

Sign of the Casimir force between two bodies coupled to a Dirac field

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We apply the result of Kenneth and Klich [Phys. Rev. Lett. **97**, 160401 (2006)] to derive a theorem for two objects coupled to a Dirac field. We demonstrate that a sufficient condition for the Casimir interaction to be *repulsive* is for their geometric configuration to be *odd* under the same reflection operation considered in that work.

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I. INTRODUCTION

The study of the relation between the properties of the Casimir effect [1] and the geometry (i.e., shapes and positions) of the bodies involved is a captivating area of research, with significant implications to both experimental and theoretical physics. Understanding this relation is crucial for Casimir experiments; indeed, one could, for instance, be interested in discovering the optimal geometric configuration to maximize the Casimir force, as this could have applications in nanotechnology and microelectromechanical systems (MEMS) [2]. Additionally, identifying shapes that exhibit stability under small deformations is vital for ensuring the reliability and robustness of devices that exploit the Casimir effect [3]. Furthermore, investigating the impact of surface corrugations and other irregularities on the Casimir force is essential for predicting its behavior in real-world scenarios [4].

On the theoretical front, an important challenge lies in developing effective methods to cope with the geometries that often arise in Casimir effect studies. Perturbation theory, a commonly used tool in theoretical physics, is generally not suitable for tackling nontrivial shapes [5], except in specific cases, like an expansion in powers of the deformation of the geometry with respect of a highly symmetric one, which is exactly solvable. A related but different approach is the derivative expansion [6].

This calls for the development of alternative theoretical frameworks and computational techniques that can accurately model the Casimir interaction for a wide range of geometric configurations. In this context, there is an important theoretical result regarding the relation between

geometry and the Casimir force: a sufficient condition for this force between two dielectric or conducting objects to be attractive is for the geometry of the system to have a reflection symmetry with respect to a plane [7]. In other words, when the function representing the electromagnetic properties of the two media as a whole is invariant (i.e., even) under the symmetry operation, the force between them is attractive.

In this work, we apply that theorem to derive another one, this time for objects coupled to a Dirac field. The result may be phrased as follows: a sufficient condition for the Casimir interaction to be *repulsive* is for their geometric configuration to be *odd* under the same geometric operation considered in the bosonic case.

This paper is organized as follows: since our approach to the Dirac field case relies heavily upon the scalar field result, we briefly review the latter in Sec. II, albeit in the notation and conventions we afterward use for the Dirac field. Besides, we focus here on the properties that are more relevant to the Dirac field case, which is dealt with in Sec. III, where we derive the result about the sign of the force in that kind of system.

In Sec. IV, we present some remarks about the general result and its application, and a brief summary of our conclusions.

II. SCALAR FIELD

We shall begin by introducing the main concepts and definitions for the case of a real scalar field, briefly reviewing the known result for the nature of the force, in terms of our approach, notation, and conventions. We deal with a vacuum massless real scalar field $\varphi(x)$, in $d + 1$ space-time dimensions, in the presence of two objects, denoted by L and R . To obtain the Casimir interaction energy, we shall work in the context of the functional integral framework, using the imaginary-time formalism; hence, it is convenient to introduce the system in terms of its

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Euclidean action. This object is a functional $\mathcal{S} = \mathcal{S}(\varphi; U)$ where U denotes a function that accounts for the relevant properties of the two bodies. We assume $\mathcal{S}$ to have the structure

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(I)}, \quad (1)$$

where $\mathcal{S}^{(0)}$ is the free action,

$$\mathcal{S}^{(0)}(\varphi) = \frac{1}{2} \int_x \partial_\mu \varphi(x) \partial_\mu \varphi(x), \quad (2)$$

and we have adopted a shorthand notation for the integration, namely, $\int_x \dots \equiv \int d^{d+1}x \dots$. The Einstein convention on summation over repeated indices has been assumed. Indices from the middle of the Greek alphabet run from 0 to d , and they correspond to space-time coordinates. We shall use $x = (x_0, x_1, \dots, x_d)$ to denote these coordinates, x_0 is the imaginary time (the speed of light, $c \equiv 1$), and $\mathbf{x} = (x_1, \dots, x_d)$ the spatial coordinates. The $\mathcal{S}^{(I)}$ term, on the other hand, deals with the coupling between the field and the two bodies, and is given by

$$\mathcal{S}^{(I)}(\varphi; U) = \int_{x_0, \mathbf{x}} \varphi(x_0, \mathbf{x}) U(x_0 - x'_0; \mathbf{x}) \varphi(x'_0, \mathbf{x}). \quad (3)$$

Working with the Fourier transformed versions, with respect to the time coordinate, of the objects involved¹:

$$\varphi(x_0, \mathbf{x}) = \int_{\phi} e^{i\omega x_0} \tilde{\varphi}(\omega, \mathbf{x}), \quad U(x_0; \mathbf{x}) = \int_{\phi} e^{i\omega x_0} \tilde{U}(\omega, \mathbf{x}), \quad (4)$$

the action decomposes into an integral over independent modes, one for each frequency. Namely,

$$\mathcal{S}(\varphi; U) = \int_0^\infty \frac{d\omega}{2\pi} \mathcal{S}_\omega(\tilde{\varphi}^*, \tilde{\varphi}; \tilde{U}) \quad (5)$$

where $\mathcal{S}_\omega = \mathcal{S}_\omega^{(0)} + \mathcal{S}_\omega^{(I)}$, such that

$$\begin{aligned} \mathcal{S}_\omega^{(0)} &= \int_{\mathbf{x}} \tilde{\varphi}^*(\omega, \mathbf{x}) (-\nabla^2 + \omega^2) \tilde{\varphi}(\omega, \mathbf{x}), \\ \mathcal{S}_\omega^{(I)} &= \int_{\mathbf{x}} \tilde{\varphi}^*(\omega, \mathbf{x}) \tilde{U}(\omega, \mathbf{x}) \tilde{\varphi}(\omega, \mathbf{x}), \end{aligned} \quad (6)$$

with $*$ denoting complex conjugation. Note that $\tilde{U}(\omega, \mathbf{x}) = \omega^2 \chi(\omega, \mathbf{x})$, where χ is the susceptibility.

In order to obtain the vacuum energy, we introduce the Euclidean vacuum transition amplitude $\mathcal{Z}$, and the effective action Γ :

$$\begin{aligned} \mathcal{Z}(U) &= e^{-\Gamma(U)} = \int \mathcal{D}\varphi e^{-\mathcal{S}(\varphi; U)} = \prod_{\omega=0}^\infty \mathcal{Z}_\omega(\tilde{U}), \\ \mathcal{Z}_\omega(\tilde{U}) &\equiv \int \mathcal{D}\tilde{\varphi}^* \mathcal{D}\tilde{\varphi} e^{-\mathcal{S}_\omega}. \end{aligned} \quad (7)$$

The vacuum energy $E(U)$, measured with respect to the empty space vacuum, is then given by

$$E(U) = \lim_{T \rightarrow \infty} \frac{1}{T} [\Gamma_T(U) - \Gamma_T(0)], \quad (8)$$

where Γ_T is formally identical to Γ , but defined for a finite time interval. We then see that

$$E(U) = - \int_0^\infty \frac{d\omega}{2\pi} \log \left[\frac{\mathcal{Z}_\omega(\tilde{U})}{\mathcal{Z}_\omega(0)} \right]. \quad (9)$$

To proceed, we now deal with the properties of the two objects, L and R . We shall assume that

$$\tilde{U}(\omega, \mathbf{x}) = \tilde{U}_L(\omega, \mathbf{x}) + \tilde{U}_R(\omega, \mathbf{x}), \quad (10)$$

such that, denoting by L and R the support of U_L and U_R , respectively, one has $L \cap R = \emptyset$. Besides, to be consistent with the assumption of the susceptibility being proportional to $\tilde{U}$, both $\tilde{U}_L$ and $\tilde{U}_R$ must be real, non-negative functions.

Taking into account those assumptions, we may produce an equivalent representation of the functional integral for $\mathcal{Z}_\omega$, this time introducing complex auxiliary fields λ_L and λ_R , with domains L and R , respectively, and their Fourier transforms. This amounts to replacing $\mathcal{S}_\omega^{(I)}$ by $\mathcal{S}_\omega^{(\text{aux})}$, so that

$$\mathcal{Z}_\omega(\tilde{U}) = \int \mathcal{D}\tilde{\varphi}^* \mathcal{D}\tilde{\varphi} \mathcal{D}\tilde{\lambda}_L^* \mathcal{D}\tilde{\lambda}_L \mathcal{D}\tilde{\lambda}_R^* \mathcal{D}\tilde{\lambda}_R e^{-(\mathcal{S}_\omega^{(0)} + \mathcal{S}_\omega^{(\text{aux})})}, \quad (11)$$

with

$$\begin{aligned} \mathcal{S}_\omega^{(\text{aux})} &= \int_{\mathbf{x} \in L} \tilde{\lambda}_L^*(\mathbf{x}) \tilde{U}_L(\mathbf{x}) \tilde{\lambda}_L(\mathbf{x}) + \int_{\mathbf{x} \in R} \tilde{\lambda}_R^*(\mathbf{x}) \tilde{U}_R(\mathbf{x}) \tilde{\lambda}_R(\mathbf{x}) \\ &\quad - i \int_{\mathbf{x} \in L} [\tilde{\lambda}_L^*(\mathbf{x}) \tilde{U}_L(\mathbf{x}) \tilde{\varphi}(\mathbf{x}) + \tilde{\varphi}^*(\mathbf{x}) \tilde{U}_L(\mathbf{x}) \tilde{\lambda}_L(\mathbf{x})] \\ &\quad - i \int_{\mathbf{x} \in R} [\tilde{\lambda}_R^*(\mathbf{x}) \tilde{U}_R(\mathbf{x}) \tilde{\varphi}(\mathbf{x}) + \tilde{\varphi}^*(\mathbf{x}) \tilde{U}_R(\mathbf{x}) \tilde{\lambda}_R(\mathbf{x})]. \end{aligned} \quad (12)$$

For the sake of clarity, we have omitted writing explicitly the dependence on ω of the objects above.

To proceed, we integrate out $\tilde{\varphi}^*, \tilde{\varphi}$. The result of this step may be put in the following form:

$$\frac{\mathcal{Z}_\omega(\tilde{U})}{\mathcal{Z}_\omega(0)} = \int \mathcal{D}\tilde{\lambda}_L^* \mathcal{D}\tilde{\lambda}_L \mathcal{D}\tilde{\lambda}_R^* \mathcal{D}\tilde{\lambda}_R e^{-\mathcal{S}_{\text{eff}}} \quad (13)$$

¹We use the notation: $\int_{\phi} \dots \equiv \int \frac{d\omega}{2\pi} \dots$

$$\mathcal{S}_{\text{eff}} = \sum_{A,B=L}^R \int_{\mathbf{x} \in A, \mathbf{x}' \in B} \tilde{\lambda}_A^*(\mathbf{x}) K_{AB}(\mathbf{x}, \mathbf{x}') \tilde{\lambda}_B(\mathbf{x}'), \quad (14)$$

where we have introduced the coordinate-space kernels:

$$K_{AB}(\mathbf{x}, \mathbf{x}') = \delta_{AB} \delta(\mathbf{x} - \mathbf{x}') \tilde{U}_A(\mathbf{x}) + \tilde{U}_A(\mathbf{x}) G_{AB}^{(0)}(\mathbf{x} - \mathbf{x}') \tilde{U}_B(\mathbf{x}'), \quad (15)$$

with

$$G_{AB}^{(0)}(\mathbf{x} - \mathbf{x}') \equiv \langle \mathbf{x} | \hat{G}_{AB}^{(0)} | \mathbf{x}' \rangle = \langle \mathbf{x} | (-\nabla^2 + \omega^2)^{-1} | \mathbf{x}' \rangle, \quad (16)$$

$$\mathbf{x} \in A, \quad \mathbf{x}' \in B.$$

We now introduce some notation to distinguish an operator from its coordinate kernel. For instance, to the operator $\hat{K}_{AB}$ corresponds the kernel $K_{AB}(\mathbf{x}, \mathbf{x}') = \langle \mathbf{x} | \hat{K}_{AB} | \mathbf{x}' \rangle$. This is an operator acting from the space of functions $H_A = L^2(A)$ the space of square integrable functions with support A , into those with support B . We will adopt the convention that indices like A, B , are assumed to run over the values that label the two objects, namely, L or R .

Using standard formulas for the evaluation of Gaussian integrals, and noting that both $\hat{K}_{LL}$ and $\hat{K}_{RR}$ are non-singular, we get

$$\frac{\mathcal{Z}_\omega(\tilde{U})}{\mathcal{Z}_\omega(0)} = [\det(\hat{K}_{LL})]^{-1} [\det(\hat{K}_{RR})]^{-1} (\det[\hat{I}_L - (\hat{K}_{LL})^{-1} \times \hat{K}_{LR}(\hat{K}_{RR})^{-1} \hat{K}_{RL}])^{-1}, \quad (17)$$

where $\hat{I}_A$ is the identity operator for functions with support on A .

Finally, using the assumption that each function $\tilde{U}_A$ is nonnegative, its operator $\hat{U}_A$, such that $\langle \mathbf{x} | \hat{U}_A | \mathbf{x}' \rangle = \tilde{U}_A(\mathbf{x}) \delta(\mathbf{x} - \mathbf{x}')$, has an inverse.

Then, introducing

$$\hat{T}_A = (\hat{I}_A + \hat{U}_A \hat{G}_{AA}^{(0)})^{-1} \hat{U}_A, \quad (18)$$

we see that

$$\frac{\mathcal{Z}_\omega(\tilde{U})}{\mathcal{Z}_\omega(0)} = (\det \hat{K}_{LL})^{-1} (\det \hat{K}_{RR})^{-1} \times (\det[\hat{I}_L - \hat{T}_L \hat{G}_{LR}^{(0)} \hat{T}_R \hat{G}_{RL}^{(0)}])^{-1}. \quad (19)$$

Thus, subtracting terms that survive when the two bodies are infinitely far apart, and therefore do not contribute to the interaction energy, we get

$$E_I = \int_0^\infty \frac{d\omega}{2\pi} \log \det[\hat{I}_L - \hat{T}_L \hat{G}_{LR}^{(0)} \hat{T}_R \hat{G}_{RL}^{(0)}] = \int_0^\infty \frac{d\omega}{2\pi} \text{Tr} \log[\hat{I}_L - \hat{T}_L \hat{G}_{LR}^{(0)} \hat{T}_R \hat{G}_{RL}^{(0)}]. \quad (20)$$

This is, in our conventions, equivalent to Eq. (5) of Ref. [7], which is used to prove the theorem on the attractive nature of the force when there is a reflection symmetry. Up to now, no assumption about the existence of that symmetry in the system has been made. Let us now assume the existence of such a symmetry. Assuming x_d is the coordinate that changes under the reflection, and denoting the transformation by a function $\mathcal{P}$, we have, at the level of the coordinates, $\mathbf{x} \rightarrow \mathcal{P}(\mathbf{x})$, where, for $\mathbf{x} = (x_1, x_2, \dots, x_d)$, $\mathcal{P}(\mathbf{x}) = (x_1, x_2, \dots, x_{d-1}, a - x_d)$. With this convention, the reflection plane is $x_d = \frac{a}{2}$, the region invariant under the transformation. One has $\mathcal{P}^2 = I$ and that this operation is linearly represented on the space of fields by a unitary and self-adjoint operator $\hat{\mathcal{P}}$:

$$\hat{\mathcal{P}}|\mathbf{x}\rangle = |\mathcal{P}(\mathbf{x})\rangle. \quad (21)$$

The action of $\hat{\mathcal{P}}$ on functions $\psi(\mathbf{x})$ of $H = H_L \oplus H_R$ is thus

$$\langle \mathbf{x} | \hat{\mathcal{P}} | \psi \rangle = \langle \mathcal{P}(\mathbf{x}) | \psi \rangle = \psi(\mathcal{P}(\mathbf{x})). \quad (22)$$

Besides, $\mathcal{P}$ interchanges L and R : $\mathcal{P}(L) = R$, and $\mathcal{P}(R) = L$, ($\mathcal{P}(A)$ denotes the image of A). This property is inherited in the functional spaces: $\hat{\mathcal{P}}(H_L) = H_R$, and $\hat{\mathcal{P}}(H_R) = H_L$.

Reflection symmetry implies that²

$$\hat{U}_R = \hat{\mathcal{P}} \hat{U}_L \hat{\mathcal{P}}^\dagger, \quad \hat{I}_R = \hat{\mathcal{P}} \hat{I}_L \hat{\mathcal{P}}^\dagger, \quad \hat{G}_{RR}^{(0)} = \hat{\mathcal{P}} \hat{G}_{LL}^{(0)} \hat{\mathcal{P}}^\dagger, \quad \hat{G}_{LR}^{(0)} = \hat{\mathcal{P}} \hat{G}_{RL}^{(0)} \hat{\mathcal{P}}^\dagger, \quad (23)$$

and as a consequence: $\hat{\mathcal{P}} \hat{U} \hat{\mathcal{P}}^\dagger = \hat{U}$.

These properties allow one to prove that

$$E_I = \int_0^\infty \frac{d\omega}{2\pi} \log[\det(\hat{I}_L - \hat{Y}^2)], \quad \hat{Y} \equiv \sqrt{\hat{T}_L \hat{G}_{LR}^{(0)} \hat{\mathcal{P}}} \sqrt{\hat{T}_L}. \quad (24)$$

Following [7], one can use (24) to find the sign of F , the force along the x_d direction. First, E_I is expressed in terms of the eigenvalues λ_n of the Hermitian operator $\hat{Y}$:

²We recall that $\hat{\mathcal{P}}^\dagger = \hat{\mathcal{P}}$.

$$E_I = \int_0^\infty \frac{d\omega}{2\pi} \sum_n \log(1 - \lambda_n^2)$$

$$F = -\partial_a E_I = 2 \int_0^\infty \frac{d\omega}{2\pi} \sum_n \frac{\lambda_n \partial_a \lambda_n}{1 - \lambda_n^2}. \quad (25)$$

The sign of the force is then that of the product $\lambda_n \partial_a \lambda_n$ (since $\lambda_n^2 < 1 \forall n$).

Introducing $|\phi_n\rangle$, the normalized eigenvectors of $\hat{Y}$, $\lambda_n = \langle \phi_n | \hat{Y} | \phi_n \rangle$. Thus,

$$\lambda_n \partial_a \lambda_n = \langle \phi_n | \hat{Y} | \phi_n \rangle \langle \phi_n | \partial_a \hat{Y} | \phi_n \rangle, \quad (26)$$

where the Hellmann-Feynman theorem [8,9] has been applied.

Since $\hat{T}$ is independent of a , it is convenient to introduce $|\psi_n\rangle \equiv \sqrt{\hat{T}_L} |\phi_n\rangle$, and $\hat{Z} = \hat{G}_{LR}^{(0)} \hat{P}$ to obtain

$$\text{sign}(\lambda_n \partial_a \lambda_n) = \text{sign}[\langle \psi_n | \hat{Z} | \psi_n \rangle] \text{sign}[\langle \psi_n | \partial_a \hat{Z} | \psi_n \rangle]. \quad (27)$$

In [7], it is shown that $\hat{Z}$ is positive, while $\partial_a \hat{Z}$ is negative. This, of course has the consequence that $\text{sign}(\lambda_n \partial_a \lambda_n) = -1$, for all n , and the force is then negative, i.e., attractive. There is an important remark we wish to make here: In order to have a negative force, it is sufficient to have opposite signs for $\langle \psi_n | \hat{Z} | \psi_n \rangle$ and $\langle \psi_n | \partial_a \hat{Z} | \psi_n \rangle$, for each n . To be more precise, the signs have to be opposite for the same n , but the sign of each factor could depend on n , as long as the sign of the product is (-1) for all n . As we shall see, this remark is relevant to the Dirac field case:

We conclude this section by recalling the assumption for the functions $\tilde{U}_A$ to be positive definite. Note also that both the symmetry and positive definiteness can be achieved, with no change in the theorem, by an $\tilde{U}$ which has a matrix structure, assuming a multicomponent scalar field. By this, we mean that

$$\mathcal{S}^{(0)}(\varphi) = \frac{1}{2} \int_x \partial_\mu \varphi^{(\alpha)}(x) \partial_\mu \varphi^{(\alpha)}(x), \quad (28)$$

with $\alpha = 1, \dots, N$, and

$$\mathcal{S}^{(I)}(\varphi; U) = \int_{x_0, x'_0, \mathbf{x}} \varphi^{(\alpha)}(x_0, \mathbf{x}) U^{(\alpha\beta)}(x_0 - x'_0; \mathbf{x}) \varphi^{(\beta)}(x'_0, \mathbf{x}), \quad (29)$$

where U is positive.

As in the single-component case, we use the same time Fourier representation and introduce a column vector $\tilde{\varphi} \equiv (\tilde{\varphi}^{(\alpha)})$, as well as a matrix $\tilde{U} = [\tilde{U}^{(\alpha\beta)}]$. We again have a decoupling of the action into independent terms, one for each frequency:

$$\mathcal{S}_\omega^{(0)} = \int_{\mathbf{x}} \tilde{\varphi}^\dagger(\omega, \mathbf{x}) (-\nabla^2 + \omega^2) \tilde{\varphi}(\omega, \mathbf{x}),$$

$$\mathcal{S}_\omega^{(I)} = \int_{\mathbf{x}} \tilde{\varphi}^\dagger(\omega, \mathbf{x}) \tilde{U}(\omega, \mathbf{x}) \tilde{\varphi}(\omega, \mathbf{x}). \quad (30)$$

In this case, a generalization of the single-component one, essentially the same proof, allows one to conclude that the force is attractive under the assumption of the parity: $\hat{U}_R = \hat{P} \hat{U}_L \hat{P}^\dagger$. The $\hat{T}_A$ operators now also have a matrix structure, but the rest of the conclusion that when there is symmetry the force is attractive remains unaltered.

III. DIRAC FIELD

Let us now deal with a vacuum massless Dirac field $\psi, \bar{\psi}$, in $d+1$ space-time dimensions, in the presence of two objects, L and R , which impose boundary conditions on the field fluctuations. The regions are assumed to be disjoint, as in the scalar case.

We shall first relate the calculation of the vacuum energy to the case of a real scalar field and then study the role of the reflection transformation in this case.

The Euclidean action is now a functional $\mathcal{S} = \mathcal{S}(\bar{\psi}, \psi; V)$, where V denotes a function that accounts for the two bodies. We also assume $\mathcal{S}$ to have the structure

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_I, \quad (31)$$

where $\mathcal{S}_0$ denotes the free action for the Dirac field,

$$\mathcal{S}_0(\bar{\psi}, \psi) = \int_x \bar{\psi}(x) \not{\partial} \psi(x). \quad (32)$$

Dirac's γ matrices are assumed to be Hermitian, $\gamma_\mu^\dagger = \gamma_\mu$, and to satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. Also, $\not{\partial} \equiv \gamma_\mu \partial_\mu$.

The $\mathcal{S}_I$ term, on the other hand, is given by

$$\mathcal{S}_I(\bar{\psi}, \psi; V) = \int_{x_0, x'_0, \mathbf{x}} \bar{\psi}(x_0, \mathbf{x}) V(x_0 - x'_0; \mathbf{x}) \psi(x'_0, \mathbf{x}). \quad (33)$$

Introducing time Fourier transformations,

$$\psi(x_0, \mathbf{x}) = \int_{\not{\partial}} e^{i\omega x_0} \tilde{\psi}(\omega, \mathbf{x}), \quad \bar{\psi}(x_0, \mathbf{x}) = \int_{\not{\partial}} e^{-i\omega x_0} \tilde{\bar{\psi}}(\omega, \mathbf{x}),$$

$$V(x_0; \mathbf{x}) = \int_{\not{\partial}} e^{i\omega x_0} \tilde{V}(\omega, \mathbf{x}), \quad (34)$$

the action decomposes again into an integral over independent modes

$$\mathcal{S} = \int_{\phi} \mathcal{S}_{\omega}, \quad \mathcal{S}_{\omega} = \int_{\mathbf{x}} \tilde{\psi}(\omega, \mathbf{x}) [\nabla + i\gamma_0 \omega + \tilde{V}(\omega, \mathbf{x})] \tilde{\psi}(\omega, \mathbf{x}). \quad (35)$$

The properties of the two objects, are again determined by the form of V :

$$\tilde{V}(\mathbf{x}) = \tilde{V}_L(\mathbf{x}) + \tilde{V}_R(\mathbf{x}). \quad (36)$$

A crucial difference between this case and the bosonic one is that the $\tilde{V}_A$ functions do not need to be positive. Indeed, on physical ground, the bosonic $\tilde{U}$ plays the role of the square of a mass, unlike $\tilde{V}$. We do assume, however, that the $\tilde{V}_A$ functions do not vanish inside the respective regions, $A = L, R$, irrespective of their sign.

Then we see that, as in the bosonic case,

$$e^{-\Gamma(V)} = \frac{\mathcal{Z}(V)}{\mathcal{Z}(0)}, \quad (37)$$

where

$$\mathcal{Z}(V) = \int \mathcal{D}\tilde{\psi} \mathcal{D}\psi e^{-\mathcal{S}(\tilde{\psi}, \psi; V)}. \quad (38)$$

Thus,

$$E(V) = - \left[\frac{\Gamma(V)}{T} \right]_{T \rightarrow \infty} = - \int_{\phi} \log \frac{\det(\nabla + i\gamma_0 \omega + \tilde{V})}{\det(\nabla + i\gamma_0 \omega)}. \quad (39)$$

Here, we have used the notation $\nabla \equiv \gamma_j \partial_j$, and the determinant is meant to be over functional as well as Dirac space. In (39), we take advantage of the fact that the energy is real, and that the determinant of the adjoint of an operator is the conjugate of the determinant of the original operator, to make

$$E(V) = E^*(V) = - \int_{\phi} \log \frac{\det[-\nabla - i\gamma_0 \omega + \tilde{V}]}{\det[-\nabla - i\gamma_0 \omega]}, \quad (40)$$

and then

$$\begin{aligned} E(V) &= - \frac{1}{2} \int_{\phi} \log \frac{\det[(-\nabla - i\gamma_0 \omega + \tilde{V})(\nabla + i\gamma_0 \omega + \tilde{V})]}{\det[(-\nabla - i\gamma_0 \omega)(\nabla + i\gamma_0 \omega)]} \\ &= - \frac{1}{2} \int_{\phi} \log \frac{\det[-\nabla^2 + \omega^2 + \tilde{V}^2 - (\nabla \tilde{V})]}{\det(-\nabla^2 + \omega^2)}. \end{aligned} \quad (41)$$

This expression is identical *except for an important global factor* $(-\frac{1}{2})$ to the one of the multicomponent scalar field considered in the previous section. The number of components depends, of course, on the dimension of the representation of the Clifford algebra generated by the Dirac matrices.

In the mapping to the scalar field case, we make the identification:

$$\tilde{U} = \tilde{V}^2 - (\nabla \tilde{V}). \quad (42)$$

Hence, the resulting $\tilde{U}$ is Hermitian and has support on the same regions L and R defined by $\tilde{V}$.

Regarding the positive definiteness of $\tilde{U}$, it is determined by the relative magnitude of the norms of $\tilde{V}^2$ and $(\nabla \tilde{V})$. We shall assume $\tilde{V}$, or what is the same, and each one of its components $\tilde{V}_L$ and $\tilde{V}_R$ are such that the $\tilde{U}$ is positive.

In the Dirac case, we then have

$$\begin{aligned} E_I &= - \frac{1}{2} \int_{\phi} \log \det [\hat{I}_L - \hat{T}_L \hat{G}_{LR}^{(0)} \hat{T}_R \hat{G}_{RL}^{(0)}] \\ &= - \frac{1}{2} \int_{\phi} \text{Tr} \log [\hat{I}_L - \hat{T}_L \hat{G}_{LR}^{(0)} \hat{T}_R \hat{G}_{RL}^{(0)}], \end{aligned} \quad (43)$$

with $\hat{G}_{AB}^{(0)}$ diagonal in Dirac space and $\hat{T}_A$ identical to the ones of the (multicomponent) bosonic case, with $\tilde{U}_A$ determined by $\tilde{V}_A$, as in (42). Note the important difference posed by the global $(-\frac{1}{2})$ factor, which is crucial for determining the sign of the force.

Let us now consider the issue of symmetry under reflection. Albeit the conclusion will be identical, we need to distinguish between two cases: $d = 1$ and $d > 1$:

A. $d = 1$

For $d = 1$, it is quite clear that if

$$\hat{\mathcal{P}} \tilde{V}_L \hat{\mathcal{P}}^\dagger = -\tilde{V}_R, \quad \hat{\mathcal{P}} \tilde{V}_R \hat{\mathcal{P}}^\dagger = -\tilde{V}_L \Rightarrow \hat{\mathcal{P}} \tilde{V} \hat{\mathcal{P}}^\dagger = -\tilde{V}, \quad (44)$$

then $\hat{\mathcal{U}}_R = \hat{\mathcal{P}} \hat{\mathcal{U}}_L \hat{\mathcal{P}}^\dagger$. Thus, under this antisymmetry assumption for $\tilde{V}$, we may apply the same arguments of the bosonic case for an U which is parity symmetric. Therefore, $\text{sign}(\lambda_n \partial_a \lambda_n) = -1, \forall n$.

Thus, because of the $-\frac{1}{2}$ in (43),

$$\text{sign}(F) = +1, \quad (45)$$

which is a repulsive Casimir interaction. Again, note that the conclusion does not change if the Dirac field has many flavors, and therefore, the previous antisymmetry is realized for the resulting Hermitian matrices $\tilde{U}_L$ and $\tilde{U}_R$.

B. $d > 1$

For $d > 1$, we postulate the same antisymmetry as for $d = 1$, namely, that (44) holds. This yields, for $\hat{\mathcal{U}}$:

$$\hat{\mathcal{P}} \gamma_d \tilde{U}_L \gamma_d \hat{\mathcal{P}}^\dagger = \tilde{U}_R, \quad \hat{\mathcal{P}} \gamma_d \tilde{U}_R \gamma_d \hat{\mathcal{P}}^\dagger = \tilde{U}_L. \quad (46)$$

Since $\hat{\mathcal{P}} \gamma_d \tilde{T}_L \gamma_d \hat{\mathcal{P}}^\dagger = \tilde{T}_R$, we find

$$E_I = -\frac{1}{2} \int_{\mathcal{D}} \log[\det(\hat{I}_L - \hat{Y}_d^2)], \quad \hat{Y}_d \equiv \sqrt{\hat{T}_L} \hat{G}_{LR}^{(0)} \hat{\mathcal{P}} \gamma_d \sqrt{\hat{T}_L}. \quad (47)$$

Now, following the same procedure as for the real scalar field case, we introduce $|\phi_n\rangle$, now the normalized eigenvectors of $\hat{Y}_d$ (note that they cannot be chosen to be, simultaneously, eigenvectors of γ_d) and arrive to the study of the sign of

$$\langle \phi_n | \hat{Z}_d | \phi_n \rangle \langle \phi_n | \partial_a \hat{Z}_d | \phi_n \rangle, \quad (48)$$

with $\hat{Z}_d = \hat{Z} \gamma_d$, where $\hat{Z} = \hat{G}_{LR}^{(0)} \hat{\mathcal{P}}$.

Again, we introduce $|\psi_n\rangle \equiv \sqrt{\hat{T}_L} |\phi_n\rangle$, and also assume that a representation has been used for the Dirac matrices such that γ_d is diagonal, each block having $+1$ or -1 as eigenvalue. Denoting by $|\psi_n^{(\pm)}\rangle$ the component of the respective block of γ_d , we get

$$\begin{aligned} \text{sign}(\lambda_n \partial_a \lambda_n) &= \text{sign} \left[\langle \psi_n^{(+)} | \hat{Z} | \psi_n^{(+)} \rangle - \langle \psi_n^{(-)} | \hat{Z} | \psi_n^{(-)} \rangle \right] \\ &\quad \times \text{sign} \left[\langle \psi_n^{(+)} | \partial_a \hat{Z} | \psi_n^{(+)} \rangle - \langle \psi_n^{(-)} | \partial_a \hat{Z} | \psi_n^{(-)} \rangle \right]. \end{aligned} \quad (49)$$

Then we notice that the structure of each one of the averages above is as follows (see [7]):

$$\begin{aligned} \langle \psi_n^{(\pm)} | \hat{Z} | \psi_n^{(\pm)} \rangle &\equiv I_n^{(\pm)}(a) \\ &= \int \frac{d^{d-1} \mathbf{k}_\perp}{(2\pi)^{d-1}} \frac{e^{-a\sqrt{\mathbf{k}_\perp^2 + \omega^2}}}{2\sqrt{\mathbf{k}_\perp^2 + \omega^2}} f_n^{(\pm)}(\mathbf{k}_\perp, \omega), \end{aligned} \quad (50)$$

where $f_n^{(\pm)}$ is a positive function (determined by $|\phi_n\rangle$) which form we do not need here.

Then,

$$\begin{aligned} \text{sign}(\lambda_n \partial_a \lambda_n) &= \text{sign}[I_n^{(+)}(a) - I_n^{(-)}(a)] \\ &\quad \times \text{sign}[\partial_a I_n^{(+)} - \partial_a I_n^{(-)}]. \end{aligned} \quad (51)$$

Assuming $I_n^{(+)}(a) > I_n^{(-)}(a)$, and noting that, if that is the case, then, for $\varepsilon > 0$,

$$I_n^{(+)}(a + \varepsilon) - I_n^{(-)}(a + \varepsilon) < I_n^{(+)}(a) - I_n^{(-)}(a). \quad (52)$$

Hence, $\partial_a I_n^{(+)}(a) - \partial_a I_n^{(-)}(a) < 0$. Therefore, $\lambda_n \partial_a \lambda_n$ is negative.

On the other hand, if $I_n^{(+)}(a) < I_n^{(-)}(a)$, the same argument, after $(+) \leftrightarrow (-)$, leads to an identical conclusion about the sign.

Since, for the Dirac field, the force in terms of the eigenvalues λ_n is

$$F = - \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \sum_n \frac{\lambda_n \partial_a \lambda_n}{1 - \lambda_n^2}, \quad (53)$$

we again conclude that the force is positive, i.e., repulsive.

IV. DISCUSSION AND CONCLUSIONS

It is appropriate to discuss here at least one example of a $\tilde{V}$ function, in order to show how the resulting condition of the positivity for the operator $\mathbb{U}$ resulting from (42) is realized. Positivity of $\mathbb{U}$ is equivalent to positivity of both its L and R components. Therefore, it is sufficient to consider it for a single body.

Let us do that for a situation that has been extensively studied: that of singular δ potentials [10], which may be used to enforce bag-model boundary conditions.

A regularized version $\tilde{V}_\varepsilon$ of $\tilde{V}$ for a body located at $x_1 = 0$, say, is

$$\tilde{V}_\varepsilon(\omega, x_1) = \frac{g(\omega)}{\varepsilon} \theta\left(\frac{\varepsilon}{2} - |x_1|\right), \quad (54)$$

such that $\tilde{V}_\varepsilon \rightarrow \tilde{V}$ for $\varepsilon \rightarrow 0$. Then, for a normalized, continuous ψ ,

$$\begin{aligned} \langle \tilde{\mathbb{U}}_\varepsilon \rangle &= \langle \psi | \tilde{\mathbb{U}}_\varepsilon | \psi \rangle \\ &= \left(\frac{g}{\varepsilon}\right)^2 + \frac{g}{\varepsilon} \left[(\psi^\dagger \gamma_1 \psi) \left(\frac{\varepsilon}{2}\right) - (\psi^\dagger \gamma_1 \psi) \left(-\frac{\varepsilon}{2}\right) \right], \end{aligned} \quad (55)$$

an expression which is clearly positive for a sufficiently small ε .

For this kind of potential, one can use the natural generalization of the method used in [10] to prove that an odd potential leads to a repulsive Casimir interaction. As an explicit realization of this framework, we may consider the case of two bodies modeled by δ -like potentials with opposite coupling constants g and $-g$, separated by a distance a along the x_d direction:

$$V(x) = g[\delta(x_d) - \delta(x_d - a)]. \quad (56)$$

This potential satisfies the required oddness condition for $\tilde{V}$ and allows for an exact computation of the interaction energy. The result can be written as an integral over the frequency and momentum components perpendicular to x_d , denoted by $\mathbf{k}_\perp = (k_1, \dots, k_{d-1})$. For spatial dimensions $d = 1, 2, 3$, the vacuum energy is given by

$$\begin{aligned}
E(a) &= -2^{\lfloor \frac{d+1}{2} \rfloor - 1} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^{d-1} \mathbf{k}_{\perp}}{(2\pi)^{d-1}} \log \left(1 - \frac{16g^2}{(4+g^2)^2} e^{-2a\sqrt{\mathbf{k}_{\perp}^2 + \omega^2}} \right) \\
&= \frac{1}{a^d} \frac{2^{\lfloor \frac{d+1}{2} \rfloor}}{(4\pi)^{\frac{d+1}{2}}} \Gamma\left(\frac{d+1}{2}\right) \text{PolyLog}_{d+1}\left(\frac{16g^2}{(4+g^2)^2}\right),
\end{aligned} \tag{57}$$

which is a positive, monotonically decreasing function of the separation a , leading to a repulsive Casimir force.

We conclude by summarizing some of the aspects of the result we present in this article. First, we wish to remark that the property of the media being described by an odd function is a sufficient condition for repulsion, although not a necessary one. For instance, one also has repulsion in one of the systems considered in [11], which corresponds to an atom facing a wall that imposes bag boundary conditions. The force is repulsive, but there is no odd function $\tilde{V}$.

A relevant comment we wish to convey is about the relation of the result we present and its dependence on the known result for bosonic fields: That dependence is at the level of the proof, namely, the formal analogy between two objects (functional determinants) is what makes it possible to use one result to prove the other. It is worth pointing out that the different statistics is what leads to repulsion (rather than attraction) when relating both functional determinants.

We also wish to elaborate on another important aspect of the approach we followed: Note that we have expressed the Casimir energy for a Dirac field in the presence of space-dependent mass [the function $\tilde{V}$ in (35)] from the determinant of a fluctuation operator $\mathcal{H}$, defined as follows:

$$\mathcal{H} \equiv \mathcal{D}^{\dagger} \mathcal{D}, \quad \mathcal{D} = \not{\nabla} + i\gamma_0 \omega + \tilde{V}. \tag{58}$$

However, one could certainly have considered the operator $\mathcal{H}' \equiv \mathcal{D} \mathcal{D}^{\dagger}$, which has the same spectrum as $\mathcal{H}$, except for the existence of zero modes. Indeed, that is the content of the Atiyah–Patodi–Singer index theorem [12–14], which can be used to relate the spectral asymmetry between $\mathcal{H}$ and $\mathcal{H}'$ to topological properties of the Dirac operator. In cases where such asymmetry exists, in order to evaluate the

contribution of the fermionic modes, one should take the average of the contributions to the energy due to the operators $\mathcal{H}$ and $\mathcal{H}'$ [15–17]. For the system we are considering, that would correspond to taking the average of the energies due to two systems, defined with two values of the object defined in (42), namely: $\tilde{\mathcal{U}}^{\pm} = \tilde{V}^2 \pm (\not{\nabla} \tilde{V})$. Equivalently, one should take the average of the energies resulting from L and R in their original positions and the energy when they are reflected. For the kind of background we are considering, a space-dependent mass could produce zero modes for domain-wall mass profiles and in an odd number of spacetime dimensions, as in the Callan and Harvey mechanism [18]. Note, however, that that would require a rather different kind of configuration: The mass we are considering here has two spatial lumps (since we are dealing with a Casimir interaction phenomenon), and it vanishes except for the region occupied by the bodies.

We conclude by pointing out an interesting problem regarding this kind of result, about the sign of the Casimir interaction: For the bosonic version, it would be to know how far a departure from a perfectly symmetric configuration can go without changing the conclusion, i.e., attractive force. The equivalent question arises for the Dirac field case and departures from a perfectly odd configuration.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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