

Nonlinear Regge trajectories of quarkonia from holography

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We propose a holographic model for quarkonia using the WKB approximation with the Langer correction to properly reproduce nonlinear Regge trajectories of the form $m_n^2 = \beta(n + c_0)^{2/3} + c_1$. This form is expected from previous studies involving the solution of Cornell potential for heavy quark-antiquark interactions using a model based on the quadratic form of the spinless Salpeter-type equation. The model fits experimental masses with very good accuracy. As a by-product, the corresponding decay constants also show a reasonable agreement with the results obtained from experimental data.

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I. INTRODUCTION

From semiclassical arguments, supported by the phenomenology of light mesons, it is expected that the squared masses of such particles should grow linearly with their radial excitation quantum number n [1]. For heavy mesons, on the other hand, the authors of [2], by employing the Bohr-Sommerfeld quantization approach to a model based on the quadratic form of the spinless Salpeter-type equation (QSSE) with the Cornell potential, show that the squared masses of quarkonia should grow with $n^{2/3}$ when n is large. The experimental masses of charmonium and bottomonium states are fitted in Ref. [2] with an equation of the form

$$m_n^2 = \beta(n + c_0)^{2/3} + c_1, \quad (1)$$

with a good agreement with experimental values, especially when compared to the linear fit (see Ref. [3] for a similar form for the mass spectra for quarkonium).

The AdS/CFT correspondence [4] motivated many different AdS/QCD holographic models with the purpose of describing hadronic properties. The general idea is to modify the gravitational background with the introduction of mass scales. Some of the first models that appeared are the hard-wall [5], that was used to calculate masses of glueballs [6], and the soft-wall model, that produces linear Regge trajectories for light mesons [7]. More recently, the tangent model was introduced in order to fit not only masses, but also the decay constants of heavy mesons.

This framework was used in order to describe the thermal dissociation of such particles in a quark-gluon plasma with temperature, density, magnetic field, and angular momentum [8–13].

In this work, we use the WKB approximation, considering the Langer correction, to build an AdS/QCD model that reproduces a Regge trajectory of the form (1). The model is shown to provide also decay constants with a reasonable agreement with the results obtained from experimental data.

The structure of the present article is as follows: In Sec. II we review the general form of holographic models with a dilaton background and the particular case of the soft wall. Then, in the following sections we show a sequence of steps aimed to reproduce a Regge trajectory of the form (1). As the first building block of the model, in Sec. III we explain how to add a term of mass shift that will generate the parameter c_1 . In Sec. IV, we make some remarks on the application of the WKB method in holography and develop a method to approximate the integral that appears in the WKB quantization condition for large values of masses. These remarks will be important through the following sections. Then we reproduce the exponent of $2/3$ using the WKB approach explained before. At the end of this section we introduce a new parameter that will be responsible for reproducing the parameter c_0 in (1). In Sec. V we discuss the calculation of the decay constants and add the last building block of the model. Finally, in Secs. VI and VII, we present the numerical results and conclusions, respectively.

II. HOLOGRAPHIC ADS/QCD MODELS WITH A DILATON BACKGROUND

A. A quick review

In the type of AdS/QCD models considered here, vector mesons are represented by an Abelian vector field V_m in

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five dimensions described by the action

$$S = -\frac{1}{4g_5^2} \int d^4x \int_0^\infty dz \sqrt{-g} e^{-\phi(z)} F^{mn} F_{mn}, \quad (2)$$

where F_{mn} is related to the field V_m through $F_{mn} = \partial_m V_n - \partial_n V_m$ and the indices are contracted with the metric of an AdS_5 space, given by

$$ds^2 = \frac{R^2}{z^2} (-dt^2 + d\mathbf{x}^2 + dz^2), \quad (3)$$

where R is the radius of the AdS and the function $\phi(z)$ is referred to as the dilaton. In the soft-wall model,

$$\phi(z) = \phi_{\text{sw}}(z) \equiv \kappa^2 z^2. \quad (4)$$

It is important to remark that the parameter κ represents a smooth infrared energy cutoff, that breaks conformal invariance and makes it possible to represent hadronic states.

Taking a Fourier component of the field V_m representing a particle at rest, we write $V_m = \varepsilon_m v(z) e^{-i\omega t}$, where ε_m is a unitary vector representing the polarization of the field and ω is the energy of the particle. Since the particle is at rest, we identify the energy of the particle with its mass m , so that $\omega = m$. The equations of motion produced by the action (2) in this case is

$$\omega^2 v(z) - \left(\frac{1}{z} + \phi'(z) \right) v'(z) + v''(z) = 0. \quad (5)$$

It is convenient to define the function ψ as

$$\psi(z) = \sqrt{\frac{R}{z}} e^{-\phi(z)/2} v(z), \quad (6)$$

so that the equation of motion (5) assumes the form of a Schrödinger-like equation with eigenvalues m_n^2 :

$$-\psi''(z) + V(z)\psi(z) = m_n^2 \psi(z), \quad (7)$$

with

$$V(z) = \frac{3}{4z^2} + \frac{1}{2z} \phi'(z) + \frac{1}{4} \phi'(z)^2 - \frac{1}{2} \phi''(z). \quad (8)$$

Equations (5) and (7) are valid for any dilaton ϕ . When a dilaton is given, the simple evaluation of the right-hand side of (8) gives the potential V . At this point there are two different approaches that one can follow. Either choose the form of the dilaton and calculate the corresponding function $V(z)$ or follow the opposite procedure.

For the particular case of the soft-wall model, ϕ is given by (4), so that

$$V_{\text{sw}}(z) = \frac{3}{4z^2} + \kappa^4 z^2. \quad (9)$$

The normalized¹ solutions of (7) in the soft-wall case are

$$\psi_n(z) = \sqrt{\frac{2\kappa}{n+1}} (\kappa z)^{3/2} e^{-\kappa^2 z^2/2} L_n^1(\kappa^2 z^2), \quad (10)$$

with the masses of the states with radial excitation number n given by

$$m_n^2 = 4\kappa^2(n+1), \quad (n=0, 1, 2, \dots). \quad (11)$$

B. Heavy mesons

The soft-wall model provides Regge trajectories of the form $m_n^2 \sim n$. However, since this model has only one parameter, κ , the squared mass of the ground state, $m_0^2 = 4\kappa^2$, is not independent of the slope of the Regge trajectory $m_{n+1}^2 - m_n^2 = 4\kappa^2$. For light mesons, it is not a problem, but heavy mesons like charmonium or bottomonium do not respect this property. The main reason is that the quark masses are not negligible in the case of heavy mesons. One can incorporate this issue by considering a relation of the form $m_n^2 = a + bn$ or, with the appropriate parameters:

$$m_n^2 = M_q^2 + 4\kappa^2(n+1), \quad (n=0, 1, 2, \dots). \quad (12)$$

In order to interpret the parameter M_q , note that the mass of a bound state of two quarks can be written as the sum of the masses of its constituents, $\sum_q m_q$, plus the binding energy E_n between them. Therefore,

$$m_n^2 = M_q^2 + 4\kappa^2(n+1) = \left(\sum_q m_q + E_n \right)^2. \quad (13)$$

Now, consider that we could turn the binding energy off, making $E_n = 0$. In this case, the masses of all states should collapse into one single value independent of n : the sum of the masses of the constituent quarks. For the expression $M_q^2 + 4\kappa^2(n+1)$ not to depend on n , κ must be zero. Therefore,

$$M_q|_{\kappa=0} = \sum_q m_q, \quad (14)$$

meaning that, when $\kappa = 0$, the parameter M_q is the sum of the mass of the constituent quarks. We cannot keep this

¹By norm we mean the conserved charge associated to the field V_m . Therefore, saying that the field is normalized is equivalent to saying that the value of the integral $\int (R/z) e^{-\phi(z)} |v(z)|^2 dz = \int |\psi(z)|^2 dz$ is 1.

assertion if κ is not zero, but we can assume that M_q and $\sum_q m_q$ should still be related also in this case.

As a first step toward a holographic model for heavy mesons, we analyze in the next section how to find a new dilaton that is capable of reproducing the spectrum given by Eq. (12). The tangent model of Refs. [8–15] attempts to shift the masses and does this with satisfactory success. In that model, a term of the form Mz is added to the dilaton. This term helps with shifting the predicted masses, making them closer to experimental values. This method, however, is not accurate. In particular, the masses of bottomonium and charmonium states present a considerable discrepancy with respect to the experimental data. The success of that model lies on the reasonable fit of the decay constants of charmonium and bottomonium states, which is necessary to describe the thermal properties of these particles. In the following section we show an improved way of shifting the masses based on Eq. (12), solving the so-called Schrödinger inverse problem. That means: we show how to find a potential that generates a given mass spectrum.

III. INCORPORATING THE HEAVY QUARK MASSES IN THE HOLOGRAPHIC APPROACH

The field equation for the soft-wall model has the form

$$-\psi''_n + V_{\text{SW}}\psi_n = 4\kappa^2(n+1)\psi_n. \quad (15)$$

We want to find a dilaton ϕ_I that leads to a spectrum of squared masses with the form $M_q^2 + 4\kappa^2(n+1)$, appropriate to heavy mesons, so that the corresponding field equation must have the form

$$-\psi''_n + V_I\psi_n = (M_q^2 + 4\kappa^2(n+1))\psi_n, \quad (16)$$

where V_I is the new Schrödinger-like potential generated by the this new dilaton ϕ_I .

Equations (15) and (16) have the same form. They are equal if one imposes

$$V_I(z) = V_{\text{SW}}(z) + M_q^2 = \frac{3}{4z^2} + M_q^2 + \kappa^4 z^2. \quad (17)$$

Using Eq. (8) one finds the dilaton ϕ_I that generates the potential above. The general solution is

$$\begin{aligned} \phi_I(z) = & \kappa^2 z^2 - 2 \ln[U(M_q^2/4\kappa^2, 0, \kappa^2 z^2) \\ & + C_1 L(-M_q^2/4\kappa^2, -1, \kappa^2 z^2)] + C_2, \end{aligned} \quad (18)$$

where C_1 and C_2 are constants, $U(a, b, x)$ is the Tricomi confluent hypergeometric function, and $L(n, \alpha, x) = L_n^{(\alpha)}(x)$ is the generalized Laguerre function.

This dilaton goes to a constant when z goes to zero. According to the appendix of [13], this constant can be incorporated into the AdS radius, making the value of $\phi(0)$

arbitrary. For convenience, we choose C_2 such that $\phi(0) = 0$. This can be done by using the relation² $U(M_q^2/4\kappa^2, 0, 0) = \Gamma(1 + M_q^2/4\kappa^2)^{-1}$. On the other hand, the Laguerre function satisfies³ $L(-M_q^2/4\kappa^2, -1, 0) = 0$. So, in order to have $\phi(0) = 0$, one chooses $C_2 = -2 \ln[\Gamma(1 + M_q^2/4\kappa^2)]$. This gives

$$\begin{aligned} \phi_I(z) = & \kappa^2 z^2 - 2 \ln[\Gamma(1 + M_q^2/4\kappa^2)(U(M_q^2/4\kappa^2, 0, \kappa^2 z^2) \\ & + C_1 L(-M_q^2/4\kappa^2, -1, \kappa^2 z^2))]. \end{aligned} \quad (19)$$

For recovering the soft-wall model when M_q is zero, we set $C_1 = 0$, which gives

$$\phi_I(z) = \kappa^2 z^2 - 2 \ln[\Gamma(1 + M_q^2/4\kappa^2)U(M_q^2/4\kappa^2, 0, \kappa^2 z^2)]. \quad (20)$$

Now that we found a way to incorporate the heavy quark masses in the holographic approach, in the following section we discuss how to incorporate the nonlinearity of the mass spectra.

IV. NONLINEAR REGGE TRAJECTORIES IN THE HOLOGRAPHIC APPROACH

Considering a model based on the quadratic form of the spinless Salpeter-type equation with a Cornell potential, it was found in Ref. [2] that the radial Regge trajectories for quarkonia are not linear. For large n they behave as $m_n^2 \sim n^{2/3}$. Masses of charmonium and bottomonium were then fitted to the expression:

$$m_n^2 = \beta(n + c_0)^{2/3} + c_1. \quad (21)$$

It was shown in [2] that the mass spectra obtained from this model finds good agreement with experimental results, even for light mesons.

We show in this section how to use the WKB approximation to build a holographic model that reproduces the same Regge trajectory (21). Other approaches to nonlinear Regge trajectories in holography, some of them also using the WKB approximation, can be found in [17–19]. In this paper, however, we will look not only to the leading order in large n ($m_n^2 \simeq \beta n^{2/3}$ in our case), but, instead, we will use the Langer correction to properly obtain the terms c_0 and c_1 from (21).

A. WKB approximation

In the first-order WKB approximation with two turning points, the quantization condition for the equation

$$-\psi''(z) + (V(z) - m^2)\psi(z) = 0 \quad (22)$$

²See Eq. 13.2.21 at [16].

³This is proven by Eqs. 18.9.13, 18.5.12, and 16.2.1 at [16].

is given by

$$\int_b^a \sqrt{-Q(z)} dz = (n + 1/2)\pi, \quad (23)$$

where $Q(z) = V(z) - m^2$, n is an integer, and the points a and b are, respectively, the largest and the smallest turning points, i.e., the values of z for which $Q(z) = 0$.

However, because of the presence of the term $3/4z^2$ in the Schrödinger-like potential (8), the WKB approximation is not directly applicable. The WKB procedure produces a solution for z close to zero and another solution for z close to the turning point b . The behaviors of these two parts should match in between.⁴ The derivation of Eq. (23) depends on this match, known as the connection condition. In the present case, however, these behaviors do not agree. Indeed, the $3/4z^2$ term is analogous to the angular momentum term $\ell(\ell + 1)/r^2$ in the quantum two-body problem, where the same issue occurs. In order to make the WKB approximation applicable to the quantum two body problem, one introduces the Langer transformation [20], which modifies the radial variable, mapping the origin $r = 0$ into $-\infty$. The wave function is also transformed in order to keep (22) with the form of a Schrödinger equation. After the changes, the regularity condition at the origin $r = 0$ becomes a normalization condition at $-\infty$. In the notation of our problem, the Langer transformation reads

$$z = e^x/\kappa \quad \text{and} \quad \psi(z) = e^{x/2} \tilde{\psi}(x), \quad (24)$$

where κ is a parameter from V with dimension of mass.

With this, Eq. (22) becomes

$$-\tilde{\psi}''(x) + \left(\frac{1}{4} + \frac{e^{2x}}{\kappa^2} V(e^x/\kappa) - \frac{e^{2x}}{\kappa^2} m^2 \right) \tilde{\psi}(x) = 0. \quad (25)$$

The quantization condition for the new problem is

$$\int_{\ln(\kappa b)}^{\ln(\kappa a)} \sqrt{-\tilde{Q}(x)} dx = (n + 1/2)\pi, \quad (26)$$

with

$$\tilde{Q}(x) = \frac{1}{4} + \frac{e^{2x}}{\kappa^2} (V(e^x/\kappa) - m^2), \quad (27)$$

and with $\ln(\kappa a)$ and $\ln(\kappa b)$ being the turning points of $\tilde{Q}(x)$. By reverting the change of coordinates defined in

⁴Formally, the large z behavior of the WKB solution at small z should coincide with the small argument behavior of the solution close to the turning point b .

(24), the quantization condition takes the same form of Eq. (23), but now with

$$Q(z) = U(z) - m^2, \quad \text{and} \quad U(z) = V(z) + \frac{1}{4z^2}. \quad (28)$$

Note that a and b are the turning points of $Q(z)$ in its new form (28) and that the “corrected” potential U differs from the potential V by $1/4z^2$. This difference is known as the Langer correction.

As the integral in the left-hand side of (23) may not be solvable with a closed form in general, it will be useful to expand it in powers of $1/m$, that is considered to be small in the WKB approach. For this reason, we will make use of the series expansion

$$\sqrt{1-x} = \sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} x^j, \quad (29)$$

where $(a)_j = \Gamma(a+j)/\Gamma(j)$ is the Pochhammer symbol, to write

$$\int_b^a \sqrt{-Q(z)} dz = \sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} m \int_b^a \frac{U^j}{m^{2j}} dz. \quad (30)$$

As an example that will be useful in the following sections, let α be a positive constant and consider a potential of the form

$$V(z) = \frac{3}{4z^2} + \kappa^{2+\alpha} z^\alpha \quad \Rightarrow \quad U(z) = \frac{1}{z^2} + \kappa^{2+\alpha} z^\alpha. \quad (31)$$

In this case,

$$\begin{aligned} \int_b^a \sqrt{-Q(z)} dz &= \sum_{j=0}^{\infty} \sum_{k=0}^j \frac{(-1/2)_j}{j!} \binom{j}{k} \frac{1}{m^{2j-1}} \int_b^a \left(\frac{1}{z^2} \right)^k (\kappa^{2+\alpha} z^\alpha)^{j-k} dz \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^j \frac{(-1/2)_j}{j!} \binom{j}{k} \frac{\kappa^{(2+\alpha)(j-k)}}{m^{2j-1}} (F_{j,k}(a) - F_{j,k}(b)), \end{aligned} \quad (32)$$

where

$$F_{j,k}(z) = \begin{cases} \frac{z^{\alpha j - (2+\alpha)k + 1}}{\alpha j - (2+\alpha)k + 1}, & \text{if } \alpha j - (2+\alpha)k + 1 \neq 0, \\ \ln \kappa z, & \text{if } \alpha j - (2+\alpha)k + 1 = 0 \end{cases} \quad (33)$$

is a primitive of $z^{\alpha j - (2+\alpha)k}$. The turning points a and b are solutions of $Q(z) = U(z) - m^2 = 0$, so that there are powers of $1/m$ hidden in them. Indeed, the largest and

smallest turning points can be expanded in powers of $1/m$, respectively, as

$$\begin{aligned} a &\simeq \frac{1}{\kappa} \left(\frac{m}{\kappa}\right)^{2/\alpha} \left[1 - \frac{1}{\alpha} \left(\frac{\kappa}{m}\right)^{2+4/\alpha}\right] \quad \text{and} \\ b &\simeq \frac{1}{m} \left[1 + \frac{1}{2} \left(\frac{\kappa}{m}\right)^{2+\alpha}\right]. \end{aligned} \quad (34)$$

Given this, collecting the powers of m in (32) is not easy in general, but we can find the leading orders by inspection, and keep the integral up the order m^0 . The leading contribution from $F_{j,k}(a)$ can be found by setting $k = 0$, which maximizes the exponent in $a^{\alpha j - (2+\alpha)k+1}$. Therefore, we write

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} \binom{j}{k} \frac{\kappa^{(2+\alpha)(j-k)}}{m^{2j-1}} F_{j,k}(a) \Big|_{k=0} \\ &\simeq \sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} \frac{1}{\alpha j + 1} \left(\frac{m}{\kappa}\right)^{1+2/\alpha} + \mathcal{O}(m^{-1-2/\alpha}). \end{aligned} \quad (35)$$

To calculate the series, we substitute $\frac{1}{\alpha j + 1} = \int_0^1 x^{\alpha j} dx$ and read Eq. (29) from the right to the left. This gives

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} \binom{j}{k} \frac{\kappa^{(2+\alpha)(j-k)}}{m^{2j-1}} F_{j,k}(a) \Big|_{k=0} \\ &\simeq \left(\frac{m}{\kappa}\right)^{1+2/\alpha} \int_0^1 \sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} (x^\alpha)^j dx \\ &= \left(\frac{m}{\kappa}\right)^{1+2/\alpha} \int_0^1 \sqrt{1-x^\alpha} dx = \frac{B(1/\alpha, 3/2)}{\alpha} \left(\frac{m}{\kappa}\right)^{1+2/\alpha}, \end{aligned} \quad (36)$$

since

$$\begin{aligned} \int_0^1 \sqrt{1-x^\alpha} dx &= \frac{1}{\alpha} \int_0^1 t^{1/\alpha-1} (1-t)^{3/2-1} dt \\ &= \frac{B(1/\alpha, 3/2)}{\alpha}, \end{aligned} \quad (37)$$

where B is the beta function.

The next contribution from a comes from $k = 1$, but $m^{1-2j} F_{j,1}(a)$ is already of an order smaller than m^0 .

The first contribution from b comes from setting $k = j$, which minimizes the exponent of $b^{\alpha j - (2+\alpha)k+1}$. These terms give

$$\begin{aligned} &-\sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} \binom{j}{k} \frac{\kappa^{(2+\alpha)(j-k)}}{m^{2j-1}} F_{j,k}(b) \Big|_{k=j} \\ &\simeq \sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} \frac{1}{2j-1} m^0 + \mathcal{O}(m^{-2-\alpha}). \end{aligned} \quad (38)$$

Now we separate the $j = 0$ term and substitute

$$\frac{1}{2j-1} = \int_0^1 x^{2j-2} dx \quad (j \neq 0) \quad (39)$$

to obtain

$$\begin{aligned} &-\sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} \binom{j}{k} \frac{\kappa^{(2+\alpha)(j-k)}}{m^{2j-1}} F_{j,k}(b) \Big|_{k=j} \\ &\simeq -1 + \int_0^1 \frac{1}{x^2} \left(\sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} (x^2)^j - 1 \right) dx \\ &= -1 + \int_0^1 \frac{1}{x^2} (\sqrt{1-x^2} - 1) dx = -\frac{\pi}{2}. \end{aligned} \quad (40)$$

The next order in b comes from setting $k = j - 1$, but it is clearly smaller than m^0 .

Using the results (32), (36), and (40), one approximates the quantization condition as

$$\begin{aligned} \int_b^a \sqrt{-Q(z)} dz &\simeq \frac{B(1/\alpha, 3/2)}{\alpha} \left(\frac{m}{\kappa}\right)^{1+2/\alpha} - \frac{\pi}{2} \\ &= \left(n + \frac{1}{2}\right) \pi. \end{aligned} \quad (41)$$

Therefore,

$$m_n = \left[\frac{\alpha \pi}{B(1/\alpha, 3/2)} (n+1) \right]^{\alpha/(2+\alpha)} \kappa \quad (n = 0, 1, 2, \dots). \quad (42)$$

As a consistency check for the procedure, note from Eq. (31) that when $\alpha = 2$, we recover the soft-wall model (9). And, indeed, Eq. (42) agrees with the exact soft-wall result found in Eq. (11) when $\alpha = 2$.

B. Regge trajectories with power 2/3

From Eq. (42), one notes that a Schrödinger-like potential of the form $V(z) = 3/4 z^2 + \kappa^{2+\alpha} z^\alpha$ produces a Regge trajectory that goes as $m \sim (n+1)^{\alpha/(2+\alpha)}$. So, a trajectory of the form $m^2 \sim (n+1)^{2/3}$ is obtained by choosing $\alpha = 1$. This gives

$$V_{\text{II}}(z) = \frac{3}{4z^2} + \kappa^3 z \quad (43)$$

and

$$m_n^2 \simeq \left(\frac{3\pi}{2}\right)^{2/3} (n+1)^{2/3} \kappa^2. \quad (44)$$

At this point we have the correct large n behavior for the trajectory, but not yet the shift of masses in the first state of

heavy mesons due to the heavy quark masses. In Sec. III we showed how to obtain a shift in the masses. We just have to add a term M_q^2 to the Schrödinger-like potential. So, we change the potential to

$$V_{\text{III}}(z) = \frac{3}{4z^2} + \kappa^3 z + M_q^2, \quad (45)$$

so that

$$m_n^2 \simeq \left(\frac{3\pi}{2}\right)^{2/3} (n+1)^{2/3} \kappa^2 + M_q^2. \quad (46)$$

C. Adjusting the parameter c_0

Recall that our objective is to obtain a Regge trajectory of the form

$$m_n^2 = \beta(n + c_0)^{2/3} + c_1. \quad (47)$$

At this point, we have $\beta = (3\pi/2)^{2/3} \kappa$, $c_1 = M_q^2$ and c_0 is the fixed and universal parameter $c_0 = 1$. However, the value $c_0 = 1$ does not agree with its value from the model in [2]. The final step in order to build a model that yields $m_n^2 = \beta(n + c_0)^{2/3} + c_1$ will be to add a new term to the Schrödinger-like potential, thus promoting c_0 to an adjustable parameter.

The term that does this in the potential must be of the form $1/\sqrt{z}$, so that the potential becomes

$$V_{\text{IV}}(z) = \frac{3}{4z^2} + \frac{A\kappa^2}{\sqrt{\kappa z}} + M_q^2 + \kappa^3 z, \quad (48)$$

where A is a dimensionless constant.

We will set M_q to zero and recover it later by summing it to the squared masses at the end, shifting the masses of the model with $M_q = 0$, as we saw in Sec. III.

Now we must update the WKB calculations of Sec. IV A. The integral that appears in the quantization condition now is

$$\begin{aligned} \int_b^a \sqrt{-Q(z)} dz &= \sum_{j=0}^{\infty} \sum' \frac{(-1/2)_j}{k_1! k_2! k_3!} \frac{1}{m^{2j-1}} \int_b^a \left(\frac{1}{z^2}\right)^{k_3} \left(\frac{A\kappa^{3/2}}{\sqrt{z}}\right)^{k_2} (\kappa^3 z)^{k_1} dz \\ &= \sum_{j=0}^{\infty} \sum' \frac{(-1/2)_j}{k_1! k_2! k_3!} \frac{A^{k_2} \kappa^{k_1+3k_2/2}}{m^{2j-1}} (F_{j;k_1,k_2,k_3}(a) - F_{j;k_1,k_2,k_3}(b)), \end{aligned} \quad (49)$$

where

$$F_{j;k_1,k_2,k_3}(z) = \begin{cases} \frac{z^{-2k_3-k_2/2+k_1+1}}{-2k_3-k_2/2+k_1+1}, & \text{if } -2k_3-k_2/2+k_1+1 \neq 0, \\ \ln \kappa z, & \text{if } -2k_3-k_2/2+k_1+1 = 0. \end{cases} \quad (50)$$

We used the multinomial theorem, so that the summation with the prime is performed over k_1 , k_2 , and k_3 ranging from 0 to j and with the constraint $k_1 + k_2 + k_3 = j$. The expressions for the turning points change to

$$a \simeq \frac{m^2}{\kappa^3} - \frac{A}{m} \quad \text{and} \quad b \simeq \frac{1}{m} + \frac{A\kappa^{3/2}}{2m^{7/2}}. \quad (51)$$

The first contribution from a to the integral (53) comes from setting $k_1 = j$ and $k_2 = k_3 = 0$. Its calculation is very similar to what was done in (36), but the order $1/m$ in the expression for a now leads to a contribution of order m^0 in $m^{1-2j} F_{j;j,0,0}(a)$. This smaller contribution, however, vanishes in the final summation. The result is

$$\begin{aligned} \sum_{j=0}^{\infty} \sum' \frac{(-1/2)_j}{k_1! k_2! k_3!} \frac{A^{k_2} \kappa^{k_1+3k_2/2}}{m^{2j-1}} F_{j;k_1,k_2,k_3}(a) \Big|_{k_1=j, k_2=k_3=0} \\ \simeq \sum_{j=0}^{\infty} \frac{(-1/2)_j}{j!} \left(\frac{1}{j+1} \frac{m^3}{\kappa^3} - A m^0 \right) = \frac{2m^3}{3\kappa^3}. \end{aligned} \quad (52)$$

A new contribution from a is now present, it comes from $k_1 = j-1$, $k_2 = 1$, and $k_3 = 0$:

$$\begin{aligned} \sum_{j=0}^{\infty} \sum' \frac{(-1/2)_j}{k_1! k_2! k_3!} \frac{A^{k_2} \kappa^{k_1+3k_2/2}}{m^{2j-1}} F_{j;k_1,k_2,k_3}(a) \Big|_{k_1=j-1, k_2=1, k_3=0} \\ \simeq \sum_{j=1}^{\infty} \frac{(-1/2)_j}{(j-1)!} \frac{2A}{2j-1} \frac{m^3}{\kappa^3} = -\frac{A\pi}{2}. \end{aligned} \quad (53)$$

The contribution from b is $-\pi/2$, exactly the same as the one found in Eq. (40). Then, we write the quantization condition as

$$\int_b^a \sqrt{-Q(z)} dz \simeq \frac{2m^3}{3\kappa^3} - \frac{A\pi}{2} - \frac{\pi}{2} = \left(n + \frac{1}{2}\right)\pi. \quad (54)$$

Therefore, recovering the term M_q^2 , the first order WKB approximation for the spectrum of squared masses is

$$m_n^2 = \left[\frac{3\pi}{2} (n+1 + A/2) \right]^{2/3} \kappa^2 + M_q^2 \quad (n = 0, 1, 2, \dots). \quad (55)$$

Hence, in order to reproduce the Regge trajectory of Eq. (47) one chooses:

$$\beta = (3\pi/2)^{2/3} \kappa^2, \quad c_0 = 1 + A/2, \quad \text{and} \quad c_1 = M_q^2. \quad (56)$$

D. Comparison with the QSSE model

In Ref. [2] one finds an expression for the parameter β from Eq. (47). However, there are no expressions for the parameters c_0 and c_1 . From the WKB approach with the Langer correction and with the method of collecting the leading orders in powers of $1/m$ explained here, one can write equations analogous to (32) and (49) for the QSSE model, and find expressions for the QSSE versions of the parameters β , c_0 , and c_1 as well. This procedure gives

$$m_n^2 = [12\pi m_q \sigma (n + 3/4)]^{2/3} + 4m_q^2 \quad (n = 0, 1, 2, \dots), \quad (57)$$

so that we identify

$$\beta = (12\pi m_q \sigma)^{2/3}, \quad c_0 = 3/4, \quad \text{and} \quad c_1 = 4m_q^2, \quad (58)$$

where m_q is the constituent quark mass and σ is the string tension found in the linear term of the Cornell potential.

Comparison of (56) with (58) allows us to interpret our holographic parameters κ and M_q^2 . This confirms that M_q can, indeed, be interpreted as the sum of the masses of the constituent quarks, since

$$c_1 = M_q^2 = 4m_q^2 \Rightarrow M_q = 2m_q. \quad (59)$$

Additionally, we see that κ is related to the string tension through

$$\beta = (12\pi m_q \sigma)^{2/3} = (3\pi/2)^{2/3} \kappa^2 \Rightarrow \kappa = (8m_q \sigma)^{1/3}. \quad (60)$$

Finally, for the large m behavior of our model to perfectly match the one given by the Cornell potential in the QSSE, we set $A = -1/2$, so that

$$c_0 = 1 + A/2 = 3/4. \quad (61)$$

V. DECAY CONSTANTS

In the type of holographic models considered here, decay constants of meson states are calculated from the formula [8]

$$f_n = \frac{1}{g_5 m_n} e^{-\phi(0)} \lim_{z \rightarrow 0} \frac{R}{z} v'(\omega, z), \quad (62)$$

where v is the field representing the meson and the constant,

$$g_5 = \sqrt{\frac{12\pi^2}{N_c} R e^{-\phi(0)}} = 2\pi \sqrt{R} e^{-\phi(0)/2}, \quad (63)$$

is fixed by imposing that the holographic propagator associated with the field v matches the perturbative QCD result [21].

The authors of [22] pointed out that by strengthening the well in the potential-like function V , one is capable of fitting charmonium's decay constants. In their model, they introduced some parameters to their potential in order to achieve this. Here, we found that adding a term of the form $-B\kappa^2/\sqrt{\kappa z}$ to the small z behavior of the potential, one can find good results of decay constants for both charmonium and bottomonium. To make this term to affect only the small z behavior of the potential, we multiplied it by $e^{-\kappa z}$, so that its influence at large z values decreases exponentially. The new and final form of the potential is

$$V(z) = \frac{3}{4z^2} - \frac{1}{2} \frac{\kappa^2}{\sqrt{\kappa z}} - \frac{B\kappa^2}{\sqrt{\kappa z}} e^{-\kappa z} + M_q^2 + \kappa^3 z. \quad (64)$$

The dilaton that produces this potential is the solution of (8), with V given by (64). This time, however, the solution to (8) is not analytic, but we can find its large and small z behaviors, which are

$$\begin{aligned} \phi(z) \simeq & -2 \ln [C_1 \sqrt{z} \text{Ai}(\kappa z + M_q^2/\kappa^2) \\ & + C_2 \sqrt{z} \text{Bi}(\kappa z + M_q^2/\kappa^2)] \end{aligned} \quad (65)$$

for z large, and

$$\phi(z) \simeq -2 \ln \left\{ C_3 \left[1 + \frac{4}{3} \left(B - \frac{1}{2} \right) (\kappa z)^{3/2} \right] \right\} \quad (66)$$

for small z . The symbols Ai and Bi represent the Airy functions.

We will choose the solution with $C_2 = 0$. This way we avoid problems, that also appear in the soft-wall model with the negative sign for the dilaton, such as the appearance of a null mass state. The large z behavior of the dilaton is, then

$$\phi(z) \simeq \frac{4}{3} (\kappa z)^{3/2} + \frac{2M_q^2}{\kappa^2} \sqrt{\kappa z} - \frac{1}{2} \ln(\kappa z), \quad (67)$$

where we expanded the Airy function for large values of its argument.

As in Sec. III, we can set $\phi(0) = 0$ for convenience. We do this by choosing $C_3 = 1$, so that

$$\phi(z) \simeq -\frac{8}{3} \left(B - \frac{1}{2} \right) (\kappa z)^{3/2} \quad (68)$$

for small z .

This analysis allows us to find the dilaton by numerically solving Eq. (8) using the fact that the dilaton should match the large and small z behaviors of Eqs. (67) and (68) as boundary conditions.

VI. NUMERICAL RESULTS

In this section we show the results found for the best fits and the values of the normalized root mean square error (NRMSE). In the literature, one finds this metric defined as

$$\text{NRMSE} = 100\% \sqrt{\frac{1}{N - N_p} \sum_{i=1}^N \left(\frac{y_i - \hat{y}_i}{\text{mean}(\hat{y}_i)} \right)^2}, \quad (69)$$

where the m 's are the masses, the f 's are the decay constants, N_m is the number of measured masses, N_f is the number of measured decay constants, $N = N_m + N_f$ is the total number of experimental points and, as before, $N_p = 3$ is the number of parameters.

Note that the presence of the number of parameters N_p in the denominator imposes a penalty to a model with too many parameters, as compared to the number of data points. For example, if a model with N_p parameters is used to fit a set of $N = N_p$ experimental points, the value of the NRMSE would be infinite, which would suggest that this hypothetical model should be rejected.

All the experimental values were taken from the Particle Data Group (PDG) [23] and we followed [24] to use $\psi(4660)$ as the $5S$ state of the vector charmonium, but we do not have an experimental value for its decay constant.

A. Best fit parameters

The parameters that provided the best fits of masses and decay constants for charmonium and bottomonium are presented in Table I. The bootstrap method was used in order to find the value of the parameters and their uncertainties. We created a sample of 2000 sets of masses and decay constants for each particle assuming that the distribution of each experimental value reproduce a Gaussian distribution with average being the expected value from PDG [23] and with standard deviation identified

TABLE I. Parameters that give the best fit of charmonium and bottomonium masses and decay constants.

Charmonium and bottomonium parameters for the best fits		
Parameter	Charmonium values (GeV)	Bottomonium values (GeV)
κ	1.161 ± 0.008	2.109 ± 0.061
M	3.12 ± 0.04	9.24 ± 0.39
B	7.42 ± 0.15	11.20 ± 0.47

where N is the number of experimental points, N_p is the number of parameters, the y_i 's are the predicted values of the quantity fitted, and the $\hat{y}_i$'s are its measured values.

Here, however, we are fitting two different quantities. For this reason, to calculate the errors for the combined sets of masses and decay constants of quarkonia, we define the combined normalized root mean square error (CNRMSE) as

$$\text{CNRMSE} = 100\% \sqrt{\frac{1}{N - N_p} \left[\sum_{n=0}^{N_m-1} \left(\frac{m_n - \hat{m}_n}{\text{mean}(\hat{m}_n)} \right)^2 + \sum_{n=0}^{N_f-1} \left(\frac{f_n - \hat{f}_n}{\text{mean}(\hat{f}_n)} \right)^2 \right]}, \quad (70)$$

with the PDG's uncertainties.⁵ We then fitted each of these 2000 sets of masses and decay constants. This gave us a distribution of values for the three parameter κ , M_q , and B . Finally, we identified the value of κ and its uncertainty with, respectively, the average and standard deviation of values of best-fitting κ 's. The same was done for M_q and B .

The value of the CNRMSE found for the best fitting parameters are 14.8% for charmonium and 11.9% for bottomonium. We also calculated the errors of masses and decay constants considered separately. We obtained 3% for charmonium's masses, 17.7% for charmonium's decay constants, 2.3% for bottomonium masses, and 11.8% for bottomonium decay constants.

B. Masses and decay constants

With each of the 2000 triads of parameters (κ, M, B) found in the fittings described in Sec. VI A, we calculated the values of masses and decay constants predicted by the model. Again, we calculated the average predicted values with respective uncertainties from the average and standard deviation of the generated distributions. The results are shown in Tables II and III, as well as in Figs. 1 and 2.

⁵For charmonium, each set is made of 9 experimental points: 5 masses and 4 decay constants, while for bottomonium, each set consists of 12 points: 6 masses and 6 decay constants.

TABLE II. Comparison of charmonium experimental masses and decay constants with the ones given by the best fit of the holographic model.

Charmonium masses and decay constants				
State	Experimental masses (GeV)	Predicted masses (GeV)	Experimental decay constants (MeV)	Predicted decay constants (MeV)
1S	3.09690(0 ± 6)	3.210 ± 0.053	415 ± 4	401 ± 11
2S	3.6860(97 ± 10)	3.744 ± 0.048	290 ± 9	260 ± 5
3S	4.04(0 ± 4)	4.037 ± 0.046	187 ± 21	221 ± 3
4S	4.41(5 ± 5)	4.262 ± 0.045	125 ± 25	202 ± 3
5S	4.6(23 ± 10)	4.454 ± 0.045	—	—

TABLE III. Comparison of bottomonium experimental masses and decay constants with the ones given by the best fit of the holographic model.

Bottomonium masses and decay constants				
State	Experimental masses (GeV)	Predicted masses (GeV)	Experimental decay constants (MeV)	Predicted decay constants (MeV)
1S	9.460(4 ± 1)	8.92 ± 0.14	702 ± 14	747 ± 18
2S	10.023(4 ± 5)	9.84 ± 0.13	497 ± 29	451 ± 8
3S	10.355(1 ± 5)	10.26 ± 0.13	430 ± 28	369 ± 6
4S	10.57(94 ± 12)	10.57 ± 0.12	371 ± 24	336 ± 5
5S	10.88(52 ⁺²⁶ ₋₁₆)	10.84 ± 0.12	370 ± 50	316 ± 4
6S	11.00(0 ± 4)	11.07 ± 0.12	240 ± 60	303 ± 4

Using Eqs. (59) and (60) and the values of the parameters κ and M_q , we find the charm and bottom quark masses and the string tensions for each meson in this case. From charmonium parameters, we have

$$m_c = \frac{M_c}{2} = (1.55 \pm 0.03) \text{ GeV} \quad \text{and} \quad \sigma_c = \frac{\kappa_c^3}{4M_c} = (0.125 \pm 0.004) \text{ GeV}^2, \quad (71)$$

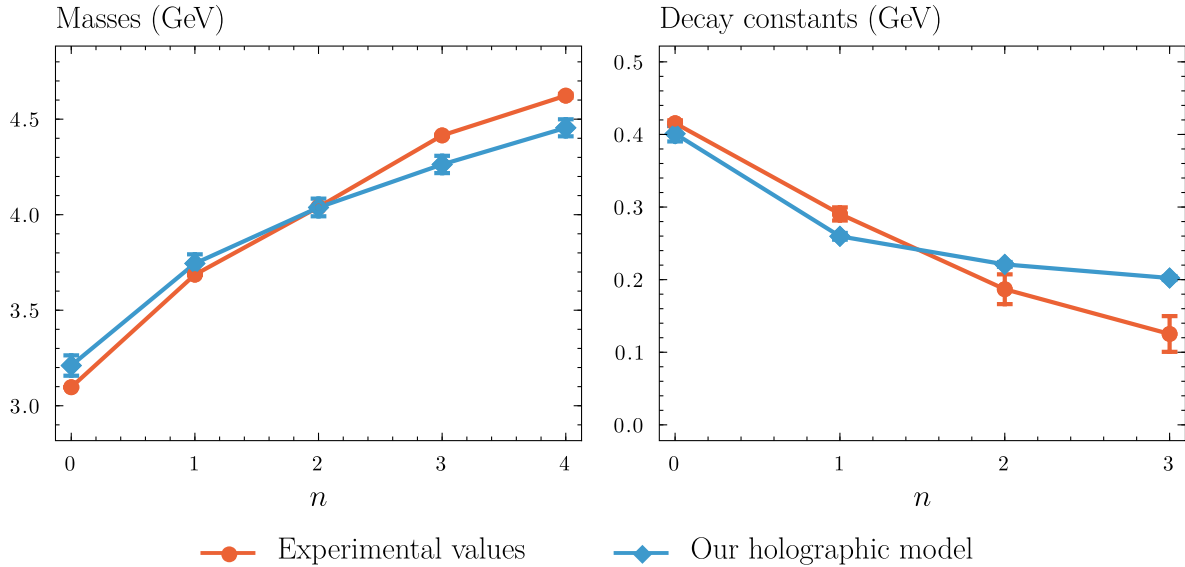


FIG. 1. Comparison of charmonium experimental masses and decay constants with the ones given by the best fit of the holographic model.

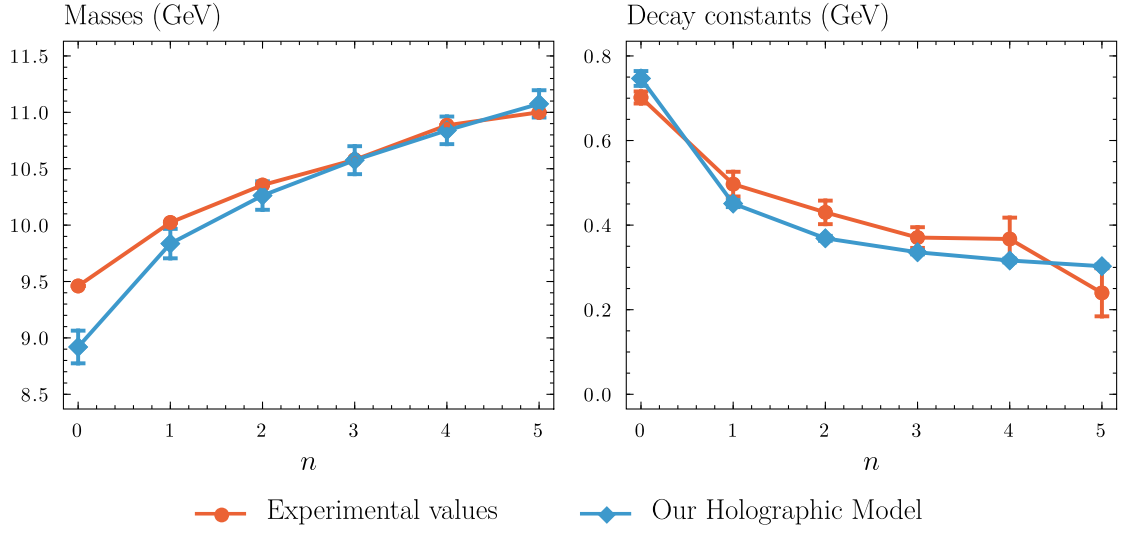


FIG. 2. Comparison of bottomonium experimental masses and decay constants with the ones given by the best fit of the holographic model.

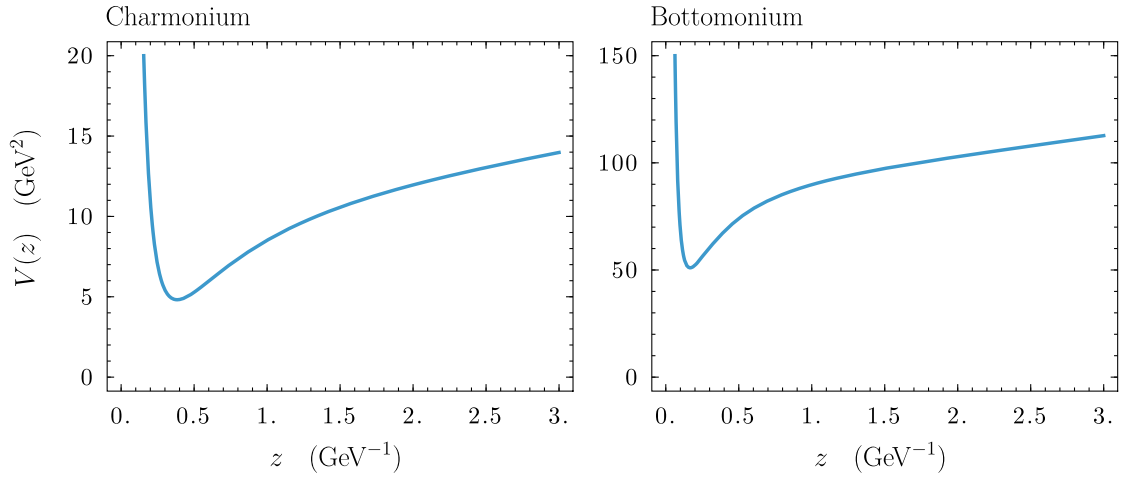


FIG. 3. Schrödinger-like potentials V of Eq. (64) for the model with the parameter values presented in Table I.

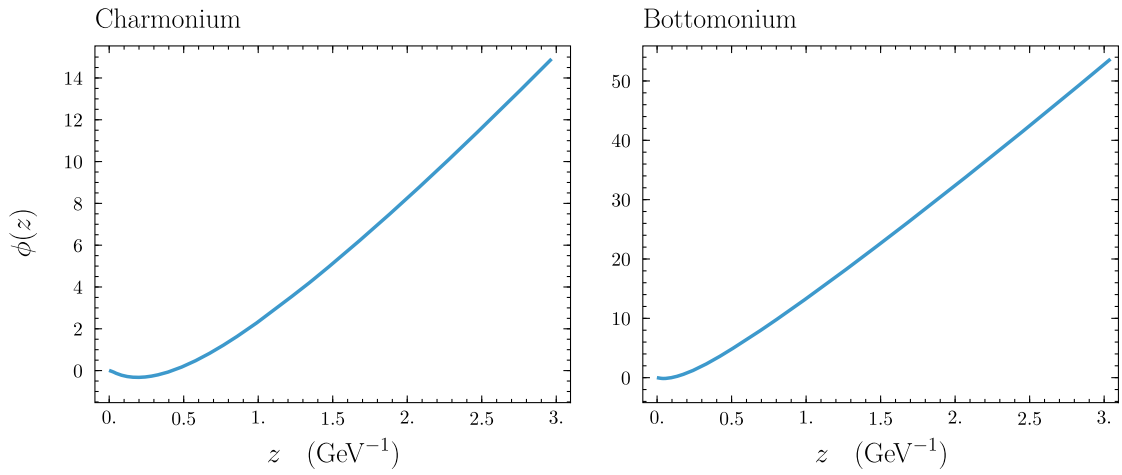


FIG. 4. Dilatons for the model with the parameter values presented in Table I.

while for bottomonium we obtain

$$m_b = \frac{M_b}{2} = (4.62 \pm 0.20) \text{ GeV} \quad \text{and} \quad \sigma_b = \frac{\kappa_b^3}{4M_b} = (0.254 \pm 0.025) \text{ GeV}^2, \quad (72)$$

which are of the same order of magnitude of the expected values. In [23] one finds that charm and bottom masses in the $\overline{\text{MS}}$ scheme are $m_c = 1.27 \text{ GeV}$ and $m_b = 4.18 \text{ GeV}$, while the string tension is found to be about $\sigma = 0.2 \text{ GeV}^2$ in models based on the Cornell potential for both charmonium and bottomonium.

C. Dilatons and potentials

In Figs. 3 and 4 we plot the potentials V of Eq. (64) for charmonium and bottomonium and corresponding dilatons with the parameters of Table I that provide the best fits.

VII. CONCLUSIONS

In this work, we developed a holographic model for heavy vector mesons that incorporates features that are present in the phenomenology of quarkonia states. Starting from the soft-wall model, we discussed in Sec. III how to introduce a mass shift to account for the constituent quark masses. Then, in Sec. IV we presented a procedure that makes it possible to find a Regge trajectory increasing with a factor $n^{2/3}$ for large n radial excitations. This is achieved by investigating, via the WKB approximation, what kind of term in the potential leads to such a behavior.

Then we investigated the changes that should be made on the potential in order to reproduce the phenomenological expression of Eq. (21). Finally, in order to fit the decay constants, that depend on the behavior of the potential at small z , we included a term proportional to $e^{-\kappa z}/\sqrt{\kappa z}$.

The results obtained in this work represent significant improvements over previous models. In particular, there are two differences with respect to previous holographic works. One is the fact that we used the Langer correction (24) in order to find the proper form of the Regge trajectories; the other is the fact that we found a way to fit the decay constants without spoiling the mass fits.

For the masses and decay constants of charmonium states we obtained a combined error CNRMSE of 14.8%. For the bottomonium states the CNRMSE obtained is 11.9%. The parameters M_q and κ have interesting physical interpretations as the sum of constituent quark masses and a quantity related to the string tension, respectively, with values comparable to those from potential models.

Future extensions could explore the thermal dissociation of quarkonia in a quark-gluon plasma, incorporating finite temperature, density, magnetic fields, or angular momentum.

Overall, the model presented here provides a bottom-up AdS/QCD framework for studying heavy mesons, representing a bridge between the QSSE [2] approach and gauge/gravity duality.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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