

Quantum entanglement in electron-nucleus collisions: Role of the linearly polarized gluon distribution

Michael Fucilla *National Centre for Nuclear Research, Pasteura 7, Warsaw 02-093, Poland*Yoshitaka Hatta *Physics Department, Brookhaven National Laboratory, Upton, New York 11973, USA
and RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA*Bo-Wen Xiao *School of Science and Engineering, The Chinese University of Hong Kong (Shenzhen),
Longgang, Shenzhen, Guangdong, 518172,
People's Republic of China and Southern Center for Nuclear-Science Theory (SCNT),
Institute of Modern Physics, Chinese Academy of Sciences, Huizhou, Guangdong 516000, China*

(Received 19 April 2026; accepted 14 July 2026; published 14 August 2026)

We calculate the spin density matrix of a back-to-back quark-antiquark pair inclusively produced in electron-nucleus scattering, taking into account the gluon saturation effect and the linearly polarized gluon distribution. We then investigate concurrence and stabilizer Rényi entropy, quantifying entanglement, Bell nonlocality, and nonstabilizerness. We find that the linearly polarized gluon distribution tends to enhance the entanglement of a heavy quark pair when the total and relative transverse momenta of the pair are orthogonal.

DOI: [10.1103/6c93-2tgg](https://doi.org/10.1103/6c93-2tgg)

I. INTRODUCTION

The interface between quantum information science (QIS) and high-energy collider physics has recently evolved into a concrete experimental program. The observation of spin correlations in top-antitop ($t\bar{t}$) production at the LHC by the ATLAS [1] and CMS [2] collaborations has demonstrated that quantum correlations among quarks can be experimentally accessed. These measurements have stimulated renewed theoretical interest in characterizing entanglement and Bell nonlocality in QCD processes [3–5]. While the top quark provides a particularly clean laboratory at hadron-hadron colliders, because it decays before hadronization, lighter quark systems can offer complementary opportunities in lepton-hadron scattering.

In the context of entanglement observables in DIS, earlier work in collinear factorization has investigated spin correlations and Bell nonlocality in inclusive $q\bar{q}$ -pair production at leading order and highlighted the prospects for maximal entanglement in certain kinematic regimes [6]

(see also [7–11]). In the high-energy limit, however, the appropriate theoretical framework is no longer collinear factorization but k_T -factorization, where the transverse momentum of small- x gluons is treated explicitly, and the relevant observables can be expressed in terms of transverse-momentum-dependent gluon distributions. In particular, deep inelastic scattering (DIS) at the future Electron-Ion Collider (EIC) will enable precision studies of quark-antiquark pair production in the small- x regime, where gluon densities become large and nonlinear QCD dynamics sets in. This regime is naturally described within the color glass condensate (CGC) effective theory [12], which provides a systematic description of the phenomena of gluon saturation. This latter constitutes one of the central pillars of the physics program of the Electron-Ion Collider. From this perspective, the investigation of $q\bar{q}$ spin-correlations in a CGC framework paves the way for a novel research direction uniting the physics of dense gluonic matter with the quantum-informational characterization of spin entanglement.¹ In a recent work, we analyzed spin-spin entanglement in exclusive diffractive $q\bar{q}$ production, where the interaction proceeds via color-singlet exchange [7,10]. At high-energy, where the color-singlet exchange

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International](https://creativecommons.org/licenses/by/4.0/) license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

¹For previous works on different types of entanglement in the context of high-energy QCD, see Refs. [13–28].

is described in terms of the celebrated hard (Balitsky-Fadin-Kuraev-Lipatov) Pomeron of QCD [29–31], it was observed that the $q\bar{q}$ diffractively produced always exhibits entanglement and Bell nonlocality [7]. In the context of the collinear factorization framework based on generalized parton distributions (GPDs), the same spin correlations were investigated in [10], showing a more complex pattern of entanglement between the quark and the antiquark.

The main aim of the present paper is the natural continuation of this program, i.e., the computation of spin correlations in inclusive $q\bar{q}$ production in electron-nucleus collisions in the small- x factorization framework. Compared to the exclusive (diffractive) channel, the inclusive process benefits from significantly larger event rates, which enhances its experimental viability. At present, access to spin correlations in heavy-quark production relies primarily on the fragmentation of heavy quarks into heavy baryons, followed by their semileptonic decays. This strategy is intrinsically limited by small branching fractions and reduced reconstruction efficiencies.² The inclusive channel at the EIC and the LHC (in ultra peripheral collisions, UPCs) offers a substantially more favorable environment, both in terms of luminosity and statistical precision, making it a particularly promising avenue for future studies of spin entanglement in DIS. In this work, we will consider the production of the $q\bar{q}$ -pair in the back-to-back limit. We will be particularly interested in how the linearly polarized gluon distribution [33,34] affects the spin density matrix, and consequently, the degrees of entanglement, Bell nonlocality, and nonstabilizerness.

II. THE SPIN-DENSITY MATRIX FOR THE INCLUSIVE DIJET PRODUCTION AT SMALL- x

The total cross section of the inclusive dijet production in DIS or photoproduction has long been considered in the small- x literature [35], including both next-to-leading logarithmic [36–41] and next-to-eikonal corrections [42–46]. In the dipole picture, it can be visualized as in Fig. 1. Here, a fast-moving photon of momenta p_γ in the x^3 -direction fluctuates into a quark-antiquark pair of momenta p_q and $p_{\bar{q}}$, respectively. The projectile dipole then interacts with the dense gluonic target (either a proton or nuclei), moving ultrarelativistically in the opposite direction with momenta p_t . At very high energies, the interaction is eikonal, i.e., it occurs with a small deflection angle. Introducing the standard Sudakov decomposition $k^\mu = k_1^+ n_1^\mu + k_2^+ n_2^\mu + k_T^\mu$, where n_1^μ and n_2^μ specify the $+/ -$ directions and such that $n_1 \cdot n_2 = 1$, we will can parametrize the 4-momenta as

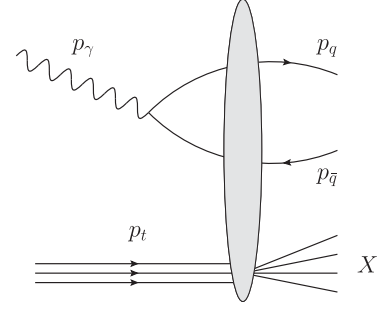


FIG. 1. Schematic representation of the inclusive dijet production in DIS or photoproduction at small- x in the dipole picture: a high-energy photon fluctuates into a quark–antiquark dipole, which then scatters eikonally off the dense gluonic target. There is no rapidity gap between the $q\bar{q}$ -pair and the undetected system X .

$$\begin{aligned} p_\gamma^\mu &= p_\gamma^+ n_1^\mu - \frac{Q^2}{2p_\gamma^+} n_2^\mu, & p_t^\mu &= p_t^- n_2^\mu, \\ p_q^\mu &= z p_\gamma^+ n_1^\mu + \frac{-p_{qT}^2 + m^2}{2z p_\gamma^+} n_2^\mu + p_{qT}^\mu, \\ p_{\bar{q}}^\mu &= \bar{z} p_\gamma^+ n_1^\mu + \frac{-p_{\bar{q}T}^2 + m^2}{2\bar{z} p_\gamma^+} n_2^\mu + p_{\bar{q}T}^\mu, \end{aligned} \quad (1)$$

where m is the current quark mass, and we introduced the longitudinal momentum fractions z and $\bar{z} \equiv 1 - z$. The target mass has been neglected. We denote with $p_\perp^i$ the Euclidian counterpart of the Minkowskian 4-vector p_T^μ , i.e., $p_\perp^2 = -p_T^2$.

In the following, we study this process within the framework of the semiclassical small- x effective field theory, commonly referred to as the CGC. In this approach, the fast partons in the target are treated as static classical color sources generating a strong background field, while the slow gluons are described as dynamical gauge fields propagating in this background. The interaction of the projectile dipole with the target is encoded in Wilson lines,

$$\mathcal{U}(x_\perp) = \mathcal{P} \exp \left[ig_s \int dx^+ T_R \cdot A^-(x^+, x_\perp) \right], \quad (2)$$

which resum multiple eikonal scatterings of the quark and antiquark off the target color field. Observables such as the inclusive dijet cross section can then be expressed in terms of correlators of Wilson lines, whose small- x evolution is governed by the corresponding nonlinear evolution equations.

A. General structure of the spin-density matrix

The cross section for inclusive dijet production in the CGC framework is given by [35] (see also Appendix A for more details)

²See, however, Ref. [32] for an alternative method to measure spin correlations in the light-quark sector.

$$\frac{d\sigma_{\alpha\alpha'\beta\beta'}^{L/T}}{dzd^2p_{q\perp}d^2p_{\bar{q}\perp}} = N_c \alpha_{em} e_f^2 p_\gamma^+ \int \frac{d^2x_{1\perp}}{(2\pi)^2} \int \frac{d^2x_{2\perp}}{(2\pi)^2} \int \frac{d^2x_{1'\perp}}{(2\pi)^2} \int \frac{d^2x_{2'\perp}}{(2\pi)^2} e^{-ip_{q\perp} \cdot x_{1'\perp}} e^{-ip_{\bar{q}\perp} \cdot x_{2'\perp}} \times \psi_{\beta\beta'}^{L/T(\lambda)*}(x_{1'2'\perp}) \psi_{\alpha\alpha'}^{L/T(\lambda)}(x_{12\perp}) \left(1 + S^{(4)}(x_{1\perp}, x_{2\perp}; x_{2'\perp}, x_{1'\perp}) - S^{(2)}(x_{1\perp}, x_{2\perp}) - S^{(2)}(x_{2'\perp}, x_{1'\perp}) \right), \quad (3)$$

where the dipole and quadrupole matrix elements are

$$S^{(2)}(x_{1\perp}, x_{2\perp}) = \frac{1}{N_c} \langle \text{Tr}_c [\mathcal{U}_F(x_{1\perp}) \mathcal{U}_F^\dagger(x_{2\perp})] \rangle_A, \quad (4)$$

$$S^{(4)}(x_{1\perp}, x_{2\perp}; x_{2'\perp}, x_{1'\perp}) = \frac{1}{N_c} \langle \text{Tr}_c [\mathcal{U}_F(x_{1\perp}) \mathcal{U}_F^\dagger(x_{2\perp}) \mathcal{U}_F(x_{2'\perp}) \mathcal{U}_F^\dagger(x_{1'\perp})] \rangle_A. \quad (5)$$

α and α' are the spinor indices of the quark and antiquark in the amplitude, respectively, and β and β' are those for the complex-conjugate amplitude. The unpolarized cross section is obtained by setting $\alpha = \beta$ and $\alpha' = \beta'$ and summing over these indices, but we keep them open for the purpose of computing the spin density matrix. The $\gamma \rightarrow q\bar{q}$ splitting wave functions $\psi^{L/T}$ are well known [47,48], see below.

In order to perform the coordinate integrations in (3), it is convenient to introduce the dijet relative and total transverse momenta as

$$P_\perp = \bar{z}p_{q\perp} - zp_{\bar{q}\perp}, \quad q_\perp = p_{q\perp} + p_{\bar{q}\perp}, \quad (6)$$

and their conjugate coordinates

$$\begin{aligned} u_\perp &= x_{1\perp} - x_{2\perp}, & u'_\perp &= x_{1'\perp} - x_{2'\perp}, \\ v_\perp &= zx_{1\perp} + \bar{z}x_{2\perp}, & v'_\perp &= zx_{1'\perp} + \bar{z}x_{2'\perp}. \end{aligned} \quad (7)$$

This transformation has a unit Jacobian

$$\begin{aligned} & \int d^2x_{1\perp} d^2x_{2\perp} d^2x_{1'\perp} d^2x_{2'\perp} e^{-ip_{q\perp}(x_{1\perp}-x_{1'\perp}) - ip_{\bar{q}\perp}(x_{2\perp}-x_{2'\perp})} \dots \\ &= \int d^2u_\perp d^2u'_\perp d^2v_\perp d^2v'_\perp e^{-iP_\perp(u-u') - iq_\perp(v-v')} \dots \end{aligned} \quad (8)$$

Naively, by Fourier transforming in $u_\perp$ and $u'_\perp$, one finds the product $\psi^*(P_\perp)\psi(P_\perp)$, where

$$\begin{aligned} \psi_{\alpha\alpha'}^L(P_\perp) &= \int d^2x_{12\perp} e^{-ix_{12\perp} \cdot P_\perp} \psi_{\alpha\alpha'}^{L/T(\lambda)}(x_{12\perp}) \\ &= (2\pi)^2 \sqrt{\frac{1}{p_q^+ p_{\bar{q}}^+ p_\gamma^+ P_\perp^2 + z\bar{z}Q^2 + m^2}} \frac{z\bar{z}Q}{\dots} \\ &\quad \times \bar{u}_\alpha(p_q) \gamma^+ v_{\alpha'}(p_{\bar{q}}), \end{aligned} \quad (9)$$

$$\psi_{\alpha\alpha'}^{T(\lambda)}(P_\perp) = (2\pi)^2 \sqrt{\frac{1}{8p_q^+ p_{\bar{q}}^+ p_\gamma^+ P_\perp^2 + z\bar{z}Q^2 + m^2}} \frac{\epsilon_\lambda^i}{\dots} \bar{u}_\alpha(p_q) \left((1-2z)P_\perp^i + i\epsilon^{ij}P_\perp^j \gamma_5 - m\gamma_\perp^i \right) \gamma^+ v_{\alpha'}(p_{\bar{q}}), \quad (10)$$

representing a $q\bar{q}$ pair with relative momentum $P_\perp$. However, this is complicated by the fact that the dipole and quadrupole amplitudes depend nontrivially on $u_\perp, u'_\perp, v_\perp, v'_\perp$. To avoid this complication, for simplicity, we consider the correlation limit, corresponding to a kinematical configuration in which the two jets are back-to-back. More precisely, the individual transverse momenta of the two jets are much larger than their sum $|P_\perp| \simeq |p_{q\perp}| \simeq |p_{\bar{q}\perp}| \gg |p_{q\perp} + p_{\bar{q}\perp}| \simeq |q_\perp|$. In coordinate space, this corresponds to the configuration $u_\perp, u'_\perp \ll v_\perp, v'_\perp$. We can then expand the target matrix elements around $u_\perp, u'_\perp \simeq 0$ and find [35]

$$\begin{aligned} & [1 + S^{(4)}(x_{1\perp}, x_{2\perp}; x_{2'\perp}, x_{1'\perp}) - S^{(2)}(x_{1\perp}, x_{2\perp}) - S^{(2)}(x_{2'\perp}, x_{1'\perp})] \\ &= -u_{\perp i} u'_{\perp j} \frac{1}{N_c} \langle \text{Tr}_c [(\partial_i \mathcal{U}(v_\perp)) \mathcal{U}^\dagger(v'_\perp) (\partial_j \mathcal{U}(v'_\perp)) \mathcal{U}^\dagger(v_\perp)] \rangle_A \equiv u_{\perp i} u'_{\perp j} G_{ij}(v_\perp, v'_\perp). \end{aligned} \quad (11)$$

G_{ij} can be identified with the Weiszäcker-Williams (WW) gluon distribution in the small- x limit

$$G_{ij}(v_\perp, v'_\perp) = \frac{g_s^2}{N_c} \int_{-\infty}^{\infty} dv^+ dv'^+ \langle \text{Tr}_c [F^{i-}(v_\perp) \mathcal{U}^{[+]\dagger} F^{j-}(v'_\perp) \mathcal{U}^{[+]}] \rangle, \quad (12)$$

where $\mathcal{U}^{[+]}$ is the half-infinite staple-shaped Wilson line extending to future infinity $x^+ = \infty$. The factor $u_\perp$ can be traded for the derivative with respect to $P_\perp$ acting on the photon wave function, leading to

$$\begin{aligned} \frac{d\sigma_{\alpha'\beta\beta'}^{L/T}}{dzd^2P_\perp d^2q_\perp} &= \frac{N_c \alpha_{em} e_f^2 P_\gamma^+}{(2\pi)^8} \int d^2u d^2u' e^{-iP_\perp(u-u')_\perp} u_{\perp i} u'_{\perp j} \psi_{\beta\beta'}^{L/T(\lambda)*}(u'_\perp) \psi_{\alpha\alpha'}^{L/T(\lambda)}(u_\perp) \int d^2v_\perp d^2v'_\perp G_{ij}(v_\perp, v'_\perp) e^{-iq_\perp(v-v')_\perp} \\ &= \frac{N_c \alpha_{em} e_f^2 P_\gamma^+}{(2\pi)^8} \frac{\partial}{\partial P_\perp^i} \frac{\partial}{\partial P_\perp^j} \psi_{\beta\beta'}^{L/T(\lambda)*}(P'_\perp) \psi_{\alpha\alpha'}^{L/T(\lambda)}(P_\perp) \Big|_{P'_\perp=P_\perp} \left[\frac{\delta_{ij}}{2} G_0(q_\perp) + \left(\frac{q_\perp^i q_\perp^j}{q_\perp^2} - \frac{\delta_{ij}}{2} \right) G_2(q_\perp) \right], \end{aligned} \quad (13)$$

where we have decomposed the WW gluon TMD $G_{ij}(q_\perp)$ into the unpolarized part G_0 and the linearly polarized part G_2 [33]. In deriving (13), we have observed that the Fourier transform and also the derivative in $P_\perp$ do not act on the spinors $\bar{u}(p_q)$ and $v(p_{\bar{q}})$. This can be justified in the shockwave (dipole) frame in which the quark and the antiquark are moving fast in the same direction. Technically, it is because only the “good” components

of the spinors contribute [49]. If one works in generic frames, in particular in the center-of-mass (CM) frame of the $q\bar{q}$ pair, the $P_\perp$ -dependence of the spinors must be fully taken into account. We emphasize that the clean factorization of the $P_\perp$ and $q_\perp$ dependencies in (13) occurs only in the correlation limit. This greatly simplifies the computation of the spin density matrix which we now turn to.

We shall evaluate the hard factor in the following form

$$\begin{aligned} &\left[\frac{\partial}{\partial P_\perp^i} \psi_{\beta\beta'}^{L/T(\lambda)*}(P_\perp) \right] \left[\frac{\partial}{\partial P_\perp^j} \psi_{\alpha\alpha'}^{L/T(\lambda)}(P_\perp) \right] \\ &= \xi_\alpha^\dagger \eta_{\alpha'}^\dagger \left[\delta^{ij} (A_{L/T}^{(0)} \mathbb{I} \otimes \mathbb{I} + (\tilde{C}_{L/T}^{(0)})_{ab} \sigma^a \otimes \sigma^b) + \left(\frac{2P_\perp^i P_\perp^j}{P_\perp^2} - \delta^{ij} \right) (A_{L/T}^{(2)} \mathbb{I} \otimes \mathbb{I} + (\tilde{C}_{L/T}^{(2)})_{ab} \sigma^a \otimes \sigma^b) \right] \xi_\beta \eta_{\beta'}, \end{aligned} \quad (14)$$

where $\mathbb{I}$ is the 2×2 unit matrix and σ^a are the Pauli matrices. These matrices are defined in a special frame in which the quark and the antiquark have back-to-back momenta, see Fig. 2

$$\begin{aligned} \tilde{p}_q^\mu &= (\tilde{p}_q^0, \tilde{p}_q^n, \tilde{p}_q^r, \tilde{p}_q^k) = \frac{M}{2} (1, 0, 0, \beta), \\ \tilde{p}_{\bar{q}}^\mu &= \frac{M}{2} (1, 0, 0, -\beta), \end{aligned} \quad (15)$$

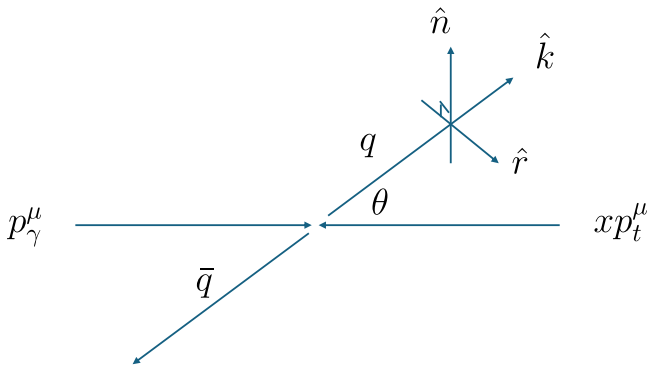


FIG. 2. Center-of-mass (CM) frame of the quark-antiquark pair at $q_\perp \approx 0$. $x = \frac{Q^2 + M^2}{2p_r \cdot p_l}$ is the momentum fraction of the target entering the hard scattering.

where

$$\begin{aligned} M &= \sqrt{\frac{P_\perp^2 + m^2}{z\bar{z}}}, \quad \beta = \sqrt{1 - \frac{4m^2}{M^2}}, \\ \cos \theta &= \frac{2z - 1}{\beta}. \end{aligned} \quad (16)$$

θ is the scattering angle in the pair’s CM frame. This frame can be reached from the original “shockwave frame” as follows. First, we perform the $P_\perp$ -derivatives in (14). As discussed above, the external spinors $\bar{u}(p_q)$, $v(p_{\bar{q}})$ do not depend on $P_\perp$, see (B1). We then boost along the x^3 -direction (assuming $q_\perp \approx 0$) to reach the CM frame of the pair. Finally, we perform a 3-dimensional rotation by angle θ of the coordinate system $(\hat{x}^1, \hat{x}^2, \hat{x}^3) \rightarrow (\hat{n}, \hat{r}, \hat{k})$ such that $\hat{k}$ coincides with the direction of the quark and $\hat{n}$ is normal to the reaction plane. ξ_α and η_β with $\alpha, \beta = \pm$ are the two-component spinors for the quark and the antiquark, respectively, with the sign $\pm$ referring to the projection of their spins along the $\hat{k}$ direction (i.e., the same as helicity for the quark and opposite to helicity for the antiquark.) They act on the first and second Pauli matrices $\sigma^{a=n,r,k}$, namely, $\xi_\alpha^\dagger \eta_\beta^\dagger (\sigma^a \otimes \sigma^b) \xi_\alpha \eta_\beta \equiv \xi_\alpha^\dagger \sigma^a \xi_\alpha \eta_\beta^\dagger \sigma^b \eta_\beta$.

Inserting (14) into (13), we obtain

$$\frac{d\sigma_{\alpha\alpha'\beta\beta'}^{L/T}}{dzd^2P_\perp d^2q_\perp} = \frac{N_c \alpha_{em} e_f^2 p_\gamma^+}{(2\pi)^8} \xi_\alpha^\dagger \eta_{\alpha'}^\dagger \left\{ (G_0 A_{L/T}^{(0)} + \cos(2\phi_{P,q}) G_2 A_{L/T}^{(2)}) \mathbb{I} \otimes \mathbb{I} \right. \\ \left. + (G_0 (\tilde{C}_{L/T}^{(0)})_{ab} + \cos(2\phi_{P,q}) G_2 (\tilde{C}_{L/T}^{(2)})_{ab}) \sigma^a \otimes \sigma^b \right\} \xi_\beta \eta_{\beta'},$$

where $\phi_{P,q} = \phi_P - \phi_q$ is the relative angle between ϕ_P and ϕ_q . The physical cross section/density matrix is a linear combination of the longitudinal and transverse contributions

$$\frac{d\sigma_{\alpha\alpha'\beta\beta'}}{dzd^2P_\perp d^2q_\perp} = \frac{N_c \alpha_{em} e_f^2 p_\gamma^+}{(2\pi)^8} \mathcal{A}_{\xi_\alpha^\dagger \eta_{\alpha'}^\dagger} (\mathbb{I} \otimes \mathbb{I} + C_{ab} \sigma^a \otimes \sigma^b) \xi_\beta \eta_{\beta'} \quad (17)$$

where

$$\mathcal{A} = G_0 (A_T^{(0)} + \varepsilon A_L^{(0)}) + \cos(2\phi_{P,q}) G_2 (A_T^{(2)} + \varepsilon A_L^{(2)}), \\ C_{ab} = \frac{1}{\mathcal{A}} \left[G_0 (\tilde{C}_T^{(0)} + \varepsilon \tilde{C}_L^{(0)}) + \cos(2\phi_{P,q}) G_2 (\tilde{C}_T^{(2)} + \varepsilon \tilde{C}_L^{(2)}) \right]_{ab}, \quad (18)$$

and $\varepsilon \approx \frac{1-y}{1-y+\frac{y^2}{2}}$, with $y = \frac{Q^2}{xs}$ being the standard DIS variable, is the ratio of the transverse and longitudinal photon fluxes. (In UPC, $\varepsilon = 0$.) By calculating the coefficients $C^{(0)}$ and $C^{(2)}$ for arbitrary photon's virtuality and polarization, we are able to consider both dijet photoproduction ($Q = 0$) and dijet electroproduction ($Q \neq 0$).

If we consider the angular-averaged cross section

$$\int_0^{2\pi} d\phi_{P,q} \frac{d\sigma^{L/T}}{dzd^2P_\perp d^2q_\perp} \sim [\mathbb{I} \otimes \mathbb{I} + C_{ab}^{(0)} \sigma^a \otimes \sigma^b], \quad C_{ab}^{(0)} = \frac{(\tilde{C}_T^{(0)} + \varepsilon \tilde{C}_L^{(0)})_{ab}}{A_T^{(0)} + \varepsilon A_L^{(0)}}, \quad (19)$$

the density matrix does not depend on G_0 and G_2 , namely, it does not depend on the structure of the target. By staying differential in the relative angle $\phi_{P,q}$, we can study how the density matrix is influenced by the gluon distributions of the target.

The simplification of the correlation limit is now clear. Because the $q\bar{q}$ pair is (approximately) back-to-back in the transverse plane, one can easily boost along the $\hat{x}^3$ -axis to reach the pair's CM frame and find a common axis (the $\hat{k}$ -axis) to quantize the spins of the quark and the antiquark. Away from the correlation limit, one has to either perform more complicated Lorentz transformations to reach the CM frame, or quantize the quark and antiquark spins along different directions, which makes the interpretation of the spin density matrix obscure. The same problem should arise in the diffractive case [7,10] if the target momentum transfer t becomes large and comparable to $P_\perp$. These are, however, technical and interpretational issues rather than a fundamental obstruction. We leave this problem to future work.

B. Longitudinal photon

We start by considering the case in which the incoming gluon is longitudinally polarized. From (9), we find

$$\left[\frac{\partial}{\partial P_\perp^i} \psi_{\beta\beta'}^{L*}(P_\perp) \right] \left[\frac{\partial}{\partial P_\perp^j} \psi_{\alpha\alpha'}^L(P_\perp) \right] \\ = \frac{(2\pi)^4}{p_q^+ p_{\bar{q}}^+ p_\gamma^+ (P_\perp^2 + z\bar{z}Q^2 + m^2)^4} \\ \times \bar{v}_{\beta'}(p_q) \gamma^+ u_\beta(p_{\bar{q}}) \bar{u}_\alpha(p_q) \gamma^+ v_{\alpha'}(p_{\bar{q}}). \quad (20)$$

As we noted below (10), the $P_\perp$ -derivative does not act on the spinors. We immediately notice that, in the notation of (14),

$$A_L^{(0)} = A_L^{(2)} \quad \tilde{C}_L^{(0)} = \tilde{C}_L^{(2)}. \quad (21)$$

Summing over the helicity indices, we reproduce the known unpolarized cross section

$$A_L = \frac{4(2\pi)^4 z^2 \bar{z}^2 Q^2 P_\perp^2}{p_\gamma^+ (P_\perp^2 + z\bar{z}Q^2 + m^2)^4}, \quad (22)$$

$$\frac{d\sigma^L}{dzd^2P_\perp d^2q_\perp} = \frac{16N_c \alpha_{em} e_f^2 z^2 \bar{z}^2 Q^2 P_\perp^2}{(2\pi)^4 (P_\perp^2 + z\bar{z}Q^2 + m^2)^4} \\ \times (G_0(q_\perp) + \cos(2\phi_{P,q}) G_2(q_\perp)). \quad (23)$$

The spinor product with open helicity indices in (20) can be evaluated by following the same procedure as in [7]. The result is

$$\bar{v}_{\beta'}(p_q)\gamma^+u_{\beta}(p_{\bar{q}})\bar{u}_{\alpha}(p_q)\gamma^+v_{\alpha'}(p_{\bar{q}}) = \frac{8z\bar{z}(p_{\gamma}^+)^2}{4}\xi_{\alpha}^{\dagger}\eta_{\alpha'}^{\dagger}(\delta_{\alpha\beta}\delta_{\alpha'\beta'} + C_{ab}^L\xi_{\alpha}^{\dagger}\sigma^a\xi_{\beta}\eta_{\alpha'}^{\dagger}\sigma^b\eta_{\beta'})\xi_{\beta}\eta_{\beta'} \quad (24)$$

with

$$\begin{aligned} C_{nn}^L &= 1, & C_{rr}^L &= -C_{kk}^L = -\frac{1 - (2 - \beta^2)\cos^2\theta}{1 - \beta^2\cos^2\theta} = -\frac{P_{\perp}^2 - (1 - 2z)^2m^2}{P_{\perp}^2 + (1 - 2z)^2m^2}, \\ C_{rk}^L &= C_{kr}^L = -\frac{\sqrt{1 - \beta^2}\sin 2\theta}{1 - \beta^2\cos^2\theta} = \frac{2(1 - 2z)\sqrt{P_{\perp}^2}m}{P_{\perp}^2 + (1 - 2z)^2m^2}, \end{aligned} \quad (25)$$

and the other components vanish (due to parity [50]). The velocity β and the angle in the CM frame are as defined in (16). We immediately notice that the C -matrix (25) is independent of the target, and is actually the same as in [6,7,10]. In other words, for the longitudinally polarized virtual photon, the same spin density matrix appears in (i) one-gluon exchange [6], (ii) two-gluon and quark-antiquark color-singlet exchange [10], (iii) multiple gluon color-singlet (Pomeron) exchange [7], and (iv) multiple gluon exchange in inclusive production (this work). Point (i) is understandable, since in the correlation limit, the hard matrix element is effectively that of one-gluon exchange. But the agreement with the diffractive (color-singlet) case is nontrivial. The reason behind this “universality” is interesting to explore. As observed in [6], the states represented by the density matrix (25) are maximally entangled pure states with the corresponding concurrence $\mathcal{C}[\rho] \equiv 1$. In the relativistic limit $\beta \rightarrow 1$, or for the symmetric configuration $z = 1/2$ (meaning $\cos\theta = 0$ in the CM frame), or along the line $z = \frac{1}{2} \pm \frac{P_{\perp}}{2m}$ [10], it reduces to one of the Bell states.

We now turn our attention to another quantum information theory quantity, the so-called nonstabilizerness (often referred to as “magic”). This latter quantifies the extent to which a quantum state departs from the class of stabilizer states, which can be efficiently simulated on a classical computer. In high-energy physics, this concept has recently been introduced as a tool to characterize the intrinsic quantum complexity of states produced in scattering processes. Nonstabilizerness can be quantified through the stabilizer Rényi entropy [51,52] (see Appendix C), which, in this case, can be written as a function of the ratio $y = m/|P_{\perp}|$ [10],

$$S_{\text{R}}^{\text{stab}}(y, z) = -\ln\left(\frac{1 + 14y^4(1 - 2z)^4 + y^8(1 - 2z)^8}{(1 + y^2(1 - 2z)^2)^4}\right). \quad (26)$$

This function is minimal (indeed zero) at $z = 1/2$, as expected for a Bell state, independently of the value of y .

For a massless quark, the nonstabilizerness is always zero. For a massive quark, it is governed by the ratio of its mass to $P_{\perp}$. In general, at fixed y , $M^L(z)$ is a function that vanishes at $\pm\infty$ and exhibits four local maxima and three local minima. Since the physical domain is restricted to $0 < z < 1$, for relatively small values of the ratio y ($y \leq 0.4$) only the local minimum at $z = 1/2$ lies within the physical range, and the global maximum is attained at the end points ($z = 0, 1$). For larger values, $0.4 < y < 1$, two of the local maxima also fall within the physical domain, where they constitute the absolute maxima.

C. Transverse photon

The transverse photon case is much more involved. The details of the calculation are provided in Appendix B, while here, we limit ourselves to reporting the result. First, we discuss the spin-density matrix elements proportional to the total cross sections. We have

$$A_T^{(0)} = \frac{2(2\pi)^4}{p_{\gamma}^+} \frac{(z^2 + \bar{z}^2)(P_{\perp}^4 + \bar{Q}^4) + 2P_{\perp}^2m^2}{(P_{\perp}^2 + \bar{Q}^2)^4}, \quad (27)$$

and

$$A_T^{(2)} = \frac{4(2\pi)^4}{p_{\gamma}^+} \frac{P_{\perp}^2(m^2 - (z^2 + \bar{z}^2)(m^2 + z\bar{z}Q^2))}{(P_{\perp}^2 + \bar{Q}^2)^4}, \quad (28)$$

where $\bar{Q}^2 \equiv z\bar{z}Q^2 + m^2$. Then, the unpolarized cross section summed over quark helicities is given by

$$\frac{d\sigma^T}{dzd^2P_{\perp}d^2q_{\perp}} = \frac{N_c\alpha_{em}e_f^2p_{\gamma}^+}{(2\pi)^8} \left(A_T^{(0)}G_0 + A_T^{(2)}G_2\cos(2\phi_{P,q}) \right). \quad (29)$$

As for the spin density matrix, our result for the angular independent part $C_{Tab}^{(0)} = \bar{C}_{Tab}^{(0)}/A_T^{(0)}$ is

$$C_{Tnn}^{(0)} = -\frac{2z\bar{z}(P_\perp^4 + \bar{Q}^4)}{2m^2P_\perp^2 + (z^2 + \bar{z}^2)(P_\perp^4 + \bar{Q}^4)}, \quad (30)$$

$$C_{Trr}^{(0)} = \frac{-2z\bar{z}}{(m^2 + P_\perp^2)(m^2(1-2z)^2 + P_\perp^2)((z^2 + \bar{z}^2)(\bar{Q}^4 + P_\perp^4) + 2m^2P_\perp^2)} \\ \times \left\{ m^4((1-2z)^2\bar{Q}^4 - (3-2z)(1+2z)P_\perp^4) \right. \\ \left. + 2m^2P_\perp^2(2\bar{Q}^2P_\perp^2 + (z^2 + \bar{z}^2)\bar{Q}^4 + (z^2 + \bar{z}^2 - 2)P_\perp^4) - P_\perp^4(\bar{Q}^4 + P_\perp^4) \right\}, \quad (31)$$

$$C_{Tkk}^{(0)} = \frac{1}{(m^2 + P_\perp^2)(m^2(1-2z)^2 + P_\perp^2)((z^2 + \bar{z}^2)(\bar{Q}^4 + P_\perp^4) + 2m^2P_\perp^2)} \\ \times \left\{ -m^4(1-2z)^2(4\bar{Q}^2P_\perp^2 + (z^2 + \bar{z}^2)\bar{Q}^4 - (3+2\bar{z}z)P_\perp^4) \right. \\ \left. + 2m^2P_\perp^2(-2(z^2 + \bar{z}^2)\bar{Q}^2P_\perp^2 - 4\bar{z}^2z^2\bar{Q}^4 + (1-4(z^4 - 2z^3 + z))P_\perp^4) \right. \\ \left. + (z^2 + \bar{z}^2)P_\perp^4(\bar{Q}^4 + P_\perp^4) + 2m^6(1-2z)^2P_\perp^2 \right\}, \quad (32)$$

$$C_{Trk}^{(0)} = \frac{2m(1-z)|P_\perp|}{(m^2 + P_\perp^2)(m^2(1-2z)^2 + P_\perp^2)((z^2 + \bar{z}^2)(\bar{Q}^4 + P_\perp^4) + 2m^2P_\perp^2)} \\ \times \left\{ -m^4(1-2z)((1-2z)\bar{Q}^2 + (1+2z)P_\perp^2) + m^2(-4z^2\bar{Q}^2P_\perp^2 + (1-2z)^2\bar{Q}^4 + (8z^2 - 4z - 1)P_\perp^4) \right. \\ \left. + (4z^2 - 2z + 1)\bar{Q}^4P_\perp^2 + \bar{Q}^2P_\perp^4 - 2z(1-2z)P_\perp^6 \right\}, \quad (33)$$

$$C_{Tkr}^{(0)} = \frac{-2zm|P_\perp|}{(m^2 + P_\perp^2)(m^2(1-2z)^2 + P_\perp^2)((z^2 + \bar{z}^2)(\bar{Q}^4 + P_\perp^4) + 2m^2P_\perp^2)} \\ \times \left\{ m^4(2z-1)((1-2z)\bar{Q}^2 + (2z-3)P_\perp^2) + m^2(-4(1-z)^2\bar{Q}^2P_\perp^2 + (1-2z)^2\bar{Q}^4 + (3-4z(3-2z))P_\perp^4) \right. \\ \left. + (4z^2 - 6z + 3)\bar{Q}^4P_\perp^2 + \bar{Q}^2P_\perp^4 + (4z^2 - 6z + 2)P_\perp^6 \right\}, \quad (34)$$

and all other vanishing. Remarkably, the matrix $C_T^{(0)}$ exactly agrees with that of the one-gluon exchange approximation $\gamma^* + g \rightarrow q + \bar{q}$ calculated in the collinear factorization framework [6]. Our derivation based on the photon light-cone wave function demonstrates the robustness of this matrix against higher order corrections (multiple gluon exchanges), and at the same time provides a highly nontrivial check of the result [6]. However, unlike in the longitudinal case, $C_T^{(0)}$ does not agree with the corresponding matrix in the color-singlet (Pomeron) exchange [7,10]. In the massless case, the matrix $C_T^{(0)}$ drastically simplifies and becomes diagonal

$$C_T^{(0)} = \begin{pmatrix} -\frac{2z\bar{z}}{z^2 + \bar{z}^2} & 0 & 0 \\ 0 & \frac{2z\bar{z}}{z^2 + \bar{z}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (35)$$

Note that $C_{Tnn}^{(0)} = -C_{Trr}^{(0)}$. This result leads to the corresponding concurrence $\mathcal{C}[\rho] = \frac{2z\bar{z}}{z^2 + \bar{z}^2}$. We find that the mass correction to the concurrence is always negative.

For the linearly polarized part, the remaining normalized coefficients $C_{Tab}^{(2)} \equiv \tilde{C}_{Tab}^{(2)}/A_T^{(2)}$ are³

$$C_{Tnn}^{(2)} = \frac{2(m^2 + z\bar{z}Q^2)}{2m^2 - Q^2(z^2 + \bar{z}^2)}, \quad (36)$$

³To avoid confusion, it should be noted that $C_T^{(2)}$ itself is not a physical spin correlation matrix. In practice we only need $\tilde{C}_T^{(2)} = A_T^{(2)} C_T^{(2)}$ when computing the physical density matrix (18).

$$C_{Trr}^{(2)} = \frac{2z\bar{z}(m^2((1-2z)^2\bar{Q}^2 + 2P_\perp^2) - \bar{Q}^2P_\perp^2)}{(m^2 - (z^2 + \bar{z}^2)\bar{Q}^2)(m^2(1-2z)^2 + P_\perp^2)}, \quad (37)$$

$$C_{Tkk}^{(2)} = \frac{m^2(1-2z)^2(P_\perp^2 - (1+2z\bar{z})\bar{Q}^2) - (z^2 + \bar{z}^2)\bar{Q}^2P_\perp^2 + m^4(1-2z)^2}{(m^2 - (z^2 + \bar{z}^2)\bar{Q}^2)(m^2(1-2z)^2 + P_\perp^2)}, \quad (38)$$

$$C_{Trk}^{(2)} = -\frac{m(1-z)(\bar{Q}^2(m^2(1-2z)^2 + (8z^2 - 4z + 1)P_\perp^2) + m^2(1-4z^2)P_\perp^2 + P_\perp^4)}{|P_\perp|(m^2 - (z^2 + \bar{z}^2)\bar{Q}^2)(m^2(1-2z)^2 + P_\perp^2)}, \quad (39)$$

$$C_{Tkr}^{(2)} = \frac{mz(\bar{Q}^2(m^2(1-2z)^2 + (4z(2z-3) + 5)P_\perp^2) - m^2(3 + 4z(z-2))P_\perp^2 + P_\perp^4)}{|P_\perp|(m^2 - (z^2 + \bar{z}^2)\bar{Q}^2)(m^2(1-2z)^2 + P_\perp^2)}, \quad (40)$$

and all other vanishing. These results are new. In the massless limit, one can check that $C_T^{(2)} = C_T^{(0)}$ as given by (35), and therefore [cf., (18)]

$$C_T \equiv \frac{G_0\tilde{C}_T^{(0)} + \cos(2\phi_{P,q})G_2\tilde{C}_T^{(2)}}{G_0A_T^{(0)} + \cos(2\phi_{P,q})G_2A_T^{(2)}} = C_T^{(0)}. \quad (41)$$

In the massless limit, after taking the linearly polarized gluon distribution into account, the transverse spin density matrix (without the longitudinal part) gives the same concurrence $\mathcal{C}[\rho] = \frac{2z\bar{z}}{z^2 + \bar{z}^2}$. Once we turn on the quark mass, we again find that the concurrence decreases irrespective of $\phi_{P,q}$ and the ratio G_2/G_0 (as long as $|G_2/G_0| \leq 1$, as required by the positivity bound). Note that Eq. (41) no longer holds once we include the longitudinal contribution with $Q \neq 0$.

Moreover, in the massless limit, the matrix $[C_T^{(0)}]^T C_T^{(0)}$ has only two independent eigenvalues

$$\mu_1 = 1, \quad \mu_2 = \frac{4(1-z)^2z^2}{[z^2 + (1-z)^2]^2}, \quad (42)$$

whose sum is greater than one. This immediately implies a violation of the Bell-CHSH (Clauser-Horne-Shimony-Holt) inequality and an entangled state. When $z = 1/2$, we have $\mu_1 + \mu_2 = 2$, implying a maximal violation of the Bell-CHSH inequality, and thus a maximally entangled state. The Rényi entropy is vanishing at the end points ($z = 0, 1$) and in $z = 1/2$, while it reaches its maximum value

$$S_{R,\max}^{\text{stab}} \simeq 0.188, \quad (43)$$

for $z \simeq 0.267$ and $z \simeq 0.732$.

III. NUMERICAL ANALYSIS

In the case of light quarks, the spin density matrices (longitudinal and transverse) are very simple, and we analyzed their main features in the previous section.

Also, the impact of the linearly polarized gluon distribution is minor. Therefore, in the following, we will consider the case of massive quark pair production, which is considerably more complex. To characterize the main features of the density matrix, we will study concurrence, Bell nonlocality and stabilizer Rényi entropy. Their basic definitions are reviewed in Appendix C.

A. $\phi_{P,q}$ -averaged case

First consider the $\phi_{P,q}$ -averaged spin-density matrix (19). In this case, the target function G_0 decouples and our result is fully equivalent to the one obtained in collinear factorization [6]. We take this opportunity to complement and extend the analysis of Ref. [6]. In order to facilitate the comparison, we rewrite the spin-density matrix elements in terms of the three variables $\cos\theta$, β , and $\alpha \equiv Q^2/M^2$ [cf., Eqs. (16) and (25)], and study how these elements vary with respect to these variables.

In Ref. [6], the transverse and longitudinal photon cases were studied separately. However, in DIS at $Q \neq 0$, the measured cross section is the linear combination of the transverse and longitudinal contributions. Unfortunately, it is difficult to experimentally separate the two contributions at the EIC, since typically $\varepsilon \sim 1$ at high energy colliders where $y = \frac{Q^2}{xs}$ tends to be small. While it is possible to reach, say, $\varepsilon \sim 0.5$ in the smallest bins of x for a given value of Q^2 [53], for simplicity we neglect the small variation of $\varepsilon \sim \mathcal{O}(1)$ and set $\varepsilon = 1$ in the following. In Fig. 3, we show the resulting spin-density matrix of heavy-quark production in DIS. The pair is not entangled (separable) in the region between the two solid curves. Bell's inequality cannot be violated in the region between the two dashed curves. (Outside, the pair exhibits Bell nonlocality, see Appendix C.) From Fig. 3 of [6], we see that the region where the pair is entangled is reduced compared to the case of the transversely polarized photon alone. This is interesting because the pair is always entangled in the purely longitudinal case [6]. Naively, adding the longitudinal component makes entanglement stronger, but this turns out to be not the case. Nonetheless, the entangled region

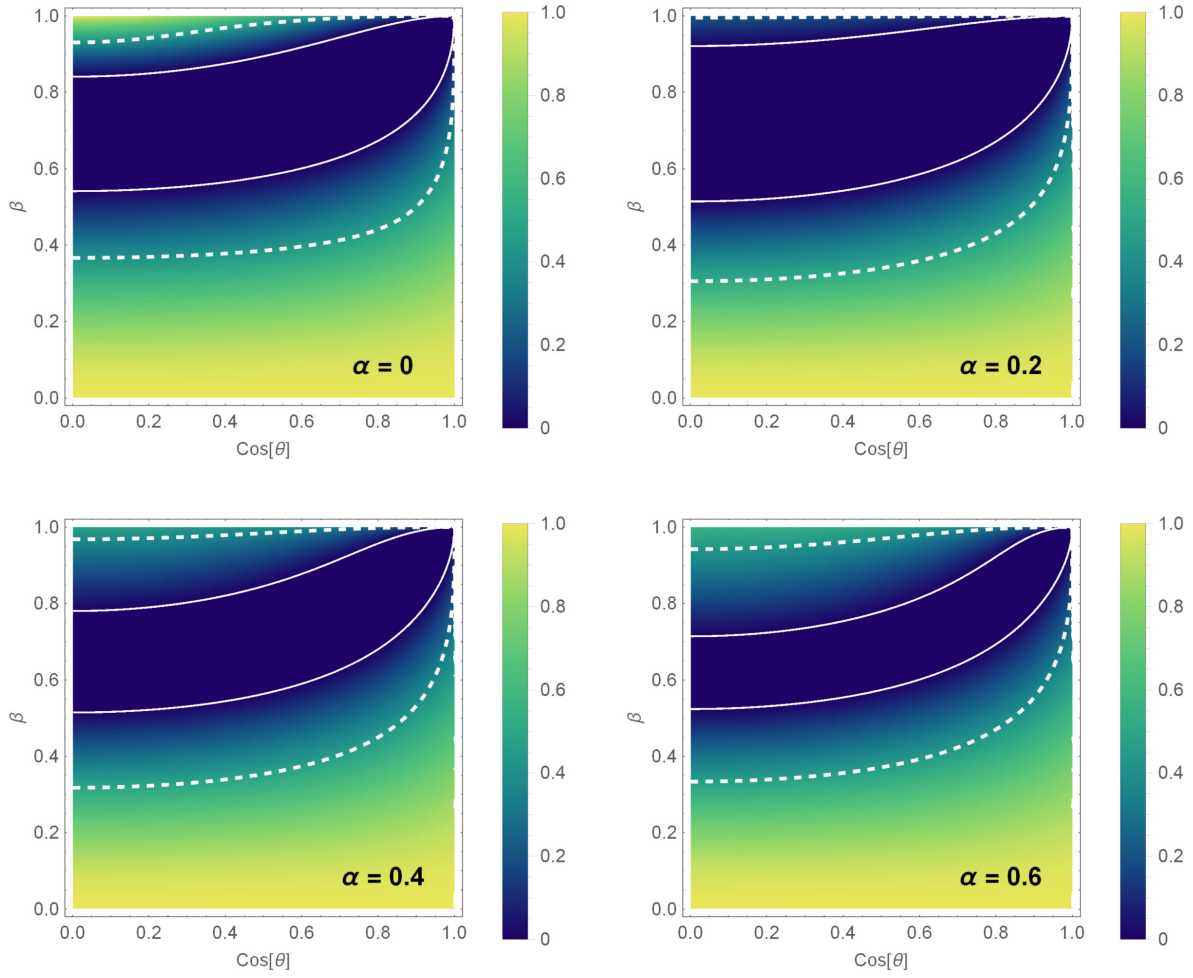


FIG. 3. Density plot of the concurrence $\mathcal{C}[\rho]$ associated to the spin density matrix in Eq. (19) with $\varepsilon = 1$, as functions of quark velocity β and $\cos \theta$, at given values of the virtuality parameter $\alpha = Q^2/M^2$. The value of the concurrence is identified by the color according to the vertical bar, and the continuous and dashed white lines indicate the boundaries for entanglement and Bell nonlocality, respectively. Namely, the $q\bar{q}$ pair is not entangled between the two solid curves and Bell's inequality cannot be violated between the two dashed curves.

remains substantial and is always larger than the region where Bell nonlocality is seen. This is in accordance with the general expectation (Bell nonlocality) $\subset$ (entanglement).

In Fig. 4, we the stabilizer Rényi entropy of the produced system. The results exhibit a characteristic pattern. The entropy is suppressed when the pair is either not entangled (i.e., between the solid curves in Fig. 5), or strongly entangled (i.e., $\beta \rightarrow 1$ or $\beta \rightarrow 0$). Instead, it is nonvanishing in intermediate regions between weak and strong entanglement. Such a tendency has been observed in other reactions [10,52,54,55]. The dependence on Q^2 appears to be non-monotonic, though regions of enhanced entropy tend to widen as Q^2 increases. The largest value of S_R^{stab} that we find in this study is about 0.45. In comparison, the theoretical upper limit for a two-qubit system is $S_R^{\text{stab}} \leq \ln(16/7) \simeq 0.827$ [56], which is reduced to [10]

$$S_R^{\text{stab}} \leq \ln \frac{9}{5} \simeq 0.588, \quad (44)$$

in systems with vanishing polarization [i.e., if $B_a = \bar{B}_b = 0$ in Eq. (C1)].

B. Dependence of entanglement and Rényi entropy on $\phi_{P,q}$

The fully $\phi_{P,q}$ -dependent spin-density matrix of heavy-quark production in DIS is given by Eq. (17). In this case, our analytical result is novel and depends on both the angle $\phi_{P,q}$ and the imbalance $q_\perp$. Furthermore, as in the diffractive case of Refs. [7,10], the coefficients $\mathcal{C}_{ab}$, depend on the gluon distributions of the target proton/nucleus. In particular, at high energy, the result can be sensitive to the small- x dynamics of the target, including the gluon saturation effect. Leaving detailed phenomenological

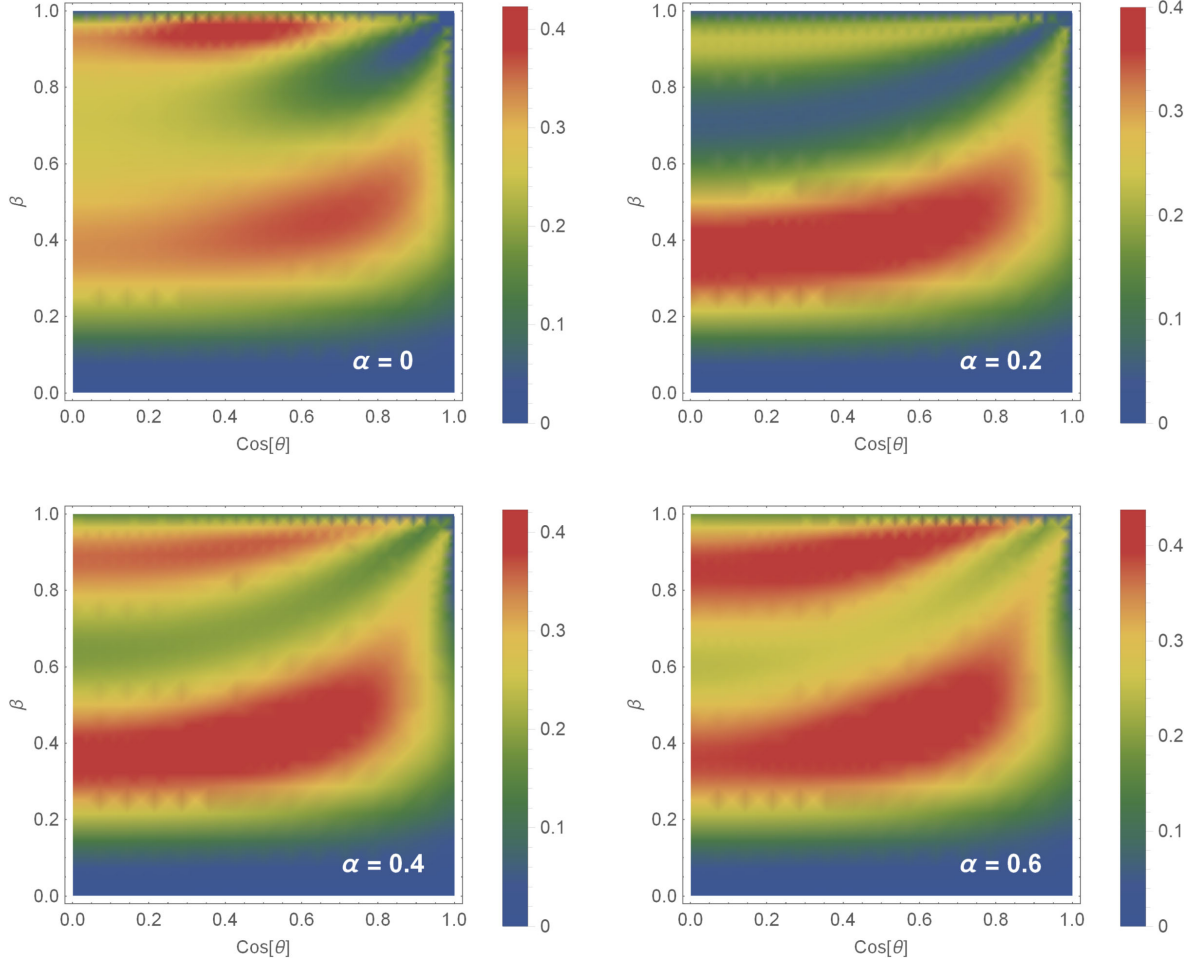


FIG. 4. Density plot of the stabilizer Rényi entropy S_R^{stab} associated to the spin density matrix in Eq. (19) with $\varepsilon = 1$, as functions of quark velocity β and $\cos \theta$, at given values of the virtuality parameter $\alpha = Q^2/M^2$. The value of S_R^{stab} is identified by the color according to the vertical bar.

investigations for future work, here we focus on understanding the general features of the results, in particular whether entanglement and Rényi entropy exhibit anisotropy with respect to the angle $\phi_{P,q}$.

We employ the McLerran-Venugopalan model [57] for the WW gluon distribution G_0 and the linearly polarized gluon distribution G_2 [58–60].

$$G_0(q_\perp) = 8\pi S_\perp \frac{C_F}{N_c} \int_0^\infty d|r_\perp| \frac{J_0(|q_\perp||r_\perp|)}{|r_\perp|} \left[1 - \exp\left(-\frac{Q_s^2 |r_\perp|^2}{4} \ln \frac{1}{|r_\perp|^2 \Lambda_{\text{IR}}^2}\right) \right], \quad (45)$$

and

$$G_2(q_\perp) = 8\pi S_\perp \frac{C_F}{N_c} \int_0^\infty d|r_\perp| \frac{J_2(|q_\perp||r_\perp|)}{|r_\perp| \ln \frac{1}{|r_\perp|^2 \Lambda_{\text{IR}}^2}} \left[1 - \exp\left(-\frac{Q_s^2 |r_\perp|^2}{4} \ln \frac{1}{|r_\perp|^2 \Lambda_{\text{IR}}^2}\right) \right], \quad (46)$$

where Q_s is the saturation momentum. The spin-density matrix elements C_{ab} clearly depend only on the ratio $G_2(q_\perp)/G_0(q_\perp)$ [see Eq. (17)]. We will follow the implementation of Ref. [60], taking $Q_s^2 \simeq 1 \text{ GeV}^2$ and

regulating the logarithmic behavior when $r\Lambda > 1$ through the replacement: $\ln(1/|r_\perp|^2 \Lambda_{\text{IR}}^2) \rightarrow \ln(1/|r_\perp|^2 \Lambda_{\text{IR}}^2 + e)$. Since $G_2(q_\perp)/G_0(q_\perp) \propto q_\perp^2$ as $q_\perp \rightarrow 0$, to see the impact of G_2 , one typically needs $q_\perp > Q_s$. On the other

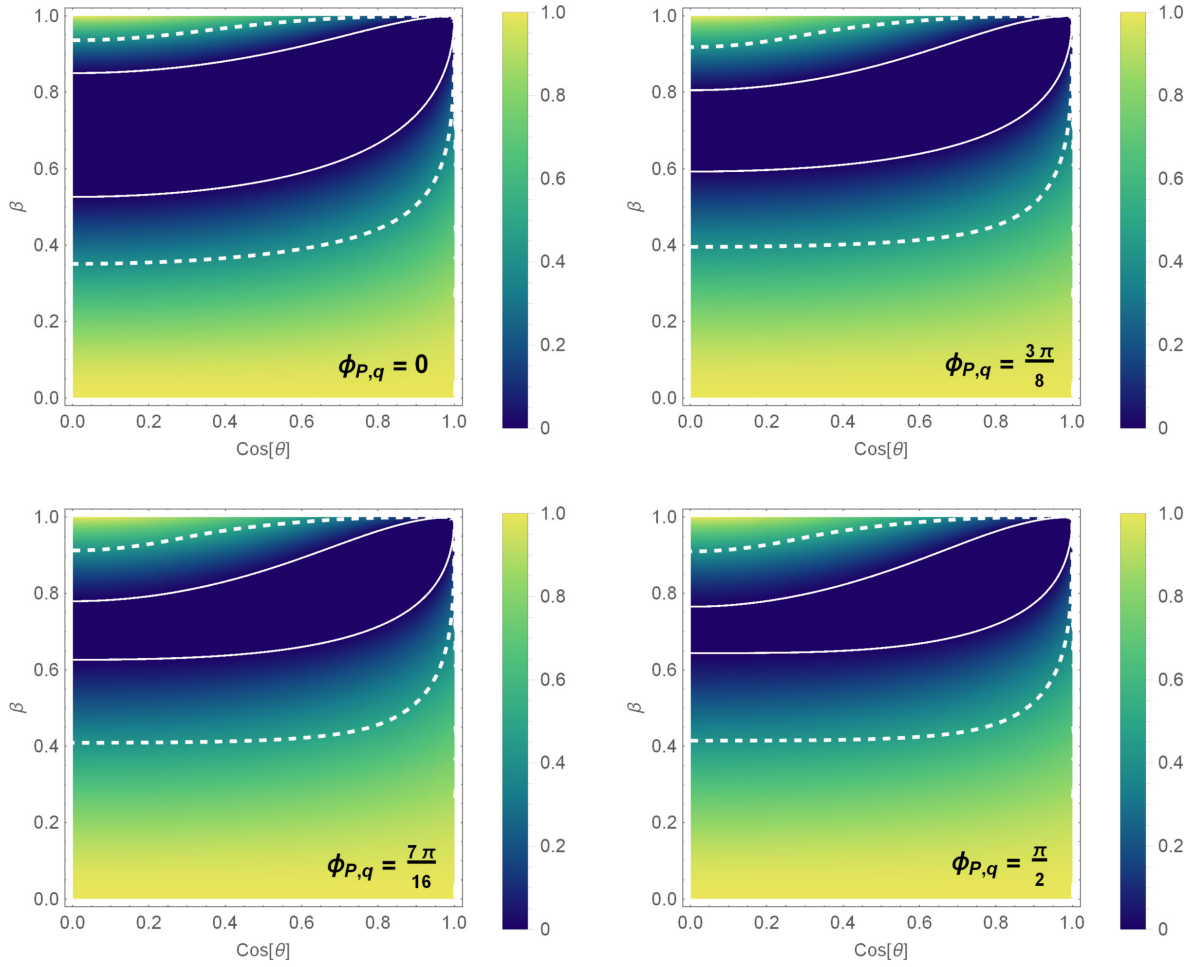


FIG. 5. Density plot of the concurrence $\mathcal{C}[\rho]$ associated to the spin density matrix in Eq. (17) for $\alpha = 0$ ($Q^2 = 0$) and $q_\perp = 2.5$ GeV, as functions of quark velocity β and $\cos \theta$. The value of the concurrence is identified by the color according to the vertical bar, and the continuous and dashed white lines indicate the boundaries for entanglement and Bell nonlocality, respectively.

hand, the present calculation assumes that the pairs are back-to-back

$$P_\perp^2 = m^2 \beta^2 \frac{1 - \cos^2 \theta}{1 - \beta^2} \gg q_\perp^2. \quad (47)$$

When $q_\perp$ is a few GeV, and for the charm quark $m \sim 1.5$ GeV, the condition (47) is satisfied only when $\beta \approx 1$, i.e., the relativistic (massless) limit. However, for the bottom quark $m \sim 4.2$ GeV, we find a reasonable phase space, say, $0.6 < \beta < 1$ with a decent window in $\cos \theta$.

The results for concurrence and Bell nonlocality are plotted in Figs. 5 and 6 for $q_\perp = 2.5$ GeV and $q_\perp = 4$ GeV, respectively, and for different values of $\phi_{P,q}$. (Note that the effect of the linearly polarized gluon vanishes at $\phi_{P,q} = \frac{\pi}{4}$). Although the plots cover the entire phase space in $(\cos \theta, \beta)$ assuming arbitrary values of m , in practice we have in mind bottom quark

pairs, and only the region $0.6 \lesssim \beta$ and $\cos \theta$ not too close to unity is physically relevant. Fortunately, interesting effects take place within this reduced phase space. We have found that when $\cos 2\phi_{P,q}$ is positive, meaning $0 \leq \phi_{P,q} \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \phi_{P,q} \leq \pi$, there is almost no dependence on $\phi_{P,q}$. Concurrence is actually reduced, but only by a very small amount of order 10^{-2} . However, as soon as $\cos 2\phi_{P,q}$ becomes negative, namely when $\frac{\pi}{4} < \phi_{P,q} < \frac{3\pi}{4}$, we observe that the region bounded by the solid curve noticeably shrinks, meaning that G_2 tends to enhance the concurrence/entanglement. The effect is more pronounced at higher $q_\perp$ because, at a fixed value of the saturation scale Q_s , the ratio $G_2(q_\perp)/G_0(q_\perp)$ monotonically increases as $q_\perp$ increases. The maximal effect is observed when $P_\perp$ and $q_\perp$ make a right angle $\phi_{P,q} = \frac{\pi}{2}$, as seen most clearly in the lower plots of Fig. 6. In contrast, the

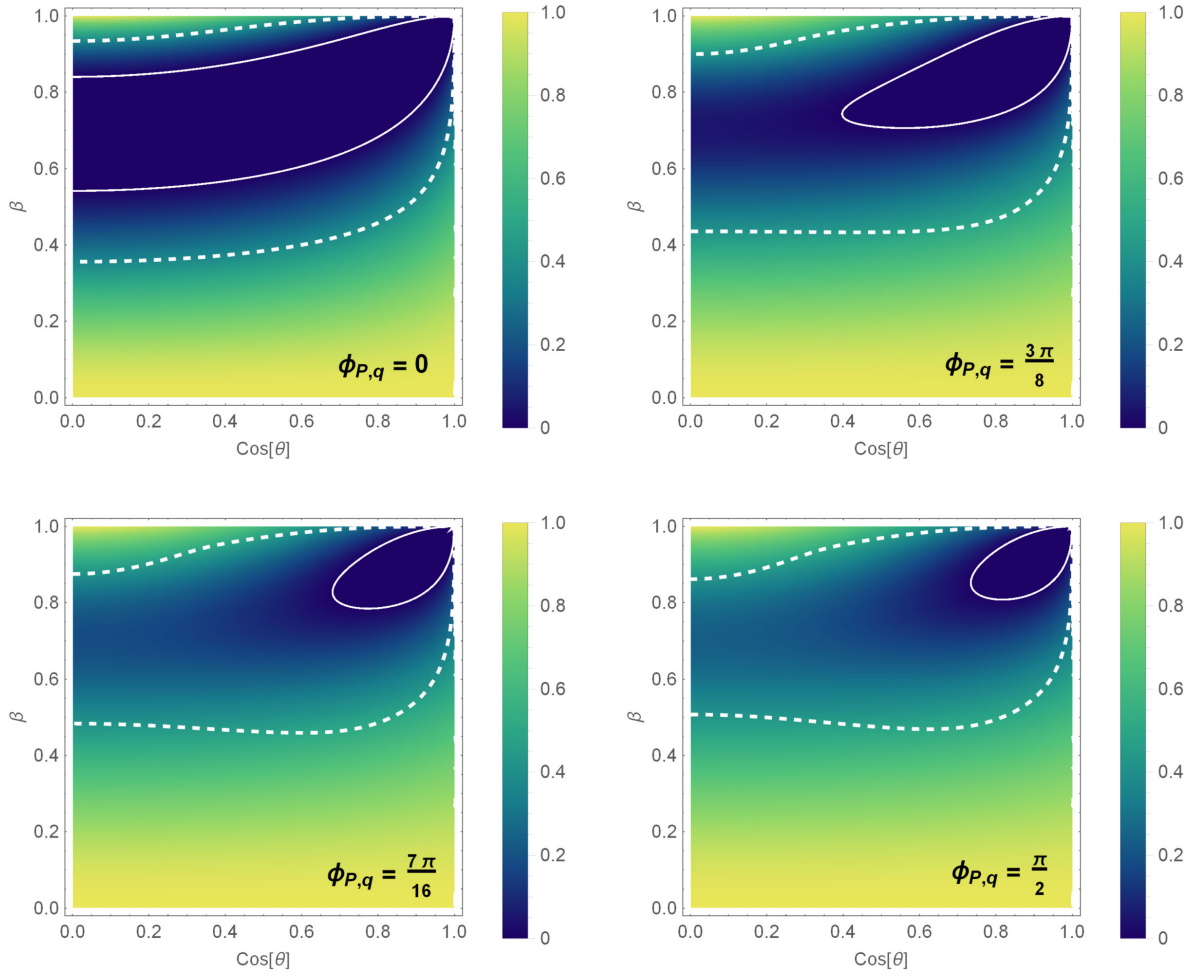


FIG. 6. Density plot of the concurrence $\mathcal{C}[\rho]$ associated with the spin density matrix in Eq. (17) for $\alpha = 0$ ($Q^2 = 0$) and $q_\perp = 4$ GeV, as functions of quark velocity β and the scalar projection $\cos \theta$. The value of the concurrence is identified by the color according to the vertical bar, and the continuous and dashed white lines indicate the boundaries for entanglement and Bell nonlocality, respectively.

boundary of Bell nonlocality is only weakly affected by the presence of G_2 .

From Fig. 7, we observe that the Rényi entropy is also sensitive to $\phi_{P,q}$. When $\cos 2\phi_{P,q} > 0$ (see the upper left plot), the entropy seems to decrease overall, but in some regions it actually increases. On the other

hand, when $\cos 2\phi_{P,q} < 0$, the entropy increases toward $\phi_{P,q} \rightarrow \frac{\pi}{2}$. This shows that the novel kinematical effects due to the intrinsic transverse momentum of the target, absent in pure collinear factorization, introduces an additional handle for generating quantum resources.

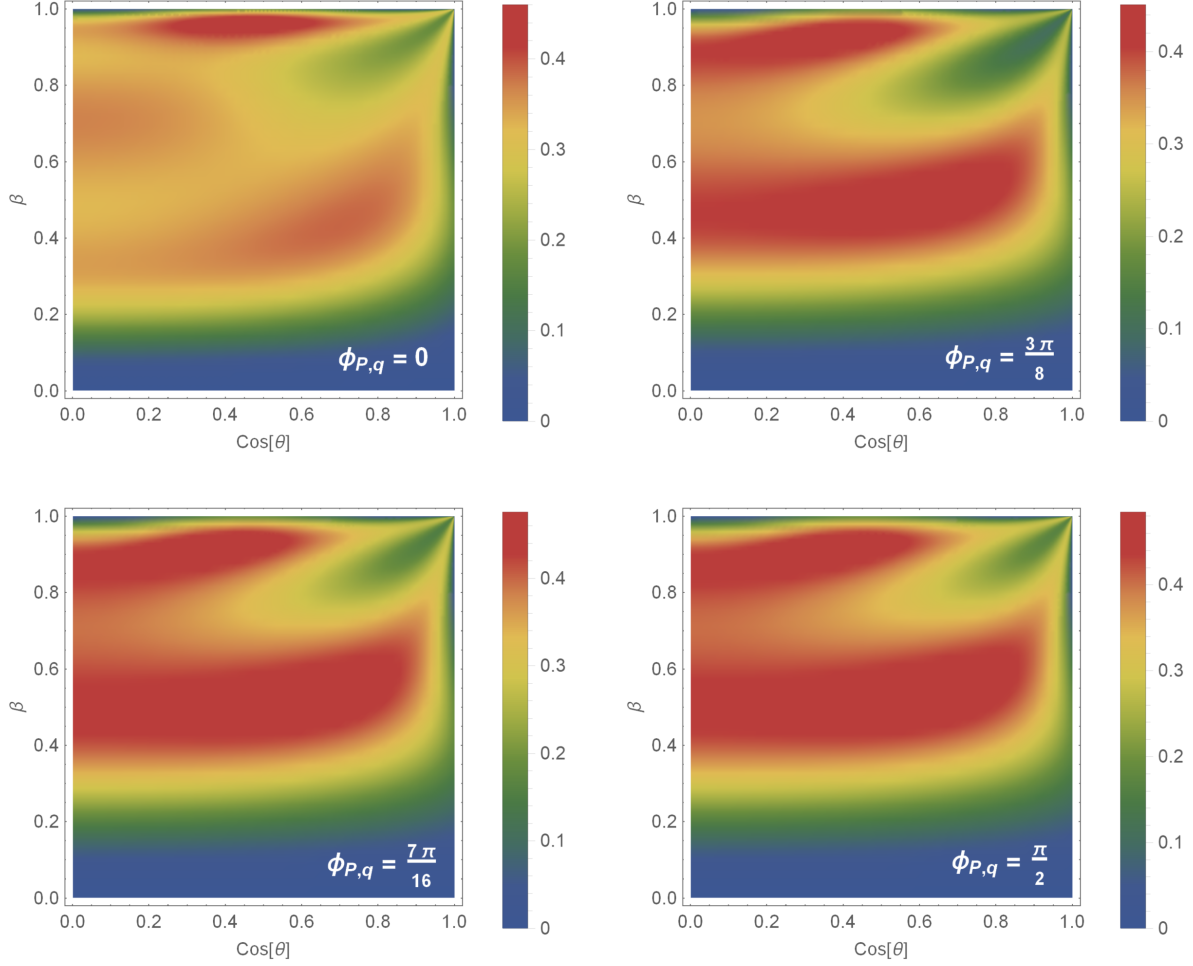


FIG. 7. Density plot of the stabilizer Rényi entropy S_R^{stab} associated to the spin density matrix in Eq. (17) for $\alpha = 0$ ($Q^2 = 0$) and $q_\perp = 4$ GeV, as functions of quark velocity β and the scalar projection $\cos \theta$. The value of S_R^{stab} is identified by the color according to the vertical bar.

IV. CONCLUSION AND OUTLOOK

In this work, we calculated the spin density matrix of a back-to-back quark-antiquark pair in electron-nucleus scattering in the k_T -factorization approach. We have shown that, by integrating over the azimuthal angle $\phi_{P,q}$ in the back-to-back limit, the density matrix obtained reduces to that of the one-gluon exchange for both the transversely and longitudinally polarized photons. In other words, multiple gluon exchanges do not modify the density matrix, at least in the correlation limit. As for differential observables in $\phi_{P,q}$, the spin density matrix becomes sensitive to both the unpolarized and linearly polarized WW gluon distributions. This is in contrast to the collinear factorization framework where the spin density matrix calculated from $2 \rightarrow 2$ processes depends only on energy and scattering angle. In DIS, in the k_T -factorization framework, it additionally depends on Q^2 , ε , $q_\perp$ and $\phi_{P,q}$, and also on possible saturation effects in the target. As a result, we have found rich and complex patterns of entanglement in the

final state, especially for massive quarks. In particular, the dependence on $\phi_{P,q}$ is strong and intriguing from QIS perspective. In bottom quark pair production, we have observed the largest amount of entanglement generated at $\phi_{P,q} = \frac{\pi}{2}$. This provides an interesting new perspective on this special kinematical point. Moreover, for the first time, we have confirmed that states with nonvanishing Rényi entropy are produced in inclusive production, although the largest value of $S_R^{\text{stab}} \sim 0.45$ found in this work is somewhat less than in the case of exclusive production $S_R^{\text{stab}} \sim 0.58$ [10].

We end with general comments about experimental measurements, leaving quantitative studies for future work. The results shown in Figs. 3 and 4 do not require small- x approximation and can be tested both at the LHC and the EIC. As for $\phi_{P,q}$ -dependent observables, the azimuthal correlation between the two momenta $P_\perp$ and $q_\perp$ in dijet production in photon-lead UPC has been measured by the CMS collaboration [61]. Such measurements are sensitive

to the small- x region of the nucleus. For example, if $\sqrt{s_{\gamma N}} = 500$ GeV, the region $x \sim M^2/s_{\gamma N} < 10^{-2}$ can be probed with $P_\perp \sim 20$ GeV. A feasibility study of the corresponding measurement at the EIC is yet to be done, but we expect it to be more challenging, especially due to the limitation in $P_\perp$ to ensure the condition (47). A fraction of these dijet events contain the fragmentation of massive quarks into heavy baryons and their subsequent weak decays such as $s\bar{s} \rightarrow \Lambda\bar{\Lambda} \rightarrow (p\pi^-)(\bar{p}\pi^+)$ [6,7,62]. In principle, the spin density matrix elements C_{ij} can be extracted from the angular correlation of decay products ($\pi^\pm$ in this case) with respect to the $\hat{i}$ and $\hat{j}$ axes. To study entanglement, Bell nonlocality, and nonstabilizerness, all the five components of the C -matrix are necessary. Such an analysis has been successfully carried out for the $t\bar{t}$ system at the LHC [63], but the extension of this method to lighter quark pairs still needs to be demonstrated. For massless (u, d) quarks, weak decay channels cannot be used, and one must seek alternative methods, e.g., [32]. One advantage of inclusive production as compared to exclusive production [7,10] is the higher cross section. The price to pay, however, is that the final states are more “contaminated” and one has to account for various effects such as Sudakov logarithms [64–67] before presenting a viable case for phenomenological studies.

ACKNOWLEDGMENTS

The work of M. F. is supported by the ULAM fellowship program of NAWA No. BNI/ULM/2024/1/00065 “Color glass condensate effective theory beyond the eikonal approximation.” Y. H. is supported by the U.S. Department of Energy under Contract No. DE-SC0012704, by LDRD funds from Brookhaven Science Associates, and also by the framework of the Saturated Glue (SURGE) Topical Theory Collaboration. B. X. is supported in part by the Ministry of Science and Technology of China under Grant No. 2024YFA1611004, by the Natural Science Foundation of Guangdong Province under Grant No. 2026A1515011242.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: INCLUSIVE DIJET PRODUCTION IN THE COLOR GLASS CONDENSATE

1. Impact factors

The S -matrix element for the inclusive dijet production in DIS reads

$$S_{\gamma^* \rightarrow q_1 q_2} = -ie e_f e_\mu^\lambda(p_\gamma) \int d^4 z \theta(-z^+) e^{ip_\gamma z} \bar{u}_\alpha^{\text{shock}}(p_q, z) \gamma^\mu v_\alpha^{\text{shock}}(p_{\bar{q}}, z) \quad (\text{A1})$$

where the effective quark/antiquark spinors through the shockwave reads

$$\begin{aligned} \bar{u}_\alpha^{\text{shock}}(p_q, z) &= \theta(p_q^+) \int \frac{d^2 k_\perp}{(2\pi)^2} \int d^2 x_{1\perp} e^{-ix_{1\perp} \cdot (k_1 - k)_\perp} \mathcal{U}_F(x_{1\perp}) \\ &\times e^{iz^+ \frac{k_\perp^2 + m^2}{2p_q^+} + iz^- p_q^+ - iz_\perp \cdot k_\perp} \bar{u}_\alpha(p_q) \gamma^+ \frac{\gamma^- p_q^+ - k_\perp + m}{2p_q^+}, \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} v_\alpha^{\text{shock}}(p_{\bar{q}}, z) &= -\theta(-p_{\bar{q}}^+) \int \frac{d^2 p_\perp}{(2\pi)^2} \int d^2 x_{2\perp} e^{-ix_{2\perp} \cdot (k_2 - p)_\perp} \mathcal{U}_F^\dagger(x_{2\perp}) \\ &\times e^{iz^+ \frac{p_\perp^2 + m^2}{2p_{\bar{q}}^+} + iz^- p_{\bar{q}}^+ - iz_\perp \cdot p_\perp} \frac{\gamma^- p_{\bar{q}}^+ - p_\perp - m}{2p_{\bar{q}}^+} \gamma^+ v_\alpha(p_{\bar{q}}). \end{aligned} \quad (\text{A3})$$

Integrating over z^+, z^- and $z_\perp$ in Eq. (A1), one can easily get

$$\begin{aligned} \mathcal{M}_{\gamma^* \rightarrow q_1 q_2} &= \frac{(-i) S_{\gamma^* \rightarrow q_1 q_2}}{2p_\gamma^+ (2\pi) \delta(p_\gamma^+ - p_q^+ - p_{\bar{q}}^+)} \\ &= (-i) e e_f e_\mu^\lambda(p_\gamma) \int d^2 x_{1\perp} \int d^2 x_{2\perp} e^{-ix_{1\perp} \cdot p_{q\perp}} e^{-ix_{2\perp} \cdot p_{\bar{q}\perp}} \Phi^\mu(x_{12\perp}) [\mathcal{U}_F(x_{1\perp}) \mathcal{U}_F^\dagger(x_{2\perp}) - 1], \end{aligned} \quad (\text{A4})$$

where

$$\begin{aligned}\Phi^\mu(x_{12\perp}) &= \frac{p_q^+(p_\gamma^+ - p_q^+)}{(p_\gamma^+)^2} \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{e^{ik_\perp \cdot x_{12\perp}}}{k_\perp^2 + z\bar{z}Q^2 + m^2} \\ &\times \bar{u}_\alpha(p_q) \gamma^+ \frac{\gamma^- p_q^+ - \not{k}_\perp + m}{2p_q^+} \gamma^\mu \frac{\gamma^- (p_\gamma^+ - p_q^+) + \not{k}_\perp - m}{2(p_\gamma^+ - p_q^+)} \gamma^+ v_{\alpha'}(p_{\bar{q}}).\end{aligned}\quad (\text{A5})$$

For a longitudinal photon, we can set

$$\varepsilon_\mu^\lambda(q) \rightarrow \varepsilon_\mu^+(q) \equiv \frac{Q}{q^+} g_\mu^+ \quad (\text{A6})$$

and we obtain

$$\Phi_{\alpha\alpha'}^L(x_{12\perp}) \equiv \varepsilon_\mu^+(q) \Phi^\mu(x_{12\perp}) = \frac{z\bar{z}Q}{q^+} \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{e^{ik_\perp \cdot x_{12\perp}}}{k_\perp^2 + z\bar{z}Q^2 + m^2} \bar{u}_\alpha(p_q) \gamma^+ v_{\alpha'}(p_{\bar{q}}), \quad (\text{A7})$$

where we defined $z = p_q^+/p_\gamma^+$ and $\bar{z} = 1 - z$. In the transverse case, we obtain

$$\Phi_{\alpha\alpha'}^{T(\lambda)}(x_{12\perp}) \equiv \varepsilon_{\perp\lambda}^i \Phi^i(x_{12\perp}) = \frac{1}{2q^+} \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{e^{ik_\perp \cdot x_{12\perp}}}{k_\perp^2 + z\bar{z}Q^2 + m^2} \bar{u}_\alpha(p_q) [(1 - 2z)k_\perp^i + i\epsilon^{ij}k_\perp^j \gamma^5 - m\gamma_\perp^i] \gamma^+ v_{\alpha'}(p_{\bar{q}}). \quad (\text{A8})$$

The $\Phi_{\alpha\alpha'}^{T(\lambda)}(x_{12\perp})$ are Fourier transform, i.e.,

$$\Phi_{\alpha\alpha'}^{L/T(\lambda)}(x_{12\perp}) = \int \frac{d^2 k_\perp}{(2\pi)^2} e^{ix_{12\perp} \cdot k_\perp} \Phi_{\alpha\alpha'}^{L/T(\lambda)}(k_\perp), \quad (\text{A9})$$

of the momentum space impact factors

$$\Phi_{\alpha\alpha'}^L(P_\perp) = \frac{z\bar{z}Q}{p_\gamma^+} \frac{1}{P_\perp^2 + z\bar{z}Q^2 + m^2} \bar{u}_\alpha(p_q) \gamma^+ v_{\alpha'}(p_{\bar{q}}), \quad (\text{A10})$$

and

$$\Phi_{\alpha\alpha'}^{T(\lambda)}(P_\perp) = \frac{1}{2p_\gamma^+} \frac{\varepsilon_\lambda^i}{P_\perp^2 + z\bar{z}Q^2 + m^2} \bar{u}_\alpha(p_q) \left((1 - 2z)P_\perp^i + i\epsilon^{ij}P_\perp^j \gamma_5 - m\gamma_\perp^i \right) \gamma^+ v_{\alpha'}(p_{\bar{q}}). \quad (\text{A11})$$

2. Cross section of the inclusive dijet production

The CGC event-by-event amplitude can be written

$$\mathcal{M}_{\gamma^* \rightarrow q_1 q_2}^{L/T(\lambda)} = (-i) e e_f \int d^2 x_{1\perp} \int d^2 x_{2\perp} e^{-ix_{1\perp} \cdot p_{q\perp}} e^{-ix_{2\perp} \cdot p_{\bar{q}\perp}} \Phi_{\alpha\alpha'}^{L/T(\lambda)}(x_{12\perp}) [\mathcal{U}_F(x_{1\perp}) \mathcal{U}_F^\dagger(x_{2\perp}) - 1], \quad (\text{A12})$$

The longitudinal and transverse cross sections are respectively obtained as

$$\frac{d\sigma^L}{dP.S.} = 2p_\gamma^+ (2\pi) \delta(p_\gamma^+ - p_q^+ - p_{\bar{q}}^+) \sum_{\text{hel. col.}} \langle |\mathcal{M}_{\gamma^* \rightarrow q_1 q_2}^L|^2 \rangle_A, \quad (\text{A13})$$

$$\frac{d\sigma^T}{dP.S.} = 2p_\gamma^+ (2\pi) \delta(p_\gamma^+ - p_q^+ - p_{\bar{q}}^+) \frac{1}{2} \sum_{\text{hel. col.}} \langle |\mathcal{M}_{\gamma^* \rightarrow q_1 q_2}^T|^2 \rangle_A, \quad (\text{A14})$$

where $dP.S.$ denotes the two-particle Lorentz-invariant phase space, $\langle \dots \rangle_A$ is the CGC averaging, and, in the transverse case, we average over the initial photon polarization λ . We thus obtain

$$\frac{d\sigma^{L/T}}{dz d^2 p_{q\perp} d^2 p_{\bar{q}\perp}} = N_c \alpha_{em} e_f^2 p_\gamma^+ \int \frac{d^2 x_{1\perp}}{(2\pi)^2} \int \frac{d^2 x_{2\perp}}{(2\pi)^2} \int \frac{d^2 x_{1'\perp}}{(2\pi)^2} \int \frac{d^2 x_{2'\perp}}{(2\pi)^2} e^{-ip_{q\perp} \cdot x_{11'\perp}} e^{-ip_{\bar{q}\perp} \cdot x_{22'\perp}} \\ \times \psi_{\alpha\alpha'}^{L/T(\lambda)}(x_{12\perp}) \psi_{\beta\beta'}^{L/T(\lambda)*}(x_{1'2'\perp}) (1 + S^{(4)}(x_{1\perp}, x_{2\perp}; x_{1\perp}, x_{2\perp}) - S^{(2)}(x_{1\perp}, x_{2\perp}) - S^{(2)}(x_{2'\perp}, x_{1'\perp})), \quad (\text{A15})$$

where we defined the longitudinal and transverse splitting wave function

$$\psi_{\alpha\alpha'}^L(x_{12\perp}) \equiv (2\pi)^2 \sqrt{\frac{p_\gamma^+}{p_q^+ p_{\bar{q}}^+}} \Phi_{\alpha\alpha'}^L(x_{12\perp}), \quad (\text{A16})$$

$$\psi_{\alpha\alpha'}^{T(\lambda)}(x_{12\perp}) \equiv (2\pi)^2 \sqrt{\frac{p_\gamma^+}{2p_q^+ p_{\bar{q}}^+}} \Phi_{\alpha\alpha'}^{T(\lambda)}(x_{12\perp}), \quad (\text{A17})$$

which exactly match the normalization of those of [35].

APPENDIX B: CALCULATION OF THE SPIN-DENSITY MATRIX IN THE TRANSVERSE PHOTON CASE

Let us start by considering

$$\frac{\partial}{\partial P_\perp^i} \psi_{\alpha\alpha'}^{T(\lambda)}(P_\perp) = \sqrt{c_T} \epsilon_\lambda^k \frac{\partial}{\partial P_\perp^j} \frac{1}{P_\perp^2 + \bar{Q}^2} \bar{u}_\alpha(p_q) [(1-2z)P_\perp^k + i\epsilon^{kl} P_\perp^l \gamma^5 - m\gamma^k] \gamma^+ v_{\alpha'}(p_{\bar{q}}), \\ \frac{\partial}{\partial P_\perp^i} \psi_{\beta\beta'}^{T(\lambda)*}(P_\perp) = \sqrt{c_T} \epsilon_\lambda^k \frac{\partial}{\partial P_\perp^j} \frac{1}{P_\perp^2 + \bar{Q}^2} \bar{v}_{\beta'}(p_{\bar{q}}) [(1-2z)P_\perp^k - i\epsilon^{kl} P_\perp^l \gamma^5 + m\gamma^k] \gamma^+ u_\beta(p_q), \quad (\text{B1})$$

where, for convenience, we introduced

$$c_T = \frac{(2\pi)^4}{8p_q^+ p_{\bar{q}}^+ p_\gamma^+}, \quad \bar{Q} = \sqrt{z\bar{z}Q^2 + m^2}. \quad (\text{B2})$$

Taking the product and summing over the two possible polarizations, we find

$$\left[\frac{\partial}{\partial P_\perp^i} \psi_{\beta\beta'}^{T(\lambda)*}(P_\perp) \right] \left[\frac{\partial}{\partial P_\perp^j} \psi_{\alpha\alpha'}^{T(\lambda)}(P_\perp) \right] \\ = c_T \left[\frac{\partial}{\partial P_\perp^i} \frac{1}{P_\perp^2 + \bar{Q}^2} \bar{v}_{\beta'}(p_{\bar{q}}) [(1-2z)P_\perp^k - i\epsilon^{kl} P_\perp^l \gamma^5 + m\gamma^k] \gamma^+ u_\beta(p_q) \right] \\ \times \left[\frac{\partial}{\partial P_\perp^j} \frac{1}{P_\perp^2 + \bar{Q}^2} \bar{u}_\alpha(p_q) [(1-2z)P_\perp^k + i\epsilon^{kn} P_\perp^n \gamma^5 - m\gamma^k] \gamma^+ v_{\alpha'}(p_{\bar{q}}) \right] \\ = c_T \left\{ \frac{4P_\perp^i P_\perp^j}{[P_\perp^2 + \bar{Q}^2]^4} \bar{v}_{\beta'}(p_{\bar{q}}) [(1-2z)P_\perp^k - i\epsilon^{kl} P_\perp^l \gamma^5 + m\gamma^k] \gamma^+ u_\beta(p_q) \right. \\ \times \bar{u}_\alpha(p_q) [(1-2z)P_\perp^k + i\epsilon^{kn} P_\perp^n \gamma^5 - m\gamma^k] \gamma^+ v_{\alpha'}(p_{\bar{q}}) \\ - \frac{2P_\perp^i}{[P_\perp^2 + \bar{Q}^2]^3} \bar{v}_{\beta'}(p_{\bar{q}}) [(1-2z)P_\perp^k - i\epsilon^{kl} P_\perp^l \gamma^5 + m\gamma^k] \gamma^+ u_\beta(p_q) \bar{u}_\alpha(p_q) [(1-2z)\delta^{kj} + i\epsilon^{kj} \gamma^5] \gamma^+ v_{\alpha'}(p_{\bar{q}}) \\ - \frac{2P_\perp^j}{[P_\perp^2 + \bar{Q}^2]^3} \bar{v}_{\beta'}(p_{\bar{q}}) [(1-2z)\delta^{ki} - i\epsilon^{ki} \gamma^5] \gamma^+ u_\beta(p_q) \bar{u}_\alpha(p_q) [(1-2z)P_\perp^k + i\epsilon^{kn} P_\perp^n \gamma^5 - m\gamma^k] \gamma^+ v_{\alpha'}(p_{\bar{q}}) \\ \left. + \frac{1}{[P_\perp^2 + \bar{Q}^2]^2} \bar{v}_{\beta'}(p_{\bar{q}}) [(1-2z)\delta^{ki} - i\epsilon^{ki} \gamma^5] \gamma^+ u_\beta(p_q) \bar{u}_\alpha(p_q) [(1-2z)\delta^{kj} + i\epsilon^{kj} \gamma^5] \gamma^+ v_{\alpha'}(p_{\bar{q}}) \right\}.$$

We now write

$$\left[\frac{\partial}{\partial P_{\perp}^i} \psi_{\beta\beta'}^{T(\lambda)*}(P_{\perp}) \right] \left[\frac{\partial}{\partial P_{\perp}^j} \psi_{\alpha\alpha'}^{T(\lambda)}(P_{\perp}) \right] = c_T \left[F_{\alpha\alpha'\beta\beta'}^{ij} + S_{\alpha\alpha'\beta\beta'}^{ij} + T_{\alpha\alpha'\beta\beta'}^{ij} \right]. \quad (\text{B3})$$

The first term on the right-hand side of Eq. (B3) reads

$$\begin{aligned} F_{\alpha\alpha'\beta\beta'}^{ij} &= \frac{4P_{\perp}^i P_{\perp}^j}{[P_{\perp}^2 + \bar{Q}^2]^4} \bar{v}_{\beta'}(p_{\bar{q}}) [(1-2z)P_{\perp}^k - i\epsilon^{kl}P_{\perp}^l \gamma^5 + m\gamma^k] \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) [(1-2z)P_{\perp}^k + i\epsilon^{kn}P_{\perp}^n \gamma^5 - m\gamma^k] \gamma^+ v_{\alpha'}(p_{\bar{q}}) \\ &= \frac{4P_{\perp}^i P_{\perp}^j}{[P_{\perp}^2 + \bar{Q}^2]^4} \left\{ P_{\perp}^2 ((1-2z)^2 \bar{v}_{\beta'}(p_{\bar{q}}) \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^+ v_{\alpha'}(p_{\bar{q}}) + \bar{v}_{\beta'}(p_{\bar{q}}) \gamma^5 \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^5 \gamma^+ v_{\alpha'}(p_{\bar{q}})) \right. \\ &\quad - m^2 \bar{v}_{\beta'}(p_{\bar{q}}) \gamma^k \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^k \gamma^+ v_{\alpha'}(p_{\bar{q}}) \\ &\quad - (1-2z)m|P_{\perp}| (\bar{v}_{\beta'}(p_{\bar{q}}) \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^1 \gamma^+ v_{\alpha'}(p_{\bar{q}}) - \bar{v}_{\beta'}(p_{\bar{q}}) \gamma^1 \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^+ v_{\alpha'}(p_{\bar{q}})) \\ &\quad \left. - im|P_{\perp}| (\bar{v}_{\beta'}(p_{\bar{q}}) \gamma^5 \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^2 \gamma^+ v_{\alpha'}(p_{\bar{q}}) + \bar{v}_{\beta'}(p_{\bar{q}}) \gamma^2 \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^5 \gamma^+ v_{\alpha'}(p_{\bar{q}})) \right\} \\ &\equiv \delta_{ij} F_{\alpha\alpha'\beta\beta'} + \left(\frac{2P_{\perp}^i P_{\perp}^j}{P_{\perp}^2} - \delta_{ij} \right) F_{\alpha\alpha'\beta\beta'}, \end{aligned}$$

where the various coefficients of the spin-density matrix $F_{\alpha\alpha'\beta\beta'}$ can be explicitly evaluated by the procedure in [7] that we also adopted for the longitudinal case.

The second term in Eq. (B3) reads

$$\begin{aligned} S_{\alpha\alpha'\beta\beta'}^{ij} &= -\frac{2P_{\perp}^i}{[P_{\perp}^2 + \bar{Q}^2]^3} \bar{v}_{\beta'}(p_{\bar{q}}) [(1-2z)P_{\perp}^k - i\epsilon^{kl}P_{\perp}^l \gamma^5 + m\gamma^k] \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) [(1-2z)\delta^{kj} + i\epsilon^{kj}\gamma^5] \gamma^+ v_{\alpha'}(p_{\bar{q}}) \\ &\quad - \frac{2P_{\perp}^j}{[P_{\perp}^2 + \bar{Q}^2]^3} \bar{v}_{\beta'}(p_{\bar{q}}) [(1-2z)\delta^{ki} - i\epsilon^{ki}\gamma^5] \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) [(1-2z)P_{\perp}^k + i\epsilon^{kn}P_{\perp}^n \gamma^5 - m\gamma^k] \gamma^+ v_{\alpha'}(p_{\bar{q}}) \end{aligned}$$

We can decompose $S_{\alpha\alpha'\beta\beta'}^{ij}$ as

$$S_{\alpha\alpha'\beta\beta'}^{ij} = \delta^{ij} S_{1;\alpha\alpha'\beta\beta'} + \left(\frac{2P_{\perp}^i P_{\perp}^j}{P_{\perp}^2} - \delta^{ij} \right) S_{2;\alpha\alpha'\beta\beta'}, \quad (\text{B4})$$

where

$$S_2 = \left(\frac{P_{\perp}^k P_{\perp}^l}{P_{\perp}^2} - \frac{\delta^{kl}}{2} \right) S^{kl} \quad (\text{B5})$$

It is easy to note that $S_{1;\alpha\alpha'\beta\beta'} = S_{2;\alpha\alpha'\beta\beta'} \equiv S_{\alpha\alpha'\beta\beta'}$, so that

$$S_{\alpha\alpha'\beta\beta'}^{ij} = \delta^{ij} S_{\alpha\alpha'\beta\beta'} + \left(\frac{2P_{\perp}^i P_{\perp}^j}{P_{\perp}^2} - \delta^{ij} \right) S_{\alpha\alpha'\beta\beta'}, \quad (\text{B6})$$

The spin-density matrix $S_{\alpha\alpha'\beta\beta'}$ is

$$\begin{aligned} S_{\alpha\alpha'\beta\beta'} &= \frac{\delta^{ij}}{2} S_{\alpha\alpha'\beta\beta'}^{ij} \\ &= -\frac{1}{[P_{\perp}^2 + \bar{Q}^2]^3} \left\{ 2P_{\perp}^2 \left((1-2z)^2 \bar{v}_{\beta'}(p_{\bar{q}}) \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^+ v_{\alpha'}(p_{\bar{q}}) + \bar{v}_{\beta'}(p_{\bar{q}}) \gamma^5 \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^5 \gamma^+ v_{\alpha'}(p_{\bar{q}}) \right) \right. \\ &\quad - (1-2z)m|P_{\perp}| \left(\bar{v}_{\beta'}(p_{\bar{q}}) \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^1 \gamma^+ v_{\alpha'}(p_{\bar{q}}) - \bar{v}_{\beta'}(p_{\bar{q}}) \gamma^1 \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^+ v_{\alpha'}(p_{\bar{q}}) \right) \\ &\quad \left. - im|P_{\perp}| \left(\bar{v}_{\beta'}(p_{\bar{q}}) \gamma^5 \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^2 \gamma^+ v_{\alpha'}(p_{\bar{q}}) + \bar{v}_{\beta'}(p_{\bar{q}}) \gamma^2 \gamma^+ u_{\beta}(p_q) \bar{u}_{\alpha}(p_q) \gamma^5 \gamma^+ v_{\alpha'}(p_{\bar{q}}) \right) \right\}. \end{aligned}$$

Again, the various coefficients of the spin-density matrix $S_{\alpha\alpha'\beta\beta'}$ can be explicitly evaluated by the procedure in [7].

Lastly, the third term in Eq. (B3) reads

$$\begin{aligned} T_{\alpha\alpha'\beta\beta'}^{ij} &= \frac{1}{[P_\perp^2 + \bar{Q}^2]^2} \bar{v}_{\beta'}(p_{\bar{q}})[(1-2z)\delta^{ki} - i\epsilon^{ki}\gamma^5]\gamma^+ u_\beta(p_q) \bar{u}_\alpha(p_q)[(1-2z)\delta^{kj} + i\epsilon^{kj}\gamma^5]\gamma^+ v_{\alpha'}(p_{\bar{q}}) \\ &= \frac{\delta^{ij}}{[P_\perp^2 + \bar{Q}^2]^2} [(1-2z)^2 \bar{v}_{\beta'}(p_{\bar{q}})\gamma^+ u_\beta(p_q) \bar{u}_\alpha(p_q)\gamma^+ v_{\alpha'}(p_{\bar{q}}) + \bar{v}_{\beta'}(p_{\bar{q}})\gamma^5 \gamma^+ u_\beta(p_q) \bar{u}_\alpha(p_q)\gamma^5 \gamma^+ v_{\alpha'}(p_{\bar{q}})] \\ &\equiv \delta^{ij} T_{\alpha\alpha'\beta\beta'}. \end{aligned}$$

Thus, we obtain

$$\left[\frac{\partial}{\partial P_\perp^i} \psi_{\beta\beta'}^{T(\lambda)*}(P_\perp) \right] \left[\frac{\partial}{\partial P_\perp^j} \psi_{\beta\beta'}^{T(\lambda)*}(P_\perp) \right] = c_T \left\{ \delta^{ij} (F_{\alpha\alpha'\beta\beta'} + S_{\alpha\alpha'\beta\beta'} + T_{\alpha\alpha'\beta\beta'}) + \left(\frac{2P_\perp^i P_\perp^j}{P_\perp^2} - \delta^{ij} \right) (F_{\alpha\alpha'\beta\beta'} + S_{\alpha\alpha'\beta\beta'}) \right\}, \quad (\text{B7})$$

which, by comparison with

$$\left[\frac{\partial}{\partial P_\perp^i} \psi_{\beta\beta'}^{T(\lambda)*}(P_\perp) \right] \left[\frac{\partial}{\partial P_\perp^j} \psi_{\beta\beta'}^{T(\lambda)}(P_\perp) \right] = \delta^{ij} [A^{(0)} + \tilde{C}_{ab}^{(0)} \sigma^a \otimes \sigma^b] + \left(\frac{2P_\perp^i P_\perp^j}{P_\perp^2} - \delta^{ij} \right) [A^{(2)} + \tilde{C}_{ab}^{(2)} \sigma^a \otimes \sigma^b],$$

enables the extraction of all coefficients in Sec. II C.

APPENDIX C: ENTANGLEMENT, BELL NONLOCALITY, AND NONSTABILIZERNESS

For the convenience of the reader, here we provide the relations useful for quantifying entanglement, Bell nonlocality, and nonstabilizerlessness from the spin-density matrix ρ . The standard form of the spin density matrix of a two-qubit system reads

$$\rho = \frac{1}{4} (\mathbb{I} \otimes \mathbb{I} + B_a \sigma^a \otimes \mathbb{I} + \bar{B}_b \mathbb{I} \otimes \sigma^b + C_{ab} \sigma^a \otimes \sigma^b). \quad (\text{C1})$$

In all the cases we consider, $B = \bar{B} = 0$. This will be assumed in the following.

1. Peres-Horodecki criterion for the entanglement

Consider the following two quantities

$$\begin{aligned} \Delta_1 &= \sqrt{(C_{rr} - C_{kk})^2 + (C_{rk} + C_{kr})^2} - 1 + C_{nn}, \\ \Delta_2 &= \sqrt{(C_{rr} + C_{kk})^2 + (C_{rk} - C_{kr})^2} - 1 - C_{nn}. \end{aligned} \quad (\text{C2})$$

According to the Peres-Horodecki criterion [68,69], if one of Δ 's is nonnegative, the $q\bar{q}$ pair is entangled [4,70]. To quantify the entanglement, we can thus use the concurrence,

$$\mathcal{C}[\rho] = \max\{\Delta_1/2, \Delta_2/2, 0\}, \quad (\text{C3})$$

which ranges between 0 and 1. $\mathcal{C}[\rho] = 0$ corresponds to a separable state, while $\mathcal{C}[\rho] = 1$ indicates a maximally entangled state.

2. Bell-CHSH inequality

Another measure of quantum correlation is the violation of the Bell-CHSH inequality [71]

$$\text{Max}_{\{\vec{n}_i\}} \left| n_1^a C_{ab} (n_2^b + n_4^b) + n_3^a C_{ab} (n_2^b - n_4^b) \right| \leq 2, \quad (\text{C4})$$

for unit vectors $|\vec{n}_i| = 1$ ($i = 1, 2, 3, 4$). Given a correlation matrix C_{ab} , this inequality is violated for certain choices of $\vec{n}_i$ if the largest two of the three eigenvalues $\mu_3 \leq \mu_2 \leq \mu_1$ of the matrix $C^T C$ (the symbol $\mathcal{T}$ denotes “transpose”) satisfy [72]

$$1 < \mu_1 + \mu_2 \leq 2. \quad (\text{C5})$$

When this is the case, we say that the pair exhibit Bell nonlocality.

3. Nonstabilizerlessness

Finally, we turn to a different entanglement-related quantifier known as nonstabilizerlessness (often called “magic”). It has been established that entanglement by itself is not sufficient to ensure a quantum computational advantage [73]. Rather, quantum computers require entangled states that also possess a nonzero amount of

nonstabilizerness in order to achieve a performance beyond classical capabilities. This observation has motivated investigations within the high-energy physics community into whether collider processes can generate particle pairs exhibiting such properties [52,54,55]. To characterize nonstabilizerness in our setup, we employ the stabilizer Rényi

entropy [51,52], which for the system under consideration is given by

$$S_R^{\text{stab}} = -\ln \left(\frac{1 + C_{nn}^4 + C_{rr}^4 + C_{kk}^4 + C_{kr}^4 + C_{rk}^4}{1 + C_{nn}^2 + C_{rr}^2 + C_{kk}^2 + C_{kr}^2 + C_{rk}^2} \right). \quad (\text{C6})$$

-
- [1] G. Aad *et al.* (ATLAS Collaboration), *Nature (London)* **633**, 542 (2024).
 - [2] A. Hayrapetyan *et al.* (CMS Collaboration), *Rep. Prog. Phys.* **87**, 117801 (2024).
 - [3] A. J. Barr, M. Fabbri, R. Floreanini, E. Gabrielli, and L. Marzola, *Prog. Part. Nucl. Phys.* **139**, 104134 (2024).
 - [4] Y. Afik and J. R. M. de Nova, *Quantum* **6**, 820 (2022).
 - [5] Y. Afik *et al.*, *Eur. Phys. J. Plus* **140**, 855 (2025).
 - [6] W. Qi, Z. Guo, and B.-W. Xiao, *Phys. Rev. D* **113**, 054048 (2026).
 - [7] M. Fucilla and Y. Hatta, *Phys. Rev. D* **113**, L031504 (2026).
 - [8] K. Cheng, T. Han, and S. Trifinopoulos, *arXiv:2510.23773*.
 - [9] X.-Q. Xi, K.-B. Chen, X.-B. Tong, S.-Y. Wei, and J. Wu, *Eur. Phys. J. C* **85**, 1458 (2025).
 - [10] Y. Hatta and J. Schoenleber, *Phys. Rev. D* **113**, 094016 (2026).
 - [11] H.-W. Zhang, X. Cao, and T.-F. Feng, *Phys. Rev. D* **113**, 114022 (2026).
 - [12] F. Gelis, E. Iancu, J. Jalilian-Marian, and R. Venugopalan, *Annu. Rev. Nucl. Part. Sci.* **60**, 463 (2010).
 - [13] A. Kovner and M. Lublinsky, *Phys. Rev. D* **92**, 034016 (2015).
 - [14] R. Peschanski and S. Seki, *Phys. Lett. B* **758**, 89 (2016).
 - [15] D. E. Kharzeev and E. M. Levin, *Phys. Rev. D* **95**, 114008 (2017).
 - [16] Y. Liu and I. Zahed, *Phys. Rev. D* **100**, 046005 (2019).
 - [17] G. S. Ramos and M. V. T. Machado, *Phys. Rev. D* **102**, 034019 (2020).
 - [18] S. Bhattacharya, R. Boussarie, and Y. Hatta, *Phys. Lett. B* **859**, 139134 (2024).
 - [19] Y. Guo, X. Liu, F. Yuan, and H. X. Zhu, *Research* **8**, 0552 (2025), <https://inspirehep.net/literature/2796551>.
 - [20] J. D. Brandenburg, H. Duan, Z. Tu, R. Venugopalan, and Z. Xu, *Phys. Rev. Res.* **7**, 013131 (2025).
 - [21] Y. Hatta and J. Montgomery, *Phys. Rev. D* **111**, 014024 (2025).
 - [22] A. Dumitru and E. Kolbusz, *Phys. Rev. D* **111**, 114033 (2025).
 - [23] H. Bloss, B. Kriesten, and T. J. Hobbs, *Phys. Lett. B* **868**, 139814 (2025).
 - [24] S. Agrawal and R. Abir, *Phys. Lett. B* **868**, 139802 (2025).
 - [25] M. Ouchen and A. Prygarin, *Phys. Rev. D* **112**, 094027 (2025).
 - [26] M. Hentschinski, H. Jung, and K. Kutak, *Phys. Rev. D* **113**, 054024 (2026).
 - [27] K. Golec-Biernat, *arXiv:2512.23102*.
 - [28] W. Zhang, W. Qian, Y. Zhou, Y. Li, and Q. Wang, *J. High Energy Phys.* **07** (2026) 061.
 - [29] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *Sov. Phys. JETP* **44**, 443 (1976).
 - [30] E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, *Sov. Phys. JETP* **45**, 199 (1977).
 - [31] Y. Y. Balitsky and L. N. Lipatov, *Sov. J. Nucl. Phys.* **28**, 822 (1978).
 - [32] K. Cheng and B. Yan, *Phys. Rev. Lett.* **135**, 011902 (2025).
 - [33] P. J. Mulders and J. Rodrigues, *Phys. Rev. D* **63**, 094021 (2001).
 - [34] D. Boer, S. J. Brodsky, P. J. Mulders, and C. Pisano, *Phys. Rev. Lett.* **106**, 132001 (2011).
 - [35] F. Dominguez, C. Marquet, B.-W. Xiao, and F. Yuan, *Phys. Rev. D* **83**, 105005 (2011).
 - [36] P. Caucal, F. Salazar, and R. Venugopalan, *J. High Energy Phys.* **11** (2021) 222.
 - [37] P. Tael, T. Altinoluk, G. Beuf, and C. Marquet, *J. High Energy Phys.* **10** (2022) 184.
 - [38] P. Caucal, F. Salazar, B. Schenke, T. Stebel, and R. Venugopalan, *Phys. Rev. Lett.* **132**, 081902 (2024).
 - [39] F. Bergabo and J. Jalilian-Marian, *Phys. Rev. D* **107**, 054036 (2023).
 - [40] E. Iancu and Y. Mulian, *J. High Energy Phys.* **07** (2023) 121.
 - [41] P. Caucal and F. Salazar, *J. High Energy Phys.* **12** (2024) 130.
 - [42] T. Altinoluk, G. Beuf, A. Czajka, and A. Tymowska, *Phys. Rev. D* **107**, 074016 (2023).
 - [43] T. Altinoluk, G. Beuf, A. Czajka, and C. Marquet, *Phys. Rev. D* **111**, 014010 (2025).
 - [44] Y. V. Kovchegov and M. Li, *J. High Energy Phys.* **08** (2025) 206.
 - [45] S. Mukherjee, V. V. Skokov, A. Tarasov, S. Tiwari, and F. Yao, *arXiv:2602.15137*.
 - [46] G. A. Chirilli, *arXiv:2603.23428*.
 - [47] N. N. Nikolaev and B. G. Zakharov, *Z. Phys. C* **49**, 607 (1991).
 - [48] S. J. Brodsky, L. Frankfurt, J. F. Gunion, A. H. Mueller, and M. Strikman, *Phys. Rev. D* **50**, 3134 (1994), <https://inspirehep.net/literature/371342>.
 - [49] G. Beuf, *Phys. Rev. D* **94**, 054016 (2016).
 - [50] W. Bernreuther, D. Heisler, and Z.-G. Si, *J. High Energy Phys.* **12** (2015) 026, <https://inspirehep.net/literature/1388866>.
 - [51] L. Leone, S. F. E. Oliviero, and A. Hamma, *Phys. Rev. Lett.* **128**, 050402 (2022).

- [52] C. D. White and M. J. White, *Phys. Rev. D* **110**, 116016 (2024).
- [53] H. T. Klest, *Phys. Rev. D* **113**, 014045 (2026).
- [54] Q. Liu, I. Low, and Z. Yin, *Phys. Rev. D* **113**, 056001 (2026).
- [55] J. Gargalionis, N. Moynihan, M. L. Reichenberg Ashby, E. N. V. Wallace, C. D. White, and M. J. White, [arXiv:2603.04148](#).
- [56] Q. Liu, I. Low, and Z. Yin, *Quantum Sci. Technol.* **11**, 015035 (2026).
- [57] L. D. McLerran and R. Venugopalan, *Phys. Rev. D* **49**, 2233 (1994).
- [58] A. Metz and J. Zhou, *Phys. Rev. D* **84**, 051503 (2011).
- [59] F. Dominguez, J.-W. Qiu, B.-W. Xiao, and F. Yuan, *Phys. Rev. D* **85**, 045003 (2012).
- [60] A. Dumitru and V. Skokov, *Phys. Rev. D* **94**, 014030 (2016).
- [61] A. Tumasyan *et al.* (CMS Collaboration), *Phys. Rev. Lett.* **131**, 051901 (2023).
- [62] M. Galanti, A. Giammanco, Y. Grossman, Y. Kats, E. Stamou, and J. Zupan, *J. High Energy Phys.* **11** (2015) 067.
- [63] A. Hayrapetyan *et al.* (CMS Collaboration), *Phys. Rev. D* **110**, 112016 (2024).
- [64] Y. Hatta, B.-W. Xiao, F. Yuan, and J. Zhou, *Phys. Rev. Lett.* **126**, 142001 (2021).
- [65] Y. Hatta, B.-W. Xiao, F. Yuan, and J. Zhou, *Phys. Rev. D* **104**, 054037 (2021).
- [66] C. Marquet, Y. Shi, and B.-W. Xiao, [arXiv:2510.18949](#).
- [67] J. Gu, S.-J. Lin, D. Y. Shao, L.-T. Wang, and S.-X. Yang, [arXiv:2510.13951](#).
- [68] A. Peres, *Phys. Rev. Lett.* **77**, 1413 (1996).
- [69] P. Horodecki, *Phys. Lett. A* **232**, 333 (1997).
- [70] Y. Afik and J. R. M. de Nova, *Eur. Phys. J. Plus* **136**, 907 (2021).
- [71] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Phys. Rev. Lett.* **23**, 880 (1969).
- [72] R. Horodecki, P. Horodecki, and M. Horodecki, *Phys. Lett. A* **200**, 340 (1995).
- [73] D. Gottesman, in *22nd International Colloquium on Group Theoretical Methods in Physics* edited by S. P. Corney, R. Delbourgo, and P. D. Jarvis (International Press, Cambridge, MA, 1999), pp. 32–43.