

Associated production of charmonia and bottomonia with color-octet channels at the Z factory, CEPC, and FCC-ee

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Within the nonrelativistic QCD framework, we investigate the associated production of charmonia and bottomonia at the future super Z-factory and at CEPC and FCC-ee. The color-octet (CO) channels in the γ^*/Z^0 -propagated processes are considered in addition to the color-singlet (CS) channels. We find that the contributions from the CO channels are substantial or even dominant for nearly all processes from the production threshold to the Z^0 pole region. However, estimates of the production rate are highly sensitive to long-distance matrix elements (LDMEs) selections, so different combinations of LDMEs produce markedly different production yields. For this reason, the study of charmonia-bottomonia production at the future super Z-factory can serve as a viable probe of the CO mechanism, and the comparison between the theoretical results and future experimental data may provide constraints on the CO matrix elements. In addition, we calculate the relativistic corrections to both the CS and CO channels, which decrease the cross sections significantly, yielding a K factor of approximately 0.5. The predicted event yields of $(J/\psi + \Upsilon, \Upsilon + \eta_c)$ production, with $J/\psi, \Upsilon$ reconstructed via lepton pairs and η_c reconstructed via hadronic decays, are (2, 4) and (18, 35) at the CEPC (2-year) and FCC-ee (4-year), respectively.

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I. INTRODUCTION

Since the discovery of the J/ψ meson in 1974 [1,2], heavy quarkonium—characterized as a nonrelativistic bound state of a heavy quark-antiquark pair—has served as an ideal laboratory for exploring QCD and provided crucial insights into the strong interaction. A foundational theoretical advance has been the development of the nonrelativistic QCD (NRQCD) effective field theory [3], within which the production cross section can be decomposed into calculable short-distance coefficients (SDCs) and universal long-distance matrix elements (LDMEs). Unlike the early color-singlet mechanism (CSM) [4–6], in which the heavy quark pair created at short distance has the same color and angular momentum as the final quarkonium, one pillar of NRQCD is the color-octet mechanism (COM), enabling the quark pair to exist in different Fock state from that of the quarkonium. This mechanism resolved the IR divergences occurring in P -wave heavy quarkonium decays [7,8] and gave the proper

prediction for charmonium production at Tevatron [9–15], marking a major triumph for NRQCD. However, several challenges to the NRQCD approach remain, such as the polarisation puzzles, where NRQCD predictions are inconsistent with experimental measurements [16–22], and the problem of the universality of LDMEs, where several theoretical groups have obtained different values [23–27]. For further details on the current status of heavy quarkonium physics, we refer the readers to Refs. [28–32].

Compared with the hadron colliders, the production mechanism in e^+e^- colliders is simpler, and the uncertainties in the theoretical calculations are smaller. Consequently, extensive work has been based on e^+e^- annihilation. One prominent area is the paired production of heavy charmonium; e.g., the large discrepancy between leading-order (LO) calculation [33,34] and measurement [35,36] for $J/\psi + \eta_c$ production has been accounted for through the inclusion of QCD correction [37–41] and relativistic correction [42,43]. However, due to the low interaction energy of the existing experiments, processes involving bottomonium, such as the associated production of charmonium and bottomonium, have not been accessible. The future Super Z-factory is a large experimental facility based on e^+e^- colliders, such as CEPC (three operation modes $\sqrt{s} \sim 91.2, 160$, and 240 GeV, corresponding to Z-factory, the W^+W^- threshold, and Higgs factory, respectively) [44] and FCC-ee ($\sqrt{s}: 90\text{--}365$ GeV) [45], and operates at the Z^0 resonance

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peak with a design luminosity up to $\mathcal{L} \simeq 10^{34-36} \text{ cm}^{-2} \text{ s}^{-1}$ [46]. Like the Super Z-factory, the GigaZ is a project based on the International Linear Collider ($\sqrt{s} \sim 250 \text{ GeV}$ with a luminosity $\mathcal{L} \simeq 0.7 \times 10^{34} \text{ cm}^{-2} \text{ s}^{-1}$) [47,48]. These facilities are expected to achieve this goal and enable the study of a wide range of processes, such as the production of B_c meson [49–53], doubly heavy baryon [54–59], doubly heavy tetraquark [60–62], fully heavy tetraquark state [63], and rare Z^0 decay [64].

The exclusive production of double heavy quarkonium in e^+e^- annihilation at the Z-factory has been studied at LO [65–69]. The next-to-leading order (NLO) QCD corrections to double heavy quarkonium [70,71] and double B_c meson [50] production have also been calculated, showing that the QCD corrections are essential. In contrast, the one-loop QCD corrections are negligible for the associated S -wave charmonium-bottomonium production (generally, $10^{-6} < \sigma_{\text{QCD}}/\sigma_{\text{EW}} < 0.1$) [72]. The contributions from the color-octet (CO) component are substantial in semi-exclusive processes [73]. In our recent work, we demonstrated that the COM plays a significant or even dominant role in the production of double heavy quarkonia around the Z-factory energy region [74], where the gluon fragmentation process is of key importance. By contrast, the associated production of charmonium and bottomonium, which also serves as an effective channel to probing the production mechanisms of heavy quarkonia [72], proceeds solely via the fragmentation mode at the tree-level diagram. It is therefore of great interest to explore the contributions from the COM to these processes, as a continuation of our previous studies. Consistent with our previous investigations, in this paper, we study the charmonium-bottomonium associated production involving double S -wave states as well as S -wave plus P -wave states, with the specific processes including $J/\psi + \Upsilon$, $J/\psi + \eta_b$, $J/\psi + \chi_{bJ}$, $J/\psi + h_b$, $\eta_c + h_b$, $\eta_c + \chi_{bJ}$, $\eta_c + \eta_b$, $\eta_c + \Upsilon$, $h_c + \Upsilon$, $\chi_{cJ} + \Upsilon$, $h_c + \eta_b$, $\chi_{cJ} + \eta_b$.

The rest of this paper is organized as follows. In Sec. II, we present the theoretical framework and computational method. Section III details the input parameters and the LDMEs used in this work. In Sec. IV, we analyze all possible production channels, present the results for (differential) cross sections, and discuss event generation as well as associated uncertainties. Finally, Sec. V concludes the paper.

II. FORMULA AND METHOD

According to the NRQCD framework, the production cross sections are factorized into a sum of products of SDCs and LDMEs [3],

$$\begin{aligned} \hat{\sigma}(e^+ + e^- \rightarrow H_1 + H_2) \\ = \sum_{mn} \frac{F_{mn}}{m_{Q1}^{d_m-4} m_{Q2}^{d_n-4}} \langle 0 | \mathcal{O}_m^{H_1} | 0 \rangle \langle 0 | \mathcal{O}_n^{H_2} | 0 \rangle. \end{aligned} \quad (1)$$

The SDC F_{mn} describes the production probability of intermediate quarks before hadronization. The LDME $\langle 0 | \mathcal{O}_{m,n}^H | 0 \rangle$ denotes the transition probability of a $Q\bar{Q}$ pair to form the final bound state H . The factor $m_Q^{d_{m,n}-4}$ is introduced to make F_{mn} dimensionless. The heavy quarkonium states have a Fock state expansion formalism up to the order of $\mathcal{O}(v^2)$:

$$\begin{aligned} |H\rangle = & \mathcal{O}(1) |Q\bar{Q}^{(2S+1)L_J^{[1]}}\rangle + \mathcal{O}(v) |Q\bar{Q}^{(2S+1)(L \pm 1)_J^{[8]}}g\rangle \\ & + \mathcal{O}(v^2) |Q\bar{Q}^{(2(S \pm 1)+1)L_J^{[8]}}g\rangle \\ & + \mathcal{O}(v^2) |Q\bar{Q}^{(2S+1)L_J^{[1,8]}}gg\rangle + \dots \end{aligned} \quad (2)$$

In Table I, the specific color-singlet (CS)/CO intermediate states for the heavy quarkonium are listed.

We also calculate the contributions from relativistic corrections in this work, for which the matrix elements of $\langle v^2 \rangle$, defined by the ratios of the NLO LDME in v^2 to the LO LDME, are introduced,

$$\langle v^2 \rangle \equiv \frac{\langle 0 | \mathcal{P}^H(2s+1)L_J | 0 \rangle}{m_Q^2 \langle 0 | \mathcal{O}^H(2s+1)L_J | 0 \rangle} \quad (3)$$

which adhere to the velocity power scaling rules and are of the order of v^2 as follows:

$$v^2 = \langle v^2 \rangle [1 + \mathcal{O}(v^4)] \quad (4)$$

In contrast to double charmonium or double bottomonium production [74], tree-level QCD diagrams cannot generate an intermediate CS $c\bar{c}$ state accompanied by a CS $b\bar{b}$ state in exclusive production as elaborated in Ref. [72]. Namely, the lowest-order QCD contributions within CSM to this process are loop processes. LO electric-weak (EW) processes in CS states are $\mathcal{O}(\alpha^4)$. LO QCD processes in CO states are $\mathcal{O}(\alpha_s^2 \alpha^2)$. Both types of processes involve mediated fragmentation mechanisms. The corresponding Feynman diagrams are shown in Fig. 1, where the initial and final momenta are defined. For the EW processes, the Z^0 -fragmentation channel is neglected, as the Z^0 -propagator is suppressed when the propagator momentum runs at meson mass scale. Therefore, one of the final mesons has to be either J/ψ or Υ , as the fragmented photon necessarily yields a $^3S_1^{[1]}$ state under this circumstance. Contributions from W-boson-involved processes (eight single-W-boson exchange s -channel diagrams, two double-W-boson exchange s -channel diagrams, and one double-W-boson exchange t -channel diagram) are also neglected, owing to the small magnitude of $|V_{bc}| (\simeq 0.046)$ and W-propagator suppression. The t -channel CS EW diagrams (two-boson exchange process) are significant for J/ψ plus Υ production via two-photon exchange. For other processes, t -channel EW diagrams are forbidden or give small contributions.

TABLE I. The color-singlet and -octet Fock states of the charmonium and bottomonium considered in this work.

Particle	Color-singlet state	Color-octet states
$J/\psi/\Upsilon$	$^3S_1^{[1]}$	$^3P_J^{[8]}, ^1S_0^{[8]}, ^3S_1^{[8]}$
η_c/η_b	$^1S_0^{[1]}$	$^1P_1^{[8]}, ^3S_1^{[8]}, ^1S_0^{[8]}$
χ_{cJ}/χ_{bJ}	$^3P_J^{[1]}$	$^3S_1^{[8]}$
h_c/h_b	$^1P_1^{[1]}$	$^1S_0^{[8]}$

The overall colorless nature of the final state requires that both recoiled $Q\bar{Q}$ pairs must be either in the CS state ($c\bar{c}_1 + b\bar{b}_1$) or in the CO state ($c\bar{c}_8 + b\bar{b}_8$). However, it should be noted that, unlike the CS channels strictly defined as two-meson exclusive processes, the LO CO channels correspond to inclusive processes involving unresolved soft radiation associated with the hadronization of the CO pair. In this case, the two final states with energy close to half of the c.m. energy recoil against each other without emitting additional light particles and thus contribute to the exclusive double quarkonia production in the experimental measurements. This specific configuration leads to the breakdown of factorization due to nonperturbative effects, which are fortunately suppressed by the inverse powers of collision energy and generally negligible. Given the coupling constants and the orders of magnitude of LDMEs, we can evaluate the enhancement and suppression factors of CO channels relative to CS channels. For the $J/\psi + \eta_b$ or $\Upsilon + \eta_c$ processes, the relevant factor reads $v_{c\bar{c}}^4 v_{b\bar{b}}^4 \alpha_s^2 / \alpha^2 \approx 0.2$. For processes where J/ψ or Υ is produced in association with a P -wave state, the corresponding maximum enhancement factors are approximately 20 and 2, determined by $v_{Q\bar{Q}}^4 \alpha_s^2 / \alpha^2$, respectively. More comprehensive inclusive CO processes feature final states consisting of CS and CO states, with the additional hard gluon serving as the $\text{NLO}(\alpha_s)$ real correction. Meanwhile, the IR divergences arising in some channels at this order are canceled by virtual corrections and NLO LDME corrections. Relevant studies have already investigated NLO CO contributions to s -channel J/ψ pair production at B-factories [75]. Furthermore, for channels involving two P -wave Fock states, uncanceled IR singularities arise from the interference of Feynman diagrams, thus causing the breakdown of factorization [76]. Nevertheless, the factorization-breaking effects are power suppressed [77,78]. For instance, factorization violations are generally suppressed by a factor $v_{c\bar{c}}^2 m_c / \sqrt{s}$ for S -wave charmonium and $m_c / \sqrt{s}$ for L -wave ($L > 0$) charmonium [79], so they can be safely neglected. The maximum enhancement factor $\alpha_s^3 / \alpha^2 \approx 20$ corresponds to the

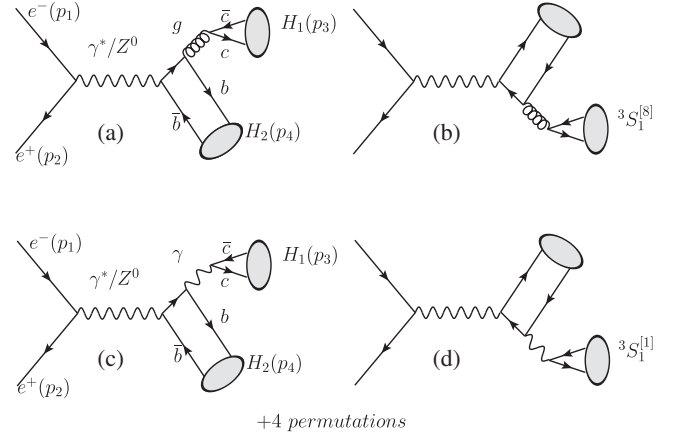


FIG. 1. Feynman diagrams for $e^-(p_1) + e^+(p_2) \rightarrow \gamma^*/Z^0 \rightarrow H_1(p_3) + H_2(p_4)$ at the tree level. H_1 and H_2 are charmonia and bottomonia, respectively. The permutation diagrams can be obtained by reversing the quark line. (a) and (b) diagrams belong to CO (QCD) channels and (c) and (d) diagrams belong to CS (EW) channels.

$Q_1\bar{Q}_1(^3S_1^{[1]}) + Q_2\bar{Q}_2(^3S_1^{[8]}) + g$ channels of $J/\psi + \chi_{bJ}$ and $\Upsilon + \chi_{cJ}$ processes, while other channels are further suppressed by higher powers of v^2 . Meanwhile, contributions from three-body final states suffer strong phase-space suppression at high energies. Since $\text{NLO}(\alpha_s)$ CO corrections lie beyond the scope of the present work, we only adopt LO CO channels for subsequent calculations and analysis.

From Fig. 1, the amplitudes can be written as

$$i\mathcal{M} = \sum_n \bar{v}(p_2) \mathcal{L}_\mu u(p_1) D^{\mu\nu} \mathcal{A}_\nu^{(n)}, \quad (5)$$

and the vertex $\mathcal{L}^\mu$ and the propagator $D_{\mu\nu}$ are

$$\begin{aligned} \mathcal{L}^\mu &= -ie\gamma^\mu, & \frac{-ie}{4\cos\theta_W \sin\theta_W} \gamma^\mu (1 - 4\sin^2\theta_W - \gamma^5) \\ \mathcal{D}_{\mu\nu} &= \frac{-ig_{\mu\nu}}{p^2}, & \frac{-ig_{\mu\nu}}{p^2 - m_Z^2 + im_Z\Gamma_Z}, \end{aligned} \quad (6)$$

where the first (second) term corresponds to the process propagated by γ^* (Z^0). $e = \sqrt{4\pi\alpha}$ is the electromagnetic coupling constant, and θ_W is the Weinberg angle. $p = p_1 + p_2$ is the 4-momentum of the propagator. m_Z and Γ_Z are the mass and the decay width of Z^0 boson.

The hadronic part of the amplitude $\mathcal{A}_\nu^{(n)} (n = a, b, c, \dots)$ is expressed as the product of traces in spinor space, with diagrams (a) and (d) provided as illustrative examples

$$\begin{aligned} \mathcal{A}_\nu^{(a)} &= -g^2 \text{Tr}[\gamma^\rho \mathbb{P}_1] \text{Tr} \left[\mathbb{P}_2 \gamma^\rho \frac{(\not{p}_{31} + \not{p}_{32} + \not{p}_{41}) + m_b}{[(-p_{31} - p_{32} - p_{41})^2 - m_b^2](-p_{31} - p_{32})^2} \mathcal{V}_b^\nu \right] \\ \mathcal{A}_\nu^{(d)} &= -\text{Tr}[\gamma^\rho \mathbb{P}_2] \text{Tr} \left[\mathbb{P}_1 \mathcal{V}_c^\nu \frac{(-\not{p}_{32} - \not{p}_{41} - \not{p}_{42}) + m_c}{[(p_{32} + p_{41} + p_{42})^2 - m_c^2](p_{41} + p_{42})^2} \gamma^\rho \right], \end{aligned} \quad (7)$$

where $p_{31} = \frac{1}{2}p_3 + q_3$, $p_{32} = \frac{1}{2}p_3 - q_3$ denote the constituent quark momenta of quarkonium H_1 (with q_3 its internal relative momentum) and $p_{41} = \frac{1}{2}p_4 + q_4$, $p_{42} = \frac{1}{2}p_4 - q_4$ denote those of H_2 (with q_4 its internal relative momentum). The vertices $\mathcal{V}^\mu$ for γ^*/Z^0 propagation are defined as

$$\begin{aligned} \mathcal{V}_c^\mu &= -ie e_c \gamma^\mu, & \frac{-ie}{4 \cos \theta_W \sin \theta_W} \gamma^\mu (-1 + 4|e_c| \sin^2 \theta_W + \gamma^5) \\ \mathcal{V}_b^\mu &= -ie e_b \gamma^\mu, & \frac{-ie}{4 \cos \theta_W \sin \theta_W} \gamma^\mu (1 - 4|e_b| \sin^2 \theta_W - \gamma^5) \end{aligned} \quad (8)$$

with $e_c = 2/3$, $e_b = -1/3$. The projector operators $\mathbb{P}_{1,2}$ in the amplitudes have the following forms for spin-singlet and spin-triplet states,

$$\mathbb{P}(00) \equiv \sum_{s_Q s_{\bar{Q}}} \sum_{kl} \langle s_Q s_{\bar{Q}}; 00 \rangle \langle 3, k; \bar{3}, l | 1, 8 \rangle v_l(p_{\bar{Q}}) \bar{u}_k(p_Q) = (-\not{p}_{\bar{Q}} + m_Q) \gamma_5 \frac{\not{p}_Q + \not{p}_{\bar{Q}} + 2E_q}{4\sqrt{2}E_q(E_q + m_Q)} (\not{p}_Q + m_Q) \otimes \pi_{1,8} \quad (9)$$

$$\mathbb{P}(1s_z) \equiv \sum_{s_Q s_{\bar{Q}}} \sum_{kl} \langle s_Q s_{\bar{Q}}; 1s_z \rangle \langle 3, k; \bar{3}, l | 1, 8 \rangle v_l(p_{\bar{Q}}) \bar{u}_k(p_Q) = (-\not{p}_{\bar{Q}} + m_Q) \not{\epsilon}(s_z) \frac{\not{p}_Q + \not{p}_{\bar{Q}} + 2E_q}{4\sqrt{2}E_q(E_q + m_Q)} (\not{p}_Q + m_Q) \otimes \pi_{1,8}, \quad (10)$$

where $\epsilon(s_z)$ is the spin polarization vector and $E_q = \sqrt{m_Q^2 + \vec{q}^2}$ is the energy of Q or $\bar{Q}$ in the meson's rest frame. In this frame, the four-dimensional total meson momentum is expressed as $(2E_q, \vec{0})$, and the four-dimensional relative momentum of Q and $\bar{Q}$ in a pair is $(0, \vec{q})$. The invariant magnitude of $\vec{q}$ associated with the effective velocity v is given by, specifically, $\vec{q}_3^2 = m_c^2 v_{c\bar{c}}^2$, $\vec{q}_4^2 = m_b^2 v_{b\bar{b}}^2$. Color projection operators $\pi_{1,8}$ (subscripts 1 and 8 denote the CS and CO states, respectively) assume the form of $\sum_{ij} \frac{\delta_{ij}}{\sqrt{N_c}}$ for the CS case and $\sqrt{2}T_{ij}^a$ ($a = 1, 2, \dots, 8$) for the CO case. For the CS channel, the total color factors of each diagram are N_c , while for the CO channel, it is $\frac{\delta_{ab}}{2}$ with a, b being the color indices of final states.

The amplitudes are expanded in terms of powers of q_3, q_4 as

$$\begin{aligned} \mathcal{A}(q_3, q_4) &= \mathcal{A}_{0,0} + q_3^{\alpha_1} \mathcal{A}_{\alpha_1,0} + q_4^{\beta_1} \mathcal{A}_{0,\beta_1} \\ &+ \frac{1}{2} q_3^{\alpha_1} q_3^{\alpha_2} \mathcal{A}_{\alpha_1 \alpha_2,0} + \frac{1}{2} q_4^{\beta_1} q_4^{\beta_2} \mathcal{A}_{0,\beta_1 \beta_2} + \dots \end{aligned} \quad (11)$$

The covariant Lorentz indices of the amplitude are omitted herein, where derivative terms of all orders for the amplitude $\mathcal{A}$ with respect to q_3, q_4 are defined as follows:

$$\begin{aligned} \mathcal{A}_{\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n} &= \sqrt{\frac{m_c}{E_c}} \sqrt{\frac{m_b}{E_b}} \left[\frac{\partial^{m+n} \mathcal{A}(q_3, q_4)}{\partial q_3^{\alpha_1} \dots \partial q_3^{\alpha_m} \partial q_4^{\beta_1} \dots \partial q_4^{\beta_n}} \right]_{q_3=q_4=0}. \end{aligned} \quad (12)$$

The factor $\sqrt{\frac{m_Q}{E_Q}}$ comes from relativistic normalization. The derivative operator changes the orbital quantum number of the amplitude by $\Delta L = \pm 1$; therefore, the LO S -wave and P -wave amplitudes in the expansion correspond to the zero-order term and the first-order derivative term, respectively. Furthermore, by virtue of the conservation of charge conjugation quantum number, the odd-order derivative terms in the expansion vanish for the S -wave, while the even-order derivative terms do so for the P -wave. We also account for the $\mathcal{O}(v^2)$ relativistic corrections which are essential in many production processes [34,42,80–84], as done in the previous work [74]. After integrating the freedoms of the relative momentum angle with the S or P state wave functions, the replacements are obtained,

$$\begin{aligned}
q_i^\mu q_i^\nu &\rightarrow \frac{|\vec{q}_i|^2}{3} \Pi_i^{\mu\nu}, \quad \text{for } S\text{-waviestates} \\
q_i^\mu &\rightarrow |\vec{q}_i| \epsilon^\mu(L_z) \\
q_i^\mu q_i^\nu q_i^\xi &\rightarrow \frac{|\vec{q}_i|^3}{5} \Pi_i^{\mu\nu\xi} = \frac{|\vec{q}_i|^3}{5} [\Pi_i^{\mu\nu} \epsilon^\xi(L_z) + \Pi_i^{\nu\xi} \epsilon^\mu(L_z) + \Pi_i^{\xi\mu} \epsilon^\nu(L_z)] \quad \text{for } P\text{-waviestates},
\end{aligned} \tag{13}$$

with $\Pi_i^{\mu\nu} = -g^{\mu\nu} + \frac{p_i^\mu p_i^\nu}{p_i^2}$, $\epsilon^\rho(L_z)$ being the orbital polarization vector of the P -wave states.

It is convenient to introduce the Lorentz invariant Mandelstam variables, which are defined as

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2, \tag{14}$$

with the relationship $s + t + u = M_{H_1}^2 + M_{H_2}^2$. The variable s is independent of $|\vec{q}|^2$, while t and u are $|\vec{q}|^2$ dependent. Let t_0, u_0 be defined as the values of t, u when $|\vec{q}_3|^2 = |\vec{q}_4|^2 = 0$. The following expansions are derived [81]:

$$\begin{aligned}
t &= t_0 + Ft(2m_c, 2m_b)|\vec{q}_3|^2 \\
&\quad + Ft(2m_b, 2m_c)|\vec{q}_4|^2 + \mathcal{O}(v^4)
\end{aligned} \tag{15}$$

$$\begin{aligned}
u &= u_0 + Fu(2m_c, 2m_b)|\vec{q}_3|^2 \\
&\quad + Fu(2m_b, 2m_c)|\vec{q}_4|^2 + \mathcal{O}(v^4)
\end{aligned} \tag{16}$$

with

$$Ft(x, y) = \frac{4[(t_0 + u_0)(t_0 + y^2) - 2y^2(t_0 + x^2)]}{s[s - 2(x^2 + y^2)] + (x^2 - y^2)^2} \tag{17}$$

$$Fu(x, y) = \frac{4[(t_0 + u_0)(u_0 + y^2) - 2y^2(u_0 + x^2)]}{s[s - 2(x^2 + y^2)] + (x^2 - y^2)^2} \tag{18}$$

III. INPUT PARAMETERS AND LDMES

We will take the following input parameters [85] in the calculation,

$$\begin{aligned}
\alpha_s &= 0.26, \quad \alpha = 1/137, \quad m_c = 1.5(1.75) \text{ GeV}, \\
m_b &= 4.7(4.94) \text{ GeV}, \quad m_Z = 91.1876 \text{ GeV}, \\
\Gamma_Z &= 2.4952 \text{ GeV}, \quad \sin^2\theta_w = 0.2312, \\
v_{c\bar{c}}^2 &= 0.23, \quad v_{b\bar{b}}^2 = 0.1.
\end{aligned} \tag{19}$$

Here, the running strong coupling constant α_s is obtained from the two-loop formula,

$$\frac{\alpha_s(\mu)}{4\pi} = \frac{1}{\beta_0 L} - \frac{\beta_1 \ln L}{\beta_0^3 L^2},$$

where $L = \ln(\mu^2/\Lambda_{\text{QCD}}^2)$ and $\Lambda_{\text{QCD}} \simeq 0.326 \text{ GeV}$. $\beta_0 = (11/3)C_A - (4/3)T_f n_f$ and $\beta_1 = (34/3)C_A^2 - 4C_F T_f n_f - (20/3)C_A T_f n_f$ denote the one-loop and two-loop coefficients of the QCD beta function, respectively. The number of active quark flavors is fixed at $n_f = 4$. $\alpha_s = 0.26$ is taken as the reference value at the renormalization scale $\mu = 2m_c$, which corresponds to the lowest hard scale in the process.

α is the EW fine structure constant. θ_w is the Weinberg angle. m_c, m_b , and m_Z are the masses of charm quark, bottom quark, and Z^0 boson, respectively. Γ_Z is the total decay width of Z^0 boson. $v_{Q\bar{Q}}^2$ is the square of the relative

TABLE II. CS LDMES of J/ψ , Υ , and χ_{Q0} obtained from potential models or phenomenological approaches.

	$\langle \mathcal{O}^{J/\psi} [^3S_1^{[1]}] \rangle \text{ (GeV}^3\text{)}$		$\langle \mathcal{O}^{\Upsilon} [^3S_1^{[1]}] \rangle \text{ (GeV}^3\text{)}$
Eichten and Quigg [87] ^a	1.2	Braaten <i>et al.</i> [88] ^a	10.9
Braaten <i>et al.</i> [18]	1.4	Braaten <i>et al.</i> [88]	12.8
Zhang <i>et al.</i> [27]	0.9	Eichten and Quigg [87]	9.28
	$\langle \mathcal{O}^{\chi_{c0}} [^3P_0^{[1]}] \rangle \text{ (GeV}^5\text{)}$		$\langle \mathcal{O}^{\chi_{b0}} [^3P_0^{[1]}] \rangle \text{ (GeV}^5\text{)}$
Eichten and Quigg [87] ^a	0.10743	Braaten <i>et al.</i> [88] ^a	2.4
Liao and Xie [89]	0.130	Anisovich <i>et al.</i> [90]	2.91
Braaten <i>et al.</i> [18]	0.089	Eichten and Quigg [87]	2.03

^aIt denotes the set of LDMES adopted in the subsequent numerical calculations.

velocity between Q and $\bar{Q}$ in heavy quarkonium, and its value is estimated via the Gremm-Kapustin relation [86]. Quark mass values enclosed in the brackets are for P -wave heavy quarkonia [$\chi_{QJ}(J = 0, 1, 2)$ and h_Q].

For the LDMEs, some typical CS LDMEs obtained from potential models or phenomenological methods are listed in Table II. We select matrix elements of moderate magnitude from these tabulated results.

For CO LDMEs, multiple parallel sets have been extracted in previous works on heavy quarkonium production at hadronic colliders using the NLO high- p_T

fits [23–27,91–97]. We summarize these values in Tables III and IV. The obvious deviations among different sets indicate the difficulty in determining accurate values of CO LDMEs, which further hinders the quantitative evaluation of CO channel contributions in our work. In subsequent numerical calculations and analyses, we evaluate the baseline CO contributions by adopting LDME sets of moderate values from the tabulated results. In addition, by employing different combinations of CO LDME sets, we also derive the maximal possible contributions of CO channels at the Z -factory energy scale.

TABLE III. The CO LDMEs of J/ψ and Υ extracted from hadronic productions.

	$\langle \mathcal{O}^{J/\psi} [^1S_0^{[8]}] \rangle (10^{-2} \text{ GeV}^3)$	$\langle \mathcal{O}^{J/\psi} [^3P_0^{[8]}] \rangle (10^{-2} \text{ GeV}^5)$	$\langle \mathcal{O}^{J/\psi} [^3S_1^{[8]}] \rangle (10^{-2} \text{ GeV}^3)$
Butenschön and Kniehl [23]	4.97	−1.61	0.224
Chao <i>et al.</i> [24]	8.9	1.26	0.30
	0	5.4	1.4
	11	0	0
Gong <i>et al.</i> [25]	9.7	−2.138	−0.46
Bodwin <i>et al.</i> [26]	9.9	1.1	1.1
Zhang <i>et al.</i> [27]	0.78	3.83	1.0
Brambilla <i>et al.</i> [96]	−0.205	1.568	0.592
	0.068 ^a	4.2278 ^a	1.050 ^a
	0.358	7.3575	1.382
	$\langle \mathcal{O}^{\Upsilon} [^1S_0^{[8]}] \rangle (10^{-2} \text{ GeV}^3)$	$\langle \mathcal{O}^{\Upsilon} [^3P_0^{[8]}] \rangle (10^{-2} \text{ GeV}^5)$	$\langle \mathcal{O}^{\Upsilon} [^3S_1^{[8]}] \rangle (10^{-2} \text{ GeV}^3)$
Gong <i>et al.</i> [92]	11.15	−15.117	−0.41
Feng <i>et al.</i> [93]	13.6	−20.983	0.61
	10.1	−5.189	0.73
	11.6	−11.056	0.47
Han <i>et al.</i> [94] ^a	0.5462	78.101	2.97
	0	81.451	3.04
	13.7	0	1.17

^aIt indicates the sets of LDMEs employed to compute the baseline CO contributions in our subsequent numerical calculations.

TABLE IV. CO LDMEs of χ_{Q0} extracted from hadronic productions.

	$\langle \mathcal{O}^{\chi_{c0}} [^3S_1^{[8]}] \rangle (10^{-2} \text{ GeV}^3)$		$\langle \mathcal{O}^{\chi_{b0}} [^3S_1^{[8]}] \rangle (10^{-2} \text{ GeV}^3)$
Ma <i>et al.</i> [91] ^a	0.215	Han <i>et al.</i> [94] ^a	0.735
Zhang <i>et al.</i> [97]	0.209	Feng <i>et al.</i> [93]	0.94
	0.235		0.91
	0.177		1.16
	0.077		
Gong <i>et al.</i> [25]	0.221		

^aIt indicates the sets of LDMEs employed to compute the baseline CO contributions in our subsequent numerical calculations.

For other LDMEs, the values could be estimated using the heavy quark spin symmetry [3],

$$\begin{aligned}
 \langle \mathcal{O}^{\chi_{\psi}} [{}^3P_J^{[1]}] \rangle &= (2J+1) \langle \mathcal{O}^{\chi_{\psi}} [{}^3P_0^{[1]}] \rangle \\
 \langle \mathcal{O}^{\chi_{\psi}} [{}^3S_1^{[8]}] \rangle &= (2J+1) \langle \mathcal{O}^{\chi_{\psi}} [{}^3S_1^{[8]}] \rangle \\
 \langle \mathcal{O}^{\eta_c} [{}^1S_0^{[1]}/{}^1S_0^{[8]}] \rangle &= \frac{1}{3} \langle \mathcal{O}^{J/\psi \text{ or } \Upsilon} [{}^3S_1^{[1]}/{}^3S_1^{[8]}] \rangle, \\
 \langle \mathcal{O}^{\eta_c} [{}^3S_1^{[8]}] \rangle &= \langle \mathcal{O}^{J/\psi \text{ or } \Upsilon} [{}^1S_0^{[8]}] \rangle \\
 \langle \mathcal{O}^{\eta_c} [{}^1P_1^{[8]}] \rangle &= 3 \langle \mathcal{O}^{J/\psi \text{ or } \Upsilon} [{}^3P_0^{[8]}] \rangle, \\
 \langle \mathcal{O}^{h_c} [{}^1P_1^{[1]}/{}^1S_0^{[8]}] \rangle &= 3 \langle \mathcal{O}^{\chi_{\psi}} [{}^3P_0^{[1]}/{}^3S_1^{[8]}] \rangle. \quad (20)
 \end{aligned}$$

IV. RESULTS AND DISCUSSION

A. Cross sections and CO contributions

In the present calculation, we only consider the t -channel double-photon fragmentation for $J/\psi + \Upsilon$ production. For the other two t -channel processes, $J/\psi + \chi_{b1}$ and $\chi_{c1} + \Upsilon$, their nonzero contributions are negligible due to suppression by the Z^0 propagator. Adopting the superscript- a -marked LDME sets from Tables II–IV, we present the production channels and corresponding cross sections at the Z^0 pole in Table V. The results in this table can be further compared with those for double charmonium and bottomonium production. It is observed that the CO contributions are dominant (basically $\geq 95\%$) for almost all processes, except for the production of $J/\psi + \Upsilon$, $J/\psi + \eta_b$, and $\Upsilon + h_c$. Unlike double charmonium/bottomonium production [74], where diagrams without fragmentation exist in the CO channels, only fragmentation ones ($g \rightarrow {}^3S_1^{[8]}$) contribute to charmonium + bottomonium production. Specifically, for the production of $\eta_c + \chi_{bJ} (J=0, 2)$, $\eta_b + \chi_{cJ} (J=0, 2)$, $\eta_c + h_b$, $\eta_b + h_c$, or $\eta_c + \eta_b$, the CS channels have no contributions because neither the photon nor Z boson can fragment into η_Q , h_Q or $\chi_{QJ} (J=0, 2)$.

One could check these results with the fragmentation function of the gluon into ${}^3S_1^{[8]}$ state at $\mathcal{O}(\alpha_s)$ in the high energy limit ($s \gg M_Q^2$),

$$\begin{aligned}
 \sigma(e^+e^- \rightarrow H_1 + g \rightarrow H_1 + H_2 + X) \\
 = \sigma(e^+e^- \rightarrow H_1 + g) \int_0^1 dz D_{g \rightarrow H_2}(z), \quad (21)
 \end{aligned}$$

where $z = 2E_2/\sqrt{s}$. At LO of $\mathcal{O}(\alpha_s)$, the fragmentation function is given by $D_{g \rightarrow H_2}({}^3S_1^{[8]})(z) = \frac{\pi\alpha_s\delta(1-z)}{3(N_c^2-1)} \frac{\langle \mathcal{O}^{H_2} [{}^3S_1^{[8]}] \rangle}{m_c^3}$ [98].

At the tree level, the gluon fragmentation to ${}^3S_1^{[8]}$ state is most important, but for the higher-order diagrams, as discussed in Ref. [74], the gluon fragmentation into other intermediate states (e.g., ${}^3P_J^{[8]}$, ${}^1S_0^{[1]}$) also yields significant contributions, which may weaken the importance of

contributions from the ${}^3S_1^{[8]}$ channels, and a definitive conclusion therefore requires more comprehensive calculations that include higher-order QCD corrections.

The cross sections varying with collision energy are illustrated in Fig. 2. We plot the CS and CS + CO cross sections at LO and NLO in the v^2 expansion, and the maximal CS + CO LO cross sections. For the LO CS cross sections, the central values are computed with the superscript- a -labeled LDMEs listed in Table II. By adopting the maximum and minimum matrix element values from Table II, we derive the cross section variations for the LO CS channel, which are displayed as the uncertainty band in the figure. The CO cross sections are also evaluated using the superscript- a -marked LDME sets in Tables III and IV. The CS + CO curves, serving as the baseline cross sections, correspond to the results given in Table V. The maximal CS + CO cross sections are constructed by summing the upper-bound cross sections of the LO CS channel (denoted as $\sigma_{\text{CS}_{\text{max}}}$) and the maximal LO CO cross sections (denoted as $\sigma_{\text{CO}_{\text{max}}}$). $\sigma_{\text{CO}_{\text{max}}}$ are determined by enumerating all possible combinations of charmonium and bottomonium LDME sets in Tables III and IV to maximize the CO cross section at the Z^0 pole. As shown in the figure, once the sensitivities of CO LDMEs are taken into account, the CO contributions can be significantly enhanced, and even rise by several orders of magnitude compared with baseline CO predictions for multiple production channels. This indicates that if future high-precision experimental measurements yield cross sections substantially larger than pure CSM predictions the COM framework will become essential for theoretical interpretation. Meanwhile, since CO channel contributions are highly sensitive to LDME magnitudes, future experimental measurements can impose stringent constraints on these matrix element values.

In the low-energy region, the γ -mediated processes dominate, while the Z^0 -mediated processes prevail in the Z -factory energy region due to the Z^0 resonance effects. Relativistic corrections yield sizable negative contributions for both CS and CO channels. The fractions of CO contributions relative to the total cross sections at LO and NLO in the v^2 expansion are plotted against the c.m. energy in Fig. 3. The maximum CO fractions are also plotted as purple lines in comparison. Except for the $J/\psi + \Upsilon$, $J/\psi + \eta_b$, and $h_c + \Upsilon$ processes, the maximum ratios can reach surprisingly large values, typically larger than 0.8, across nearly all low and high energy ranges. The $J/\psi + \Upsilon$ process is entirely governed by the CSM. For $J/\psi + \eta_b$, and $h_c + \Upsilon$ processes, the maximum ratios can amount to approximately 0.6 and 0.2,

¹It is worth emphasizing that some LDME combinations can also produce negative CO contributions, which further leads to negative total cross sections at the Z^0 -boson energy scale. For instance, $\sigma_{\text{CS+CO}}(\eta_c + \Upsilon) = -2.0 \times 10^{-4} \text{ fb}$ when using “Brambilla I-Gong (b)” LDME combination, and $\sigma_{\text{CS+CO}}(\eta_c + \eta_b) = -1.6 \times 10^{-5} \text{ fb}$ for the “Brambilla I-Han III” combination at the Z^0 pole.

TABLE V. Production channels and corresponding LO cross sections in units of 10^{-4} fb at the Z^0 pole. Percentages in brackets denote the fraction of the CO contribution to the total cross sections. Cross section values are omitted for channels with negligible contributions ($< 0.1 \times 10^{-4}$ fb).

H_1		H_2									
J/ψ	Υ	384.8 (5.2%)	χ_{bJ}	1.1 ($J=0$) (86.0%)	3.8 ($J=1$) (73.7%)	5.0 ($J=2$) (93.2%)	η_b	31.7 (40.9%)	h_b	4.0 (75.0%)	
$^3S_1^{[1]}$	$^3S_1^{[1]}$	364.9	$^3P_J^{[1]}$	0.2	1.0	0.3	$^1S_0^{[1]}$	18.7	$^1P_1^{[1]}$	1.0	
$^3S_1^{[8]}$	$^3S_1^{[8]}$	3.4	$^3S_1^{[8]}$	0.8	2.3	3.9	$^3S_1^{[8]}$	0.6	$^1S_0^{[8]}$	3.0	
	$^1S_0^{[8]}$						$^1S_0^{[8]}$	1.4			
	$^3P_J^{[8]}$	15.8					$^1P_1^{[8]}$	10.8			
$^1S_0^{[8]}$	$^3S_1^{[8]}$		$^3S_1^{[8]}$				$^3S_1^{[8]}$				
$^3P_J^{[8]}$	$^3S_1^{[8]}$	0.7	$^3S_1^{[8]}$	0.1	0.4	0.7	$^3S_1^{[8]}$	0.1			
η_c	η_b	1.1 (100.0%)	χ_{bJ}	0.4 ($J=0$) (100.0%)	1.1 ($J=1$) (100.0%)	1.9 ($J=2$) (100.0%)	h_b	0.2 (100.0%)	Υ	3.4 (83.2%)	
$^1S_0^{[1]}$	$^1S_0^{[1]}$		$^3P_J^{[1]}$				$^1P_1^{[1]}$		$^3S_1^{[1]}$	0.6	
$^3S_1^{[8]}$	$^3S_1^{[8]}$		$^3S_1^{[8]}$		0.2	0.3	$^1S_0^{[8]}$	0.2	$^3S_1^{[8]}$	0.2	
	$^1S_0^{[8]}$								$^1S_0^{[8]}$		
	$^1P_1^{[8]}$	0.7							$^3P_J^{[8]}$	1.0	
$^1S_0^{[8]}$	$^3S_1^{[8]}$		$^3S_1^{[8]}$						$^3S_1^{[8]}$		
$^1P_1^{[8]}$	$^3S_1^{[8]}$	0.5	$^3S_1^{[8]}$	0.3	1.0	1.6			$^3S_1^{[8]}$	1.5	
η_b	h_c	(100.0%)	χ_{cJ}	1.6 ($J=0$) (100.0%)	4.9 ($J=1$) (100.0%)	8.2 ($J=2$) (100.0%)					
$^1S_0^{[1]}$	$^1P_1^{[1]}$		$^3P_J^{[1]}$								
$^3S_1^{[8]}$	$^1S_0^{[8]}$		$^3S_1^{[8]}$		0.2	0.4					
$^1S_0^{[8]}$			$^3S_1^{[8]}$	0.2	0.6	0.9					
$^1P_1^{[8]}$			$^3S_1^{[8]}$	1.4	4.1	6.9					
Υ	h_c	0.3 (20.0%)	χ_{cJ}	2.6 ($J=0$) (99.4%)	7.8 ($J=1$) (98.9%)	12.9 ($J=2$) (99.8%)					
$^3S_1^{[1]}$	$^1P_1^{[1]}$	0.3	$^3P_J^{[1]}$								
$^3S_1^{[8]}$	$^1S_0^{[8]}$		$^3S_1^{[8]}$	0.4	1.3	2.2					
$^1S_0^{[8]}$			$^3S_1^{[8]}$	0.1	0.3	0.5					
$^3P_J^{[8]}$			$^3S_1^{[8]}$	2.0	6.1	10.2					

respectively. Meanwhile, the energy dependence of these ratios clearly differs from that observed in double charmonium or double bottomonium production [74], where the CO ratios rise gradually with increasing $E_{\text{c.m.}}$. This arises from the absence of nonfragmentation mechanisms for the charmonium-bottomonium associated production. If we disregard the nonfragmentation diagrams, i.e., the (a), (b), (e), and (f) diagrams in Fig. 1 of Ref. [74], the energy variation of the CO ratios agrees with our present results. This behavior can be explained from the perspectives of scattering amplitudes and cross sections. For double charmonium/bottomonium production, e.g., the $J/\psi + \eta_c$ process, asymptotically the total cross section falls off with increasing energy as

$\sigma_{\text{tot}} \sim \frac{1}{s^2} (A + B + \mathcal{O}(\frac{1}{s^2}))$, where $A \sim B \sim \mathcal{O}(1)$. The main contributions A and B stem from gluon fragmentation CO processes [(c) and (d) diagrams in Fig. 1 of Ref. [74]] and the photon fragmentation CS processes [(g) and (h) diagrams in Fig. 1 of Ref. [74]], respectively. The last $\mathcal{O}(\frac{1}{s^2})$ term²

²The amplitudes of the QCD nonfragmentation CS process and QED fragmentation CS process share the identical Lorentz structure $\mathcal{A} \sim \epsilon_\nu^{J/\psi} p_{3\rho} p_{4\sigma} \epsilon^{\mu\nu\rho\sigma}$, and the amplitude ratio between them is $\frac{\mathcal{A}_{\text{QCD}}}{\mathcal{A}_{\text{QED}}} \big|_{s \rightarrow \infty} \sim \frac{1}{s}$. Therefore, the CS cross section $\sigma_{\text{CS}} \simeq \sigma_{\text{QCD}} + \sigma_{\text{QED}}$ asymptotically falls off with increasing energy as $\sigma_{\text{CS}} \sim \frac{1}{s^2} (1 + \mathcal{O}(\frac{1}{s}))$, where the main contribution originates from the QED fragmentation process.

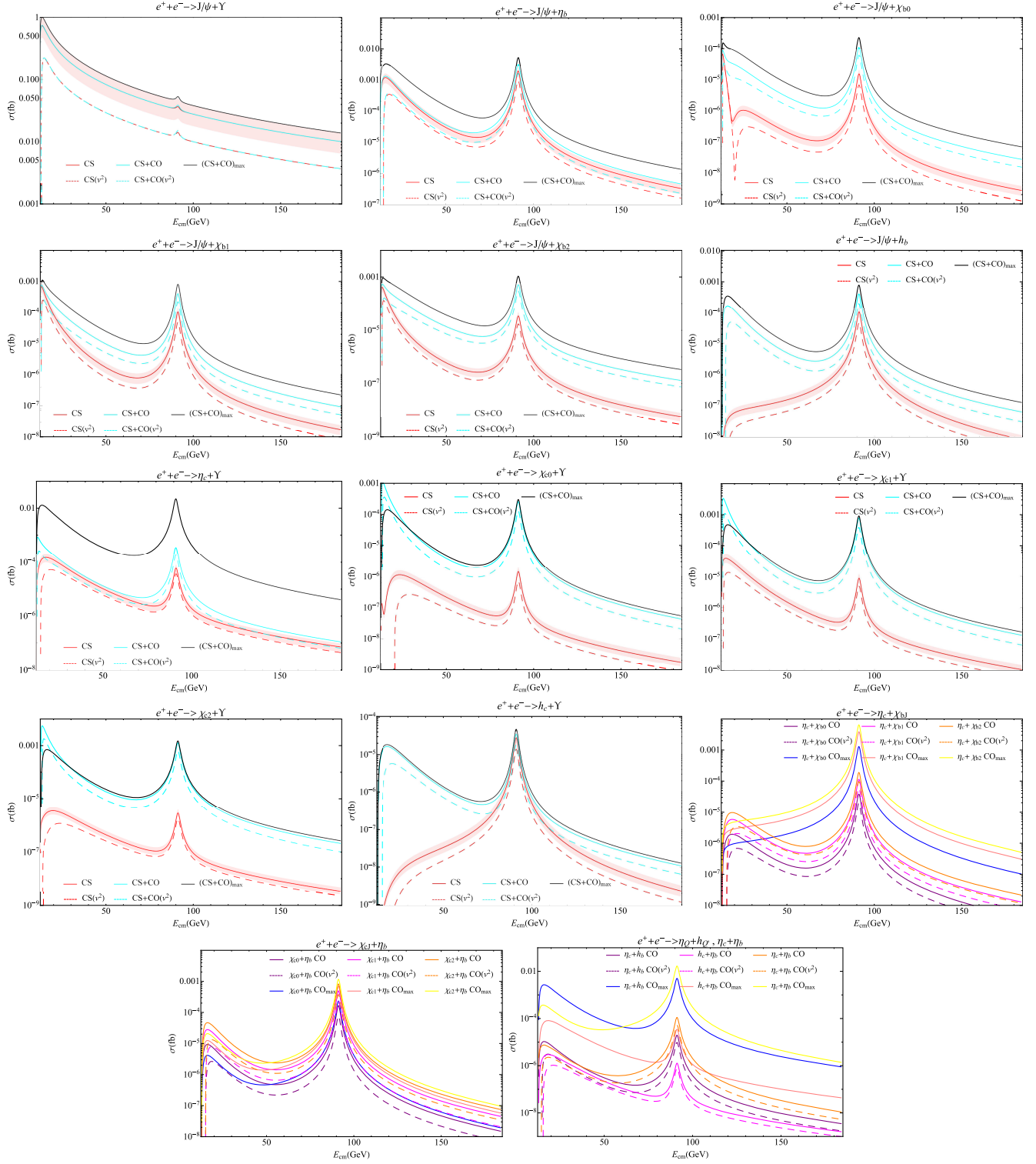


FIG. 2. The production cross sections as a function of c.m. energy. The solid red and cyan lines denote our $\mathcal{O}(v^0)$ color-singlet and CS + CO results, respectively, and their corresponding dashed lines are the $\mathcal{O}(v^2)$ results. The light red uncertainty band comes from the CS LDMEs. The black lines represent the $\mathcal{O}(v^0)$ results obtained by enumerating all possible combinations of the LDME sets to maximize the total cross section at the Z^0 pole.

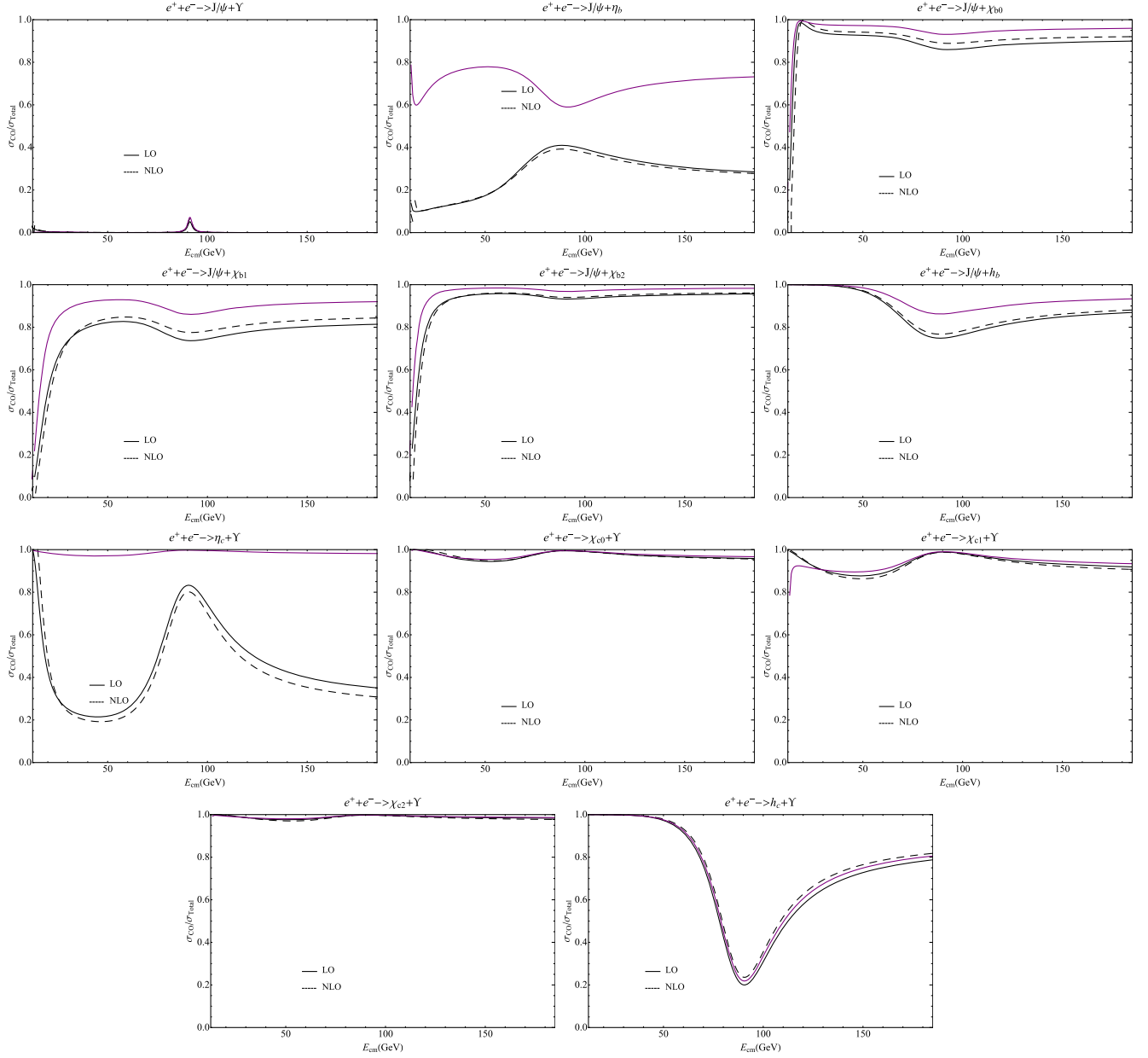


FIG. 3. $\sigma_{CO}/\sigma_{Total}$ as a function of c.m. energy $E_{c.m.}$. The black solid and dashed lines represent LO and NLO results in v^2 , respectively. The purple lines represent the LO maximum CO fractions.

TABLE VI. CO fractions in the high-energy limit ($\sqrt{s} \gg m_Z$). Values outside and inside the parentheses correspond to the results at LO and NLO in v^2 , respectively.

Process	CO	CO _{max}
$J/\psi + \Upsilon$	0.00(0.00)	0.00(0.00)
$J/\psi + \eta_b$	0.25(0.25)	0.69(0.73)
$J/\psi + \chi_{b0}$	0.91(0.93)	0.95(0.96)
$J/\psi + \chi_{b1}$	0.83(0.86)	0.90(0.92)
$J/\psi + \chi_{b2}$	0.96(0.97)	0.98(0.98)
$J/\psi + h_b$	0.90(0.91)	0.93(0.94)
$\eta_c + \Upsilon$	0.29(0.25)	0.97(0.96)
$\chi_{c0} + \Upsilon$	0.95(0.94)	0.94(0.94)
$\chi_{c1} + \Upsilon$	0.90(0.88)	0.89(0.88)
$\chi_{c2} + \Upsilon$	0.98(0.97)	0.98(0.97)
$h_c + \Upsilon$	0.86(0.88)	0.83(0.85)

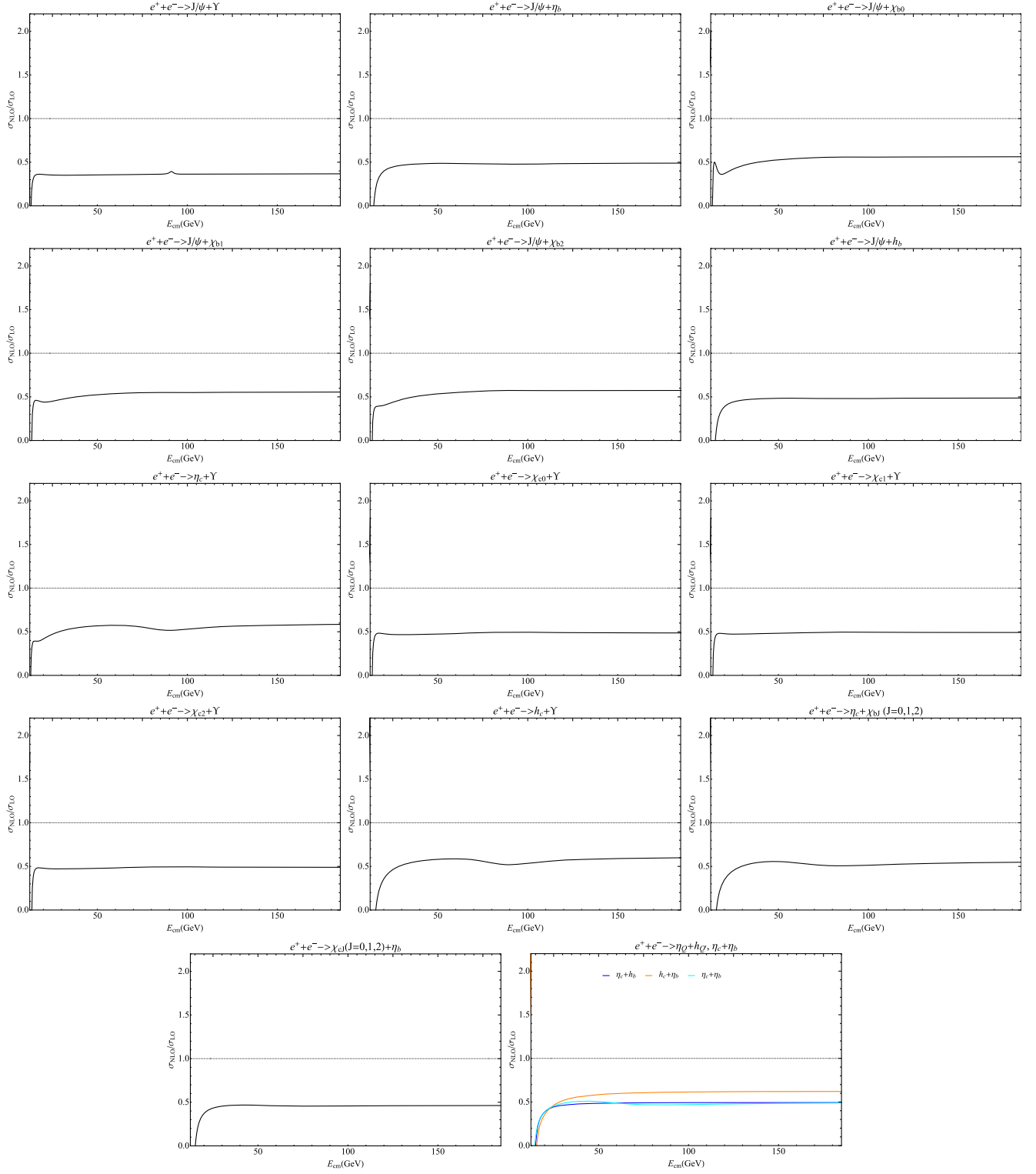
FIG. 4. The K factors ($\sigma_{\text{NLO}(v^2)}/\sigma_{\text{LO}}$) as functions of c.m. energy $E_{\text{c.m.}}$.

TABLE VII. Production cross sections (units: $\times 10^{-4}$ fb) at $\sqrt{s} = 91.1876$ GeV. “...” denotes forbidden processes. LO and NLO(v^2) denote the leading-order and next-to-leading order in the v^2 expansion, respectively. CS and CO represent the cross sections for the CS channel and the sum of all CO channels, respectively.

	CS		CS + CO			CS		CS + CO	
	LO	NLO(v^2)	LO	NLO(v^2)		LO	NLO(v^2)	LO	NLO(v^2)
$J/\psi + \Upsilon$	364.9	141.6	384.8	151.5	$\eta_c + \eta_b$	...	...	1.1	0.5
$J/\psi + \eta_b$	18.7	9.2	31.7	15.1	$\eta_c + \Upsilon$	0.6	0.4	3.4	1.8
$J/\psi + \chi_{b0}$	0.2	0.1	1.1	0.6	$\chi_{c0} + \Upsilon$	0.014	0.007	2.6	1.3
$J/\psi + \chi_{b1}$	1.0	0.5	3.8	2.1	$\chi_{c1} + \Upsilon$	0.09	0.05	7.8	3.9
$J/\psi + \chi_{b2}$	0.3	0.2	5.0	2.9	$\chi_{c2} + \Upsilon$	0.03	0.02	12.9	6.4
$J/\psi + h_b$	1.0	0.5	4.0	1.9	$h_c + \Upsilon$	0.3	0.1	0.3	0.2
$\eta_c + h_b$	...	...	0.2	0.1	$h_c + \eta_b$	...	...	0.013	0.008
$\eta_c + \chi_{b0}$	...	...	0.4	0.2	$\chi_{c0} + \eta_b$	...	...	1.6	0.8
$\eta_c + \chi_{b1}$	...	...	1.1	0.6	$\chi_{c1} + \eta_b$	...	...	4.9	2.3
$\eta_c + \chi_{b2}$	...	...	1.9	1.0	$\chi_{c2} + \eta_b$	...	...	8.2	3.8

originates from the QCD non-fragmentation CS process [(a) and (b) diagrams in Fig. 1 of Ref. [74]]. As a result, the QCD nonfragmentation process dominates in the low-energy region, while as the c.m. energy increases, it becomes negligible; then, the comparable gluon fragmentation and photon fragmentation processes are dominant, and the CO ratio $\frac{\sigma_{\text{CO}}}{\sigma_{\text{total}}} \sim \frac{A}{A+B}$ yields an approximately constant value (see Fig. 4 of Ref. [74]). However, for charmonium + bottomonium production, e.g., the $\eta_c + \Upsilon$ production, the amplitude ratio between the CO and CS processes, determined by coupling vertices and color factors, is $\frac{A_{\text{CO}}}{A_{\text{CS}}} \sim \frac{9\alpha_s \sqrt{C_A C_F}}{2\sqrt{2}N_c \alpha}$. Therefore, the total cross section $\sigma_{\text{tot}}|_{s \rightarrow \infty} \sim \frac{1}{s^2}(A+B)$, where A and B originate from the CO and CS channels, respectively. Further accounting for abundant CO channels and suppressed CO LDMEs, the cross section ratio $\frac{\sigma_{\text{CO}}}{\sigma_{\text{total}}}$ behaves approximately as a constant $\frac{A}{A+B}$. The CO ratios $\frac{\sigma_{\text{CO}}}{\sigma_{\text{total}}}$ and the maximum CO ratios at the high-energy limit ($\sqrt{s} \gg m_Z$) are summarized in Table VI. One can see that the maximum CO ratios are smaller than our CO ratios for several production processes. Thus, the maximum CO ratios are not universally larger across the full energy range, as also seen for $\chi_{cJ} + \Upsilon$ and $\eta_c + \chi_{bJ}$ production at low-energy region in Fig. 2.

The K factors ($\sigma_{\text{NLO}(v^2)}/\sigma_{\text{LO}}$) as a function of $\sqrt{s}$ are shown in Fig. 4. The relativistic corrections are significant, which reduce the LO cross sections by approximately 50%.

In the Appendix, we provide the ratios of the SDCs between the RC and LO in the high-energy limit ($\sqrt{s} \gg m_Q$), which also illustrate the effects of the relativistic corrections. The numerical results of the total cross section at the Z^0 peak are shown in Table VII, where we list the CS channel cross sections and those for the CO channels in both $\mathcal{O}(v^0)^3$ and $\mathcal{O}(v^2)$. Figure 5 depicts the differential cross sections $d\sigma/d\cos\theta$, where θ is the angle between the electron (p_1) and the charmonium (p_3). We find that the angular distributions exhibit a hollow structure for all processes in both the CS and CO channels.

The differential cross section $\frac{d\sigma}{dp_t}$ can be written as

$$\frac{d\sigma}{dp_t} = \left| \frac{d\cos\theta}{dp_t} \right| \left(\frac{d\sigma}{d\cos\theta} \right) = \frac{p_t}{|\vec{p}_3| \sqrt{\vec{p}_3^2 - p_t^2}} \left(\frac{d\sigma}{d\cos\theta} \right), \quad (22)$$

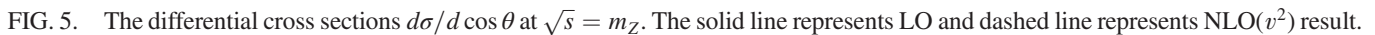
where $\vec{p}_3$ is the momentum of the H_1 charmonium; we have

$$\vec{p}_3 = \frac{\sqrt{\lambda[s, (2E_{q3})^2, (2E_{q4})^2]}}{2\sqrt{s}}, \quad \lambda[x, y, z] = x^2 + y^2 + z^2 - 2(xy + xz + yz). \quad (23)$$

Combining Eqs. (22) and (23), we can also get an $\mathcal{O}(v^2)$ expression:

$$\frac{d\sigma}{dp_t} = \left\{ \frac{4p_t s}{[(A - 4p_t^2 s)A]^{1/2}} + \frac{32[m_c^2 v_{c\bar{c}}^2 (4m_b^2 - 4m_c^2 + s) + m_b^2 v_{b\bar{b}}^2 (4m_c^2 - 4m_b^2 + s)](A - 2p_t^2 s)p_t s}{[(A - 4p_t^2 s)A]^{3/2}} \right\} \left(\frac{d\sigma}{d\cos\theta} \right) + \mathcal{O}(v^4) \quad (24)$$

³Our LO CS cross sections of processes $J/\psi + \eta_b$, $\Upsilon + \eta_c$, and $J/\psi + \Upsilon$ are in agreement with the findings presented in the latest arXiv version of Ref. [72], when the same input parameters are adopted.



In Fig. 7, we show the angular and transverse momentum distributions separately for $J/\psi + \eta_b$ and $\eta_c + \Upsilon$ production. As shown in Fig. 3, the CO

fractions for these two channels are the most sensitive to variations in CO LDMEs, and we explore how they respond to different CO LDME combinations. Clearly, the $\eta_c + \Upsilon$ process relies heavily on CO LDMEs, with the resulting discrepancies reaching several orders of magnitude.



Consistent with previous research [74], we also explore uncertainties originating from the renormalization scale, collision energy deviation from m_Z , heavy quark masses. The cross section variations related to LDME sensitivity have been partially analyzed in the previous section.

Uncertainties arising from the renormalization scale μ are presented in Fig. 8. In the figure, μ ranges from 2.0 to 90.0 GeV, yielding $\alpha_s \in (0.31, 0.11)$. As illustrated in the figure, observable strong scale dependence occurs at $\mu < 20$ GeV, and such dependence gradually weakens with increasing μ .

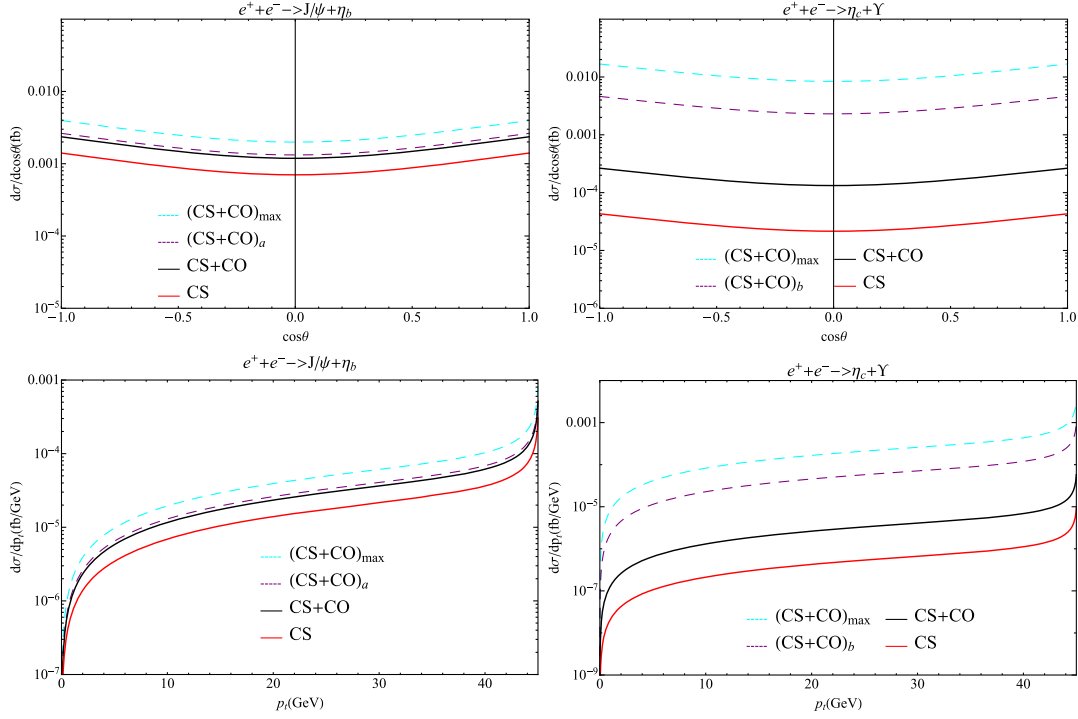


FIG. 7. The $\mathcal{O}(v^0)$ differential angular and transverse momentum distributions for $J/\psi + \eta_b$ (left) and $\eta_c + \Upsilon$ (right) production. The red solid line represents the CS result, and the black solid line represents the baseline result using the “BrambillaII-HanI” combination. The dashed cyan line represents the maximum result $\text{CS}_{\text{max}} + \text{CO}_{\text{max}}$. The dashed purple lines, denoted by the subscripts a and b, correspond to the combinations “Bodwin-Gong(b)” and “Butenschoen-Gong(b),” respectively.

In Table VIII, we show the sensitivity of the total cross sections to the collision energy around the Z^0 peak. Here, we estimate the uncertainties in the cross sections by assuming a 3% deviation of c.m. energy at the Z-factory with respect to the Z^0 pole. This analysis method is taken from Ref. [73]. The total cross sections are reduced to 15–20% of the peak values, a result consistent with Tables VIII and IX in Ref. [73] and Tables VII and VIII in Ref. [74].

Taking a ± 0.15 GeV variation in m_Q ($m_c = (1.5 \pm 0.15)$ GeV, $m_b = (4.7 \pm 0.15)$ GeV for S -wave heavy quarkonia and $m_c = (1.75 \pm 0.15)$ GeV, $m_b = (4.94 \pm 0.15)$ GeV for P -wave heavy quarkonia), we summarize the resulting uncertainties due to heavy quark masses in Table IX. For total cross sections, the uncertainties are 12–14% for $h_c + \eta_b$, 21–28% for $h_c + \Upsilon$; 21–32% for $\chi_{cJ} + \Upsilon/\eta_b$, 25–38% for $J/\psi + \chi_{bJ}/\eta_b/h_b$ and $\eta_c + h_b/\chi_{bJ}$; 27–40% for $J/\psi + \Upsilon$; and 13–45% for $\eta_c + \eta_b$, and 33–46% for $\eta_c + \Upsilon$ production.

C. Events

In Table X, the events are obtained for the CEPC (for its Z-factory mode with two interaction points, the designed integrated luminosity over two years is $\sim 16 \text{ ab}^{-1}$ [44]) and for the FCC-ee (which operates as a super Z-factory in its first four years with a designed integrated luminosity of 150 ab^{-1} [45]). For most

production processes, event yields are dominated by the COM. As the CS mechanism alone gives nearly zero events, future precise measurements could provide valuable evidence for the COM.

Heavy quarkonia are reconstructed via their decay products and the reconstructed events can be estimated as

$$N = N[H_1 + H_2] \times Br_1 \times Br_2 \times \epsilon_1 \times \epsilon_2, \quad (25)$$

where Br_i and ϵ_i ($i = 1, 2$) are the branching fractions and detection efficiency, respectively. Precise measurements of the Higgs boson and the electroweak sector require the detector to identify leptons, photons, tau leptons, jets, and flavors efficiently and precisely [44,45]; therefore, we assume the detection efficiency is 100%. Using the branching fractions of Ref. [85], the reconstructed events for processes $\{J/\psi + \Upsilon, \eta_c + \chi_{b1}, \eta_c + \chi_{b2}, \text{ and } \Upsilon + \eta_c\}$ are given in Table XI. Here, J/ψ and Υ are reconstructed from leptonic decays, η_c is reconstructed from hadronic decays, and χ_{bJ} are reconstructed via the decay chain $\chi_{bJ} \rightarrow \gamma + \Upsilon \rightarrow \gamma + l^+ l^-$. With the combined uncertainties from the heavy quark mass and renormalization scale, the total events for processes $\{J/\psi + \Upsilon, \Upsilon + \eta_c\}$ are $\{4^{+1.46}_{-1.00}(1^{+0.57}_{-0.39}), 0^{+0.05}_{-0.08}(0^{+0.03}_{-0.05})\}$ and $\{34^{+13.7}_{-9.33}(13^{+5.35}_{-3.66}), 1^{+0.50}_{-0.77}(1^{+0.25}_{-0.44})\}$ for the two colliders, respectively. In addition, production rate estimates show a strong dependence on CO LDME selections. Accordingly, we also

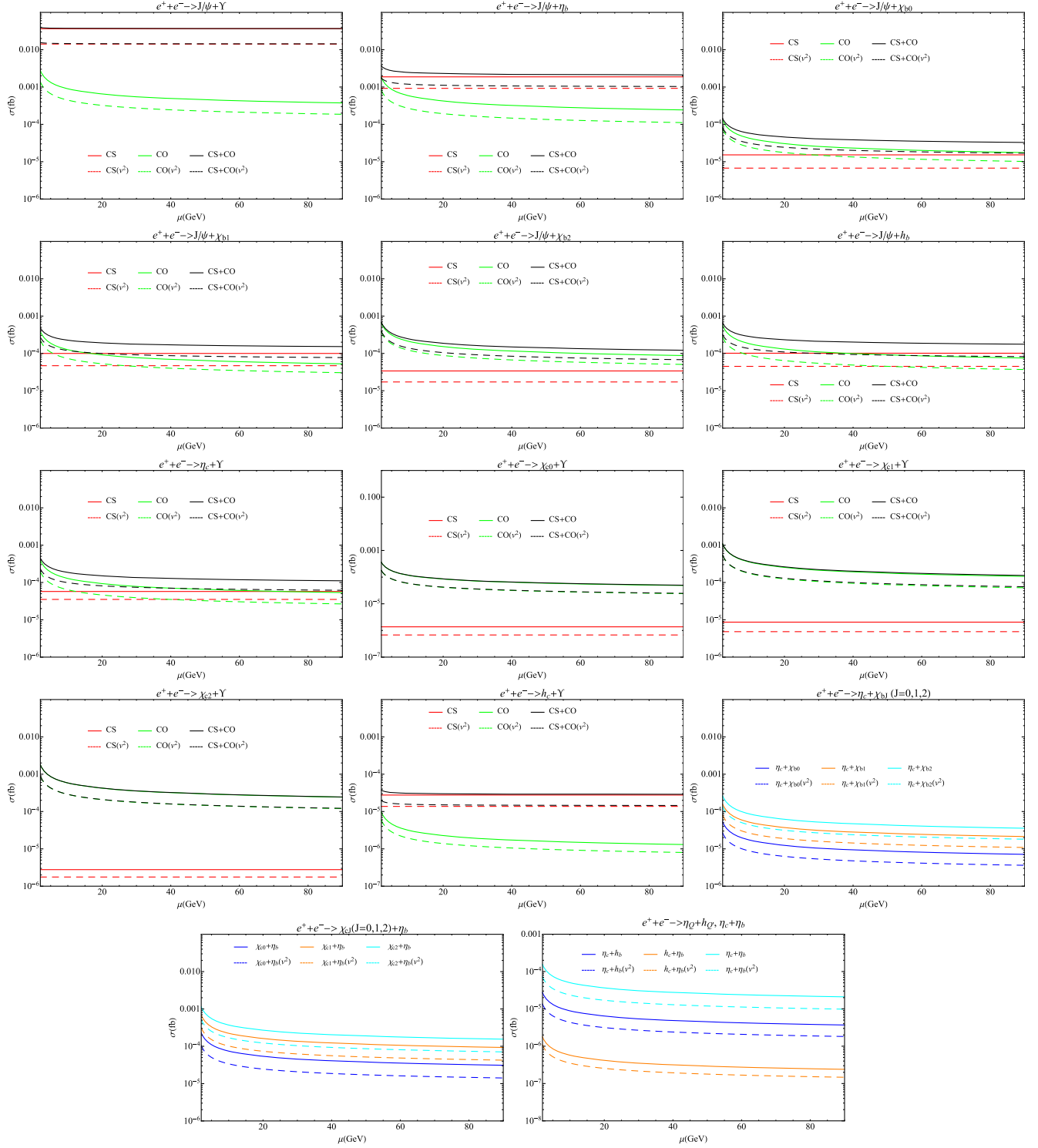


FIG. 8. Cross section σ versus renormalization scale μ at $\sqrt{s} = m_Z$. The solid line represents the LO result, and the dashed line represents the NLO(v^2) result.

present the corresponding maximum event yields in the table. Note that in this work we did not use the CO LDMEs fitted at LO in α_s [99]; using them instead would lead to a larger maximum production rate estimation. For example,

the largest CO LDME of $\chi_{bJ} \langle \mathcal{O}^{\chi_{b0}} [{}^3S_1^{[8]}] \rangle$ would change from 0.0116 [93] to 0.1008 GeV³ [99], and then for $\eta_c + \chi_{b1,2}$ production at FCC-ee, the maximum reconstructed events would be ~ 20 .

TABLE VIII. Total cross sections (units: $\times 10^{-4} fb$) up to $\mathcal{O}(v^2)$ for the charmonia + bottomonia production, with varying values of $E_{\text{c.m.}}$. “...” denotes forbidden processes. The ratios $R_{\pm}$, with definitions as $R_- = \frac{\sigma(E_{\text{c.m.}}=97\%m_Z)}{\sigma(E_{\text{c.m.}}=m_Z)}$, $R_+ = \frac{\sigma(E_{\text{c.m.}}=103\%m_Z)}{\sigma(E_{\text{c.m.}}=m_Z)}$, characterize the variation of the cross sections with $E_{\text{c.m.}}$.

	CS			CO			Total		
	$97\%m_Z$	$103\%m_Z$	$\begin{pmatrix} R_- \\ R_+ \end{pmatrix}$	$97\%m_Z$	$103\%m_Z$	$\begin{pmatrix} R_- \\ R_+ \end{pmatrix}$	$97\%m_Z$	$103\%m_Z$	$\begin{pmatrix} R_- \\ R_+ \end{pmatrix}$
$J/\psi + \Upsilon$	128.0	113.6	$\begin{pmatrix} 90\% \\ 80\% \end{pmatrix}$	1.7	1.7	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	129.8	115.3	$\begin{pmatrix} 86\% \\ 76\% \end{pmatrix}$
$J/\psi + \eta_b$	1.6	1.6	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	1.0	1.0	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	2.7	2.6	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$J/\psi + \chi_{b0}$	0.012	0.012	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	0.1	0.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.1	0.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$J/\psi + \chi_{b1}$	0.08	0.08	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.3	0.3	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.4	0.4	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$J/\psi + \chi_{b2}$	0.03	0.03	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.5	0.5	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.5	0.5	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$J/\psi + h_b$	0.08	0.08	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.3	0.3	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.3	0.3	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\eta_c + h_b$	...	...	...	0.02	0.02	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.02	0.02	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\eta_c + \chi_{b0}$	...	...	...	0.03	0.03	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.03	0.03	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\eta_c + \chi_{b1}$	...	...	...	0.1	0.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.1	0.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\eta_c + \chi_{b2}$	...	...	...	0.2	0.2	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.2	0.2	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\eta_c + \eta_b$	...	...	...	0.1	0.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.1	0.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\eta_c + \Upsilon$	0.06	0.07	$\begin{pmatrix} 18\% \\ 19\% \end{pmatrix}$	0.8	0.8	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.9	0.9	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\chi_{c0} + \Upsilon$	0.001	0.001	$\begin{pmatrix} 18\% \\ 19\% \end{pmatrix}$	0.2	0.2	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.2	0.2	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\chi_{c1} + \Upsilon$	0.009	0.009	$\begin{pmatrix} 18\% \\ 19\% \end{pmatrix}$	0.7	0.7	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.7	0.7	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\chi_{c2} + \Upsilon$	0.003	0.003	$\begin{pmatrix} 18\% \\ 19\% \end{pmatrix}$	1.1	1.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	1.1	1.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$h_c + \Upsilon$	0.02	0.02	$\begin{pmatrix} 17\% \\ 17\% \end{pmatrix}$	0.008	0.008	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.03	0.03	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$h_c + \eta_b$	...	...	...	0.001	0.001	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.001	0.001	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\chi_{c0} + \eta_b$	...	...	...	0.1	0.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.1	0.1	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\chi_{c1} + \eta_b$	...	...	...	0.9	0.8	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	0.9	0.8	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$
$\chi_{c2} + \eta_b$	...	...	...	1.5	1.4	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$	1.5	1.4	$\begin{pmatrix} 18\% \\ 17\% \end{pmatrix}$

TABLE IX. Total cross sections (units: $\times 10^{-4}$ fb) up to $\mathcal{O}(v^2)$ at $\sqrt{s} = 91.1876$ GeV with varying m_Q . “...” denotes forbidden processes. As noted in the previous text, the central values are $m_c = 1.5$ GeV for S -wave charmonia and 1.75 GeV for P -wave charmonia, and $m_b = 4.7$ GeV for S -wave bottomonia and 4.94 GeV for P -wave bottomonia. Here, $m_{c\pm}$ denotes cross sections obtained by fixing m_b and shifting m_c by ± 0.15 GeV, and $m_{b\pm}$ denotes cross sections obtained by fixing m_c and shifting m_b by ± 0.15 GeV. Each cross section has two uncertainties arising from ± 0.15 GeV deviations in m_c and m_b , and the total uncertainty is combined from these two uncertainties.

Process	CS					CS + CO				
	m_{c-}	m_{c+}	m_{b-}	m_{b+}	Uncertainty $\Delta_{\pm}$	m_{c-}	m_{c+}	m_{b-}	m_{b+}	Uncertainty $\Delta_{\pm}$
$J/\psi + \Upsilon$	197.2	105.0	155.3	129.5	$^{+57.33}_{-38.55}$	210.8	112.4	166.1	138.6	$^{+61.07}_{-41.11}$
$J/\psi + \eta_b$	12.67	6.914	9.532	8.927	$^{+3.463}_{-2.324}$	20.80	11.36	15.99	14.37	$^{+5.723}_{-3.861}$
$J/\psi + \chi_{b0}$	0.093	0.050	0.074	0.061	$^{+0.026}_{-0.018}$	0.818	0.464	0.639	0.576	$^{+0.215}_{-0.146}$
$J/\psi + \chi_{b1}$	0.644	0.351	0.513	0.429	$^{+0.181}_{-0.124}$	2.821	1.591	2.209	1.974	$^{+0.746}_{-0.507}$
$J/\psi + \chi_{b2}$	0.236	0.129	0.188	0.158	$^{+0.067}_{-0.045}$	3.866	2.195	3.015	2.733	$^{+1.009}_{-0.685}$
$J/\psi + h_b$	0.620	0.337	0.494	0.412	$^{+0.175}_{-0.120}$	2.672	1.458	2.036	1.860	$^{+0.734}_{-0.494}$
$\eta_c + h_b$	...	...	...	...	...	0.133	0.073	0.100	0.094	$^{+0.036}_{-0.024}$
$\eta_c + \chi_{b0}$	...	...	...	...	...	0.262	0.146	0.211	0.176	$^{+0.072}_{-0.049}$
$\eta_c + \chi_{b1}$	...	...	...	...	...	0.786	0.439	0.634	0.529	$^{+0.215}_{-0.148}$
$\eta_c + \chi_{b2}$	...	...	...	...	...	1.310	0.731	1.056	0.881	$^{+0.359}_{-0.247}$
$\eta_c + \eta_b$	...	...	...	...	...	0.715	0.392	0.572	0.477	$^{+0.233}_{-0.067}$
$\eta_c + \Upsilon$	0.391	0.320	0.389	0.319	$^{+0.054}_{-0.079}$	2.321	1.388	1.945	1.608	$^{+0.584}_{-0.807}$
$\chi_{c0} + \Upsilon$	0.009	0.005	0.008	0.006	$^{+0.002}_{-0.002}$	1.672	1.006	1.394	1.184	$^{+0.405}_{-0.294}$
$\chi_{c1} + \Upsilon$	0.063	0.038	0.053	0.044	$^{+0.016}_{-0.011}$	5.054	3.040	4.212	3.579	$^{+1.225}_{-0.887}$
$\chi_{c2} + \Upsilon$	0.023	0.014	0.020	0.016	$^{+0.006}_{-0.004}$	8.341	5.018	6.950	5.908	$^{+2.021}_{-1.464}$
$h_c + \Upsilon$	0.179	0.107	0.152	0.124	$^{+0.045}_{-0.033}$	0.226	0.146	0.200	0.162	$^{+0.050}_{-0.038}$
$h_c + \eta_b$	...	...	...	...	...	0.009	0.007	0.009	0.007	$^{+0.001}_{-0.001}$
$\chi_{c0} + \eta_b$	...	...	...	...	...	0.986	0.587	0.820	0.692	$^{+0.243}_{-0.176}$
$\chi_{c1} + \eta_b$	...	...	...	...	...	2.957	1.760	2.460	2.076	$^{+0.730}_{-0.528}$
$\chi_{c2} + \eta_b$	...	...	...	...	...	4.928	2.934	4.101	3.460	$^{+1.216}_{-0.880}$

TABLE X. The events of charmonia + bottomonia production at $\sqrt{s} = 91.1876$ GeV for the CEPC and FCC-ee. In each cell, the values outside/inside the brackets are for leading order and next-to-leading order results in the v^2 expansions, respectively.

CEPC (16 ab $^{-1}$)							
	CS	CS + CO	(CS + CO) $_{\max}$		CS	CS + CO	(CS + CO) $_{\max}$
$J/\psi + \Upsilon$	584(227)	616(242)	846(333)	$\eta_c + \eta_b$	0(0)	2(1)	269(156)
$J/\psi + \eta_b$	30(15)	51(24)	84(45)	$\eta_c + \Upsilon$	1(1)	5(3)	356(177)
$J/\psi + \chi_{b0}$	0(0)	2(1)	3(2)	$\chi_{c0} + \Upsilon$	0(0)	4(2)	5(2)
$J/\psi + \chi_{b1}$	2(1)	6(3)	12(6)	$\chi_{c1} + \Upsilon$	0(0)	13(6)	15(7)
$J/\psi + \chi_{b2}$	1(0)	8(5)	17(10)	$\chi_{c2} + \Upsilon$	0(0)	21(10)	24(12)
$J/\psi + h_b$	2(1)	6(3)	12(6)	$h_c + \Upsilon$	0(0)	1(0)	1(0)
$\eta_c + h_b$	0(0)	0(0)	81(40)	$h_c + \eta_b$	0(0)	0(0)	1(0)

(Table continued)

TABLE X. (Continued)

CEPC (16 ab ⁻¹)							
	CS	CS + CO	(CS + CO) _{max}		CS	CS + CO	(CS + CO) _{max}
$\eta_c + \chi_{b0}$	0(0)	1(0)	20(12)	$\chi_{c0} + \eta_b$	0(0)	3(1)	4(2)
$\eta_c + \chi_{b1}$	0(0)	2(1)	62(36)	$\chi_{c1} + \eta_b$	0(0)	8(4)	11(7)
$\eta_c + \chi_{b2}$	0(0)	3(2)	103(60)	$\chi_{c2} + \eta_b$	0(0)	13(6)	18(11)
FCC-ee (150 ab ⁻¹)							
$J/\psi + \Upsilon$	5473(2124)	5772(2272)	7927(3123)	$\eta_c + \eta_b$	0(0)	17(8)	2518(1459)
$J/\psi + \eta_b$	281(138)	475(227)	788(423)	$\eta_c + \Upsilon$	9(5)	51(26)	3340(1664)
$J/\psi + \chi_{b0}$	2(1)	16(9)	34(19)	$\chi_{c0} + \Upsilon$	0(0)	39(19)	45(22)
$J/\psi + \chi_{b1}$	15(7)	57(31)	113(63)	$\chi_{c1} + \Upsilon$	1(1)	117(58)	136(68)
$J/\psi + \chi_{b2}$	5(3)	75(43)	160(93)	$\chi_{c2} + \Upsilon$	0(0)	194(96)	224(112)
$J/\psi + h_b$	15(7)	61(29)	117(56)	$h_c + \Upsilon$	4(2)	5(3)	7(4)
$\eta_c + h_b$	0(0)	3(1)	753(371)	$h_c + \eta_b$	0(0)	0(0)	5(3)
$\eta_c + \chi_{b0}$	0(0)	6(3)	194(113)	$\chi_{c0} + \eta_b$	0(0)	25(11)	35(20)
$\eta_c + \chi_{b1}$	0(0)	17(9)	582(338)	$\chi_{c1} + \eta_b$	0(0)	74(34)	105(62)
$\eta_c + \chi_{b2}$	0(0)	28(14)	970(563)	$\chi_{c2} + \eta_b$	0(0)	124(56)	175(102)

TABLE XI. The reconstructed events of charmonia + bottomonia production from their decay products at $\sqrt{s} = 91.1876$ GeV for the CEPC and FCC-ee. In each cell, the values outside/inside the brackets are for leading-order and next-to-leading-order results in the v^2 expansions, respectively. The efficiency is selected as 100%.

$H_1 + H_2$	H_1		H_2		CEPC (16 ab ⁻¹)			FCC (150 ab ⁻¹)		
	Decay	Br (%)	Decay	Br (%)	CS	CS + CO	(CS + CO) _{max}	CS	CS + CO	(CS + CO) _{max}
$J/\psi + \Upsilon$	$l^+ l^- (l = e, \mu)$	12	$l^+ l^- (l = e, \mu)$	4.86	3(1)	4(1)	5(2)	32(12)	34(13)	46(18)
$\eta_c + \chi_{b1}$	Hadron decay	43	$\gamma l^+ l^- (\gamma \Upsilon)$	1.71	0(0)	0(0)	0(0)	0(0)	0(0)	4(2)
$\eta_c + \chi_{b2}$	Hadron decay	43	$\gamma l^+ l^- (\gamma \Upsilon)$	0.87	0(0)	0(0)	0(0)	0(0)	0(0)	4(2)
$\eta_c + \Upsilon$	Hadron decay	43	$l^+ l^- (l = e, \mu)$	4.86	0(0)	0(0)	7(4)	0(0)	1(1)	70(35)

V. CONCLUSIONS

In this paper, we study the associated charmonium-bottomonium production in e^+e^- annihilation via γ^*/Z^0 propagators at Z -factories. Our calculations cover extensive double S -wave and $S - P$ mixed processes, including both CS and CO channels up to NLO(v^2). Among these processes, the $J/\psi + \Upsilon$ channel is predominantly governed by the CSM through double photon t -channel diagrams. For all remaining channels, CO contributions originating from gluon fragmentation into the $^3S_1^{[8]}$ state are considerable and even dominant, both at the Z^0 pole and around the production threshold. In contrast, double charmonium and double bottomonium production is dominated by the CSM near the threshold due to nonfragmentation effects. However, production rate estimates are highly sensitive to

LDME choices, such that substantial discrepancies in LDMEs from different studies yield drastically different event yields. On the other hand, future high-precision experimental measurements at Z factories can test the COM and constrain the magnitude of CO LDMEs and may provide crucial data to resolve the long-standing issue of LDME universality. We also include relativistic corrections, which are found to be significant and reduce the LO cross sections by roughly 50%. Besides the uncertainties stemming from LDMEs, we discuss other relevant uncertainty sources in detail. Finally, we estimate the expected event yields for the CEPC and FCC-ee. Based on final-state particle reconstruction, the event yields for $J/\psi + \Upsilon$ and $\Upsilon + \eta_c$ are (2, 4) for the CEPC (2-year operation) and (18, 35) for the FCC-ee (4-year operation) under Z -factory mode.

Given that the one-loop QCD corrections within CSM are negligible [72], it is crucial to study QCD corrections within COM. This motivation is reinforced by a recent calculation of the next-to-next-to-leading-order (NNLO) QCD corrections for the processes $e^+e^- \rightarrow 2J/\psi$ and 2Υ [100]. The results show that the LO cross section can grow by roughly 1 order of magnitude. If a similar NNLO enhancement occurs in associated charmonium-bottomonium production, combined with the strong LDME sensitivity of production rates, the number of reconstructed events at future e^+e^- colliders can reach $\mathcal{O}(10^2)$.

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DATA AVAILABILITY

Wolfram Language code with analytical expressions of all short-distance coefficients is publicly available in a permanent repository [101].

APPENDIX

Like previous work [74], we do not present the analytical expressions of the SDCs in this appendix for simplicity. Instead, we provide the ratios of SDCs from relativistic corrections (denoted as G) to LO SDCs (denoted as F) in the high c.m. energy limit ($m_Q^2 \ll s$) in Table XII.

We find these ratios to be consistent with Ref. [74], and they could also be verified by Refs. [81,83,84]. The slightly different R ratios correspond to the $^3S_1^{[1/8]}$ + $^3S_1^{[1/8]}$ channel, which depend on the heavy quark masses. Moreover, when we take $m_c = m_b$, these ratios become -1 , which is consistent with the results for double charmonium/bottomonium production [74].

TABLE XII. Ratios of relativistic correction SDCs (denoted as G) to LO SDCs (denoted as F) in the high c.m. energy limit ($m_Q^2 \ll s$) for $e^+e^- \rightarrow (c\bar{c})[^{2S_1+1}L_{1J_1}^{c_1}] + (b\bar{b})[^{2S_2+1}L_{2J_2}^{c_2}]$. In each cell, we define, $R_i \equiv G[n_i]/F[n_1 + n_2]$, $c = 1, 8$ for CS or CO states. The first four rows are for CS channels, the second three rows are for CO channels, and the last row is for the t -channel processes.

$^3S_1^{[1]} + ^1S_0^{[1]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{5}{6}$	$^3S_1^{[1]} + ^3S_1^{[1]}$	$R_1 = -\frac{11m_b^2+m_c^2}{6(m_b^2+m_c^2)}$	$R_2 = -\frac{m_b^2+11m_c^2}{6(m_b^2+m_c^2)}$	$^3S_1^{[1]} + ^1P_1^{[1]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{13}{10}$
$^3S_1^{[1]} + ^3P_0^{[1]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{13}{10}$	$^3S_1^{[1]} + ^3P_1^{[1]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{11}{10}$	$^3S_1^{[1]} + ^3P_2^{[1]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{7}{10}$
$^1S_0^{[1]} + ^3S_1^{[1]}$	$R_1 = -\frac{5}{6}$	$R_2 = -\frac{11}{6}$	$^1P_1^{[1]} + ^3S_1^{[1]}$	$R_1 = -\frac{13}{10}$	$R_2 = -\frac{11}{6}$	$^3P_0^{[1]} + ^3S_1^{[1]}$	$R_1 = -\frac{13}{10}$	$R_2 = -\frac{11}{6}$
$^3P_1^{[1]} + ^3S_1^{[1]}$	$R_1 = -\frac{11}{10}$	$R_2 = -\frac{11}{6}$	$^3P_2^{[1]} + ^3S_1^{[1]}$	$R_1 = -\frac{7}{10}$	$R_2 = -\frac{11}{6}$			
$^1S_0^{[8]} + ^3S_1^{[8]}$	$R_1 = -\frac{5}{6}$	$R_2 = -\frac{11}{6}$	$^3P_J^{[8]} + ^3S_1^{[8]}$	$R_1 = -\frac{31}{30}$	$R_2 = -\frac{11}{6}$	$^3S_1^{[8]} + ^1S_0^{[8]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{5}{6}$
$^3S_1^{[8]} + ^3P_J^{[8]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{31}{30}$	$^3S_1^{[8]} + ^3S_1^{[8]}$	$R_1 = -\frac{11m_b^2+m_c^2}{6(m_b^2+m_c^2)}$	$R_2 = -\frac{m_b^2+11m_c^2}{6(m_b^2+m_c^2)}$	$^3S_1^{[8]} + ^1P_1^{[8]}$	$R_1 = -\frac{11}{6}$	$R_2 = -\frac{13}{10}$
$^1P_1^{[8]} + ^3S_1^{[8]}$	$R_1 = -\frac{13}{10}$	$R_2 = -\frac{11}{6}$						
$^3S_1^{[1]} + ^3S_1^{[1]}$	$R_1 = R_2 = -\frac{11}{6}$							

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