution and rapidities. The evolved TMDs at higher resolutions and rapidities show substantial shift to lower parton x . Interpolating the pQCD results at small $b_\perp$ and the ILM results at large $b_\perp$ for the rho TMDs allows the extraction of the TMDs in transverse space for fixed parton x .

The rho meson TMDs integrated over $k_\perp$ yield its four PDFs. The resulting rho tensor is about 1 at low resolution, which is consistent with the constituent quark model, as well as lattice [57] and model estimate [22]. The quark spin sum or axial charge is also about 1 at low resolution, in contrast to lattice [57] and model estimates [22] which puts it near $\frac{1}{2}$. This is notable as it implies consistency with the constituent quark model for a rho meson in the ILM. This may not be surprising, given the $LL + RR$ character of this meson, which does not mix with the LR fluctuations strongly induced by the instantons and anti-instantons (but mix with their topologically trivial molecules), which are a major source of intrinsic spin in the pion and nucleon. This point deserves further investigations, namely on a lattice with different resolutions.

Indeed, the emergent tomographic picture of the rho meson at different resolutions is encouraging for future comparison with lattice simulations, assuming that the rho meson state can be numerically separated from the mixing with the low-lying multipion states. Using the gradient flow technique, a direct comparison to our TMDs at low resolution would be very useful.

Finally, the extension of the present analysis to include the four-Fock space contribution with possible P-wave admixing may prove important in understanding the possible role of the orbital contribution in the rho meson TMDs.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: POLARIZATION AND SPIN FOUR-VECTORS

For a spin- j particle with a mass m in the rest frame, the spin operator $\vec{S}$ is defined by a $(2j+1) \times (2j+1)$ matrix in the irreducible representation of the spin algebra, with the unit vector $\vec{s}$ as the (arbitrary) spin quantization axis. In a boosted frame, the rest frame four-vector $(0, \vec{s})$ transforms to

$$S^\mu = \left(\frac{\vec{p} \cdot \vec{s}}{m}, \vec{s} + \frac{(\vec{p} \cdot \vec{s})\vec{p}}{m(E_{\vec{p}} + m)} \right) \quad (A1)$$

with the condition $p \cdot S = 0$ and $S^2 = -1$ alongside. The spin operator $\vec{S}$ is then the Pauli-Lubanski pseudovector Σ^μ :

$$\Sigma^\mu = \frac{1}{2m} \epsilon^{\mu\nu\rho\sigma} p_\nu J_{\rho\sigma}, \quad (A2)$$

where $J_{\mu\nu}$ are the generators of the Lorentz group.

For a spin-1 particle with a mass m_ρ , the spin operator is a 3×3 matrix with entries $(\Sigma^k)_{ij} = -i\epsilon_{kij}$, defining the spin-1 eigenstates $\vec{e}_\lambda$ by

$$-is^k \epsilon_{kij} e_\lambda^j = \lambda e_\lambda^i \quad (A3)$$

with the polarization $\lambda = 0, \pm 1$. If we set the spin quantization axis in the z direction, the polarization vectors are $\vec{e}_\pm = (1, \pm i, 0)/\sqrt{2}$ and $\vec{e}_0 = (0, 0, 1)$.

The covariant generalization of (A3) is

$$\frac{i}{m_\rho} \epsilon_{\mu\nu\rho\sigma} S^\mu p^\nu \epsilon_\lambda^\sigma(p) = \lambda \epsilon_\lambda^\rho(p) \quad (A4)$$

with a four-vector generalization of the spin-1 eigenstate $\vec{e}_\lambda$. The polarization four-vectors of a spin-1 particle are given by

$$\epsilon_\lambda^\mu(p) = \left(\frac{\vec{p} \cdot \vec{\epsilon}_\lambda}{m_\rho}, \vec{\epsilon}_\lambda + \frac{\vec{p}(\vec{p} \cdot \vec{\epsilon}_\lambda)}{m_\rho(E_{\vec{p}} + m_\rho)} \right) \quad (\text{A5})$$

with the Ward identity $p \cdot \epsilon_\lambda(p) = 0$ and normalization condition $\epsilon_\pm^* \cdot \epsilon_\pm = \epsilon_0^* \cdot \epsilon_0 = -1$. The following relation between the polarization and spin four-vectors (A8) is then

$$\frac{i}{m_\rho} \epsilon^{\mu\nu\rho\sigma} p_\nu \epsilon_{\lambda\rho}^*(p) \epsilon_{\lambda\sigma}(p) = \lambda S^\mu. \quad (\text{A6})$$

It is convenient to recast the covariant form of the spin-1 density matrix $\epsilon_\lambda^i \epsilon_\lambda^{j*}$ for pure spin states with some fixed polarization λ , using the spin and momentum four-vectors. The covariant spin density matrix with a fixed λ pure state can be expressed as

$$\begin{aligned} \epsilon_\lambda^{\mu*}(p) \epsilon_\lambda^\nu(p) = & \frac{1}{3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m_\rho^2} \right) - \frac{i\lambda}{2m_\rho} \epsilon^{\mu\nu\rho\sigma} p_\rho S_\sigma \\ & - \frac{3\lambda^2 - 2}{2} \left[S^\mu S^\nu - \frac{1}{3} \left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{m_\rho^2} \right) \right]. \end{aligned} \quad (\text{A7})$$

The identity is achieved by decomposing the spin-1 density matrix into symmetric traceless and antisymmetric Lorentz tensors and trace using (A6) and the completeness relation of polarization vector $\epsilon_\lambda^\mu(p)$ [73,74].

For simplicity, here we choose the three-momentum of the target along the z direction ($p_\perp = 0$) and denote the Cartesian components of the spin quantization unit vector by $\vec{s} = (s_\perp, s_L)$, where $s_\perp = (s^1, s^2)$ is normal to the hadron target momentum and s_L is the z component. In the light-cone signature, the spin vector can be written as

$$S^\mu = s_L \frac{p^+}{m_\rho} \bar{n}^\mu - s_L \frac{m_\rho}{2p^+} n^\mu + s_\perp^\mu, \quad (\text{A8})$$

where s_L can be viewed as the target averaged helicity and the transverse spin is defined by $s_\perp^2 = 1 - s_L^2$.

Generally, for any given direction $\vec{S}$, the spin of the particle can be projected onto this direction. We note that longitudinal polarization means that $s_\perp = 0$ and $|s_L| = 1$. There are $2j + 1$ spin projections for a spin- j particle along the direction of the momentum. Transverse polarization means that $s_L = 0$ and $|s_\perp| = 1$, and that there are $2j + 1$ spin projections along the direction $s_\perp$ perpendicular to the momentum.

APPENDIX B: SPIN-DEPENDENT WAVE FUNCTIONS ON THE LIGHT FRONT

Throughout this paper, our conventions of the light front frame follows Kogut-Soper convention based on the Weyl chiral basis of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \quad (\text{B1})$$

The light front components are normalized to

$$\gamma^\pm = \frac{\gamma^0 \pm \gamma^3}{\sqrt{2}} \quad (\text{B2})$$

with the light front projectors defined as

$$\mathcal{P}_+ = \frac{1}{2} \gamma^- \gamma^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{B3})$$

and $\mathcal{P}_- = 1 - \mathcal{P}_+$.

The spin-dependent wave functions denote the spin states in the creation of a quark-antiquark pair. The spin wave functions for each channel can be computed by the light front spinors for the quarks and antiquarks. The spinors are defined as

$$u_s(k) = \frac{1}{\sqrt{\sqrt{2}k^+}} (\gamma^- k^+ - \gamma_\perp \cdot k_\perp + M) \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix} \quad (\text{B4})$$

and

$$v_s(k) = \frac{1}{\sqrt{\sqrt{2}k^+}} (\gamma^- k^+ - \gamma_\perp \cdot k_\perp - M) \begin{pmatrix} -\eta_s \\ \eta_s \end{pmatrix}, \quad (\text{B5})$$

respectively, with ξ_s (η_s), a quark two-spinor (antiquark two-spinor) with a spin pointing in the z direction, and M the constituent quark mass. The antiquark two-spinor is related to a quark two-spinor by charge conjugation $\eta_s = i\sigma_2 \xi_s^* = -2s \xi_{-s}$. The corresponding ρ meson spin wave functions with a meson momentum p along z direction read

$$\begin{aligned} & \bar{u}_{s_1}(k_1) \frac{\gamma^1 + i\gamma^2}{\sqrt{2}} v_{s_2}(k_2) \\ &= \frac{-1}{\sqrt{x\bar{x}}} \xi_{s_1}^\dagger \left[\sqrt{2} M \sigma^+ + \left(\bar{x} \frac{1 + \sigma_z}{\sqrt{2}} + x \frac{1 - \sigma_z}{\sqrt{2}} \right) k_R \right] \eta_{s_2} \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} & \bar{u}_{s_1}(k_1) \frac{\gamma^1 - i\gamma^2}{\sqrt{2}} v_{s_2}(k_2) \\ &= \frac{-1}{\sqrt{x\bar{x}}} \xi_{s_1}^\dagger \left[\sqrt{2} M \sigma^- - \left(\bar{x} \frac{1 - \sigma_z}{\sqrt{2}} + x \frac{1 + \sigma_z}{\sqrt{2}} \right) k_L \right] \eta_{s_2} \end{aligned} \quad (\text{B7})$$

$$\bar{u}_{s_1}(k_1)\gamma^+v_{s_2}(k_2) = 2\sqrt{x\bar{x}}p^+\xi_{s_1}^\dagger\sigma_z\eta_{s_2} \quad (\text{B8})$$

$$\bar{u}_{s_1}(k_1)\gamma^-v_{s_2}(k_2) = -\sqrt{x\bar{x}}\left(\frac{k_\perp^2 + M^2}{x\bar{x}p^+}\right)\xi_{s_1}^\dagger\sigma_z\eta_{s_2}, \quad (\text{B9})$$

where $\sigma^\pm = (\sigma^1 \pm i\sigma^2)/2$ and $k_{L,R} = k_\perp^1 \pm ik_\perp^2$.

APPENDIX C: SUMMARY OF THE RHO WAVE FUNCTIONS IN THE ILM

The twist-2 distribution amplitude (DA) for the longitudinally polarized rho meson is

$$\varphi_{\rho,\parallel}(x) = \frac{1}{f_\rho} \int_{-\infty}^{+\infty} \frac{dz^-}{2\pi} e^{ixp^+z^-} \times \langle 0 | \bar{q}(0) \frac{\tau^a}{2} \gamma^+ W[0, z^-] q(z^-) | \rho(\lambda = 0) \rangle. \quad (\text{C1})$$

The twist-3 distribution amplitude (DA) for the transversely polarized rho meson is

$$\varphi_{\rho,\perp}(x) \epsilon_\perp^i(p) = \frac{p^+}{m_\rho f_\rho} \int_{-\infty}^{+\infty} \frac{dz^-}{2\pi} e^{ixp^+z^-} \times \langle 0 | \bar{q}(0) \frac{\tau^a}{2} \gamma_\perp^i W[0, z^-] q(z^-) | \rho(\lambda = \pm 1) \rangle. \quad (\text{C2})$$

Here $f_\rho \approx 155$ MeV [75] is the rho electroweak decay constant. The explicit form of the rho DAs in the ILM read [32]

$$\varphi_{\rho,\parallel}(x) = \frac{\sqrt{N_c} C_{\rho_L}}{m_\rho f_\rho 8\pi^2} \theta(x\bar{x}) \int_0^\infty dk_\perp^2 \frac{4(k_\perp^2 + M^2)}{k_\perp^2 + M^2 - x\bar{x}m_\rho^2} \times \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right) \quad (\text{C3})$$

$$\varphi_{\rho,\perp}(x) = \frac{\sqrt{N_c} C_{\rho_T}}{m_\rho f_\rho 8\pi^2} \frac{\theta(x\bar{x})}{x\bar{x}} \int_0^\infty dk_\perp^2 \frac{(1 - 2x\bar{x})k_\perp^2 + M^2}{k_\perp^2 + M^2 - x\bar{x}m_\rho^2} \times \mathcal{F}^2\left(\frac{k_\perp}{\lambda_\rho \sqrt{x\bar{x}}}\right). \quad (\text{C4})$$

Here $\bar{x} = 1 - x$ and the unit step function $\theta(x\bar{x}) = 1$ constrains the DAs to the physical domain $0 < x < 1$. The parameter $\lambda_\rho = 1.260$ is chosen so that the DAs are properly normalized at $\mu = 1/\rho$, for a fixed decay constant f_ρ .

APPENDIX D: DGLAP EVOLUTION

The quark distribution $q(x)$ and gluon distribution $g(x)$ evolves with energy scale,

$$\begin{aligned} \frac{dq(x)}{d\ln\mu^2} &= \int_x^1 \frac{dx'}{x'} P_{qq'}\left(\frac{x}{x'}\right) q(x') + P_{qg}\left(\frac{x}{x'}\right) g(x') \\ \frac{dg(x)}{d\ln\mu^2} &= \int_x^1 \frac{dx'}{x'} P_{gq'}\left(\frac{x}{x'}\right) q(x') + P_{gg}\left(\frac{x}{x'}\right) g(x'). \end{aligned} \quad (\text{D1})$$

Partons q or g come out from the hadronic bound state carrying the longitudinal momentum $k^+ = x'p^+$ and split into a parton pair carrying momentum $(x/x')k^+$ and $(1 - x/x')k^+$, respectively. The evolution kernel can be written in terms of a perturbative expansion,

$$p_{ij}(x) = \sum_n \left(\frac{\alpha_s(\mu)}{4\pi} \right)^{n+1} p_{ij}^{(n)}(x). \quad (\text{D2})$$

For the parton distribution $q(x)$ and $g(x)$, the splitting functions are defined as

$$\begin{aligned} p_{qq}^{(0)}(x) &= 2C_F \left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right) \\ p_{qg}^{(0)}(x) &= N_f(x^2 + (1-x)^2) \\ p_{gq}^{(0)}(x) &= 2C_F \left(\frac{1+(1-x)^2}{x} \right) \\ p_{gg}^{(0)}(x) &= 4C_A \left(\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x) \right) \\ &\quad + \beta_0 \delta(1-x), \end{aligned} \quad (\text{D3})$$

where the one-loop beta function coefficient is $\beta_0 = \frac{11}{3}C_A - \frac{2}{3}N_f$, $C_F = \frac{N_c^2-1}{2N_c}$, and $C_A = N_c C_A = N_c$.

For the helicity distribution $\Delta q(x)$ and $\Delta g(x)$, the evolution is of the same form with similar splitting functions defined by [76]

$$\begin{aligned} \Delta p_{qq}^{(0)}(x) &= 2C_F \left(\frac{2}{(1-x)_+} - 1 - x + \frac{3}{2}\delta(1-x) \right) \\ \Delta p_{qg}^{(0)}(x) &= 2N_f(2x-1) \\ \Delta p_{gq}^{(0)}(x) &= 2C_F(2-x) \\ \Delta p_{gg}^{(0)}(x) &= 2C_A \left(\frac{2}{(1-x)_+} + 2 - 4x \right) + \beta_0 \delta(1-x). \end{aligned} \quad (\text{D4})$$

For the transversity distribution $\delta q(x)$, with no corresponding gluon distribution, the evolution does not mix

with the gluon, leaving the evolution kernel with one splitting function defined by [77–79]

$$\delta p_{qq}^{(0)}(x) = 2C_F \left(\frac{2}{(1-x)_+} - 2 + \frac{3}{2} \delta(1-x) \right). \quad (\text{D5})$$

APPENDIX E: SUBTRACTED TMD DISTRIBUTIONS IN b SPACE

Here we summarize the Fourier transformation for each of the structure functions in the subtracted TMD distributions in (12):

$$\begin{aligned} \tilde{f}_1^{q/\rho}(x, b) &= 2\pi \int_0^\infty dk k f_1^{q/\rho}(x, k) J_0(kb) \\ \tilde{f}_{1LL}^{q/\rho}(x, b) &= 2\pi \int_0^\infty dk k f_{1LL}^{q/\rho}(x, k) J_0(kb) \\ \tilde{f}_{1LT}^{q/\rho}(x, b) &= \frac{2\pi}{m_\rho^2 b} \int_0^\infty dk k^2 f_{1LT}^{q/\rho}(x, k) J_1(kb) \\ \tilde{f}_{1TT}^{q/\rho}(x, b) &= \frac{2\pi}{m_\rho^4 b^2} \int_0^\infty dk k^3 f_{1TT}^{q/\rho}(x, k) J_2(kb) \end{aligned} \quad (\text{E1})$$

$$\begin{aligned} \tilde{g}_1^{q/\rho}(x, b) &= 2\pi \int_0^\infty dk k g_1^{q/\rho}(x, k) J_0(kb) \\ \tilde{g}_{1T}^{q/\rho}(x, b) &= \frac{2\pi}{m_\rho^2 b} \int_0^\infty dk k^2 f_{1T}^{q/\rho}(x, k) J_1(kb) \end{aligned} \quad (\text{E2})$$

$$\begin{aligned} \tilde{h}_1^{q/\rho}(x, b) &= 2\pi \int_0^\infty dk k h_1^{q/\rho}(x, k) J_0(kb) \\ \tilde{h}_{1L}^{q/\rho}(x, b) &= \frac{2\pi}{m_\rho^2 b} \int_0^\infty dk k^2 h_{1L}^{q/\rho}(x, k) J_1(kb) \\ \tilde{h}_{1T}^{q/\rho}(x, b) &= \frac{2\pi}{m_\rho^4 b^2} \int_0^\infty dk k^3 h_{1T}^{q/\rho}(x, k) J_2(kb). \end{aligned} \quad (\text{E3})$$

APPENDIX F: CS KERNEL

The integrability of the RG evolution equations guarantees

$$\frac{dK_{\text{CS}}(b_\perp, \mu)}{d \ln \mu^2} = -\Gamma_{\text{cusp}}(\alpha_s(\mu)). \quad (\text{F1})$$

The solution to the CSS renormalization group equation (F1) reads

$$K_{\text{CS}}(b_\perp, \mu) = \begin{cases} K_{\text{CS}}^{(\text{pert})}(b_\perp, \mu_b) - 2 \int_{\mu_b}^\mu \frac{d\mu'}{\mu'} \Gamma_{\text{cusp}}, & b_\perp \ll b_{\text{NP}} \\ K_{\text{CS}}^{(\text{inst})}(b_\perp), & b_\perp \gg b_{\text{NP}}. \end{cases} \quad (\text{F2})$$

The Collins-Soper kernel is split into a perturbative contribution plus a nonperturbative remainder. The splitting is enforced by a smooth cutoff $b_{\text{NP}} \sim \rho$ with a transverse scale

$$\mu_b = \frac{2e^{-\gamma_E}}{b_\perp}$$

to ensure that the evolution remains within the perturbative regime.

The coefficients for each α_s order of the perturbative part of $K_{\text{CS}}^{(\text{pert})}$ with minimal logarithmic dependence can be found in G. The perturbative gluon radiation at large transverse distance is suppressed, and the nonperturbative Collins-Soper kernel compensates the gluon radiation nonperturbatively. The μ -independent nonperturbative correction to the Collins-Soper kernel can be estimated in the ILM, as we have done recently in [12].

APPENDIX G: ANOMALOUS DIMENSIONS AND WILSON COEFFICIENTS FOR TMD EVOLUTION

Here we record the perturbative anomalous dimensions and the Wilson coefficients, for the TMD evolution up to next-to-next-to-next-to-leading order (N³LO) [69,80,81].

First, the light-cone cusp anomalous dimension is given by

$$\Gamma_{\text{cusp}}(\alpha_s) = \sum_{n=1}^{\infty} \Gamma_{\text{cusp}}^{(n)}(\alpha_s), \quad (\text{G1})$$

where each order is defined as

$$\Gamma_{\text{cusp}}^{(n)}(\alpha_s) = \left(\frac{\alpha_s}{4\pi} \right)^n \Gamma_n \quad (\text{G2})$$

with the n th order coefficients defined as

$$\Gamma_1 = 4C_F \quad (\text{G3})$$

$$\begin{aligned} \Gamma_2 &= 8C_F \left[C_A \left(\frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{5}{9} N_f \right] \\ &= \frac{1072}{9} - \frac{16\pi^2}{3} - \frac{160}{27} N_f \end{aligned} \quad (\text{G4})$$

$$\begin{aligned} \Gamma_3 &= 352\zeta(3) + \frac{176\pi^4}{15} - \frac{2144\pi^2}{9} + 1960 \\ &+ N_f \left(-\frac{832\zeta(3)}{9} + \frac{320\pi^2}{27} - \frac{5104}{27} \right) - \frac{64}{81} N_f^2 \end{aligned} \quad (\text{G5})$$

$$\begin{aligned}\Gamma_4 = & \left(-1536\zeta(3)^2 - 704\pi^2\zeta(3) + 28032\zeta(3) - \frac{34496\zeta(5)}{3} + \frac{337112}{9} - \frac{178240\pi^2}{27} + \frac{3608\pi^4}{5} - \frac{32528\pi^6}{945} \right) \\ & + N_f \left(\frac{1664\pi^2\zeta(3)}{9} - \frac{616640\zeta(3)}{81} + \frac{25472\zeta(5)}{9} - \frac{1377380}{243} + \frac{51680\pi^2}{81} - \frac{2464\pi^4}{135} \right) \\ & + N_f^2 \left(\frac{16640\zeta(3)}{81} + \frac{71500}{729} - \frac{1216\pi^2}{243} - \frac{416\pi^4}{405} \right) + N_f^3 \left(\frac{256\zeta(3)}{81} - \frac{128}{243} \right).\end{aligned}\quad (G6)$$

The collinear anomalous dimension for the TMD operator is given by

$$\gamma_f(\alpha_s) = \sum_{n=1}^{\infty} \gamma_f^{(n)}(\alpha_s), \quad (G7)$$

where each order is defined as

$$\gamma_f^{(n)}(\alpha_s) = \left(\frac{\alpha_s}{4\pi} \right)^n \gamma_n \quad (G8)$$

with the n th order coefficients defined as

$$\gamma_1 = 6C_F \quad (G9)$$

$$\begin{aligned}\gamma_2 = & C_F^2(3 - 4\pi^2 + 48\zeta(3)) \\ & + C_F C_A \left(\frac{961}{27} + \frac{11\pi^2}{3} - 52\zeta(3) \right) \\ & + C_F N_f \left(-\frac{130}{27} - \frac{2\pi^2}{3} \right)\end{aligned}\quad (G10)$$

$$\begin{aligned}\gamma_3 = & C_F^2 N_f \left(-\frac{2953}{27} + \frac{26\pi^2}{9} + \frac{28\pi^4}{27} - 512\zeta(3) \right) + C_F N_f^2 \left(-\frac{4834}{729} + \frac{20\pi^2}{27} + \frac{16\zeta(3)}{27} \right) \\ & + C_F^3 \left(29 + 6\pi^2 + \frac{16\pi^4}{5} + 136\zeta(3) - \frac{32\pi^2\zeta(3)}{3} - 480\zeta(5) \right) \\ & + C_F^2 C_A \left(\frac{139345}{1458} + \frac{7163\pi^2}{243} + \frac{83\pi^4}{27} - \frac{7052\zeta(3)}{9} + \frac{88\pi^2\zeta(3)}{9} + 272\zeta(5) \right) \\ & + C_A C_F N_f \left(\frac{17318}{729} - \frac{2594\pi^2}{243} - \frac{22\pi^4}{27} + \frac{1928\zeta(3)}{27} \right) \\ & + C_A C_F^2 \left(\frac{151}{2} - \frac{410\pi^2}{9} - \frac{494\pi^4}{135} + \frac{1688\zeta(3)}{3} + \frac{16\pi^2\zeta(3)}{3} + 240\zeta(5) \right).\end{aligned}\quad (G11)$$

The 2D evolution of the transverse momentum dependence is driven by the anomalous dimensions γ_F and Γ_{cusp} . They are identical for TMD PDFs and TMD fragmentation functions to all orders [81]. The anomalous dimension γ_F only varies by the quark operator, as defined in the TMD functions.

The perturbative part of the Collins-Soper kernel at minimal logarithmic scale $K_{\text{CS}}^{(\text{pert})}(b, \mu = \mu_b)$ is defined as

$$K_{\text{CS}}^{(\text{pert})}(b, \mu_b) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^n k_n \quad (G12)$$

$$k_1 = 0 \quad (G13)$$

$$\begin{aligned}k_2 = & 8C_F \left[\left(\frac{7}{2}\zeta(3) - \frac{101}{27} \right) C_A + \frac{14}{27} N_f \right] \\ = & -2 \left(-56\zeta(3) + \frac{1616}{27} - \frac{224}{81} N_f \right)\end{aligned}\quad (G14)$$

$$\begin{aligned}k_3 = & -2 \left(\frac{176\pi^2\zeta(3)}{3} - \frac{24656\zeta(3)}{9} + 1152\zeta(5) \right. \\ & \left. + \frac{594058}{243} - \frac{6392\pi^2}{81} - \frac{154\pi^4}{45} \right) \\ & - 2N_f \left(\frac{7856\zeta(3)}{81} - \frac{166316}{729} + \frac{824\pi^2}{243} + \frac{4\pi^4}{405} \right) \\ & - 2N_f^2 \left(\frac{64\zeta(3)}{27} + \frac{3712}{2187} \right).\end{aligned}\quad (G15)$$

With the above in mind, the OPE convolution in the small- b ($b \ll b_{\text{NP}}$) TMD evolution (G16) can be written as

$$\begin{aligned}\tilde{f}_1^{q/\rho}(x, b; \mu_b, \mu_b^2) &\simeq \int_x^1 \frac{dx'}{x'} C_{q/q}^f\left(\frac{x}{x'}, b, \mu_b\right) f_1^{q'/\rho}(x', \mu_b) \\ \tilde{g}_1^{q/\rho}(x, b; \mu_b, \mu_b^2) &\simeq \int_x^1 \frac{dx'}{x'} C_{q/q}^g\left(\frac{x}{x'}, b, \mu_b\right) g_1^{q'/\rho}(x', \mu_b) \\ \tilde{h}_1^{q/\rho}(x, b; \mu_b, \mu_b^2) &\simeq \int_x^1 \frac{dx'}{x'} C_{q/q}^h\left(\frac{x}{x'}, b, \mu_b\right) h_1^{q'/\rho}(x', \mu_b).\end{aligned}\quad (\text{G16})$$

The Wilson coefficient for each type of TMD distribution is given in [82]

$$\begin{aligned}C_{q/q}^{f,g}(x, b, \mu_b) \\ = \delta(1-x) + \frac{\alpha_s}{4\pi} C_F \left[-\frac{\pi^2}{6} \delta(1-x) + 2(1-x) \right]\end{aligned}\quad (\text{G17})$$

and

$$C_{q/q}^h(x, b, \mu_b) = \delta(1-x) + \frac{\alpha_s}{4\pi} C_F \left[-\frac{\pi^2}{6} \delta(1-x) \right]. \quad (\text{G18})$$

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- [1] E. C. Aschenauer, I. Borsa, R. Sassot, and C. Van Hulse, *Phys. Rev. D* **99**, 094004 (2019).
 - [2] R. Boussarie *et al.*, arXiv:2304.03302.
 - [3] J. Blumlein, *Prog. Part. Nucl. Phys.* **69**, 28 (2013).
 - [4] A. Bacchetta, V. Bertone, C. Bissolotti, G. Bozzi, F. Delcarro, F. Piacenza, and M. Radici, *J. High Energy Phys.* **117** (2020) 07.
 - [5] A. Vladimirov, *J. High Energy Phys.* **10** (2019) 090.
 - [6] I. Scimemi and A. Vladimirov, *Eur. Phys. J. C* **78**, 89 (2018).
 - [7] M. Cerutti, L. Rossi, S. Venturini, A. Bacchetta, V. Bertone, C. Bissolotti, and M. Radici (MAP (Multi-dimensional Analyses of Partonic distributions) Collaboration), *Phys. Rev. D* **107**, 014014 (2023).
 - [8] A. Bacchetta, F. Delcarro, C. Pisano, M. Radici, and A. Signori, *J. High Energy Phys.* **06** (2017) 081; **06** (2019) 51.
 - [9] A. Bacchetta, V. Bertone, C. Bissolotti, G. Bozzi, M. Cerutti, F. Piacenza, M. Radici, and A. Signori (MAP (Multi-dimensional Analyses of Partonic distributions) Collaboration), *J. High Energy Phys.* **10** (2022) 127.
 - [10] V. Moos, I. Scimemi, A. Vladimirov, and P. Zurita, *J. High Energy Phys.* **05** (2024) 036.
 - [11] W.-Y. Liu and I. Zahed, arXiv:2502.09789.
 - [12] W.-Y. Liu, I. Zahed, and Y. Zhao, *Phys. Rev. D* **111**, 074022 (2025).
 - [13] M.-H. Chu *et al.* (Lattice Parton Collaboration), *Phys. Rev. D* **109**, L091503 (2024).
 - [14] M.-H. Chu *et al.* (Lattice Parton (LPC) Collaboration), *Phys. Rev. D* **106**, 034509 (2022).
 - [15] Z.-F. Deng, W. Wang, and J. Zeng, *J. High Energy Phys.* **09** (2022) 046.
 - [16] M. A. Ebert, I. W. Stewart, and Y. Zhao, *Phys. Rev. D* **99**, 034505 (2019).
 - [17] A. Avkhadiev, P. E. Shanahan, M. L. Wagman, and Y. Zhao, *Phys. Rev. D* **108**, 114505 (2023).
 - [18] M. Schlemmer, A. Vladimirov, C. Zimmermann, M. Engelhardt, and A. Schäfer, *J. High Energy Phys.* **08** (2021) 004.
 - [19] T. C. Rogers, *Eur. Phys. J. A* **52**, 153 (2016).
 - [20] Q.-A. Zhang *et al.* (Lattice Parton Collaboration), *Phys. Rev. Lett.* **125**, 192001 (2020).
 - [21] B. Pasquini, S. Rodini, and A. Bacchetta, *Phys. Rev. D* **100**, 054039 (2019).
 - [22] Y. Ninomiya, W. Bentz, and I. C. Cloët, *Phys. Rev. C* **96**, 045206 (2017).
 - [23] C. Michael and P. S. Spencer, *Nucl. Phys. B, Proc. Suppl.* **42**, 261 (1995).
 - [24] C. Michael and P. S. Spencer, *Phys. Rev. D* **52**, 4691 (1995).
 - [25] D. B. Leinweber, in *Workshop on Light-Cone QCD and Nonperturbative Hadron Physics* (1999), pp. 138–143, arXiv:hep-lat/0004025.
 - [26] I. Bakas, *J. Phys. Conf. Ser.* **283**, 012004 (2011).
 - [27] J. C. Biddle, W. Kamleh, and D. B. Leinweber, *Proc. Sci., LATTICE2018* (2018) 256 [arXiv:1903.07767].
 - [28] A. Hasenfratz and O. Witzel, *Phys. Rev. D* **101**, 034514 (2020).
 - [29] A. Athenodorou, P. Boucaud, F. De Soto, J. Rodríguez-Quintero, and S. Zafeiropoulos, *J. High Energy Phys.* **02** (2018) 140.
 - [30] J. C. Biddle, W. Kamleh, and D. B. Leinweber, *EPJ Web Conf.* **245**, 06010 (2020).
 - [31] D. Diakonov and V. Y. Petrov, *Nucl. Phys.* **B272**, 457 (1986).
 - [32] W.-Y. Liu, E. Shuryak, and I. Zahed, *Phys. Rev. D* **109**, 074029 (2024).
 - [33] W.-Y. Liu, E. Shuryak, and I. Zahed, *Phys. Rev. D* **107**, 094024 (2023).
 - [34] E. Shuryak and I. Zahed, *Phys. Rev. D* **107**, 034027 (2023).
 - [35] E. Shuryak and I. Zahed, *Phys. Rev. D* **107**, 034026 (2023).
 - [36] E. Shuryak and I. Zahed, *Phys. Rev. D* **107**, 034024 (2023).
 - [37] E. Shuryak and I. Zahed, *Phys. Rev. D* **107**, 034023 (2023).
 - [38] W.-Y. Liu, E. Shuryak, and I. Zahed, *Phys. Rev. D* **110**, 054005 (2024).
 - [39] W.-Y. Liu, E. Shuryak, C. Weiss, and I. Zahed, *Phys. Rev. D* **110**, 054021 (2024).
 - [40] I. Zahed, *Symmetry* **14**, 932 (2022).
 - [41] I. Zahed, *Phys. Rev. D* **104**, 054031 (2021).
 - [42] D. Diakonov, *Prog. Part. Nucl. Phys.* **51**, 173 (2003).
 - [43] T. Schäfer and E. V. Shuryak, *Rev. Mod. Phys.* **70**, 323 (1998).
 - [44] W.-Y. Liu, arXiv:2501.07776.

- [45] S. Kumano and Q.-T. Song, *Phys. Rev. D* **103**, 014025 (2021).
- [46] R. Angeles-Martinez *et al.*, *Acta Phys. Pol. B* **46**, 2501 (2015).
- [47] M. Grosse Perdekamp and F. Yuan, *Annu. Rev. Nucl. Part. Sci.* **65**, 429 (2015).
- [48] C. A. Aidala, S. D. Bass, D. Hasch, and G. K. Mallot, *Rev. Mod. Phys.* **85**, 655 (2013).
- [49] V. Barone, F. Bradamante, and A. Martin, *Prog. Part. Nucl. Phys.* **65**, 267 (2010).
- [50] U. D'Alesio and F. Murgia, *Prog. Part. Nucl. Phys.* **61**, 394 (2008).
- [51] C. Shi, J. Li, M. Li, X. Chen, and W. Jia, *Phys. Rev. D* **106**, 014026 (2022).
- [52] V. Moos and A. Vladimirov, *J. High Energy Phys.* **12** (2020) 145.
- [53] S. Kaur, C. Mondal, and H. Dahiya, *J. High Energy Phys.* **01** (2021) 136.
- [54] P. Hoodbhoy, R. L. Jaffe, and A. Manohar, *Nucl. Phys.* **B312**, 571 (1989).
- [55] G. A. Miller, *Phys. Rev. C* **89**, 045203 (2014).
- [56] S. Liuti and K. Kathuria, *J. Phys. Conf. Ser.* **543**, 012005 (2014).
- [57] J.-L. Zhang and J. Wu, *Eur. Phys. J. C* **85**, 13 (2025).
- [58] F. E. Close and S. Kumano, *Phys. Rev. D* **42**, 2377 (1990).
- [59] A. Efremov and O. Teryaev, *Sov. J. Nucl. Phys.* **36**, 557 (1982).
- [60] A. Y. Umnikov, *Phys. Lett. B* **391**, 177 (1997).
- [61] H. He and X.-D. Ji, *Phys. Rev. D* **52**, 2960 (1995).
- [62] J. Collins, *Foundations of Perturbative QCD*, Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology Vol. 32 (Cambridge University Press, Cambridge, England, 2023).
- [63] X. Ji, J. Ma, and F. Yuan, *Phys. Rev. D* **71**, 034005 (2005).
- [64] D. Boer, *Int. J. Mod. Phys. Conf. Ser.* **40**, 1660014 (2016).
- [65] J. Collins, *EPJ Web Conf.* **85**, 01002 (2015).
- [66] J. Collins and T. Rogers, *Phys. Rev. D* **91**, 074020 (2015).
- [67] I. Scimemi and A. Vladimirov, *J. High Energy Phys.* **03** (2017) 002.
- [68] U. D'Alesio, M. G. Echevarria, S. Melis, and I. Scimemi, *J. High Energy Phys.* **11** (2014) 098.
- [69] J. C. Collins, D. E. Soper, and G. F. Sterman, *Nucl. Phys.* **B250**, 199 (1985).
- [70] T. Rogers, F. Aslan, M. Boglione, T. Rainaldi, A. Simonelli, and J. O. Gonzalez-Hernandez, *Proc. Sci., Transversity2024* (2024) 030 [arXiv:2408.07170].
- [71] D. Boer and W. J. den Dunnen, *Nucl. Phys.* **B886**, 421 (2014).
- [72] J. Collins, L. Gamberg, A. Prokudin, T. C. Rogers, N. Sato, and B. Wang, *Phys. Rev. D* **94**, 034014 (2016).
- [73] R. L. Jaffe, H. Meyer, and G. Piller, Spin, twist and hadron structure in deep inelastic processes, in *Lectures on QCD: Applications*, edited by F. Lenz, H. Griebhammer, and D. Stoll (Springer, Berlin, Heidelberg, 1997), pp. 178–249.
- [74] R. L. Jaffe and A. Manohar, *Nucl. Phys.* **B321**, 343 (1989).
- [75] M. Tanabashi *et al.* (Particle Data Group), *Phys. Rev. D* **98**, 030001 (2018).
- [76] S. Moch, J. A. M. Vermaseren, and A. Vogt, *Nucl. Phys.* **B889**, 351 (2014).
- [77] W. Vogelsang, *Phys. Rev. D* **57**, 1886 (1998).
- [78] S. V. Mikhailov and A. A. Vladimirov, *Phys. Lett. B* **671**, 111 (2009).
- [79] X. Artru and M. Mekhfi, *Z. Phys. C* **45**, 669 (1990).
- [80] C. T. H. Davies and W. J. Stirling, *Nucl. Phys.* **B244**, 337 (1984).
- [81] J. Collins and T. C. Rogers, *Phys. Rev. D* **96**, 054011 (2017).
- [82] D. Gutiérrez-Reyes, I. Scimemi, and A. A. Vladimirov, *Phys. Lett. B* **769**, 84 (2017).