

Higher spin fields and the field strength multicopy

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We discuss the generalization of the Weyl double copy to higher spin “multicopies,” showing how the natural linearized higher spin field strengths can be related to sums of powers of the Maxwell tensor. The tracelessness of the field strength involves the appropriate Fronsdal equations of motion for the higher spin field. We work with spacetimes admitting Kerr-Schild coordinates and give a number of examples in different dimensions. We note that the multicopy is particularly transparent in four dimensions if one uses spinor descriptions of the fields, relating this to the Penrose transform. The higher dimensional spinor multicopy is also explored and reveals some interesting new features arising from the little group based identification of higher spin field strengths and Maxwell tensor types. We then turn to the vector superspace formalism describing higher spin and “continuous” spin representations given by Schuster and Toro, based on symmetric tensor fields. Here the Kerr-Schild higher spin fields we have used earlier naturally package into a simple expression involving an arbitrary function, when the continuous spin scale ρ is set to zero. Further, we discuss the case of an anti-de Sitter background, where there is also a vector space formalism given by Segal and we clarify this approach using a different definition of the covariant derivative. We give a general solution of Kerr-Schild type and finally we describe some of the obstacles to a continuous spin formulation.

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I. INTRODUCTION

The double copy has proven a fertile source of new ideas and results over recent years. It was first applied to scattering amplitudes, initially relating amplitudes in Yang-Mills theory and gravity [1,2], where it continues to influence progress in increasingly diverse areas (see, for example, the SAGEX series [3] for comprehensive reviews). Subsequently it has been explored in the context of field theory solutions, where similar progress has ensued (see, for example, the book [4] and references therein). One particularly interesting approach to the double copy is the Weyl double copy, first explored in [5].¹ Here the Weyl tensor in four dimensions is related to an expression

quadratic in the Maxwell tensor. Subsequent work has expanded on this in a number of different directions [9–15].

In this paper we would like to explore the generalization of the Weyl double copy to higher spin fields, that we will call the field strength multicopy. This relates the higher spin $s > 2$ field strengths to products of the Maxwell tensor. We will develop this in various dimensions and for both tensor and spinor formulations. We will also explore higher spin fields in the continuous spin formulation of [16], with a related Kerr-Schild construction, applying this to anti-de Sitter space.

We begin in Sec. II with the definition of the higher spin Kerr-Schild fields and the Fronsdal equations of motion, and give the recursion relation satisfied by these. We then turn in Sec. III to define the field strength and gauge invariance for the spin s fields, relating its trace and divergence to the Fronsdal equations. This allows us to formulate the field strength multicopy, which relates the spin s field strength to sums of powers of s copies of the Maxwell tensor which share the same symmetry and trace properties. We give a number of examples: the Schwarzschild and Eguchi-Hanson metrics in four dimensions, the Tangherlini metric in five and six dimensions, and the Myers-Perry singly rotating solution in five dimensions.

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¹With some related earlier work in the general relativity literature [6–8].

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In Sec. IV we describe the spinor description of the multicopy. This avoids many of the complications of the tensor analysis and provides new insights. In four dimensions, the spin s multicopy is easily derived using the Penrose transform. This also provides simple formulas for “mixed” multicopies whereby the spin s spinor field strength is expressed in terms of field strengths of lower spin. This includes a simple recursion relation linking the spin s field strength to the product of the spin $s - 1$ field strength and the Maxwell spinor. After a brief review of the spinor version of the Petrov-type classification of spacetimes, we describe how the multicopy leads to the definition of higher spin quantities which are linked to those in the classification of Maxwell spinors. This then relates the types of higher spin field strengths which can be generated from the differing Maxwell types via the multicopy. We then explore higher dimensional versions of the spinor multicopy, where the little group is nontrivial and plays an important role in the analysis. We discuss the five-dimensional case, and after summarizing briefly the recently discovered Petrov-type classification, we describe how the spinor multicopy relates the higher spin little group-valued quantities to the Maxwell spinors. This again then relates the types of higher spin field strengths which can be generated from the differing Maxwell fields via the multicopy, albeit in a more involved way due to the presence of the little group.

In Sec. V we discuss the vector superspace formulation of higher spin and “continuous” spin particles (CSPs). Here the Kerr-Schild higher spin field that we have used earlier can be written in a unified compact form involving an arbitrary function. When the continuous spin parameter is zero we show that this unified higher spin Kerr-Schild field satisfies the vector superspace equations of motion. Finally in this section we study the AdS-Kerr case, presenting gauge invariant higher spin field strengths defined using the relevant Kerr-Schild vector and scalar, and noting examples of the tensor multicopy. The vector superspace formulation of the equations of motion and gauge invariance are then given, using a new covariant derivative. A generalized Kerr-Schild type solution is also given, and the simple extension of this formulation to a continuous spin particle which applied in flat space is shown not to yield a CSP theory in this case.

In Sec. VI we present conclusions and note some further avenues of exploration that our work suggests. Finally, the Appendix contains additional details about the covariant derivative used in Sec. V.

II. HIGHER SPIN KERR-SCHILD FIELDS

The study of metrics admitting a Kerr-Schild formulation [17,18] (see also the review [19]) has had many applications in general relativity. These metrics have more recently played a key role in the discovery of the double copy for classical field theory and gravitational

solutions [20,21].² Here we will discuss higher spin versions of the standard Kerr-Schild fields and their equations of motion, before turning to generalizations of the Weyl double copy that can be defined based on these.

Given a static vacuum Kerr-Schild metric solution

$$g_{\mu\nu} = \eta_{\mu\nu} + k_\mu k_\nu \phi, \quad (2.1)$$

with a Kerr-Schild vector k_μ satisfying

$$k^\nu \partial_\nu k^\mu = 0 = k^\mu k_\mu, \quad (2.2)$$

one can define the zeroth, single and double copy fields

$$\begin{aligned} \phi, \\ \phi_\mu = k_\mu \phi, \\ \phi_{\mu\nu} = k_\mu k_\nu \phi, \end{aligned} \quad (2.3)$$

which satisfy the spin zero, one and two equations

$$\begin{aligned} \square \phi &= 0, \\ \square \phi_\mu - \partial^\nu \partial_\mu \phi_\nu &= 0, \\ \square \phi_{\mu\nu} - 2\partial^\rho \partial_{(\mu} \phi_{\nu)\rho} + \partial_\mu \partial_\nu \phi^\rho{}_\rho &= 0. \end{aligned} \quad (2.4)$$

It is natural to then consider the higher spin Kerr-Schild field

$$\phi_{\mu_1 \dots \mu_s}^{\text{KS}} := k_{\mu_1} \dots k_{\mu_s} \phi \quad (2.5)$$

and show that it satisfies a higher spin linearized field equation, i.e., the Fronsdal equation. This has been done in [24] (with earlier work in four dimensions in [22]). Note that ϕ^{KS} is traceless as the vector k_μ is null. To summarize their result, we define an expression whose vanishing is the Fronsdal equation for a symmetric double traceless spin s field $\phi_{\mu_1 \dots \mu_s}$

$$\begin{aligned} F_{\mu_1 \dots \mu_s}^{(s)}(\phi^{(s)}) &:= \square \phi_{\mu_1 \dots \mu_s} - s \partial^\lambda \partial_{(\mu_1} \phi_{\mu_2 \dots \mu_s)\lambda} \\ &\quad + \frac{1}{2} s(s-1) \partial_{(\mu_1} \partial_{\mu_2} \phi_{\mu_3 \dots \mu_s)\lambda}{}^\lambda \end{aligned} \quad (2.6)$$

[(anti)symmetrizations are defined with unit weight, e.g., $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$]. The Fronsdal equation is invariant under the gauge transformation

$$\delta \phi_{\mu_1 \dots \mu_s}^{(s)} = \partial_{(\mu_1} \xi_{\mu_2 \dots \mu_s)} \quad (2.7)$$

where ξ is required to be traceless.

²Some earlier work on higher spin fields and the multicopy for Kerr-Schild spacetimes is [22,23].

A helpful discussion of the Fronsdal equation can be found in the recent review of higher spin theories in [25]. The subject has a long and varied history; two earlier reviews are [26,27].

We will also define $\tilde{F}_{\mu_1 \dots \mu_s}^{(s)}$ by

$$\tilde{F}_{\mu_1 \dots \mu_s}^{(s)} := F_{\mu_1 \dots \mu_s}^{(s)}(\phi^{\text{KS}}) \quad (2.8)$$

i.e., the expression on the right-hand side of Eq. (2.6) with the spin s Kerr-Schild field Φ^{KS} replacing $\phi^{(s)}$.

In Ref. [24], it is shown that if the Fronsdal equations $\tilde{F}^{(s)} = 0$ are satisfied for $s = 0, 1, 2$ [i.e., Eq. (2.4)] then they are satisfied for all values of s . This argument relies on the following recursion relation, implicit in their derivation:

$$\begin{aligned} \tilde{F}_{\mu_1 \dots \mu_s}^{(s)} &= k_{(\mu_1} \tilde{F}_{\mu_2 \dots \mu_s)}^{(s-1)} + (s-1) k_{(\mu_1} \dots k_{\mu_{s-2}} \tilde{F}_{\mu_{s-1} \mu_s)}^{(2)} \\ &\quad - (2s-3) k_{(\mu_1} \dots k_{\mu_{s-1}} \tilde{F}_{\mu_s)}^{(1)} \\ &\quad + (s-2) k_{\mu_1} \dots k_{\mu_s} \tilde{F}^{(0)}. \end{aligned} \quad (2.9)$$

Applying this repeatedly then leads to the desired result

$$\begin{aligned} \tilde{F}_{\mu_1 \dots \mu_s}^{(s)} &= \frac{1}{2} s(s-1) k_{(\mu_1} \dots k_{\mu_{s-2}} \tilde{F}_{\mu_{s-1} \mu_s)}^{(2)} \\ &\quad - s(s-2) k_{(\mu_1} \dots k_{\mu_{s-1}} \tilde{F}_{\mu_s)}^{(1)} \\ &\quad + \frac{1}{2} (s-1)(s-2) k_{\mu_1} \dots k_{\mu_s} \tilde{F}^{(0)}. \end{aligned} \quad (2.10)$$

III. THE FIELD STRENGTH MULTICOPY

Now consider field strengths associated to the above higher spin fields. The linearized field strength associated to a symmetric higher spin field $\phi_{\mu_1 \dots \mu_s}$ is the $2s$ -component tensor

$$H_{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_s \nu_s}^{(s)}(\phi^{(s)}) = \partial_{\mu_s} \partial_{\mu_{s-1}} \dots \partial_{\mu_1} \phi_{\nu_1 \nu_2 \dots \nu_s}^{(s)} \Big|_{[\mu_i \nu_i]}, \quad (3.1)$$

where the notation $\Big|_{[\mu_i \nu_i]}$ means to antisymmetrize the expression in each pair of indices (μ_i, ν_i) , for $i = 1, \dots, s$. The field strength is invariant under the gauge transformations

$$\delta \phi_{\mu_1 \dots \mu_s}^{(s)} = \partial_{(\mu_1} \Lambda_{\mu_2 \dots \mu_s)} \quad (3.2)$$

for a symmetric spin $s-1$ field Λ .³

It is straightforward to show that the trace of the field strength is related to the Fronsdal equation by

³A discussion of field strengths, gauge symmetries, (nonlocal) actions, and the equivalents of the Einstein tensor for fields in arbitrary representations of the Lorentz group can be found in [28,29].

$$\eta^{\mu_1 \nu_s} H_{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_s \nu_s}^{(s)} = -\frac{1}{4} \partial_{\mu_2} \dots \partial_{\mu_{s-1}} F_{\nu_2 \dots \nu_{s-1} \nu_1 \mu_s}^{(s)} \Big|_{[\mu_i \nu_i]}, \quad (3.3)$$

where the antisymmetrization on the right-hand side is over each pair (μ_i, ν_i) for $i = 2, \dots, s-1$. The divergence of the field strength is also related to the Fronsdal equation, by the equation

$$\partial^{\mu_1} H_{\mu_1 \nu_1 \mu_2 \nu_2 \dots \mu_s \nu_s}^{(s)} = \frac{1}{2} \partial_{\mu_2} \dots \partial_{\mu_s} F_{\nu_2 \dots \nu_{s-1} \nu_1}^{(s)} \Big|_{[\mu_i \nu_i]}, \quad (3.4)$$

where here again the antisymmetrization on the right-hand side is over each pair (μ_i, ν_i) for $i = 2, \dots, s$. If we insert the Kerr-Schild solution for the spin s field into these equations we thus deduce that the field strength for this field is traceless and divergence-free.

Now consider the linearized Weyl double copy. This concerns examples where the linearized Weyl tensor $C_{\mu\nu\rho\sigma}^{(0)}$ is proportional to a sum of terms quadratic in the Maxwell field strength tensor. Explicitly

$$\begin{aligned} C_{\mu\nu\rho\sigma}^{(0)} &= \alpha \left(F_{\mu\nu} F_{\rho\sigma} - F_{\rho\mu} F_{\nu\sigma} - \frac{6}{d-2} \eta_{\mu\rho} F_{\nu}{}^{\lambda} F_{\sigma\lambda} \right. \\ &\quad \left. + \frac{3}{(d-1)(d-2)} \eta_{\mu\rho} \eta_{\nu\sigma} F^{\lambda\pi} F_{\lambda\pi} \right) \Big|_s, \end{aligned} \quad (3.5)$$

where α is a proportionality factor, d is the number of spacetime dimensions, and the symbol $\Big|_s$ here means to antisymmetrize in the pairs of indices (μ, ν) and (ρ, σ) . Note that both sides of this equation are traceless using the flat metric. Here the Maxwell and Weyl tensor are just special cases of the higher spin field strength tensors defined in (3.1)

$$F := 2H_{\mu\nu}^{(1)}(\phi^{(1)}), \quad C_{\mu_1 \nu_1 \mu_2 \nu_2}^{(0)} := 2H_{\mu_1 \nu_1 \mu_2 \nu_2}^{(2)}(\phi^{(2)}). \quad (3.6)$$

One can then consider “multicopy” versions of the linearized Weyl double copy, where the spin s field strength can be expressed in terms sums of products of s Maxwell tensors which share the same symmetry and trace properties. For example, a triple copy where the field strength for a spin three field, given by (3.1) above for $s = 3$, is proportional to an expression cubic in the Maxwell tensor which has the same symmetries,

$$H_{\mu\nu\rho\sigma\lambda\tau}^{(3)}(\phi^{(3)}) \propto C[F]_{\mu\nu\rho\sigma\lambda\tau} := F_{\mu\nu} F_{\rho\sigma} F_{\lambda\tau} + \dots, \quad (3.7)$$

where the dots indicate terms which ensure the symmetries and tracelessness of the field strength on the left-hand side of the equation are reproduced by the combination of Maxwell tensors.

To see this explicitly, define

$$X_{\mu\nu\rho\sigma\lambda\tau} = (F_{\mu\nu}F_{\rho\sigma}F_{\lambda\tau} + 3F_{\mu\nu}F_{\rho\lambda}F_{\sigma\tau} + 2F_{\mu\rho}F_{\nu\lambda}F_{\sigma\tau})|_{\text{symm}}, \quad (3.8)$$

and the trace terms (taken with the flat metric)

$$X'_{\mu\nu\rho\sigma\lambda\tau} = \eta_{\mu\nu}X^{\alpha}_{\rho\sigma\lambda\tau}, \quad X''_{\mu\nu\rho\sigma\lambda\tau} = \eta_{\mu\nu}\eta_{\rho\sigma}X^{\alpha\beta}_{\lambda\tau\alpha\beta}, \quad (3.9)$$

where the notation $|_{\text{symm}}$ means to antisymmetrize the expression in the indices $(\mu\nu)$, $(\rho\sigma)$, $(\lambda\tau)$ and symmetrize under the interchange of any two of those pairs. The expression in (3.8) vanishes upon antisymmetrization in any three adjacent indices. Then the triple copy expression is given explicitly by

$$\begin{aligned} C[F]_{\mu\nu\rho\sigma\lambda\tau} &= X_{\mu\nu\rho\sigma\lambda\tau} - \frac{12}{d}(X'_{\mu\rho\nu\sigma\lambda\tau} + X'_{\mu\rho\lambda\tau\nu\sigma} + X'_{\lambda\tau\mu\rho\nu\sigma})|_{\text{symm}} \\ &\quad - \frac{12}{d(d+1)}(X''_{\nu\rho\mu\sigma\lambda\tau} + X''_{\nu\rho\lambda\tau\mu\sigma} + X''_{\lambda\tau\nu\rho\mu\sigma} \\ &\quad - 2(X''_{\lambda\mu\tau\sigma\nu\rho} + X''_{\lambda\mu\nu\rho\tau\sigma} + X''_{\nu\rho\lambda\mu\tau\sigma}))|_{\text{symm}}. \end{aligned} \quad (3.10)$$

A. Schwarzschild metric

To give a simple example, consider the Schwarzschild metric, written in the Kerr-Schild form

$$\begin{aligned} ds^2 &= (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\ &= \left(\eta_{\mu\nu} + \frac{\lambda}{r}k_\mu k_\nu\right)dx^\mu dx^\nu, \end{aligned} \quad (3.11)$$

with $k_\mu = (1, x/r, y/r, z/r)$ and λ a constant.

First consider the linearized double copy. The field strength is given by

$$C_{\mu\nu\rho\sigma}^{(0)} = 2\partial_\mu\partial_\rho h_{\nu\sigma}|_s, \quad (3.12)$$

and the Maxwell single copy gauge field is

$$A_\mu = \frac{\lambda}{r}k_\mu. \quad (3.13)$$

The linearized double copy is then

$$\begin{aligned} C_{\mu\nu\rho\sigma}^{(0)} &= \frac{2r}{\lambda} \left(F_{\mu\nu}F_{\rho\sigma} - F_{\rho\mu}F_{\nu\sigma} - 3\eta_{\mu\rho}F_{\nu}^{\lambda}F_{\sigma\lambda} \right. \\ &\quad \left. + \frac{1}{2}\eta_{\mu\rho}\eta_{\nu\sigma}F^{\lambda\pi}F_{\lambda\pi} \right)|_s, \end{aligned} \quad (3.14)$$

where indices are raised with the flat metric in the expression above.

For the triple copy, define the spin three field

$$\phi_{\mu\nu\rho} = \frac{\lambda}{r}k_\mu k_\nu k_\rho, \quad (3.15)$$

and its field strength

$$H_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3)} = \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi_{\nu_1\nu_2\nu_3}|_{[\mu_i,\nu_i]}, \quad (3.16)$$

where the notation indicates that the expression on the right-hand side is to be antisymmetrized in the pairs (μ_i, ν_i) for $i = 1, 2, 3$. This field strength is traceless with the flat metric.

The triple copy is then found to be

$$H_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3)} = \frac{5r^2}{2\lambda^2}C[F]_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}. \quad (3.17)$$

B. Eguchi-Hanson

Another example is the Euclidean Eguchi-Hanson metric. With coordinates (u, v, X, Y) the Kerr-Schild vector is $k_\mu = \frac{1}{uv-XY}(v, 0, 0, -X)$ and the flat metric and metric perturbation are given by

$$\eta_{\mu\nu}dx^\mu dx^\nu = 2(dudv - dXdY), \quad h_{\mu\nu} = \frac{m}{uv-XY}k_\mu k_\nu, \quad (3.18)$$

with m a constant, and the total metric given by

$$ds_{\text{EH}}^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu. \quad (3.19)$$

The Maxwell gauge field is

$$A_\mu = \frac{m}{uv-XY}k_\mu. \quad (3.20)$$

The linearized field strength is given again by

$$C_{\mu\nu\rho\sigma}^{(0)} = 2\partial_\mu\partial_\rho h_{\nu\sigma}|_s, \quad (3.21)$$

and the double copy is

$$C_{\mu\nu\rho\sigma}^{(0)} = \frac{uv-XY}{m}C[F]_{\mu\nu\rho\sigma}. \quad (3.22)$$

For the triple copy, define the spin three field

$$\phi_{\mu\nu\rho} = \frac{m}{uv-XY}k_\mu k_\nu k_\rho, \quad (3.23)$$

and its field strength

$$H_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3)} = \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\phi_{\nu_1\nu_2\nu_3}|_{[\mu_i,\nu_i]}, \quad (3.24)$$

and we find the triple copy expression

$$H_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}^{(3)} = \frac{5(uv - XY)^2}{8m^2} C[F]_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}. \quad (3.25)$$

C. $D > 4$

For examples in more than four dimensions, first consider the Tangherlini metric in five dimensions. In Kerr-Schild coordinates (t, x, y, z, w) the Kerr-Schild vector is given by $k_i = (1, x/r, y/r, z/r, w/r)$, where here $i = 1, \dots, 5$ and $r^2 = x^2 + y^2 + z^2 + w^2$. The metric is

$$ds^2 = (\eta_{ij} + h_{ij})dx^i dx^j = \left(\eta_{ij} + \frac{\lambda}{r^2} k_i k_j \right) dx^i dx^j, \quad (3.26)$$

with λ a constant.

For the linearized double copy, the field strength is given by

$$C_{ijkl}^{(0)} = 2\partial_i \partial_k h_{jl}|_s, \quad (3.27)$$

and the Maxwell single copy gauge field is

$$A_i = \frac{\lambda}{r^2} k_i. \quad (3.28)$$

The linearized double copy is then

$$C_{ijkl}^{(0)} = \frac{r^2}{\lambda} \left(F_{ij} F_{kl} - F_{ki} F_{jl} - 2\eta_{ik} F_j^m F_{lm} + \frac{1}{4} \eta_{ik} \eta_{jl} F^{mn} F_{mn} \right) \Big|_s, \quad (3.29)$$

where we antisymmetrize in (i, j) and (k, l) , and indices are raised with the flat metric in the expression above.

For the triple copy, define the spin three field

$$\phi_{ijk} = \frac{\lambda}{r} k_i k_j k_k \quad (3.30)$$

and its field strength

$$H_{i_1 j_1 i_2 j_2 i_3 j_3}^{(3)} = \partial_{i_1} \partial_{i_2} \partial_{i_3} \phi_{j_1 j_2 j_3} \Big|_{[i_a, j_a]}. \quad (3.31)$$

This is traceless with the flat metric.

The triple copy is then found to be

$$H_{i_1 j_1 i_2 j_2 i_3 j_3}^{(3)} = \frac{5r^4}{8\lambda^2} C[F]_{i_1 j_1 i_2 j_2 i_3 j_3}. \quad (3.32)$$

This can be generalized to the higher dimensional Tangherlini metrics in a straightforward way; for example the double and triple copies in six dimensions are

$$C_{ijkl}^{(0)} = -\frac{20r^3}{27\lambda} \left(F_{ij} F_{kl} - F_i F_{jk} - \frac{3}{2} \eta_{ik} F_i^m F_{lm} + \frac{3}{20} \eta_{ik} \eta_{jl} F^{mn} F_{mn} \right) \Big|_s, \quad (3.33)$$

and

$$H_{i_1 j_1 i_2 j_2 i_3 j_3}^{(3)} = \frac{70r^6}{27\lambda^2} C[F]_{i_1 j_1 i_2 j_2 i_3 j_3}, \quad (3.34)$$

where we remind the reader that $C[F]$, defined in (3.10), should be evaluated in the appropriate space time dimension.

For a different five-dimensional example, consider the singly rotating Myers-Perry solution. The Weyl double copy for this case was studied in a particular coordinate system in [12], but here we will use Kerr-Schild coordinates $x^i = (t, x_1, x_2, y_1, y_2)$ so we can set up the formalism for studying multiple copies.

The metric is given by

$$g_{ij} = \eta_{ij} + h k_i k_j \quad (3.35)$$

where the Kerr-Schild vector k_i is given by

$$k_i = \left(1, \frac{R x_1 - a y_1}{R^2 + a^2}, \frac{x_2}{R}, \frac{R y_1 + a x_1}{R^2 + a^2}, \frac{y_2}{R} \right). \quad (3.36)$$

The vector k_i is null with respect to both the flat and full metric. The inverse metric is simply

$$g^{ij} = \eta^{ij} - h k^i k^j. \quad (3.37)$$

We raise the index on k_i with either the flat or curved metric.

The parameter a is the rotation constant, and the function h in (3.35) is

$$h = \frac{\mu}{(R^2 + a^2) \left(1 - \frac{a^2(x_1^2 + y_1^2)}{(R^2 + a^2)^2} \right)}, \quad (3.38)$$

where μ is a constant. Finally, R is a solution of the equation

$$\frac{x_1^2 + y_1^2}{R^2 + a^2} + \frac{x_2^2 + y_2^2}{R^2} = 1. \quad (3.39)$$

Now consider the linearized Weyl double copy. The Maxwell field strength is given by

$$F_{ij} = \partial_i (h k_j) - \partial_j (h k_i). \quad (3.40)$$

Use this to form a tensor with the same symmetries as the Weyl tensor, which is also traceless with respect to the flat metric as

$$C[F]_{ijkl} = \left(F_{ij}F_{kl} - F_{ki}F_{jl} - 2\eta_{ik}F_j^s F_{ls} + \frac{1}{4}\eta_{ik}\eta_{jl}F^{st}F_{st} \right) \Big|_s, \quad (3.41)$$

where the notation $|_s$ means to antisymmetrize the expression within the brackets in (i, j) and (k, l) .

The linearized Weyl double copy can then be written

$$C_{ijkl}^{(0)} = \frac{1}{h} C[F]_{ijkl}. \quad (3.42)$$

Now consider the triple copy. Firstly, define the spin three field strength by

$$H_{ijklst}^{(3)} = \partial_s \partial_k \partial_i (h k_j k_l k_t) \Big|_s, \quad (3.43)$$

where here $|_s$ means to antisymmetrize the expression within the brackets in (i, j) , (k, l) , and (s, t) .

The field strength $H_{ijklst}^{(3)}$ has the following properties: it is antisymmetric in the pairs of indices (i, j) , (k, l) , and (s, t) , and symmetric under the interchange of any two of these pairs. The tensor also vanishes if any three indices are antisymmetrized. Finally, it can be checked that it is traceless using the flat metric. We find that the triple copy in this case is given by

$$H_{ijklst}^{(3)} = -\frac{5}{8h^2} C[F]_{ijklst}. \quad (3.44)$$

There are obvious generalizations of the tensor triple copy, studied in various examples above, to higher spins, defining a spin s field strength using s derivatives of the appropriate function h times s copies of the Kerr-Schild vector k_i , and showing that this is reproduced by the sum of products of s copies of the Maxwell tensor which shares the same symmetry and trace features. This will involve an increasing number of terms. We will see in the following sections that this is much simplified in the spinor approach, which will also allow a study of generalized Petrov types for the Maxwell, spin two and higher spin spinor field strengths, and show how multicopies lead to relations between these.

IV. THE SPINOR MULTICOPY

A. Four dimensions

We now show how many of the above results can be derived using a spinor-based approach. Firstly consider four dimensions. Then the spinor form of the Maxwell tensor is a two index symmetric spinor ϕ_{AB} , along with its conjugate $\phi_{A'B'}[(A, B, A', B' = (1, 2)]$ and the spinor equivalent of the Weyl tensor is a four index symmetric spinor ϕ_{ABCD} along with its conjugate $\phi_{A'B'C'D'}$. One can immediately construct a symmetric four index spinor from the

Maxwell spinor as an object proportional to $\phi_{(A'B'}\phi_{C'D')}$. Using twistor space results, an understanding of how and under what conditions this maps between solutions of the spin one and two equations and with what proportionality factor has been given in [30] (see also [14]), and we will use the results of that paper, and their notation, here.

Using the Penrose transform, one can derive the scalar, and electromagnetic and gravitational spacetime field strength spinors in terms of the spacetime principal spinors $\alpha_{A'} = (1, \xi_1), \beta_{B'} = (1, \xi_2)$ as

$$\begin{aligned} \phi &= \frac{N(x)}{\xi_1 - \xi_2}, & \phi_{A'B'} &= -\frac{N^2(x)}{(\xi_1 - \xi_2)^3} \alpha_{(A'} \beta_{B')}, \\ \phi_{A'B'C'D'} &= \frac{N^3(x)}{(\xi_1 - \xi_2)^5} \alpha_{(A'} \beta_{B'} \alpha_{C'} \beta_{D')}, \end{aligned} \quad (4.1)$$

where $N(x)$ is the normalization factor arising from the twistor/spinor incidence relation. An advantage of the twistor approach is that it identifies this function explicitly; a simple example is the Schwarzschild metric where one can derive the result $\phi \propto 1/r$ [30], as can be seen in the tensor equation (3.13) earlier. Notice also that in this construction the Weyl double copy is restricted to type D spacetimes, i.e., those with two pairs of repeated principal spinors.⁴

Using the spin s Penrose transform (see Eq. (13) of [30]), one can derive the general result

$$\phi_{A'_1 B'_1 \dots A'_s B'_s} = \frac{(-1)^s N^{s+1}(x)}{(\xi_1 - \xi_2)^{2s+1}} \alpha_{(A'_1} \beta_{B'_1} \dots \alpha_{A'_s} \beta_{B'_s)}. \quad (4.2)$$

Using this and the previous equation one can immediately write down the spin s multicopy,⁵

$$\phi_{A'_1 B'_1 \dots A'_s B'_s} = \frac{(\xi_1 - \xi_2)^{s-1}}{N^{s-1}(x)} \phi_{(A'_1 B'_1} \dots \phi_{A'_s B'_s)}. \quad (4.3)$$

One can also consider equivalent expressions which could be interpreted as a “mixed” multicopy, e.g., for spin three $\phi_{A'B'C'D'E'F'} = \frac{(\xi_1 - \xi_2)^2}{N^2(x)} \phi_{(A'B'} \phi_{C'D'} \phi_{E'F')} = \frac{\xi_1 - \xi_2}{N(x)} \phi_{(A'B'C'D'} \phi_{E'F')}$. A particular example then takes the form of a recursion relation,

$$\phi_{A'_1 B'_1 \dots A'_s B'_s} = -\frac{N(x)}{(\xi_1 - \xi_2)^2} \phi_{(A'_1 B'_1} \dots \phi_{A'_{s-1} B'_{s-1}} \phi_{A'_s B'_s)}. \quad (4.4)$$

For an explicit example, consider the Eguchi-Hanson metric of Sec. III B. In [5] it was shown that in this case the

⁴More general double copies which yield a variety of spacetime types have been found in [14] and the Weyl double copy with sources studied in [31].

⁵The spin s spinor multicopy for AdS_4 black holes was first studied in [23], and a more recent discussion of the flat and AdS cases is in [32].

Weyl spinor is proportional to the product of two Maxwell spinors. To connect with the discussion just above and the higher spin copies, define the spin- s spinor field strengths by

$$\phi_{A'_1 B'_1 \dots A'_s B'_s} = \frac{1}{2^s} \sigma^{\mu_1 \nu_1}_{(A'_1 B'_1} \dots \sigma^{\mu_s \nu_s}_{A'_s B'_s)} H_{\mu_1 \nu_1 \dots \mu_s \nu_s} \quad (4.5)$$

with

$$\hat{\alpha}_{A'} = \frac{1}{\sqrt{2}}(u - iY, u + iY), \quad \hat{\beta}_{A'} = \frac{1}{\sqrt{2}}(v + iX, -(v - iX)). \quad (4.7)$$

Then $\hat{\alpha}\hat{\beta} := \hat{\alpha}^{A'}\hat{\beta}_{A'} = -(uv - XY)$ and one can derive the results

$$\begin{aligned} \phi &= -\frac{m}{\hat{\alpha}\hat{\beta}}, & \phi_{A'B'} &= -\frac{m}{(\hat{\alpha}\hat{\beta})^3} \hat{\alpha}_{(A'}\hat{\beta}_{B')}, & \phi_{A'B'C'D'} &= -\frac{3m}{(\hat{\alpha}\hat{\beta})^5} \hat{\alpha}_{(A'}\hat{\beta}_{B'}\hat{\alpha}_{C'}\hat{\beta}_{D')}, \\ \phi_{A'B'C'D'E'F'} &= -\frac{15m}{(\hat{\alpha}\hat{\beta})^7} \hat{\alpha}_{(A'}\hat{\beta}_{B'}\hat{\alpha}_{C'}\hat{\beta}_{D'}\hat{\alpha}_{E'}\hat{\beta}_{F')}, \\ \phi_{A'B'C'D'E'F'G'H'} &= -\frac{105m}{(\hat{\alpha}\hat{\beta})^9} \hat{\alpha}_{(A'}\hat{\beta}_{B'}\hat{\alpha}_{C'}\hat{\beta}_{D'}\hat{\alpha}_{E'}\hat{\beta}_{F'}\hat{\alpha}_{G'}\hat{\beta}_{H')}, \end{aligned} \quad (4.8)$$

leading to the conjecture that

$$\phi_{A'_1 B'_1 \dots A'_s B'_s} = -\frac{m(2s-1)!!}{(\hat{\alpha}\hat{\beta})^{2s+1}} \alpha_{(A'_1} \beta_{B'_1} \dots \alpha_{A'_s} \beta_{B'_s)}. \quad (4.9)$$

This reproduces the relations (4.2) for the Eguchi-Hanson case, up to purely numerical factors, noting that $\alpha^A \beta_A = -(\xi_1 - \xi_2)$.⁷

Both the Maxwell and Weyl spinors may be assigned to different classes reflecting how algebraically special they are, using the Petrov classification (see, for example, the discussions in [33,5] and references therein). This can be presented using a Newman-Penrose tetrad of null vectors $k^\mu, n^\mu, m^\mu, \bar{m}^\mu$ which can be written in terms of a pair of two-spinors $t^A, \bar{\sigma}^A$ and their conjugates.

The Maxwell spinor ϕ_{AB} can then be described by the three scalars

$$\phi_0 := \phi_{AB} \sigma^A \sigma^B, \quad \phi_1 := \phi_{AB} \sigma^A t^B, \quad \phi_2 := \phi_{AB} t^A t^B, \quad (4.10)$$

$$H_{\mu_1 \nu_1 \dots \mu_s \nu_s} = (\nabla_{(\mu_1} \dots \nabla_{\mu_s)} \phi_{\nu_1 \dots \nu_s} - \text{traces})|_{[\mu_i, \nu_i]}, \quad (4.6)$$

where $\phi_{\nu_1 \dots \nu_s} = k_{\nu_1} \dots k_{\nu_s} \phi$ is the Kerr-Schild spin- s field and the notation indicates antisymmetrization in the pairs (μ_i, ν_i) for $i = 1 \dots s$. [Note that the spinor index symmetrization in (4.5) projects out the traces of H .] Now consider the (unscaled) Newman-Penrose spinors⁶

with

$$\phi_{AB} = \phi_0 t_A t_B + 2\phi_1 \sigma_{(A} t_{B)} + \phi_2 \sigma_A \sigma_B. \quad (4.11)$$

There are then two distinct types of nontrivial Maxwell tensor: type I where $\phi_0 = 0$ (which can always be arranged in four, but not higher, dimensions) but $\phi_1 \neq 0$, and type II, where both are zero. This can also be described in terms of spinor alignment: generically one can write $\phi_{AB} = \alpha_{(A} \beta_{B)}$ for some choice of spinors α_A, β_B . If these are proportional then the Maxwell spinor is of type II, and if not it is type I. Note that the three quantities ϕ_0, ϕ_1, ϕ_2 depend on the choice of tetrad, so that the classification of Maxwell types is more transparent using spinor alignment.

There is an analogous well-known classification of the Weyl spinor ϕ_{ABCD} in curved spacetimes. This can be described by the five complex scalars

$$\begin{aligned} \psi_0 &:= \phi_{ABCD} \sigma^A \sigma^B \sigma^C \sigma^D, & \psi_1 &:= \phi_{ABCD} \sigma^A \sigma^B \sigma^C t^D, & \psi_2 &:= \phi_{ABCD} \sigma^A \sigma^B t^C t^D, \\ \psi_3 &:= \phi_{ABCD} \sigma^A t^B t^C t^D, & \psi_4 &:= \phi_{ABCD} t^A t^B t^C t^D \end{aligned} \quad (4.12)$$

⁶Forming the null vectors $k, n, m, \bar{m}$ in the usual way from these, the only nonzero inner products are $k.n = -(uv - XY)^2$ and $m.\bar{m} = (uv - XY)^2$.

⁷These formulas also follow for the linearized case, with partial derivatives in (4.6) and the flat space sigma matrices in (4.5), giving the spinor versions of the formulas in Sec. III B.

(the NP spinors ι^A, σ^A are the curved spacetime versions here) with

$$\begin{aligned} \phi_{ABCD} = & \psi_0 \iota^A \iota^B \iota^C \iota^D + 4\psi_1 \iota^A \iota^B \iota^C \sigma^D + 6\psi_2 \iota^A \iota^B \sigma^C \sigma^D \\ & + 4\psi_3 \iota^A \sigma^B \sigma^C \sigma^D + \psi_4 \sigma^A \sigma^B \sigma^C \sigma^D. \end{aligned} \quad (4.13)$$

There are then five different Petrov classes of Weyl spinor: type I when $\psi_0 = 0$, type II when $\psi_0 = \psi_1 = 0$, type D when $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$ (here there are two distinct pairs of aligned spinors), type III when $\psi_0 = \psi_1 = \psi_2 = 0$, and type N when $\psi_0 = \psi_1 = \psi_2 = \psi_3 = 0$. As in the Maxwell case, the quantities $\psi_0, \dots, \psi_4$ depend on the choice of tetrad, and the classification of Weyl spinor types can alternatively be made by studying spinor alignment: type I when there is no alignment, type II when there is one

aligned pair, type D when there are two different aligned pairs, type III when there is a triplet of aligned spinors, and type N when there is a quartet.

The standard Weyl double copy is based on two copies of the same Maxwell tensor, with $\phi_{ABCD} \sim \phi_{(AB}\phi_{CD)}$ and this imposes restrictions on which type of Weyl spinor arises, depending on the different types of Maxwell spinor. A type II Maxwell spinor, when double copied, yields a type N Weyl spinor, for example. The double copy in this case yields examples of radiation regions of isolated gravitational systems [9]. A type I Maxwell spinor, on the other hand, yields a type D Weyl spinor under the double copy, since this yields two pairs of aligned spinors.

The triple copy analog has $\phi_{ABCDEFGH} \sim \phi_{(AB}\phi_{CD}\phi_{EF)}$ with similar conditions arising. One has seven quantities,

$$\begin{aligned} \chi_0 &:= \phi_{ABCDEFGH} \sigma^A \sigma^B \sigma^C \sigma^D \sigma^E \sigma^F, & \chi_1 &:= \phi_{ABCDEFGH} \sigma^A \sigma^B \sigma^C \sigma^D \sigma^E \iota^F, \\ \chi_2 &:= \phi_{ABCDEFGH} \sigma^A \sigma^B \sigma^C \sigma^D \iota^E \iota^F, & \chi_3 &:= \phi_{ABCDEFGH} \sigma^A \sigma^B \sigma^C \iota^D \iota^E \iota^F, \\ \chi_4 &:= \phi_{ABCDEFGH} \sigma^A \sigma^B \iota^C \iota^D \iota^E \iota^F, & \chi_5 &:= \phi_{ABCDEFGH} \sigma^A \iota^B \iota^C \iota^D \iota^E \iota^F, \\ \chi_6 &:= \phi_{ABCDEFGH} \iota^A \iota^B \iota^C \iota^D \iota^E \iota^F \end{aligned} \quad (4.14)$$

with

$$\begin{aligned} \phi_{ABCDEFGH} = & \chi_0 \iota^A \iota^B \iota^C \iota^D \iota^E \iota^F + 6\chi_1 \iota^A \iota^B \iota^C \iota^D \sigma^E \sigma^F + 15\chi_2 \iota^A \iota^B \iota^C \sigma^D \sigma^E \sigma^F + 20\chi_3 \iota^A \iota^B \sigma^C \sigma^D \sigma^E \sigma^F \\ & + 15\chi_4 \iota^A \sigma^B \sigma^C \sigma^D \sigma^E \sigma^F + 6\chi_5 \iota^A \sigma^B \sigma^C \sigma^D \sigma^E \sigma^F + \chi_6 \sigma^A \sigma^B \sigma^C \sigma^D \sigma^E \sigma^F. \end{aligned} \quad (4.15)$$

One can then define different types of spin three field strength spinors based on the vanishing of sets of the seven quantities above, or using spinor alignment: a type II Maxwell spinor will triple copy to a null type spin three field strength, with six aligned spinors, and a type I Maxwell spinor will triple copy to a field strength with two triplets of aligned spinors. This generalizes to multicopies in the obvious way. Spinor alignment will also come into play for multicopies here and in higher dimensions as there are only a finite number of independent spinors in each dimension.

B. Higher dimensions

In more than four dimensions the spinor double copy again follows from transforming the tensor version into spinor coordinates. In five dimensions, for example, the spinor indices $A, B, \dots$ run from 1 to 4, reflecting the relationship $SO(5) \sim Sp(2)$. We then have the Maxwell and Weyl spinors

$$\phi_{AB} = F_{ij} \sigma^{ij}_{AB}, \quad \phi_{ABCD} = C_{ijkl} \sigma^{ij}_{AB} \sigma^{ij}_{CD}, \quad (4.16)$$

using the Lorentz generators σ^{ij}_{AB} . Inserting a double copy formula for C_{ijkl} , such as those given in examples earlier,

into the second equation above then leads to the spinor double copy relationship $\phi_{ABCD} \sim \phi_{(AB}\phi_{CD)}$.

In dimensions greater than four the classification of the Maxwell and Weyl spinors is more detailed, since the little groups are non-Abelian (c.f. [33] and references therein). In the five-dimensional case, the little group is $SU(2)$, and we will use the two-spinor indices $a, b, \dots = 1, 2$. Here the Newman-Penrose pentad can be defined using null vectors $k^i, n^i, (i = 1, \dots, 5)$, together with ‘‘polarization’’ vectors ϵ^i_{ab} , symmetric in (a, b) . This pentad can be formulated in terms of spinors k^i_a, n^i_a , carrying little group indices. Contracting these into the field strength spinors then yields little group-valued quantities that can be used to classify the field strengths into different types.

For the Maxwell spinor ϕ_{AB} this yields the bispinors

$$\begin{aligned} \phi_{ab}^{(0)} &:= \phi_{AB} k^A_a k^B_b, & \phi_{ab}^{(1)} &:= \phi_{AB} k^A_a n^B_b, \\ \phi_{ab}^{(2)} &:= \phi_{AB} n^A_a n^B_b, \end{aligned} \quad (4.17)$$

with

$$\phi_{AB} = \phi_{ab}^{(0)} n^a_A n^b_B + 2\phi_{ab}^{(1)} n^a_{(A} k^b_{B)} + \phi_{ab}^{(2)} k^a_A k^b_B. \quad (4.18)$$

While $\phi_{ab}^{(0)}$ and $\phi_{ab}^{(2)}$ are symmetric in (a, b) and in the 3 of $SU(2)$, the bispinor $\phi_{ab}^{(1)}$ has two irreducible components: the symmetrized piece $\phi_{(ab)}^{(1)}$ and the trace $\frac{1}{2}\epsilon_{ab}\phi_{tr}^{(1)}$.

The Petrov-like classification here is then type G when none of the little group-valued bispinors in (4.17) vanish, type I when only $\phi_{ab}^{(0)} = 0$, and type II when both $\phi_{ab}^{(0)} = 0$ and $\phi_{ab}^{(1)} = 0$. However, there is a further subclassification

$$\begin{aligned}\phi_{abcd}^{(0)} &:= \phi_{ABCD}k^A{}_ak^B{}_bk^C{}_ck^D{}_d, & \phi_{abcd}^{(1)} &:= \phi_{ABCD}k^A{}_ak^B{}_bk^C{}_cn^D{}_d, \\ \phi_{abcd}^{(2)} &:= \phi_{ABCD}k^A{}_ak^B{}_bn^C{}_cn^D{}_d, & \phi_{abcd}^{(3)} &:= \phi_{ABCD}k^A{}_an^B{}_bn^C{}_cn^D{}_d, \\ \phi_{abcd}^{(4)} &:= \phi_{ABCD}n^A{}_an^B{}_bn^C{}_cn^D{}_d,\end{aligned}\tag{4.19}$$

with the expansion

$$\begin{aligned}\phi_{ABCD} &:= \phi_{abcd}^{(0)}n^a{}_An^b{}_Bn^c{}_Cn^d{}_D + 4\phi_{abcd}^{(1)}n_{(A}{}^an_{B}{}^bn_{C}{}^cn_{D)}{}^d + 6\phi_{abcd}^{(2)}n_{(A}{}^an_{B}{}^bn_{C}{}^ck_{D)}{}^d \\ &+ 4\phi_{abcd}^{(3)}n_{(A}{}^ak_{B}{}^bk_{C}{}^ck_{D)}{}^d + \phi_{abcd}^{(4)}k_A{}^ak_B{}^bk_C{}^ck_D{}^d.\end{aligned}\tag{4.20}$$

The little group-valued quantities in Eq. (4.19) may be further broken down into irreducible representations, and a detailed classification given of the separate types and subtypes of Weyl spinor depending on which of these vanish. One can also describe these, in complex Minkowski spacetime, in terms of spinor alignment [33].

For the higher spin field strengths there is a similar, albeit more complex structure of field types. For example, for spin three the field strength ϕ_{ABCDEF} will give rise to seven little group-valued quantities $\phi_{abcdef}^{(q)}$, ($q = 1, \dots, 7$) upon contracting with the n_A^a, k_A^a fields, which will then generate a range of little group irreducible components and a variety of types of field depending on which of these vanish. The double copy and multicopy will yield relationships between the types of Maxwell spinor and higher spin field strength. The Maxwell spinor for the constant electric field mentioned above, for example, is not of the most general type as $\phi_{(ab)}^{(1)} = 0$, and this is reflected in the result that the only nonzero gravitational quantity is $\phi_{abcd}^{(2)} \sim \frac{1}{2}(\epsilon_{ac}\epsilon_{bd} + \epsilon_{bc}\epsilon_{ad})(\text{Tr}\phi^{(1)})^2$. Similarly, for the Maxwell field for the Myers-Perry solution in (3.40), the symmetric part of $\phi_{ab}^{(1)}$ vanishes.

An analogous discussion to the above applies to the multicopy for a spin s field strength, based on the relationship $\phi_{A_1, \dots, A_{2s}} \sim \phi_{(A_1 A_2 \dots A_{2s-1} A_{2s})}$.

V. VECTOR SUPERSPACE AND CONTINUOUS SPIN

Returning now to four dimensions, the classification of massless particles includes a “continuous spin” particle

of type I as $\phi_{(ab)}^{(1)}$ or $\phi_{tr}^{(1)}$ may independently vanish or not. An example is a constant electric field which has $\phi_{(ab)}^{(1)} = 0$ but $\phi_{tr}^{(1)} \neq 0$, whereas a constant magnetic field has $\phi_{(ab)}^{(1)} \neq 0$ but $\phi_{tr}^{(1)} = 0$ [33].

For the Weyl spinor ϕ_{ABCD} one follows an analogous path, defining five little group-valued quantities

with nonzero spin Casimir $W^2 = -\rho^2$, for a parameter $\rho > 0$ [34]. In this case one has an infinite tower of helicity states which are mixed under Lorentz transformations, with the mixing depending on ρ . A gauge theory construction of a bosonic CSP was found in [16] (see also [35–38] for reviews). This approach uses a vector superspace, and has been generalized in various ways [36, 39–42], see also [43] for references. Alternative approaches to the CSP have been presented in [44, 45] and the relationships between the different formulations discussed in [35, 41, 46]. Interactions have also been explored in a number of papers [43, 47–54].

The vector superspace approach to the bosonic continuous spin particle uses a vector coordinate η^μ ($\mu = 1, \dots, 4$), and the continuous spin field

$$\Psi(x, \eta) = \sum_{s=0}^{\infty} \phi_{\mu_1 \dots \mu_s}^{(s)} \eta^{\mu_1} \dots \eta^{\mu_s}, \tag{5.1}$$

with spin s component spacetime fields $\phi_{\mu_1 \dots \mu_s}^{(s)}$. The CSP action, equations of motion, and gauge invariances for the field Ψ have been presented in [16], and further discussed in [47]. The theory contains the continuous spin parameter ρ , and for $\rho \rightarrow 0$ and a partial gauge-fixing the equations of motion and symmetries reduce to the Fronsdal equations (2.6) for each component of Ψ . In the following we highlight some key points of this discussion, notably from Sec. 2 of [47], before applying this to the Kerr-Schild field.

The equations of motion for the case $\rho = 0$ are

$$\delta'(\eta^2 + 1)\square\Psi - \frac{1}{2}\Delta(\delta(\eta^2 + 1)\Delta\Psi) = 0, \tag{5.2}$$

where $\Delta = \partial \cdot \tilde{\partial}$, with $\tilde{\partial}_\mu := \frac{\partial}{\partial \eta^\mu}$. The symmetries of (5.2) are

$$\delta_\epsilon \Psi = D\epsilon := \left(\eta \cdot \partial - \frac{1}{2}(\eta^2 + 1)\Delta \right) \epsilon(\eta, x), \quad (5.3)$$

where ϵ can be expressed as an expansion of coefficient fields times powers of η . Gauge invariance follows straightforwardly using the identities

$$\begin{aligned} \Delta(D\epsilon) &= \square \epsilon - \frac{1}{2}(\eta^2 + 1)\Delta^2 \epsilon, \\ \delta'(\eta^2 + 1)D\epsilon &= \frac{1}{2}\Delta(\delta(\eta^2 + 1)\epsilon). \end{aligned} \quad (5.4)$$

The formulation uses certain polynomials in the η coordinates

$$P_{(n)}^{\mu_1 \dots \mu_n} = 2^{n/2} \left(\eta^{\mu_1} \dots \eta^{\mu_n} - \frac{n(n-1)}{4} g^{(\mu_1 \mu_2} \eta^{\mu_3} \dots \eta^{\mu_n)} (\eta^2 + 1) \right) \quad (5.5)$$

(to avoid potential confusion with the η coordinates, $g_{\mu\nu}$ denotes the flat space metric in this section). It is convenient to expand the field Ψ in components as

$$\Psi = \sum_{n \geq 0} P_{(n)}(\eta) \phi^{(n)}(x). \quad (5.6)$$

This shift in the definition of the component fields has the effect of decoupling them at each level, simplifying the analysis. We similarly define the component fields of the gauge parameter as

$$\epsilon = \sum_{n \geq 0} P_{(n)}(\eta) \epsilon^{(n)}(x). \quad (5.7)$$

The equations of motion for the component fields of Ψ at each level are then obtained by applying the polynomial $P_{(n)}$ to the equations of motion (5.2) and integrating over the η coordinates. The $P_{(n)}$ satisfy certain orthogonality relations which have the effect of projecting the equations of motion onto individual and independent equations for the component fields of Ψ . The outcome is the Fronsdal equations (2.6) for each component. This requires a partial gauge-fixing which imposes a double tracelessness condition on the component fields and reduces the gauge symmetry to the standard one

$$\delta_\epsilon \Phi = \eta \cdot \partial \epsilon(\eta, x), \quad (5.8)$$

with the components of ϵ now traceless. We note in passing that the η integrals may be evaluated using the results

$$\begin{aligned} \int d^4 \eta \delta(\eta^2 + 1) F(\eta) &= \sum_{n=0} c_n (\tilde{\partial})^{2n} F(\eta)|_{\eta=0}, \\ \int d^4 \eta \delta'(\eta^2 + 1) F(\eta) &= \sum_{n=0} d_n (\tilde{\partial})^{2n} F(\eta)|_{\eta=0}, \end{aligned} \quad (5.9)$$

for any function $F(\eta)$, with $c_n = \frac{(-1)^n}{4^n n! (n+1)!}$ and $d_n = \frac{(-1)^n}{4^n (n!)^2}$.

Now define $E(\Psi)$ to be the left-hand side of the equations of motion (5.2). Then

$$E_{(s)}^{\mu_1 \dots \mu_s}(\Psi) := \int d^4 \eta P_{(s)}^{\mu_1 \dots \mu_s} E(\Psi) \quad (5.10)$$

are the spin s Fronsdal equations of motion, and the relation (2.5) then implies that $E_{(s)}^{\mu_1 \dots \mu_s}(\Phi^{\text{KS}})$ vanishes for all $s > 2$, since the equations for $s = 0, 1, 2$ are satisfied. Thus all the components of $E(\Phi^{\text{KS}})$ are zero and the Kerr-Schild field satisfies the equations of motion (5.2).

We can also translate our higher spin Kerr-Schild solutions to the superspace approach

$$\hat{\Psi} = f(k \cdot \eta) \phi, \quad (5.11)$$

where k^μ and ϕ satisfy the properties (2.2), (2.4) and f is an arbitrary function. We can show that this satisfies the higher spin field equations (5.2) directly for $\rho = 0$. First we rewrite the equation of motion (5.2) as follows:

$$\delta'(\eta^2 + 1) [\square \Psi - (\eta \cdot \partial) \Delta \Psi] - \frac{1}{2} \delta(\eta^2 + 1) \Delta^2 \Psi = 0. \quad (5.12)$$

Now we find that

$$\Delta \hat{\Psi} = \partial \cdot \tilde{\partial} f(k \cdot \eta) \phi = f'(k \cdot \eta) \partial^\nu (k_\nu \phi), \quad (5.13)$$

where we have used the fact that $(k \cdot \partial) k_\mu = 0$. Similarly, it is immediate that

$$\Delta^2 \hat{\Psi} = f''(k \cdot \eta) \partial^\mu (k_\mu \partial^\nu (k_\nu \phi)) = f''(k \cdot \eta) \partial^\mu \partial^\nu (k_\mu k_\nu \phi). \quad (5.14)$$

If we contract the flat metric $g^{\mu\nu}$ into the last relation of (2.4) we find

$$\begin{aligned} g^{\mu\nu} (\square (k_\mu k_\nu \phi) - 2 \partial^\rho \partial_{(\mu} (k_{\nu)} k_\rho \phi) + \partial_\mu \partial_\nu (k^\rho k_\rho \phi)) &= 0 \\ \Rightarrow \partial^\rho \partial^\nu (k_\nu k_\rho \phi) &= 0, \end{aligned} \quad (5.15)$$

where we have used $k^2 = 0$. Comparing (5.14) and (5.15) we find that $\Delta^2 \hat{\Psi} = 0$. Now let us consider the first term in (5.12) which is proportional to

$$\square \Psi - (\eta \cdot \partial) \Delta \Psi. \quad (5.16)$$

This also turns out to be zero for the field (5.1). For the first and second terms above we find

$$\square \hat{\Psi} = f''(k \cdot \eta) [(\partial^\mu(k \cdot \eta))(\partial_\mu(k \cdot \eta))\phi] + f'(k \cdot \eta) [\eta^\mu \square(k_\mu \phi)], \quad (5.17)$$

$$(\eta \cdot \partial) \Delta \hat{\Psi} = f''(k \cdot \eta) [(\eta \cdot \partial(k \cdot \eta))(\partial^\nu(k_\nu \phi))] + f'(k \cdot \eta) [\eta \cdot \partial \partial^\nu(k_\nu \phi)]. \quad (5.18)$$

When combined the second terms in the expressions above cancel due to the spin-one equation in (2.4) and we obtain

$$\square \hat{\Psi} - (\eta \cdot \partial) \Delta \hat{\Psi} = f(k \cdot \eta) [(\partial^\mu(k \cdot \eta))(\partial_\mu(k \cdot \eta))\phi - (\eta \cdot \partial(k \cdot \eta))(\partial^\nu(k_\nu \phi))]. \quad (5.19)$$

To simplify this, note that we can write the nonzero terms of the spin-two equation in (2.4) as

$$\eta^\mu \eta^\nu \square(k_\mu k_\nu \phi) = 2(k \cdot \eta) \eta^\mu \square(k_\mu \phi) + 2(\partial^\mu(k \cdot \eta))(\partial_\mu(k \cdot \eta))\phi, \quad (5.20)$$

$$\eta^\mu \eta^\nu 2\partial^\rho \partial_\mu(k_\nu k_\rho \phi) = 2(\eta \cdot \partial(\eta \cdot k))(\partial^\rho(k_\rho \phi)) + 2(\eta \cdot k) \eta \cdot \partial(\partial^\rho(k_\rho \phi)), \quad (5.21)$$

which allows us to express (5.19) as

$$\square \hat{\Psi} - (\eta \cdot \partial) \Delta \hat{\Psi} = f''(k \cdot \eta) \left[\frac{1}{2} \eta^\mu \eta^\nu \left(\square(k_\mu k_\nu \phi) - 2\partial^\rho \partial_\mu(k_\nu k_\rho \phi) \right) - (k \cdot \eta) \eta^\mu \left(\square(k_\mu \phi) - \partial_\mu(\partial^\rho(k_\rho \phi)) \right) \right] = 0. \quad (5.22)$$

This is zero since the terms in brackets above are exactly the spin-one and spin-two field equations from (2.4). By taking $f(k \cdot \eta) = (k \cdot \eta)^s$ we recover the result from [22,24] that the Fronsdal equations are satisfied by the higher spin copies of the Kerr-Schild fields.

Now consider the plane wave solutions discussed in [47]. These take the form

$$\begin{aligned} \Psi_{h,l} &= e^{-il \cdot x} \psi_{h,l}(\eta) := e^{-il \cdot x} (i\eta \cdot \epsilon_+)^h \quad (h \geq 0), \\ \Psi_{h,l} &= e^{-il \cdot x} \psi_{h,l}(\eta) := e^{-il \cdot x} (-i\eta \cdot \epsilon_-)^h \quad (h \leq 0) \end{aligned} \quad (5.23)$$

where h is the helicity. We use the symbol l rather than the traditional k to avoid confusion with the Kerr-Schild vector k . We use the null complex frame vectors $(l^\mu, q^\mu, \epsilon_+^\mu, \epsilon_-^\mu)$, where ϵ_-^μ is the complex conjugate of ϵ_+^μ and the nonzero inner products are $q \cdot l = 1$ and $\epsilon_+ \cdot \epsilon_- = 2$. The fields (5.23) satisfy (5.2) since both $\square$ and Δ annihilate them.

One can expand a general field $\Psi(\eta, x)$ in modes as

$$\Psi(\eta, x) = \int \frac{d^3 \mathbf{l}}{(2\pi)^3 2|\mathbf{l}|} \sum_h (a_h(\mathbf{l}) \psi_{h,l}(\eta) e^{-il \cdot x} + \text{c.c.}) \Big|_{l^0=|\mathbf{l}|}, \quad (5.24)$$

where the coefficients $a_h(\mathbf{l})$ are given by

$$a_h(\mathbf{l}) = 2|\mathbf{l}| \int d^4 \eta \delta'(\eta^2 + 1) \psi_{h,l}^*(\eta) \Psi(\eta, l) \Big|_{l^0=|\mathbf{l}|}. \quad (5.25)$$

As an example, for the Kerr-Schild solution $\Phi^{\text{KS}} := e^{k \cdot \eta} \phi$ one finds the formulas

$$\begin{aligned} a_{+h}(\mathbf{l}) &= 2|\mathbf{l}| \frac{i^h}{2^h h!} \int dx e^{-il \cdot x} (k \cdot \epsilon_+)^h \phi(x), \\ a_{-h}(\mathbf{l}) &= 2|\mathbf{l}| \frac{(-i)^h}{2^h h!} \int dx e^{-il \cdot x} (k \cdot \epsilon_-)^h \phi(x), \end{aligned} \quad (5.26)$$

where k here is the Kerr-Schild vector, which depends on the coordinates x^μ .

A. Continuous spin

Now consider the continuous spin case where $\rho \neq 0$. Here the equations of motion and gauge symmetries are obtained by simply replacing Δ by $\Delta + \rho$ [16,47], whence

$$\delta'(\eta^2 + 1) \square \Psi - \frac{1}{2} (\Delta + \rho) (\delta(\eta^2 + 1) (\Delta + \rho) \Psi) = 0, \quad (5.27)$$

and

$$\delta_\epsilon \Psi = \left(\eta \cdot \partial - \frac{1}{2} (\eta^2 + 1) (\Delta + \rho) \right) \epsilon(\eta, x). \quad (5.28)$$

This introduces interactions between the particles of different spin. Note that we can rewrite the equations of motion (5.27) as

$$\begin{aligned} \delta'(\eta^2 + 1) \left(\square \Psi - \eta \cdot \partial (\Delta + \rho) \Psi \right. \\ \left. + \frac{1}{2} (\eta^2 + 1) (\Delta + \rho)^2 \Psi \right) = 0, \end{aligned} \quad (5.29)$$

where we have used the distributional identity

$$(\eta^2 + 1)\delta'(\eta^2 + 1) = -\delta(\eta^2 + 1). \quad (5.30)$$

This also implies that $(\eta^2 + 1)^2\delta'(\eta^2 + 1) = 0$.

Using the fact that the Kerr-Schild field Φ^{KS} satisfies the $\rho = 0$ equations of motion, we would like to find a generalization of this which solves (5.27). While we have not succeeded in this, we note that there are some known solutions—these are the plane waves given in [47], which are a simple extension of the solutions (5.23) given by

$$\begin{aligned} \tilde{\Psi}_{h,l} &= e^{-il.x} \tilde{\psi}_{h,l}(\eta) := e^{-il.x} e^{-i\rho\eta.q} (i\eta \cdot \epsilon_+)^h \quad (h \geq 0), \\ \tilde{\Psi}_{h,l} &= e^{-il.x} \tilde{\psi}_{h,l}(\eta) := e^{-il.x} e^{-i\rho\eta.q} (-i\eta \cdot \epsilon_-)^h \quad (h \leq 0) \end{aligned} \quad (5.31)$$

where q^μ is one of the null frame vectors introduced above. These plane waves are annihilated by both $\square$ and $\Delta + \rho$ and so solve the equations of motion (5.27).

If a solution $\Psi^{(\rho)}$ of (5.27) has the component-field expansion

$$\Psi^{(\rho)} = \sum_{s=0} \phi_{\mu_1 \dots \mu_s}^{(\rho)} \eta^{\mu_1} \dots \eta^{\mu_s}, \quad (5.32)$$

then the Fourier modes are given by

$$\begin{aligned} a_{+h}^{(\rho)}(\mathbf{l}) &= 2|\mathbf{l}|^h \sum_{m \geq h} \int dx \frac{(-1)^m}{2^m} \frac{(i\rho)^{m-h}}{(m-h)!} \\ &\quad \times \epsilon_+^{\mu_1} \dots \epsilon_+^{\mu_h} q^{\mu_{h+1}} \dots q^{\mu_m} \phi_{\mu_1 \dots \mu_m}^{(\rho)}, \\ a_{-h}^{(\rho)}(\mathbf{l}) &= 2|\mathbf{l}|^h (-i)^h \sum_{m \geq h} \int dx \frac{(-1)^m}{2^m} \frac{(-i\rho)^{m-h}}{(m-h)!} \\ &\quad \times \epsilon_-^{\mu_1} \dots \epsilon_-^{\mu_h} q^{\mu_{h+1}} \dots q^{\mu_m} \phi_{\mu_1 \dots \mu_m}^{(\rho)}, \end{aligned} \quad (5.33)$$

where the component fields have been assumed to be traceless in the above. As would be expected, these formulas reflect the fact that the parameter ρ mediates interactions between particles of different helicities. It may be possible to take the Fourier coefficients for Φ^{KS} at $\rho = 0$ in (5.26) and multiply them by the ρ -dependent Fourier modes of (5.33) to generate a CSP version of the Kerr-Schild solution. However, we were not able to perform the inverse Fourier transform to obtain this result.

B. AdS

We now turn to consider how the discussions above may generalize to anti-de Sitter (AdS) backgrounds. (Formulations of a continuous spin particle in AdS space using the alternative light-cone, framelike, and Becchi-Rouet-Stora-Tyutin (BRST) approaches have been given in [44,52,55–62].)

Higher spin field equations with Kerr-Schild solutions for the AdS spacetime have been studied in [24,63]. We will use the notation of [24] in the following (however we will use the metric with signature $+- --$). Thus consider the d -dimensional metric $g_{\mu\nu}$ given by⁸

$$g_{\mu\nu} dx^\mu dx^\nu = (1 + \lambda r^2) dt^2 - \frac{1}{1 + \lambda r^2} dr^2 - r^2 d\Omega_{d-2}^2, \quad (5.34)$$

with $d\Omega_{d-2}^2$ the metric on the unit sphere in $(d-2)$ -dimensions and λ the cosmological constant. The AdS covariant derivative ∇_μ satisfies

$$[\nabla_\mu, \nabla_\nu] X_\rho = 2\lambda g_{\rho[\mu} X_{\nu]}, \quad (5.35)$$

for a covector X_μ . The higher spin equations on this spacetime are

$$\begin{aligned} \nabla^2 \phi_{\mu_1 \dots \mu_s} - s \nabla_{(\mu_1} \nabla^{\rho} \phi_{\mu_2 \dots \mu_s) \rho} + \frac{1}{2} s(s-1) \nabla_{(\mu_1} \nabla_{\mu_2} \phi_{\mu_3 \dots \mu_s) \rho}{}^{\rho} \\ + \lambda s(s-1) g_{(\mu_1 \mu_2} \phi_{\mu_3 \dots \mu_s) \rho}{}^{\rho} + \lambda ((s-2)(d+s-3) - s) \phi_{\mu_1 \dots \mu_s} = 0, \end{aligned} \quad (5.36)$$

with the double trace of ϕ vanishing. These have the gauge invariance

$$\delta \phi_{\mu_1 \dots \mu_s} = s \nabla_{(\mu_1} \Lambda_{\mu_2 \dots \mu_s)} \quad (5.37)$$

for a traceless gauge parameter field $\Lambda_{\mu_1 \dots \mu_{s-1}}$.

One can then consider field strengths in this AdS background. Considering four dimensions for the moment, the spin one field strength is just $H_{\mu\nu} = 2\nabla_{[\mu} \phi_{\nu]}$, while for spin two the gauge invariant field strength is

$$H_{\mu\nu\rho\sigma} = 4(\nabla_\rho \nabla_\mu \phi_{\nu\sigma} + \lambda g_{\rho\mu} \phi_{\nu\sigma})|_s, \quad (5.38)$$

where $|_s$ here means to antisymmetrize in the pairs (μ, ν) and (ρ, σ) .

The Kerr-Schild higher spin solutions are given by

$$\phi_{\mu_1 \dots \mu_s} = k_{\mu_1} \dots k_{\mu_s} \phi, \quad (5.39)$$

where the scalar field and Kerr-Schild vector are given by

⁸We have also changed the sign of λ relative to [24].

$$\phi = \frac{2}{r}, \quad k_\mu = \left(1, \frac{1}{1+\lambda r^2}, 0, 0\right). \quad (5.40)$$

The field strength (5.38) is then traceless and divergence-free when the solution $\phi_{\mu\nu} = k_\mu k_\nu \phi$ is inserted.

In a similar way one can write down the gauge invariant spin-3 field strength

$$H_{\mu\nu\rho\sigma\lambda\tau} = 8(\nabla_{(\lambda}\nabla_\rho\nabla_\mu)\phi_{\nu\sigma\tau} + 4\lambda g_{(\lambda\rho}\nabla_\mu)\phi_{\nu\sigma\tau})|_s, \quad (5.41)$$

where here $|_s$ means to antisymmetrize in the pairs (μ, ν) , (ρ, σ) , (λ, τ) . Then $H_{\mu\nu\rho\sigma\lambda\tau}$ is traceless and divergence-free when the spin-3 solution $\phi_{\nu\sigma\tau} = k_\nu k_\sigma k_\tau \phi$ is inserted into its definition. These results generalize to higher spins with additional terms coming in; for example for spin-4 the invariant on-shell traceless and divergence-free field strength is given by

$$H_{\mu\nu\rho\sigma\lambda\tau\alpha\beta} = 16(\nabla_{(\alpha}\nabla_\lambda\nabla_\rho\nabla_\mu) + 10\lambda g_{(\alpha\lambda}\nabla_\rho\nabla_\mu) + 9\lambda^2 g_{(\alpha\lambda}g_{\rho\mu})\phi_{\nu\sigma\tau\beta})|_s, \quad (5.42)$$

where here $|_s$ means to antisymmetrize in the pairs (μ, ν) , (ρ, σ) , (λ, τ) , (α, β) .

When the Kerr-Schild solutions (5.39) are inserted into these field strengths, one can derive tensor expressions for the multicopies. For example the double copy is given by

$$H_{\mu\nu\rho\sigma}(\phi^{KS}) = 2rC[H]_{\mu\nu\rho\sigma}, \quad (5.43)$$

where $C[H]_{\mu\nu\rho\sigma}$ is the expression on the right-hand side of (3.5), with $F_{\mu\nu}$ replaced by $H_{\mu\nu}$ and the AdS metric used throughout instead of the flat metric. Similarly the triple copy is given by

$$H_{\mu\nu\rho\sigma\lambda\tau}(\phi^{KS}) = 30r^2C[H]_{\mu\nu\rho\sigma\lambda\tau}, \quad (5.44)$$

where $C[H]_{\mu\nu\rho\sigma\lambda\tau}$ is the expression on the right-hand side of (3.10), with again $F_{\mu\nu}$ replaced by $H_{\mu\nu}$ and the AdS metric used instead of the flat metric. Note that these expressions, and their generalization to higher spin, are tensor forms of the spinor multicopy formula given in [23].⁹

The above results generalize to a Kerr black hole in AdS where it is useful to write the AdS metric in spheroidal coordinates [24]

$$g_{\mu\nu}dx^\mu dx^\nu = W(1 + \lambda r^2)dt^2 - Fdr^2 - \frac{r^2 + a^2 \cos^2 \theta}{1 - \lambda a^2 \cos^2 \theta}d\theta^2 - \frac{(r^2 + a^2) \sin^2 \theta}{1 - \lambda a^2}d\phi^2, \quad (5.45)$$

⁹We thank Slava Didenko for pointing this out, which led to multicopy results above being added.

with $W = \frac{1-\lambda a^2 \cos^2(\theta)}{1-\lambda a^2}$ and $F = \frac{r^2 + a^2 \cos^2(\theta)}{(1+\lambda r^2)(r^2 + a^2)}$, and with the scalar field and Kerr-Schild vector now given by

$$\phi = \frac{2r}{r^2 + a^2 \cos^2 \theta}, \quad k_\mu = \left(F, W, 0, -\frac{a \sin^2 \theta}{1 - \lambda a^2}\right). \quad (5.46)$$

The equations of motion and field strengths are then given by the formulas presented above for AdS, but using the scalar and Kerr-Schild vector from (5.46).

C. Vector superspace

Returning to d dimensions, a vector superspace equation of motion in AdS space has been given in [64], and generalized to fermionic fields in [65] (along with correcting a factor in [64]). Both [64] are [65] are based on the derivative

$$\bar{\nabla}'_\mu = \partial_\mu + \Gamma_{\mu\nu}^\rho \eta_\rho \tilde{\partial}^\nu \quad (5.47)$$

with $\Gamma_{\mu\nu}^\rho$ the spacetime connection. This derivative fails to satisfy some expected identities such as $[\bar{\nabla}'_\mu, g_{\rho\sigma}]A^{\alpha_1 \dots \alpha_n} = 0$ and $[\bar{\nabla}'_\mu, \eta_\nu]A^{\alpha_1 \dots \alpha_n} = 0$, where $A^{\alpha_1 \dots \alpha_n}$ is any function of the spacetime coordinates and the η_μ .

Let us instead define a covariant derivative as¹⁰

$$\bar{\nabla}_\mu = \nabla_\mu + \Gamma_{\mu\nu}^\rho \eta_\rho \tilde{\partial}^\nu = \nabla_\mu - \nabla_\mu(\eta_\nu) \tilde{\partial}^\nu \quad (5.48)$$

(we assume that $\partial_\mu \eta_\nu = 0$), where ∇_μ is the usual spacetime covariant derivative for some metric $g_{\mu\nu}$. In flat space the η_μ vectors are simply coordinates in an additional vector space. However, in curved space it is natural to identify η_μ with coordinates on the cotangent bundle associated with a particular coordinate patch (as noted in [64]). To be concrete, given a chart with coordinates x^μ for some manifold M there are canonical coordinates on the cotangent bundle T^*M given by (x, η) where

$$(x, \eta) \rightarrow (x^\mu, \eta_\nu dx^\nu) \in T^*M. \quad (5.49)$$

The two coordinates are independent so $\partial_\mu \eta_\nu = 0$, and any particular fixed choice of η_μ defines a one-form (at least locally) to which we can apply the usual covariant derivative, as used in (5.48). We will raise and lower indices on η_μ using the metric and, from the above definitions, under a coordinate transformation $x^\mu \rightarrow x'^\mu$ (and the induced transformation $\eta_\mu \rightarrow \eta'_\mu$) we have

¹⁰This type of covariant derivative has also appeared in [66,67].

$$\begin{aligned}\partial_{\mu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu + \left(\frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^\nu} \right) \eta_\nu \tilde{\partial}_\nu, & \eta_{\mu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \eta_\mu, \\ \tilde{\partial}^{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^\mu} \tilde{\partial}^\mu \Rightarrow \bar{\nabla}_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \bar{\nabla}_\mu.\end{aligned}\quad (5.50)$$

Note that the extra term in the transformation law of the spatial derivative ∂_μ implies that the usual derivative ∇_μ does not transform covariantly when acting on a function of x and η . Indeed, this is one motivation for introducing (5.48) in the first place (see Appendix for more details). It is also worth mentioning that (5.48) is useful even in flat space when using coordinates where $\Gamma_{\mu\nu}^\rho$ is nonzero. As will be made clear in later examples, $\bar{\nabla}_\mu$ acts on expressions with indices in the usual way *including* those indices on η_μ and $\tilde{\partial}^\mu$. (The definition (5.48) may be compared with the natural metric on the co-tangent bundle, discussed in [68,69] for example.) This definition has several nice properties which we list below¹¹:

$$[\bar{\nabla}_\mu, g_{\rho\sigma}] A^{\alpha_1 \dots \alpha_n} = 0, \quad (5.51)$$

$$[\bar{\nabla}_\mu, \eta_\nu] A^{\alpha_1 \dots \alpha_n} = 0, \quad (5.52)$$

$$[\bar{\nabla}_\mu, \tilde{\partial}^\nu] A^{\alpha_1 \dots \alpha_n} = 0, \quad (5.53)$$

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu] f = R_{\beta\mu\nu}^\alpha \eta_\alpha \tilde{\partial}^\beta f, \quad (5.54)$$

$$[\bar{\nabla}_\mu, \bar{\nabla}_\nu] X_\sigma = R_{\beta\mu\nu}^\alpha \eta_\alpha \tilde{\partial}^\beta X_\sigma + R_{\mu\nu\sigma}^\alpha X_\alpha, \quad (5.55)$$

$$\begin{aligned}\bar{\nabla}_\mu (A^{\alpha_1 \dots \alpha_n} B^{\beta_1 \dots \beta_m}) &= (\bar{\nabla}_\mu A^{\alpha_1 \dots \alpha_n}) B^{\beta_1 \dots \beta_m} \\ &+ A^{\alpha_1 \dots \alpha_n} (\bar{\nabla}_\mu B^{\beta_1 \dots \beta_m}),\end{aligned}\quad (5.56)$$

$$\bar{\nabla}_\mu (\eta_{\alpha_1} \dots \eta_{\alpha_n} A^{\alpha_1 \dots \alpha_n}) = \eta_{\alpha_1} \dots \eta_{\alpha_n} \bar{\nabla}_\mu A^{\alpha_1 \dots \alpha_n}, \quad (5.57)$$

where all of the tensors $A^{\alpha_1 \dots}, f, X^\mu, Y^\mu$ are arbitrary functions of x and η . If $\bar{\nabla}_\mu$ acts on an expression that does not depend on η it simply reduces to the usual covariant derivative.

Acting on a scalar superfield, one then has the identities

$$\begin{aligned}[\bar{\Delta}, \eta \cdot \bar{\nabla}] &= \bar{\nabla}^2 + R^\alpha_{\beta} \eta_\alpha \tilde{\partial}^\beta - R^\alpha_{\beta}{}^\mu{}_\nu \eta_\alpha \eta_\mu \tilde{\partial}^\beta \tilde{\partial}^\nu, \\ [\bar{\nabla}^2, \eta \cdot \bar{\nabla}] &= R^{\mu\nu} \eta_\mu \bar{\nabla}_\nu + \nabla_\mu (R^\alpha_{\beta}{}^{\mu\nu} \eta_\nu \eta_\alpha \tilde{\partial}^\beta \\ &+ 2R^\alpha_{\beta}{}^{\mu\nu} \eta_\nu \eta_\alpha \tilde{\partial}^\beta \bar{\nabla}_\mu), \\ [\bar{\nabla}^2, \bar{\Delta}] &= (\nabla^\nu R_{\nu\mu}{}^\alpha{}_\beta) \eta_\alpha \tilde{\partial}^\mu \tilde{\partial}^\beta - (\nabla^\nu R_{\nu\beta}) \tilde{\partial}^\beta \\ &- R_{\mu\alpha} \bar{\nabla}^\alpha \tilde{\partial}^\mu + 2R_{\nu\mu}{}^\alpha{}_\beta \eta_\alpha \bar{\nabla}^\nu \tilde{\partial}^\mu \tilde{\partial}^\beta,\end{aligned}\quad (5.58)$$

¹¹Equation (5.54) indicates that there is torsion in this super-space, if we think of $\tilde{\partial}^\beta$ as the covariant derivative in the η direction.

where $\bar{\Delta} = \tilde{\partial} \cdot \bar{\nabla}$ and $\bar{\nabla}^2 = \bar{\nabla}^\mu \bar{\nabla}_\mu$. In the AdS case these reduce to¹²

$$\begin{aligned}[\bar{\Delta}, \eta \cdot \bar{\nabla}] &= \bar{\nabla}^2 + \lambda((d-1)\eta \cdot \tilde{\partial} + \eta^\mu \eta^\nu \tilde{\partial}_\mu \tilde{\partial}_\nu - \eta^2 \tilde{\partial}^2), \\ [\bar{\nabla}^2, \eta \cdot \bar{\nabla}] &= \lambda(2\eta \cdot \bar{\nabla} \eta \cdot \tilde{\partial} - 2\eta^2 \bar{\Delta} + (d-1)\eta \cdot \bar{\nabla}), \\ [\bar{\nabla}^2, \bar{\Delta}] &= \lambda(2\eta \cdot \bar{\nabla} \tilde{\partial}^2 - (d-1)\bar{\Delta} - 2\eta \cdot \tilde{\partial} \bar{\Delta}).\end{aligned}\quad (5.59)$$

Using these, one can show that the following equation of motion¹³

$$\begin{aligned}\delta'(\eta^2 + 1) \left[\bar{\nabla}^2 \Psi - (\eta \cdot \bar{\nabla}) \bar{\Delta} \Psi + \frac{1}{2}(\eta^2 + 1) \bar{\Delta}^2 \Psi \right. \\ \left. + \lambda((\eta \cdot \tilde{\partial})^2 + (d-6)\eta \cdot \tilde{\partial} - 2(d-3) + \eta^2 \tilde{\partial}^2 + 2\tilde{\partial}^2) \Psi \right] = 0\end{aligned}\quad (5.60)$$

is invariant under the gauge transformations

$$\delta_\epsilon \Psi = \left(\eta \cdot \bar{\nabla} - \frac{1}{2}(\eta^2 + 1) \bar{\Delta} \right) \epsilon + (\eta^2 + 1)^2 \xi, \quad (5.61)$$

with the second-order gauge symmetry

$$\delta \epsilon = (\eta^2 + 1) \Lambda, \quad \delta \xi = \frac{1}{2} \bar{\Delta} \Lambda. \quad (5.62)$$

The equation of motion (5.60) can be obtained by varying the action¹⁴

$$\begin{aligned}\int d^d x d^d \eta \left[\frac{1}{2} \delta'(\eta^2 + 1) (\bar{\nabla}_\mu \Psi) (\bar{\nabla}^\mu \Psi) + \frac{1}{4} \delta(\eta^2 + 1) (\bar{\Delta} \Psi)^2 \right. \\ \left. + \delta'(\eta^2 + 1) \frac{\lambda}{2} \Psi ((\eta \cdot \tilde{\partial})^2 + (d-6)\eta \cdot \tilde{\partial} - 2(d-3) \right. \\ \left. + \eta^2 \tilde{\partial}^2 + 2\tilde{\partial}^2) \Psi \right].\end{aligned}\quad (5.63)$$

Note that since we are integrating over the cotangent bundle the integration measure no longer contains any factors of $\sqrt{|g|}$, as would be expected when just integrating over just the base manifold. This is due to the fact that, under a coordinate transformation $(x^\mu, \eta_\nu) \rightarrow (x^{\mu'}, \eta_{\nu'})$,

¹²In AdS space we have $R_{\mu\nu\rho\sigma} = \lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ and $R_{\mu\nu} = \lambda(d-1)g_{\mu\nu}$.

¹³This is equivalent to Eq. (2.25) in [65] but is based on the covariant derivative (5.48) rather than (5.47).

¹⁴We thank Mojtaba Najafizadeh for asking if our equation of motion could be obtained from an action, which led us to add the explanation below.

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} dx^{\mu}, \quad d\eta_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} d\eta_{\mu} + \left(\frac{\partial^2 x^{\nu}}{\partial x^{\mu} \partial x^{\mu'}} \right) \eta_{\nu} dx^{\mu}. \quad (5.64)$$

When we wedge dx and $d\eta$ the extra term in the transformation rule of $d\eta$ disappears and so the correct invariant measure is

$$(\sqrt{|g|} d^d x) \left(\frac{1}{\sqrt{|g|}} d^d \eta \right) = d^d x d^d \eta, \quad (5.65)$$

which is just the standard measure on phase space. This measure also has the property that the derivative (5.48) satisfies the usual integration-by-parts relations. To show this consider

$$\begin{aligned} \int d^d x d^d \eta \bar{\nabla}_{\mu}(X^{\mu}) &= \int d^d x d^d \eta \nabla_{\mu}(X^{\mu}) \\ &\quad - \int d^d x d^d \eta \nabla_{\mu}(\eta_{\nu}) \tilde{\partial}^{\nu}(X^{\mu}), \end{aligned} \quad (5.66)$$

where X^{μ} is an arbitrary vector field depending on x, η . The two terms above cancel each other, as we will now demonstrate. First we write

$$\begin{aligned} \int d^d x d^d \eta \nabla_{\mu}(\eta_{\nu}) \tilde{\partial}^{\nu}(X^{\mu}) &= \int d^d x d^d \eta \tilde{\partial}^{\nu}(\nabla_{\mu}(\eta_{\nu}) X^{\mu}) \\ &\quad - \int d^d x d^d \eta \tilde{\partial}^{\nu}(\nabla_{\mu}(\eta_{\nu})) X^{\mu} \\ &= \int d^d x d^d \eta \Gamma_{\mu\nu}^{\nu} X^{\mu}, \end{aligned} \quad (5.67)$$

where we have discarded a total derivative in η . If we combine this with the first term in (5.66) we find

$$\begin{aligned} \int d^d x d^d \eta \bar{\nabla}_{\mu}(X^{\mu}) &= \int d^d x d^d \eta \nabla_{\mu}(X^{\mu}) - \Gamma_{\mu\nu}^{\nu} X^{\mu} \\ &= \int d^d x d^d \eta \partial_{\mu}(X^{\mu}) = 0 \end{aligned} \quad (5.68)$$

where the result vanishes since it is a total spacetime derivative. This gives the integration-by-parts identity

$$\begin{aligned} \int d^d x d^d \eta \bar{\nabla}_{\mu}(X^{\mu}) &= 0 \Rightarrow \int d^d x d^d \eta f \bar{\nabla}_{\mu}(X^{\mu}) \\ &= - \int d^d x d^d \eta \bar{\nabla}_{\mu}(f) X^{\mu}. \end{aligned} \quad (5.69)$$

With this one can show explicitly the equation of motion (5.60) can be obtained by varying the action (5.63). Equivalently, one can also show that the operator in (5.60) is Hermitian.

We can now show that there is a generalized AdS Kerr-Schild solution analogous to the flat space expression (5.11) i.e.,

$$\hat{\Psi} = f(k \cdot \eta) \phi, \quad (5.70)$$

where $k^2 = 0 = k^{\mu} \nabla_{\mu} k^{\nu}$ here. Inserting this expression into (5.60), and using similar manipulations to those below (5.11), we find that these equations of motion are satisfied if

$$\begin{aligned} \nabla^2 \phi - 2(d-3)\lambda \phi &= 0, \\ \nabla^2(k^{\mu} \phi) - \nabla^{\mu} \nabla_{\nu}(k^{\nu} \phi) - (d-1)\lambda(k^{\mu} \phi) &= 0, \\ \nabla^2(k^{\mu} k^{\nu} \phi) - 2\nabla^{(\mu} \nabla_{\rho}(k^{\nu)} k^{\rho} \phi) - 2\lambda(k^{\mu} k^{\nu} \phi) &= 0. \end{aligned} \quad (5.71)$$

These are precisely the spin zero, one and two equations from (5.36) if one substitutes the Kerr-Schild fields $\phi \rightarrow \phi$, $\phi_{\mu} \rightarrow k_{\mu} \phi$ and $\phi_{\mu\nu} \rightarrow k_{\mu} k_{\nu} \phi$, with the ϕ on the right-hand sides the field in (5.70), satisfying the first equation in (5.71). If we choose $\hat{\Psi} = (k \cdot \eta)^s \phi$ we recover the result of [24] that the spin s Kerr-Schild field satisfies the AdS higher spin equation of motion (5.36), assuming the spin zero, one and two equations above are satisfied.

It is not clear if there is a continuous spin generalization of (5.60); if one replaces $\bar{\Delta} \rightarrow \bar{\Delta} + \rho$ in these equations of motion and in the gauge invariance (5.61), the order ρ^3 , ρ^2 terms cancel in the variation of the equations of motion, but there is a noncancelling order ρ term $-\rho \lambda \delta'(\eta^2 + 1)(2\eta \cdot \tilde{\partial} + d-1)(\eta^2 + 1)\epsilon$. We note that this contains no space-time derivatives, and so given the presence of these in the gauge transformations it appears unlikely that there is any local correction term that can be added to the equations of motion to obtain invariance.

Another way to explore this question is to consider the Pauli-Lubanski vector

$$W^{\mu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_{\sigma}. \quad (5.72)$$

In [16] it was shown that for their flat space action and symmetries $W^2 = -\rho^2$ up to gauge transformations, as required to describe a CSP (see also [36] for a related discussion). This result does not appear to immediately generalize to AdS.

VI. CONCLUSIONS

In this paper we have discussed a generalization of the Weyl double copy to higher spin “multicopies,” with the linearized higher spin field strengths given by sums of powers of the Maxwell tensor. This utilized Kerr-Schild formulations of the fields. The Fronsdal equations for higher spin fields guarantee that the field strengths are traceless and divergence-free. Various examples were

given. The spinor formulation provided further insights, working from the Penrose transform. The multicopy was seen to provide information on admissible spacetime types under classification schemes. We studied the Schuster-Toro vector superspace formulation of the continuous spin particle, giving a generalized Kerr-Schild solution involving an arbitrary function. We then discussed the case of an anti-de Sitter background and gave examples of the tensor multicopy, corresponding to the spinor formulations of [23]. We clarified the vector superspace formulation of the theory, using a different definition of the covariant derivative, with a generalized solution given in terms of the AdS-Kerr Kerr-Schild vector and scalar. We then described some of the obstacles to a continuous spin formulation.

Whether the multicopy generalizes beyond linearized field strengths in flat space, or the AdS case, is one question. It would be interesting to study whether imposing some self-duality constraints does allow a multicopy, notably using the spinor formalism and the Penrose transform on self-dual spaces, linking this for example to the discussion of the Eguchi-Hanson case in Sec. IV A. Regarding nonlinear extensions, multicopy formulas involving corrections to the linearized higher spin black brane have been found in [70].

A further question is the formulation of continuous spin particles in nonflat spacetimes. The AdS formulation of the Schuster-Toro theory for $\rho = 0$ does not give a CSP formulation upon shifting $\bar{\Delta} \rightarrow \bar{\Delta} + \rho$, as worked in flat space. However, continuous spin solutions in AdS have been studied in alternative approaches (see, e.g., [60,71,72]) and it would be interesting to see if these results can be translated into the superspace formalism. It was interesting that the general function in (5.11) solves the higher spin equations and it may be possible to use this freedom to explore CSP solutions. However, in the case of CSPs the equations of motion for different spins will mix and thus the general function we found for $\rho = 0$ may be constrained. It is also worth exploring possible CSP formulations on backgrounds such as the Schwarzschild metric, although we found that solving the equations iteratively in ρ did not prove possible. This is perhaps not surprising given that there are significant obstacles when coupling conventional massless higher spins to gravity beyond (A)dS backgrounds; see for example reviews given in [26,73]. However CSPs naturally incorporate a tower of higher spins, a property expected of higher spin fields which interact consistently with gravity [26,73]. Thus one might hope that the gravitational interactions of CSPs are better behaved; see [48,51] for first steps in this direction. We leave investigations of this last point to future work.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX: APPENDIX THE COTANGENT BUNDLE AND COVARIANT DERIVATIVE

Here we compile some useful properties about the cotangent bundle T^*M and its symplectic and geometric structure which we used in Sec. V C to define a covariant derivative. The material found here is covered in many standard textbooks, e.g., [74].

First let us restate our notation from the main text. Given a manifold M , in each chart (with coordinates x^μ) one can define natural coordinates on T^*M given by (x^μ, η_ν)

$$(x, \eta) \rightarrow (x^\mu, \eta_\nu dx^\nu) \in T^*M. \quad (\text{A1})$$

From the definition of η_μ under a change of coordinates $x^\mu \rightarrow x^{\mu'}$ coordinates we have

$$\eta_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \eta_\mu. \quad (\text{A2})$$

The vectors $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\tilde{\partial}^\mu = \frac{\partial}{\partial \eta_\mu}$ then naturally span the tangent space to T^*M . However, this basis of vectors does not transform in a nice way under a change of coordinates $(x^\mu, \eta^\mu) \rightarrow (x^{\mu'}, \eta_{\mu'})$,

$$\begin{aligned} \partial_{\mu'} &= \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu + \frac{\partial \eta_\mu}{\partial x^{\mu'}} \tilde{\partial}^\mu = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu + \left(\frac{\partial^2 x^\nu}{\partial x^{\mu'} \partial x^\mu} \right) \eta_\nu \tilde{\partial}^\nu, \\ \tilde{\partial}^{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^\mu} \tilde{\partial}^\mu. \end{aligned} \quad (\text{A3})$$

A better basis is given by a splitting into horizontal and vertical vectors

$$H_\mu := \partial_\mu + \Gamma_{\mu\nu}^\rho \eta_\rho \tilde{\partial}^\nu, \quad V^\mu := \tilde{\partial}^\mu, \quad (\text{A4})$$

which now have the simpler transformation properties

$$H_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} H_\mu, \quad V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} \tilde{\partial}^\mu. \quad (\text{A5})$$

For the connection coefficients $\Gamma_{\mu\nu}^\rho$ we simply choose the usual Levi-Civita connection coefficients from M . There is also a dual basis

$$\bar{H}^\mu := dx^\mu, \quad \bar{V}_\mu := d\eta_\mu - \Gamma_{\mu\nu}^\rho \eta_\rho dx^\nu, \quad (\text{A6})$$

which transforms similarly

$$\bar{V}_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \bar{V}_\mu, \quad \bar{H}^{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \bar{H}^\mu, \quad (\text{A7})$$

due to the transformation laws

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu, \quad d\eta_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} d\eta_\mu + \left(\frac{\partial^2 x^\nu}{\partial x^\mu \partial x^{\mu'}} \right) \eta_\nu dx^\mu. \quad (\text{A8})$$

We extend the covariant derivative from M to T^*M so that it is compatible with the horizontal and vertical splitting defined above,

$$\begin{aligned} \nabla_{H_\mu}(H_\nu) &:= \Gamma_{\mu\nu}^\rho H_\rho, & \nabla_{H_\mu}(\bar{V}_\mu) &:= \Gamma_{\mu\nu}^\rho \bar{V}_\rho, \\ \nabla_{H_\mu}(\bar{H}^\nu) &:= -\Gamma_{\mu\rho}^\nu \bar{H}^\rho, & \nabla_{H_\mu}(V^\nu) &:= -\Gamma_{\mu\rho}^\nu V^\rho, \end{aligned} \quad (\text{A9})$$

and

$$\nabla_{V^\mu}(H_\nu) = \nabla_{V^\mu}(\bar{V}_\mu) = 0, \quad \nabla_{V^\mu}(\bar{H}^\nu) = \nabla_{V^\mu}(V^\nu) = 0. \quad (\text{A10})$$

With this choice $\nabla_{V^\mu} = \tilde{\partial}^\mu$ and $\nabla_{H_\mu} = \bar{\nabla}_\mu$ defined in the main text. For example

$$\begin{aligned} \nabla_{H_\mu}(X^\nu H_\nu) &= (H_\mu(X^\nu) + \Gamma_{\mu\rho}^\nu X^\rho) H_\nu \\ &= (\partial_\mu(X^\nu) + \Gamma_{\mu\sigma}^\rho \eta_\rho \tilde{\partial}^\sigma X^\mu + \Gamma_{\mu\rho}^\nu X^\rho) H_\nu. \end{aligned} \quad (\text{A11})$$

This covariant derivative also preserves the canonical symplectic form

$$\omega := d\eta_\mu \wedge dx^\mu = \bar{V}_\mu \wedge \bar{H}^\mu, \quad \nabla_{H_\mu} \omega = \nabla_{V^\mu} \omega = 0 \quad (\text{A12})$$

which looks the same in any choice of coordinates, and in the second equality above we have used the torsionless condition of $\Gamma_{\mu\nu}^\rho$. We can use this to relate the horizontal/vertical basis and its dual

$$\omega(V^\mu, \cdot) = \bar{H}^\mu, \quad \omega(\cdot, H^\mu) = \bar{V}^\mu. \quad (\text{A13})$$

The metric on the manifold M can also be lifted to a Sasaki metric [75]¹⁵ on T^*M (see also [68,69]) given by

$$ds_{T^*M}^2 = g_{\mu\nu}(x) \bar{H}^\mu \bar{H}^\nu + g^{\mu\nu}(x) \bar{V}_\mu \bar{V}_\nu. \quad (\text{A14})$$

With this definition the metric component $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ transform in the usual way. In addition, the metric above has vanishing covariant derivative as expected since the connection coefficients $\Gamma_{\mu\nu}^\rho$ come from the Levi-Civita connection on M . Due to the compatibility of the covariant derivative with the symplectic form and the metric, we will write component fields as X^μ without specifying whether they belong to a vector or covector. Indeed, ∇X^μ takes the same value regardless of whether X^μ was the component of $X^\mu H_\mu$ or $X^\mu \bar{V}_\mu$, and similarly for X_μ . For concreteness, one can exclusively use either the horizontal basis or the vertical one.

¹⁵The Sasaki metric was originally defined for TM not T^*M but the structure is essentially the same.

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- [1] Z. Bern, J. J. M. Carrasco, and H. Johansson, New relations for gauge-theory amplitudes, *Phys. Rev. D* **78**, 085011 (2008).
 - [2] Z. Bern, J. J. M. Carrasco, and H. Johansson, Perturbative quantum gravity as a double copy of gauge theory, *Phys. Rev. Lett.* **105**, 061602 (2010).
 - [3] G. Travaglini *et al.*, The SAGEX review on scattering amplitudes, *J. Phys. A* **55**, 443001 (2022).
 - [4] C. D. White, *The Classical Double Copy* (World Scientific, Singapore, 2024).
 - [5] A. Luna, R. Monteiro, I. Nicholson, and D. O'Connell, Type D spacetimes and the Weyl double copy, *Classical Quantum Gravity* **36**, 065003 (2019).
 - [6] M. Walker and R. Penrose, On quadratic first integrals of the geodesic equations for type [22] spacetimes, *Commun. Math. Phys.* **18**, 265 (1970).

- [7] L. Hughston, R. Penrose, P. Sommers, and M. Walker, On a quadratic first integral for the charged particle orbits in the charged Kerr solution, *Commun. Math. Phys.* **27**, 303 (1972).
- [8] W. Dietz and R. Rüdiger, Space-times admitting Killing-Yano tensors, *Proc. R. Soc. A* **375**, 361 (1981).
- [9] H. Godazgar, M. Godazgar, R. Monteiro, D. Peinador Veiga, and C. N. Pope, Weyl double copy for gravitational waves, *Phys. Rev. Lett.* **126**, 101103 (2021).
- [10] S. Sabharwal and J. W. Dalhuisen, Anti-self-dual space-times, gravitational instantons and knotted zeros of the Weyl tensor, *J. High Energy Phys.* **07** (2019) 004.
- [11] R. Alawadhi, D. S. Berman, B. Spence, and D. Peinador Veiga, S-duality and the double copy, *J. High Energy Phys.* **03** (2020) 059.

- [12] R. Alawadhi, D. S. Berman, and B. Spence, Weyl doubling, *J. High Energy Phys.* **09** (2020) 127.
- [13] G. Elor, K. Farnsworth, M. L. Graesser, and G. Herczeg, The Newman-Penrose map and the classical double copy, *J. High Energy Phys.* **12** (2020) 121.
- [14] E. Chacón, S. Nagy, and C. D. White, The Weyl double copy from twistor space, *J. High Energy Phys.* **05** (2021) 2239.
- [15] E. Chacón, S. Nagy, and C. D. White, Alternative formulations of the twistor double copy, *J. High Energy Phys.* **03** (2022) 180.
- [16] P. Schuster and N. Toro, Continuous-spin particle field theory with helicity correspondence, *Phys. Rev. D* **91**, 025023 (2015).
- [17] R. Kerr and A. Schild, A new class of vacuum solutions of the Einstein field equations, in *Atti del convegno sulla relativita' generale; problemi dell'energia e onde gravitazionali*, edited by G. Barbera (Springer, Berlin, 1965), p. 173.
- [18] R. Kerr and A. Schild, Some algebraically degenerate solutions of Einstein's gravitational field equations, edited by R. Finn, *Proc. Sym. in Applied Math*, Vol. XVII (American Mathematical Society, Providence Rhode Island, 1965), pp. 199–209.
- [19] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 2003).
- [20] R. Monteiro, D. O'Connell, and C. D. White, Black holes and the double copy, *J. High Energy Phys.* **12** (2014) 056.
- [21] A. Luna, R. Monteiro, D. O'Connell, and C. D. White, The classical double copy for Taub–NUT spacetime, *Phys. Lett. B* **750**, 272 (2015).
- [22] V. E. Didenko, A. S. Matveev, and M. A. Vasiliev, Unfolded description of AdS₄ Kerr black hole, *Phys. Lett. B* **665**, 284 (2008).
- [23] V. E. Didenko and M. A. Vasiliev, Static BPS black hole in 4d higher-spin gauge theory, *Phys. Lett. B* **682**, 305 (2009).
- [24] V. E. Didenko and N. K. Dosmanbetov, Classical double copy and higher spin fields, *Phys. Rev. Lett.* **130**, 071603 (2023).
- [25] D. Ponomarev, Basic introduction to higher-spin theories, *Int. J. Theor. Phys.* **62**, 146 (2023).
- [26] D. Sorokin, Introduction to the classical theory of higher spins, *AIP Conf. Proc.* **767**, 172 (2005).
- [27] N. Bouatta, G. Compere, and A. Sagnotti, An Introduction to free higher-spin fields, in *1st Solvay Workshop on Higher Spin Gauge Theories* (Springer, Brussels Belgium, 2004), pp. 79–99; [arXiv:hep-th/0409068](https://arxiv.org/abs/hep-th/0409068).
- [28] P. de Medeiros and C. Hull, Exotic tensor gauge theory and duality, *Commun. Math. Phys.* **235**, 255 (2003).
- [29] P. de Medeiros and C. Hull, Geometric second order field equations for general tensor gauge fields, *J. High Energy Phys.* **05** (2003) 019.
- [30] C. D. White, Twistorial foundation for the classical double copy, *Phys. Rev. Lett.* **126**, 061602 (2021).
- [31] K. Armstrong-Williams, N. Moynihan, and C. D. White, Deriving Weyl double copies with sources, *J. High Energy Phys.* **03** (2025) 121.
- [32] V. E. Didenko and A. V. Korybut, Planar solutions of higher-spin theory. Part I. Free field level, *J. High Energy Phys.* **08** (2021) 144.
- [33] R. Monteiro, I. Nicholson, and D. O'Connell, Spinor-helicity and the algebraic classification of higher-dimensional spacetimes, *Classical Quantum Gravity* **36**, 065006 (2019).
- [34] E. Wigner, On unitary representations of the inhomogeneous Lorentz group, *Ann. Math.* **40**, 149 (1939).
- [35] V. O. Rivelles, Gauge theory formulations for continuous and higher spin fields, *Phys. Rev. D* **91**, 125035 (2015).
- [36] V. O. Rivelles, Remarks on a gauge theory for continuous spin particles, *Eur. Phys. J. C* **77**, 433 (2017).
- [37] X. Bekaert and E. D. Skvortsov, Elementary particles with continuous spin, *Int. J. Mod. Phys. A* **32**, 1730019 (2017).
- [38] M. Najafizadeh, Modified Wigner equations and continuous spin gauge field, *Phys. Rev. D* **97**, 065009 (2018).
- [39] P. Schuster and N. Toro, A new class of particle in 2 + 1 dimensions, *Phys. Lett. B* **743**, 224 (2015).
- [40] X. Bekaert, M. Najafizadeh, and M. R. Setare, A gauge field theory of fermionic continuous-spin particles, *Phys. Lett. B* **760**, 320 (2016).
- [41] M. Najafizadeh, Supersymmetric continuous spin gauge theory, *J. High Energy Phys.* **03** (2020) 027.
- [42] M. Najafizadeh, Off-shell supersymmetric continuous spin gauge theory, *J. High Energy Phys.* **02** (2022) 038.
- [43] P. Schuster and N. Toro, Quantum electrodynamics mediated by a photon with continuous spin, *Phys. Rev. D* **109**, 096008 (2024).
- [44] R. R. Metsaev, BRST-BV approach to continuous-spin field, *Phys. Lett. B* **781**, 568 (2018).
- [45] I. L. Buchbinder, V. A. Krykhtin, and H. Takata, BRST approach to Lagrangian construction for bosonic continuous spin field, *Phys. Lett. B* **785**, 315 (2018).
- [46] K. B. Alkalaev and M. A. Grigoriev, Continuous spin fields of mixed-symmetry type, *J. High Energy Phys.* **03** (2018) 030.
- [47] P. Schuster, N. Toro, and K. Zhou, Interactions of particles with “continuous spin” fields, *J. High Energy Phys.* **04** (2023) 010.
- [48] B. Bellazzini, S. De Angelis, and M. Romano, Continuous-spin particles, on shell, *J. High Energy Phys.* **05** (2025) 166.
- [49] P. Schuster, G. Sundaresan, and N. Toro, Thermodynamics of continuous spin photons, *Phys. Rev. D* **111**, 056019 (2025).
- [50] A. Reilly, P. Schuster, and N. Toro, Probing “continuous spin” QED with rare atomic transitions, *Phys. Rev. D* **113**, 015028 (2026).
- [51] S. Kundu, P. Schuster, and N. Toro, A first look at “continuous spin” gravity—time delay signatures, [arXiv:2503.03817](https://arxiv.org/abs/2503.03817).
- [52] R. R. Metsaev, Interacting massive/massless continuous-spin fields and integer-spin fields, *J. High Energy Phys.* **09** (2025) 002.
- [53] S. Kundu, A. Russo, P. Schuster, and N. Toro, Interactions of a continuous-spin field with a spin-1/2 particle, *J. High Energy Phys.* **11** (2025) 125.
- [54] A. Reilly, A. Russo, P. Schuster, and N. Toro, Hydrogen 21 cm constraints on the photon's spin scale, [arXiv:2505.15890](https://arxiv.org/abs/2505.15890).
- [55] R. R. Metsaev, Continuous spin gauge field in (A)dS space, *Phys. Lett. B* **767**, 458 (2017).

- [56] R. R. Metsaev, Fermionic continuous spin gauge field in (A)dS space, *Phys. Lett. B* **773**, 135 (2017).
- [57] R. R. Metsaev, Continuous-spin mixed-symmetry fields in AdS(5), *J. Phys. A* **51**, 215401 (2018).
- [58] M. V. Khabarov and Y. M. Zinoviev, Infinite (continuous) spin fields in the frame-like formalism, *Nucl. Phys.* **B928**, 182 (2018).
- [59] R. R. Metsaev, Light-cone continuous-spin field in AdS space, *Phys. Lett. B* **793**, 134 (2019).
- [60] R. R. Metsaev, Mixed-symmetry continuous-spin fields in flat and AdS spaces, *Phys. Lett. B* **820**, 136497 (2021).
- [61] I. L. Buchbinder, S. A. Fedoruk, A. P. Isaev, and V. A. Krykhtin, Infinite (continuous) spin particle in constant curvature space, *Phys. Lett. B* **853**, 138689 (2024).
- [62] I. L. Buchbinder, S. A. Fedoruk, A. P. Isaev, and V. A. Krykhtin, BRST construction for infinite spin field on AdS₄, *Eur. Phys. J. Plus* **139**, 621 (2024).
- [63] V. E. Didenko, Coordinate independent approach to 5d black holes, *Classical Quantum Gravity* **29**, 025009 (2012).
- [64] A. Y. Segal, A generating formulation for free higher spin massless fields, [arXiv:hep-th/0103028](https://arxiv.org/abs/hep-th/0103028).
- [65] M. Najafizadeh, Local action for fermionic unconstrained higher spin gauge fields in AdS and dS spacetimes, *Phys. Rev. D* **98**, 125012 (2018).
- [66] M. Taronna, Higher-spin interactions: Three-point functions and beyond, Ph.D. thesis, Pisa, Scuola Normale Superiore, 2012, [arXiv:1209.5755](https://arxiv.org/abs/1209.5755).
- [67] T. Nutma and M. Taronna, On conformal higher spin wave operators, *J. High Energy Phys.* **06** (2014) 066.
- [68] I. V. Gorbunov, S. L. Lyakhovich, and A. A. Sharapov, Wick quantization of cotangent bundles over Riemannian manifolds, *J. Geom. Phys.* **53**, 98 (2005).
- [69] K. P. Mok, Metrics and connections on the cotangent bundle, *Kodai Math. Sem. Rep.* **28**, 226 (1977).
- [70] V. E. Didenko and A. V. Korybut, Planar solutions of higher-spin theory. Nonlinear corrections, *J. High Energy Phys.* **01** (2022) 125.
- [71] R. R. Metsaev, Light-cone vector superspace and continuous-spin field in AdS, *Phys. Lett. B* **868**, 139778 (2025).
- [72] I. L. Buchbinder, S. A. Fedoruk, A. P. Isaev, and M. A. Podoinitsyn, On the realization of infinite (continuous) spin field representations in AdS₄ space, *Phys. Lett. B* **861**, 139226 (2025).
- [73] X. Bekaert, S. Cnockaert, C. Iazeolla, and M. A. Vasiliev, Nonlinear higher spin theories in various dimensions, in *1st Solvay Workshop on Higher Spin Gauge Theories* (Springer, Brussels Belgium, 2004), pp. 132–197; [arXiv:hep-th/0503128](https://arxiv.org/abs/hep-th/0503128).
- [74] A. Cannas da Silva, *Lectures on Symplectic Geometry* (Springer, Berlin, Heidelberg, 2001).
- [75] S. Sasaki, On the differential geometry of tangent bundles of Riemannian manifolds, *Tohoku Math. J.* **10**, 338 (1958).