

Discrete p -form symmetry and higher Coulomb phases in various theories

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We argue that a field theory with a $\mathbb{Z}_N$ p -form symmetry generically admits, in addition to a Higgs phase and a “confining” phase, a Coulomb phase in which the infrared theory contains Abelian p -form electrodynamics, similar to the behavior of the Yang-Mills theory coupled to adjoint or fundamental matter. We illustrate our claim with continuum and lattice examples including 2-group higher gauge theory, effective p -form BF theories, and lattice p -form electrodynamics.

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I. INTRODUCTION

A. Summary

Global p -form symmetries generalize familiar (i.e., zero-form) global symmetries and entail similarly important physical consequences; a prominent example is given by the role one-form symmetries have played in our understanding of confinement. Here, we show that for any theory with a $\mathbb{Z}_N$ p -form symmetry there are monopole $(d - 3 - p)$ -branes that carry magnetic p -form charge under the symmetry, which source N units of $(d - 2 - p)$ -brane center vortices. Consequently, all such theories are argued to have Coulomb, Higgs, and confining phases for $d \geq p + 3$. We give lattice and continuum higher gauge theory examples exhibiting this behavior. This generalizes the conclusions of [1] for $\mathbb{Z}_N$ one-form symmetries and includes, to the best of our knowledge, an original understanding how adjoint matter couples to a higher gauge theory. In the process, we explain how higher gauge theory entails ∞ -group generalized global symmetries, giving an explicit 2-group example, in direct analogy to the one-form center symmetry of Yang-Mills theory. This complements previous constructions of ∞ -group global symmetries, which typically proceed in the other direction starting from

the higher global symmetries themselves, as realized by extended topological operators, and then coupling to higher form (background) gauge potentials [2].

B. Background

Although global p -form symmetries (also referred to as “generalized global symmetries” or “higher form symmetries”) have previously appeared in various guises in various contexts, for example in [3–15], they were only systematically characterized in the seminal work of [2]. Subsequently, such higher form symmetries have become an increasingly central aspect of gauge theory, as reviewed in [16–20], and found to be closely related to topological phases, e.g., [1, 15, 21–23].

A well-known example of a one-form symmetry by another name is the $\mathbb{Z}_N$ “center symmetry” of Polyakov and ’t Hooft [3–5, 24]. Specifically, an $SU(N)$ gauge theory with adjoint matter admits a $\mathbb{Z}_N$ one-form symmetry and has a Higgs phase, a Coulomb phase, and a confining phase; in particular, the $\mathbb{Z}_N$ one-form symmetry plays a crucial role in confinement. See reviews [25, 26] and the references therein. Given this insight, it is natural to ask if the relationship between the one-form symmetry and the phases of Yang-Mills theory generalizes.

Indeed, it was recently shown [1] that such phases occur generically in any $d \geq 4$ -dimensional quantum field theory with a $\mathbb{Z}_N$ one-form symmetry. In contrast to the dual-superconductor picture mediated by magnetic monopole condensation, in this scenario confinement is driven by center vortices [which are $(d - 3)$ -branes] sourced by magnetic monopoles [which are $(d - 4)$ -branes] carrying N units of charge. Depending on which of these two kinds

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of defects proliferate, the theory exhibits Higgs, Coulomb, or confining behavior (cf. also [27,28]). In particular, in the Coulomb phase, one generically obtains a propagating $U(1)$ Maxwell field in the low-energy effective theory. This prediction has been confirmed numerically in lattice models [29]. Assuming a $\mathbb{Z}_N$ symmetry (which remains unbroken in the Coulomb phase), one can microscopically picture the situation as follows: there exist particles (vortices) carrying unit $\mathbb{Z}$ -valued charge in a sea of light particles (monopoles) that instead carry N units of charge. Because of the presence of this sea, the $\mathbb{Z}$ -valued conserved charge is broken to a $\mathbb{Z}_N$ -valued conserved charge. In this case, the $\mathbb{Z}$ -valued vortices then couple (magnetically, say) to a Maxwell field that provides the conservation law for the charge.

C. Results

In this paper, we discuss the higher-dimensional analog of the above scenario, where the one-form symmetry is generalized to a p -form symmetry. That is, in any theory with a $\mathbb{Z}_N$ p -form symmetry, we have monopole $(d-3-p)$ -branes that carry magnetic charge under the p -form symmetry, and these source N units of center vortices, the $(d-2-p)$ -branes. The behavior of the system then depends on which of these two defects proliferate (i.e., become light). Since each monopole is attached to N center vortices, it is impossible for a monopole to be light while center vortices are heavy. Thus, we have three possibilities:

- (i) When both monopoles and center vortices are light and hence proliferate, the system is in a confining phase: the effective potential between two $(d-3)$ -dimensional end points of the monopoles increases linearly with respect to the distance, analogously to the area law of Wilson loops, and thus provide a higher-dimensional analog of QCD strings between quark-antiquark pairs.
- (ii) When both monopoles and center vortices are heavy, the system is in a Higgs phase. The system is then described by an effective BF model in the infrared whose symmetry matches the $\mathbb{Z}_N$ p -form symmetry of the ultraviolet theory.
- (iii) When the center vortices are light but the monopoles are heavy, the system develops a light $U(1)$ $(d-2-p)$ -form electromagnetic field; this is a higher analog of the Coulomb phase of gauge theories. The mechanism in which the Coulomb phase develops an Abelian field is an instance of electromagnetic duality: the $\mathbb{Z}_N$ -valued p -form gauge field dualizes to a $U(1)$ $(d-2-p)$ -form gauge field.

This paper is organized as follows: In Sec. II, we consider the phases of an adjusted higher gauge theory coupled to matter fields of various representations. In Sec. III, we start from a BF model effective description of the Higgs phase and argue that natural deformations

away from the Higgs phase yield Coulomb and unbroken phases. Finally, in Sec. IV, we consider the emergence of the higher Coulomb phase in a Villain model of p -form electrodynamics on the lattice.

II. PHASES OF ADJUSTED HIGHER GAUGE THEORY

Yang-Mills theory coupled to matter provides a paradigmatic example of the relation between one-form symmetry (the center symmetry) and phases. In particular, when adjoint matter coupled to the gauge boson acquires a vacuum expectation value, the theory exhibits a Coulomb phase, where the low-energy spectrum contains Maxwell theory (corresponding to the Cartan subgroup that is the stabilizer subgroup of the adjoint vacuum expectation value). On the other hand, when matter sitting in a generic representation acquires a vacuum expectation value, the theory often exhibits a Higgs phase where all gauge symmetry has broken down. Finally, in the unbroken phase the strongly coupled theory often shows confinement.

Here we consider the analogous mechanism for higher gauge theory. Higher gauge theory is a higher-form generalization (categorification) of Yang-Mills theory, where the gauge symmetry is described by a Lie ∞ -group. See [30,31] for reviews. Essentially, a higher gauge theory consists of one-form, two-form, three-form, ... local gauge potentials, $A, B, C, \dots$ valued in a gauge homotopy Lie algebra (also known as an L_∞ -algebra). The corresponding two-form, three-form, four-form, ... field strengths and gauge transformations are determined by the categorified symmetry data.

For definiteness (and tractability), we take the Lie ∞ -group to be given by a strict Lie 2-group $\mathcal{G}$. In this case, the higher gauge theory consists only of a one-form gauge potential A and a two-form gauge potential B . Note that the strict 2-group $\mathcal{G}$ is the (generically non-Abelian) higher gauge (as opposed to global) symmetry of the theory. The theory is mostly fixed by the choice of $\mathcal{G}$, up to an important subtlety that implies for a *dynamical and non-Abelian* higher gauge theory the data of the gauge ∞ -group alone is insufficient. This follows from the observation that for a generic non-Abelian gauge ∞ -group one must impose the so-called fake flatness condition [32–35] in which one of the field strengths is constrained to zero for consistency of the gauge algebra. For topological or Abelian higher field theories, this is of no consequence. However, the fake-flatness condition consigns aspiring non-Abelian theories to be topological or essentially Abelian [36,37]. This has led to the erroneous assumption that there are no dynamical and genuinely non-Abelian higher gauge theories. However, a subset of Lie ∞ -groups admits a structure of *adjustment* [37–46] (see [31] for a review), which lifts the fake-flatness conditions and so allows for non-Abelian dynamical higher gauge theories.

Thus, in order for our model to exhibit dynamics, we consider an adjusted higher gauge theory.

A. Review of adjusted higher gauge theory

Let us consider an adjusted higher gauge theory associated with a strict 2-group $\mathcal{G}$ modeled by a crossed module $(H \rightarrow G)$, for H and G Lie groups. This class of (adjusted) higher gauge theories is treated in detail in [31,47], with explicit examples and applications. The corresponding local higher gauge potentials consist of a $\mathfrak{h}$ -valued two-form gauge potential B and a $\mathfrak{g}$ -valued one-form potential A ,

$$B \in \Omega^2(M, \mathfrak{h}), \quad A \in \Omega^1(M, \mathfrak{g}), \quad (1)$$

where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H , respectively.

a. The gauge Lie 2-group A strict Lie 2-group is given by the data of a group homomorphism $s: H \rightarrow G$ and an action $\rho: G \times H \rightarrow H$ satisfying

$$s(\rho(g)h_1) = hs(h_1)h^{-1} \quad \text{and} \quad \rho(s(h_1))h_2 = h_1h_2h_1^{-1}$$

for all $h_1, h_2 \in H$ and $g \in G$.

An *adjusted* strict 2-group comes equipped with an adjustment

$$\kappa: G \times \mathfrak{g} \rightarrow \mathfrak{h}. \quad (2)$$

The adjustment of a strict Lie 2-group must satisfy

$$\begin{aligned} \kappa(s(h), x) &= h(\rho(x)h^{-1}), \\ \kappa(g_2g_1, x) &= \rho(g_2)\kappa(g_1, x) + \kappa(g_2, g_1xg_1^{-1} - s(\kappa(g_1, x))), \end{aligned} \quad (3)$$

for all $g_1, g_2 \in G$, $h \in H$, and $x \in \mathfrak{g}$. Note that we have used the same symbol $\rho: G \times H \rightarrow H$ to denote the linearization of the action $\rho: G \times H \rightarrow H$. We trust the context will alleviate any ambiguity. For examples of adjusted Lie 2-groups, one can take $\mathcal{G}$ to be the loop model of the string Lie 2-algebra [40,41,48], where both G and H are non-Abelian.

b. The adjusted gauge Lie 2-algebra The strict 2-group $\mathcal{G}$ linearizes to an adjusted strict Lie 2-algebra [49] (or, equivalently, a strict two-term L_∞ -algebra) with graded vector space

$$\mathfrak{G} = \mathfrak{h}[1] \oplus \mathfrak{g}[0]. \quad (4)$$

The nontrivial graded structure maps [50] on $\mathfrak{G}$ are

$$s: \mathfrak{h}[1] \rightarrow \mathfrak{g}[0], \quad [-, -]: \mathfrak{g}[0] \otimes \mathfrak{g}[0] \rightarrow \mathfrak{g}[0], \quad (5)$$

$$\rho: \mathfrak{g}[0] \otimes \mathfrak{h}[1] \rightarrow \mathfrak{h}[1], \quad \kappa: \mathfrak{g}[0] \otimes \mathfrak{g}[0] \rightarrow \mathfrak{h}[1], \quad (6)$$

carrying degrees 1, 0, 0, and -1 , respectively. The map κ defines the adjustment, while s , ρ and $[-, -]$ are the structure maps of a strict Lie 2-algebra satisfying

$$s(\rho(x)(b)) = [x, s(b)], \quad \rho(s(a))(b) = [a, b],$$

for any $x \in \mathfrak{g}$ and $a, b \in \mathfrak{h}$. By defining the mixed Lie bracket $[-, -]: \mathfrak{g} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ via the action $[x, a] := \rho(x)(a)$ a strict Lie 2-algebra is equivalent to a differential graded (dg) Lie algebra concentrated in degrees -1 and 0 .

Specifically, a dg Lie algebra concentrated in degrees -1 and 0 has maps

$$\begin{aligned} s: \mathfrak{h}[1] &\rightarrow \mathfrak{g}[0], \\ [-, -]: \mathfrak{g}[0] \otimes \mathfrak{h}[1] &\rightarrow \mathfrak{h}[1], \\ [-, -]: \mathfrak{g}[0] \otimes \mathfrak{g}[0] &\rightarrow \mathfrak{g}[0], \end{aligned} \quad (7)$$

where $[-, -]$ is a graded-antisymmetric Lie bracket obeying the graded Jacobi identity, i.e.,

$$[z_1, z_2] = -(-1)^{|z_1||z_2|}[z_2, z_1], \quad (8)$$

$$[z_1, [z_2, z_3]] = [[z_1, z_2], z_3] + (-1)^{|z_1||z_2|}[z_2, [z_1, z_3]], \quad (9)$$

and s is a differential graded nilquadratic derivation, i.e.,

$$s^2 = 0, \quad s[z_1, z_2] = [sz_1, z_2] + (-1)^{|z_1|}[z_1, sz_2]. \quad (10)$$

In the above identities, z_i are elements of homogeneous degree in $\mathfrak{h}[1] \oplus \mathfrak{g}[0]$, and $|z|$ is the degree of $z \in \mathfrak{h}[1] \oplus \mathfrak{g}[0]$. We shall use the dg Lie algebra formulation in what follows.

c. Adjusted higher gauge theory The higher gauge theory with higher gauge algebra $\mathfrak{G}$ has (locally) a $\mathfrak{h}$ -valued two-form gauge potential $B \in \Omega^2(M, \mathfrak{h})$ and a $\mathfrak{g}$ -valued one-form potential $A \in \Omega^1(M, \mathfrak{g})$, with higher gauge transformations given by

$$\delta B = [B, \alpha] + d\Lambda + [A, \Lambda] + \kappa(\alpha, F), \quad (11)$$

$$\delta A = d\alpha + [A, \alpha] - s(\Lambda), \quad (12)$$

where $\alpha \in \Omega^0(M, \mathfrak{g})$, $\Lambda \in \Omega^1(M, \mathfrak{h})$, and it is understood that the action of s has been extended by linearity to $\Omega^1(M) \otimes \mathfrak{h}$. In (11), H and F are the corresponding adjusted field strengths

$$H := dB + [A, B] - \kappa(A, F), \quad (13)$$

$$F := dA + \frac{1}{2}[A, A] + s(B). \quad (14)$$

Here, $\kappa(F, A)$ is the adjustment of H , where for the $\mathfrak{g}$ -valued forms A and B the wedge product in $\kappa(A, B)$ is

understood. The role of this term is to allow for $F \neq 0$ with a consistent gauge-algebra (specifically, in the absence of an adjustment, $F \neq 0$ implies the gauge-algebra does not close for $\mathfrak{g}$ non-Abelian).

Under (11), the field strengths transform as

$$\delta F = [F, \alpha] + s(\kappa(\alpha, F)), \quad (15)$$

$$\delta H = [H, \alpha] + \kappa(\alpha, s(H)). \quad (16)$$

Note that both the three-form H and two-form F field strengths are generically non-Abelian. If κ is antisymmetric, however, then $\delta H = 0$ since $\kappa(s(a), x) = \rho(x)a = [x, a]$ for all $a \in \mathfrak{h}$ and $x \in \mathfrak{g}$. The Bianchi identities are then

$$dF + [A, F] + s(H + \kappa(A, F)) = 0, \quad (17)$$

$$dH + [A, H] - \kappa(A, s(H)) + \kappa(F, F) = 0. \quad (18)$$

Note that the three-form field strength H is not closed generically and hence is not the Noether current of a magnetic $(d-4)$ -form symmetry unless $\kappa = 0$, as in the case of Abelian two-form electrodynamics [51].

Finally, assuming invariant inner products $\langle -, - \rangle_{\mathfrak{h}}$ and $\langle -, - \rangle_{\mathfrak{g}}$ on $\mathfrak{h}$ and $\mathfrak{g}$, respectively, we may straightforwardly write down the higher gauge theory action

$$S_0[A, B] := \int \frac{1}{2} \langle H, \star H \rangle_{\mathfrak{h}} + \frac{1}{2} \langle F, \star F \rangle_{\mathfrak{g}}, \quad (19)$$

where the wedge product between the forms A and B in $\langle A, B \rangle$ is understood.

B. Higher form symmetries of adjusted higher gauge theory

Here we consider the global higher p -form symmetries of the adjusted higher form gauge theories discussed above. In particular, we shall determine the adjusted higher gauge theory analog of the well-known Yang-Mills one-form center symmetry.

The adjusted higher gauge theory p -form center symmetry is most readily identified in analogy to the classic derivation of the Yang-Mills center symmetry, which we briefly recall here to illustrate the parallel. Consider a Euclidean spacetime $M = \Sigma \times \mathbb{S}^1$ with periodic time $t \sim t + \beta$, where β may be interpreted as an inverse temperature. In the classical case of Euclidean Yang-Mills theory with gauge group G , if we for simplicity work with the trivial G -bundle on M , then the potential A is periodic, i.e., $A(\vec{x}, t + \beta) = A(\vec{x}, t)$. The potential transforms under a gauge transformation $g: M \rightarrow G$ as

$$A \mapsto g^{-1}Ag + g^{-1}dg. \quad (20)$$

Since g is periodic (being defined on $M = \Sigma \times \mathbb{S}^1$), A remains periodic after gauge transformation. One then considers nonperiodic gauge-like transformations formally given by (20) but now parametrized by a not-necessarily-periodic $g: \Sigma \times \mathbb{R} \rightarrow G$ satisfying the G -twisted periodicity condition

$$g(\vec{x}, t + \beta) = \gamma g(\vec{x}, t), \quad (21)$$

$$g(\vec{x}, t + \beta)^{-1}dg(\vec{x}, t + \beta) = g(\vec{x}, t)^{-1}dg(\vec{x}, t), \quad (22)$$

for some $\gamma \in G$. For this to be a symmetry of the theory, the periodicity of the gauge potential, $A(\vec{x}, t + \beta) = A(\vec{x}, t)$, must be preserved under (20); that is, we require

$$\begin{aligned} g^{-1}(t + \beta)A(t + \beta)g(t + \beta) + g(t + \beta)^{-1}dg(t + \beta) \\ = g^{-1}(t)A(t)g(t) + g(t)^{-1}dg(t), \end{aligned} \quad (23)$$

where we have suppressed the dependence on $\vec{x}$ for clarity. Since A and $g^{-1}dg$ are periodic, this reduces to

$$\gamma^{-1}g^{-1}(t)A(t)g(t)\gamma = g^{-1}(t)A(t)g(t). \quad (24)$$

Since the above must hold for arbitrary $g^{-1}(t)A(t)g(t) \in G$, the group element γ must belong to the center $Z(G)$ of the gauge group [e.g., $Z(SU(N)) = \mathbb{Z}_N$ when $G = SU(N)$]. Note that, while taking the form of a gauge transformation and leaving the Yang-Mills action invariant, Eq. (21) has a nontrivial action on certain gauge-invariant observables such as Polyakov loops and so has physical implications [true gauge transformations must be periodic, not twisted-periodic as in (21)]. This implies that pure Yang-Mills theory has a $Z(G)$ -valued one-form symmetry; in particular, that the center symmetry is a one-form symmetry is apparent from the fact that Wilson lines in the fundamental representation transform under γ , with $Z(G)$ -valued charge (see, e.g., [18]). Furthermore, the symmetry continues to exist if there are additional adjoint-valued vector fields but generically breaks if there exists matter transforming under other (e.g., fundamental) representations.

A similar analysis holds for adjusted higher gauge theory. To illustrate this analogy we need the *finite* adjusted higher gauge transformations, which are given by

$$\begin{aligned} A &\xmapsto{(g, \Lambda)} g^{-1}Ag + g^{-1}dg - s(\Lambda), \\ B &\xmapsto{(g, \Lambda)} \rho(g^{-1})B + d\Lambda + [g^{-1}Ag + g^{-1}dg - s(\Lambda), \Lambda] \\ &\quad + \frac{1}{2}[\Lambda, \Lambda] - \kappa(g^{-1}, F), \end{aligned} \quad (25)$$

where the gauge parameters are $g: M \rightarrow G$ and $\Lambda \in \Omega^1(M, \mathfrak{h})$ (see [31] [(3.35)] and references therein for further details). Similar to Yang-Mills theory, identifying the (higher form) center symmetry amounts to determining

global higher gauge transformations (g, Λ) in (25) that leave the potentials A and B periodic (assuming, again for simplicity, a trivial $\mathcal{G}$ -bundle on spacetime $M = \Sigma \times \mathbb{S}^1$):

$$\begin{aligned} g(\vec{x}, t + \beta) &= \gamma g(\vec{x}, t), \\ g(\vec{x}, t + \beta)^{-1} dg(\vec{x}, t + \beta) &= g(\vec{x}, t)^{-1} dg(\vec{x}, t), \\ \Lambda(\vec{x}, t + \beta) &= \Lambda(\vec{x}, t) + \lambda, \\ d\Lambda(\vec{x}, t + \beta) &= \Lambda(\vec{x}, t), \end{aligned} \quad (26)$$

for $\gamma \in G$ and $\lambda \in \mathfrak{h} \otimes T_{(\vec{x}, t)}^* M$. From (25), we see that gauge transformations (g, Λ) leave the potentials A and B periodic if $\lambda \in (\ker s \cap Z(\mathfrak{h})) \otimes T_{(\vec{x}, t)}^* M$ and $\gamma \in Z(G)$ and

$$\kappa(\gamma^{-1}, x) = 0 \quad \forall x \in \mathfrak{g}, \quad \rho(\gamma) = id_{\mathfrak{h}}, \quad (27)$$

where, again overloading the notation, $\rho: G \times \mathfrak{h} \rightarrow \mathfrak{h}$ is the induced action of G on $\mathfrak{h}$. Thus, the 2-group center symmetries are given by the 2-group described by the crossed module

$$\begin{aligned} Z(\mathcal{G}) &= ((\ker s) \cap Z(\mathfrak{h})) \\ &\rightarrow \{\gamma \in Z(G) | \kappa(\gamma^{-1}, \mathfrak{g}) = 0, \rho(\gamma) = id_{\mathfrak{h}}\} \end{aligned} \quad (28)$$

with trivial structure maps. Note that this 2-group is Abelian in the strongest possible sense (i.e., all structure maps are trivial) in agreement with the expectation that higher form symmetries must be Abelian due to the Eckmann-Hilton argument. For lack of a better name, we call this the *adjusted center symmetry 2-group* of $\mathcal{G}$ [52]. Note, although there is the familiar $Z(G)$ factor, $Z(\mathcal{G})$ has more structure than the typical instances of discrete p -form symmetries and arises from a non-Abelian higher gauge theory directly.

Now, just as the center symmetry group of Yang-Mills theory is, in fact, a one-form symmetry valued in a (1-)group, the center symmetry 2-group of our higher gauge theory is a pair of one-form and two-form symmetries that combine into a 2-group.

The symmetry topological defects generating the two-form center symmetry and the corresponding charged branes are a direct generalization of the one-form center symmetry for Yang-Mills theory. In the latter case, symmetry topological defects are co-dimension-2 Gukov-Witten operators that can be regarded equivalently as (20) and (21) for $\gamma \in Z(G)$ in the conventional center symmetry language or as a Wilson line operator of a $\mathbb{Z}_N$ -valued background one-connection $A_{\mathbb{Z}_N}$ (see, e.g., [17]). The charged objects are the conventional Wilson line operators.

Analogously, for the 2-group higher gauge theory symmetry, the topological defects are the co-dimension-3 Gukov-Witten-type operators that can be regarded equivalently as (25) and (26) for $(\lambda, \gamma) \in Z(\mathcal{G})$ in the conventional center symmetry language or a Wilson surface operator of

$\mathbb{Z}_N$ -valued background two-connection, locally $(A_{\mathbb{Z}_N}, B_{\mathbb{Z}_N})$. The charged objects are the Wilson surface operators taking values in the 2-group $\mathcal{G}$, given by the path-ordered exponentials of B integrated over a surface and A integrated over the associated boundary in a coherent fashion. More precisely, Wilson surface operators are two-holonomies of the two-connection (i.e., locally B and A), which are equivalent to the two-functors from the two-path groupoid on M to the 2-group $\mathcal{G}$, as described in [34] but where one has to account for the failure of fake flatness using the adjustment, as described in [41].

In principle [53], this entire discussion readily generalizes to arbitrary Lie p -group higher gauge theories, with corresponding p -group generalized global symmetry. Here, we simply assume that the highest component is a $\mathbb{Z}_N$ p -form symmetry, which we take to be our microscopic global symmetry in order to generalize the results of [1].

C. Coupling to adjoint matter: Higher gauge Coulomb phases

Let us couple the gauge potentials A and B to matter transforming in the adjoint representation of $\mathcal{G}$, formally given by the associated two-bundle. Thus, we have the fields

$$\phi \in \Omega^1(M; \mathfrak{h}), \quad \chi \in \Omega^0(M; \mathfrak{g}), \quad (29)$$

with associated covariant derivatives modeled after the field strengths:

$$D\chi = d\chi + [A, \chi] + s(\phi), \quad (30)$$

$$D\phi = d\phi + [A, \phi] + [\chi, B] + \kappa(F, \phi) + \kappa(D\chi, A). \quad (31)$$

Similarly, the higher gauge transformations are modeled after the potentials,

$$\delta\phi = [\alpha, \phi] + [\chi, \Lambda] + \kappa(\alpha, D\chi), \quad (32)$$

$$\delta\chi = [\chi, \alpha], \quad (33)$$

so that the covariant derivatives transform in the same manner as the field strengths

$$\delta D\phi = [D\phi, \alpha] + \kappa(\alpha, s(D\phi)), \quad (34)$$

$$\delta D\chi = [D\chi, \alpha] + s(\kappa(\alpha, D\chi)). \quad (35)$$

Thus, we may immediately deduce that the invariant minimally coupled action is given by

$$\begin{aligned} S[A, B, \phi, \chi] &= S_0[A, B] + \int \frac{1}{2} \langle D\phi, \star D\phi \rangle_{\mathfrak{g}} \\ &\quad + \frac{1}{2} \langle D\chi, \star D\chi \rangle_{\mathfrak{h}}. \end{aligned} \quad (36)$$

Now, assume that the scalar field χ obtains a vacuum expectation value χ_0 of order m , via spontaneous symmetry breaking upon including a potential term. In that case, we obtain mass terms for A and B of the form

$$\int \frac{1}{2} \langle [A, \chi_0], \star [A, \chi_0] \rangle + \frac{1}{2} \langle [B, \chi_0], \star [B, \chi_0] \rangle,$$

so that only the subalgebra left invariant under the action by χ_0 will remain massless; assuming semisimplicity and generic values of χ_0 , this means that the surviving subalgebras of $\mathfrak{g}$ and $\mathfrak{h}$ will be Cartan subalgebras, so the massless remnants of A and B will be purely Abelian. In this case, the global higher form (center) symmetry survives intact and will also be visible from the low-energy effective theory. We now turn to an analysis of the other possible phases taking a bottom-up approach, starting with the expected low energy effective BF model.

III. EFFECTIVE DEFORMED p -FORM BF MODELS

In the previous section we considered purely adjoint matter and argued that, generically, this leads to a higher analog of the Coulomb phase. Now suppose, instead, that we couple the system to matter fields in some arbitrary fashion (that is, with matter fields not in the adjoint representation). In this case the matter condenses and we, generically, break the entirety of the gauge symmetry $\mathcal{G}$. This process generally yields a Higgs phase described by an effective BF model, for example as witnessed in the case of Higgsed Maxwell theory [54] (reviewed in, e.g., [17,19]).

Motivated by this general picture, in this section we take a bottom-up approach supposing that microscopically we have $\mathbb{Z}_N$ p -form symmetry which is broken, resulting in an effective p -form BF model. However, as described below (see also [1]), the p -form BF model has more than the required $\mathbb{Z}_N$ p -form symmetry. Consequently, the extra symmetry must be broken by certain class-preserving deformations. For ordinary and higher effective BF models alike, partially breaking the global p -form symmetries for $d > 3$, as in [1], requires a deformation given by a minimal coupling of scalar fields to the higher p -potentials. If $p > 1$, then obviously the minimal coupling given by the naive covariant derivative is not possible. There are various possible solutions, in particular minimally coupling to higher matter fields as in Sec. II C. Here instead, we shall use the notion of Chen forms [34,55–58] (also called *iterated integrals*; for an informal introduction, see [[41], Appendix H]). to minimally couple p -forms to ordinary (zero-form) scalar matter. For simplicity, throughout the following we assume $d - p - 2 > 0$, time reversal symmetry (this forbids, e.g., Chern-Simons topological mass terms), and that the homology classes of spacetime M are finitely generated and torsion-free.

A. Chen forms

In [1], the magnetic $(d - 2)$ -form symmetry exhibited by the effective one-form BF model is explicitly broken by minimally coupling a scalar via $D\phi = (d + B)\phi$. This is required by the assumption that the microscopic theory has only a $\mathbb{Z}_N$ one-form symmetry. However, this is only straightforwardly possible in $d = 3$, where B is a one-form. To generalize this construction to p and d for which $d - p - 1 \neq 1$, we employ the Chen form construction.

Given a smooth manifold M , the (unbased) loop space $\mathcal{LM}$ is the space of smooth functions $\mathbb{S}^1 \rightarrow M$; it is an infinite-dimensional manifold. Let A be a $(p + 1)$ -form on M . Then the associated *Chen form* is a p -form $\check{A}$ on $\mathcal{LM}$ defined as

$$\check{A}|_\gamma = \int_{\mathbb{S}^1} dt \frac{\partial \gamma}{\partial t} \lrcorner ev_t^* A, \quad (37)$$

where ev_t^* is the pullback of a differential form along the evaluation map

$$\begin{aligned} ev_t: \mathcal{LM} &\rightarrow M \\ \gamma &\mapsto \gamma(t), \end{aligned} \quad (38)$$

and $\lrcorner$ is the interior product with the vector field $\partial \gamma / \partial t$ defined on $\mathcal{LM}$. Iterating this construction, one obtains from a $(p + k)$ -form $A \in \Omega^{p+k}(M)$ a p -form $\check{A} \in \Omega^p(\mathcal{L}^k(M))$.

The loop space and Chen forms are central to string field theory. A particle living on spacetime M is given by a worldline $\mathbb{R} \rightarrow M$, and second quantization yields a scalar field living on M . A string living on a spacetime M is given by a world sheet $\mathbb{R} \times \mathbb{S}^1 \rightarrow M$, which is equivalent to a map $\mathbb{R} \rightarrow \mathcal{LM}$, that is, a worldline in the loop space of spacetime; a second quantization yields a scalar field living on $\mathcal{LM}$, which is in turn equivalent to an infinite tower of fields living on M by Fourier-series expansion. More generally, a toroidal k -brane living on a spacetime M is given by a field living on $\mathcal{L}^k M$.

B. Phases of the deformed p -form BF model

We shall assume that in the deep infrared, on a spacetime manifold M , the low-energy effective action of a theory with a microscopic $\mathbb{Z}_N$ p -form symmetry should be given by a BF model [17,19]

$$S_0 = \int_M \frac{iN}{2\pi} B \wedge dA, \quad (39)$$

where A is a p -form and B is a $(d - p - 1)$ -form and the gauge transformations are given by

$$A \mapsto A + dc, \quad B \mapsto B + d\Lambda, \quad (40)$$

for $c \in \Omega^{p-1}(M)$ and $\Lambda \in \Omega^{d-p-2}(M)$.

Let us briefly summarize the discrete p -form symmetries. See [17,19] for detailed treatments of d -dimensional (continuous and discrete) p -form BF theory and its higher global symmetries. This model has a $\mathbb{Z}_N$ -valued p -form symmetry as well as a $\mathbb{Z}_N$ -valued $(d-p-1)$ -form symmetry since A and B are on an equal footing; that is, the global internal symmetry group is the Abelian ∞ -group

$$(\mathbb{Z}_N)[-p] \oplus (\mathbb{Z}_N)[1+p-d]. \quad (41)$$

The electric p -form symmetry $\mathbb{Z}_N[-p]$ is generated by the “current”

$$\star J_{\text{el}} = \frac{NB}{2\pi}, \quad (42)$$

and the usual Wilson operator for B is the symmetry defect operator, which acts on the Wilson operators for A (and vice versa).

In generalizing [1], our basic assumption is that the microscopic symmetry is only the $\mathbb{Z}_N$ -valued p -form symmetry. Thus, the p -form BF action has too much symmetry and, again following [1], we explicitly break the $(d-p-1)$ -form symmetry by deforming the action. Specifically, we break the extra symmetry by coupling B to a $(d-p-2)$ -brane with unit charge. This brane can be represented by a complex scalar field ϕ living on the iterated loop space $\mathcal{L}^{d-p-2}M$, so that we obtain

$$S_1 = \int_{\mathcal{L}^{d-p-2}M} (d + i\check{B})\phi \wedge \star((d - i\check{B})\phi^*), \quad (43)$$

where $\check{B}$ is a one-form living on $\mathcal{L}^{d-p-2}M$ obtained by the Chen form construction from B .

Having coupled a source to B , its equation of motion no longer enforces $dA = 0$, and dA becomes the Noether current for a $U(1)$ -valued $(d-p-2)$ -form symmetry. We may break this symmetry as well by introducing a kinetic term for A and coupling to magnetic currents:

$$S_2 = \int \frac{1}{2g_A^2} dA \wedge \star dA + \tilde{A} \wedge J, \quad (44)$$

where $d\tilde{A} = \star dA$ and J is a $(p+2)$ -form operator built out of matter fields.

Given the action $S = S_0 + S_1 + S_2$, let us consider the various phases:

- (i) If we additionally have mass terms or strongly coupled interactions that render the fields ϕ and J heavy at a mass scale M , then in the deep infrared $\ll M$, S_1 and S_2 become irrelevant, and we end up with the action S_0 describing the Higgs phase, where the symmetry is enhanced from $\mathbb{Z}_N[-1]$ to (41).
- (ii) Suppose that we additionally have mass terms for J so that at low energy scales J can be integrated out, but that a component of the string field ϕ condenses and

acquires vacuum expectation values. In that case, B becomes massive and may be integrated out, and similarly for J . The low-energy action then becomes

$$S = \int \frac{1}{2g_A^2} dA \wedge \star dA + \dots,$$

with no monopole operators for A . This is a Coulomb phase with a low energy $U(1)$ gauge field A .

- (iii) If both ϕ and J acquire vacuum expectation values, then we obtain a $U(1)$ gauge theory perturbed by monopole operators:

$$S = \int \frac{1}{2g_A^2} dA \wedge \star dA + \tilde{A} \wedge J + \dots$$

The term $\tilde{A} \wedge J$ may be relevant [3,4] and lead to unbroken $\mathbb{Z}_N$ one-form symmetry.

Note that, in the case $p+2=d$, then one can generically take $J \propto \phi^N + \text{c.c.}$, as noted in [1]. In such cases, if one cannot independently set ϕ and J , one might not have a Coulomb phase.

Here we deformed the BF theory by introducing a $(d-p-2)$ -brane with unit charge. It would be interesting to understand if this can be related to cocycle twist deformations appearing in various aspects of topological quantum field theories such as the Dijkgraaf-Witten theory (themselves often related to BF theories), and generalized symmetries, e.g., [59–62].

IV. COULOMB PHASES IN LATTICE p -FORM ELECTRODYNAMICS

We illustrate the claim with a lattice model, namely a modified Villain model, where the mechanism of the Coulomb phase is particularly transparent. This is a straightforward higher analog of the modified Villain model studied in [1]; see also [63] for an earlier discussion of higher form $U(1)$ gauge theory in the Villain form.

A. Cellular cochains as lattice analogs of differential forms

In a lattice model, the lattice structure equips spacetime with the structure of a cellular complex (or CW-complex), so that it is natural to use the language of cellular cohomology, such that cellular cocycles may be thought of as discrete analogs of differential forms. (For a pedagogical description of cellular homology, see [64].) Let us consider a tiling (or tessellation) [65] Λ of $\mathbb{R}^d$ —for instance, by hypercubes, or the triangular or hexagonal tiling of $\mathbb{R}^2$, or the tiling of $\mathbb{R}^3$ by truncated octahedra or more complicated polyhedra. In technical terms, this defines a cell complex structure on $\mathbb{R}^d$. We use the notation $X_k(\Lambda)$ for the k -cells of this cell complex. Thus, $X_0(\Lambda)$ is the set of vertices;

$X_1(\Lambda)$ is the set of links (edges); $X_2(\Lambda)$ is the set of plaquettes. For instance, when Λ is the hypercube tiling, we have

$$X_0(\Lambda) = \{\{v\} | v \in \mathbb{Z}^d\}, \quad (45)$$

$$X_1(\Lambda) = \{\{v + su | s \in [0, 1]\} | v \in \mathbb{Z}^d, u \in \{\mathbf{e}_1, \dots, \mathbf{e}_n\}\}, \quad (46)$$

$$X_2(\Lambda) = \{\{v + s_1 u_1 + s_2 u_2 | s_1, s_2 \in [0, 1]\} | v \in \mathbb{Z}^d, u_1 \in \{\mathbf{e}_1, \dots, \mathbf{e}_n\} \ni u_2\}, \quad (47)$$

where $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ are the canonical basis vectors of $\mathbb{R}^n$. Tilings admit a duality, generalizing that of planar graphs, which may be thought of as the “infinite-volume limit” of the duality of convex polytopes. If $\hat{\Lambda}$ is the tiling dual to Λ , then we have the canonical identifications

$$X_k(\Lambda) = X_{d-k}(\hat{\Lambda}). \quad (48)$$

For instance, the triangular tiling of $\mathbb{R}^2$ is dual to the hexagonal tiling, and the hypercube tiling of $\mathbb{R}^d$ is dual to itself. On such a tiling, we use the notation $\Omega^k(\Lambda; G)$ (where G is an Abelian group under addition) for the set of functions

$$X_k(\Lambda) \rightarrow G. \quad (49)$$

When $G = \mathbb{R}$, this should be thought of as a discrete analog of the space of differential k -forms on $\mathbb{R}^d$; hence, we denote the function evaluation as $\alpha|_x$ instead of $\alpha(x)$. On the other hand, in the discrete context now it makes sense to talk of $\mathbb{Z}$ -valued k -forms, for instance.

Let $C_k(\Lambda)$ be the free Abelian group of $\mathbb{Z}$ -linear combinations of elements of $X_k(\Lambda)$. (These are the chains in the cellular homology associated with the cell complex Λ .) Thus, $\Omega^k(\Lambda; G)$ may be equivalently thought of as the set of Abelian group homomorphisms (i.e., $\mathbb{Z}$ -linear maps)

$$C_k(\Lambda) \rightarrow G. \quad (50)$$

Mathematically, $\Omega^k(\Lambda; G)$ is the set of G -valued cellular cochains. On $C_k(\Lambda)$, there exists a standard notion of boundary, where we keep track of the orientation,

$$\partial: C_k(\Lambda) \rightarrow C_{k-1}(\Lambda), \quad (51)$$

which squares to zero, i.e., $\partial^2 = 0$; this defines cellular homology [64].

While we do not define the wedge product of the “differential forms” on Λ , we may define their exterior derivative as the Wilson hypersurfaces around each cell. That is, suppose that the boundary of a cell x is $\partial x = l_1 + l_2 + \dots + l_m$. For $\alpha \in \Omega^k(\Lambda; G)$, then $d\alpha \in \Omega^{k+1}(\Lambda; G)$

is defined as

$$d\alpha|_x = \alpha|_{l_1} + \alpha|_{l_2} + \dots + \alpha|_{l_m} \quad (52)$$

for $x \in C_{k+1}(\Lambda)$. This defines a map

$$d: \Omega^p(\Lambda; G) \rightarrow \Omega^{p+1}(\Lambda; G). \quad (53)$$

This operator squares to zero; the corresponding cohomology is cellular cohomology with coefficients in G . Finally, there also exists an analog of the Hodge dual operator

$$\star: \Omega^k(\Lambda; G) \rightarrow \Omega^{d-k}(\hat{\Lambda}; G) \quad (54)$$

defined by means of the isomorphism (48).

B. Villain model of p -form electrodynamics

In a Villain model (first introduced in [66]), one resolves the group $U(1) \cong \mathbb{R}/\mathbb{Z}$ by considering an $\mathbb{R}$ -valued gauge field and then subsequently gauging a $\mathbb{Z}$ -valued two-form symmetry. Analogously, one may realize the gauge group $\mathbb{Z}_N \cong N^{-1}\mathbb{Z}/\mathbb{Z}$ by starting with a $N^{-1}\mathbb{Z}$ -valued gauge field and then gauging an $\mathbb{Z}$ -valued two-form symmetry; similarly, a $\mathbb{Z}_N$ -valued p -form symmetry may be obtained by starting with an $N^{-1}\mathbb{Z}$ -valued p -form symmetry whose $\mathbb{Z}$ -valued subgroup is then gauged away.

Suppose that we are given a tiling Λ of $\mathbb{R}^d$ with associated dual tiling $\hat{\Lambda}$. A Villain model of a p -form electrodynamics with a discrete gauge group $\mathbb{Z}_N \cong N^{-1}\mathbb{Z}/\mathbb{Z}$ is given by the following action:

$$\begin{aligned} S(a, m; \beta) &= \frac{1}{2}\beta \sum_{x \in X_{p+1}(\Lambda)} (da + m)^2 \\ &= \frac{1}{2}\beta \sum_{x \in X_{p+1}(\Lambda)} ((da)|_x + m|_x)^2 \end{aligned} \quad (55)$$

for dynamical fields

$$a \in \Omega^p(\Lambda; N^{-1}\mathbb{Z}), \quad m \in \Omega^{p+1}(\Lambda; \mathbb{Z}), \quad (56)$$

and where $\beta \in \mathbb{R}$ is a coupling constant playing the role of inverse temperature. As explained in [1], this model now has two different kinds of defects: in addition to monopoles for a [which are $(d-p-2)$ -branes], one also has center vortices (analogs of the QCD center vortices described in [5,67,68]), which are $(d-p-1)$ -branes. Each monopole is attached to N center vortices.

Given the bestiary of defects, there are three possible phases [69]:

- (i) Higgs phase, when both monopoles and center vortices are heavy ($\beta \rightarrow \infty$).
- (ii) Confined phase, when both monopoles and center vortices are light ($\beta \rightarrow 0$).

- (iii) Coulomb phase, when monopoles are heavy but center vortices are light.

The case where monopoles are light but center vortices are heavy is not possible since each monopole is attached to N center vortices. Now, we claim that, in the Coulomb phase as defined above, there is a light $U(1)$ q -form field for $q = d - p - 2$, as the name suggests.

The argument for the existence of the Coulomb phase is as follows. To see it, we must first suppress the monopoles. We may force this by introducing a Lagrange multiplier $\tilde{a} \in \Omega^{d-p-2}(\hat{\Lambda}; \mathbb{R})$ to kill the monopole operator dm ; the resulting model is the modified Villain action [70,71]

$$S_{\text{mV}}(a, \tilde{a}, m; \beta) := \frac{1}{2}\beta \sum_{X_{p+1}(\Lambda)} (da + m)^2 - i \sum_{X_{p+1}(\Lambda)} m(\star d\tilde{a}). \quad (57)$$

Now, a Poisson resummation yields

$$\begin{aligned} & \sum_{m \in \Omega^{p+1}(\Lambda)} \exp(-S_{\text{mV}}(a, \tilde{a}, m; \beta)) \\ &= \sum_{\tilde{m} \in \Omega^{p+1}(\Lambda)} \exp(-\tilde{S}_{\text{mV}}(a, \tilde{a}, \tilde{m}; \tilde{\beta})), \end{aligned} \quad (58)$$

where $\tilde{\beta} = \beta^{-1}$ and the dual action is defined as

$$\begin{aligned} \tilde{S}_{\text{mV}}(a, \tilde{a}, \tilde{m}; \tilde{\beta}) &:= \frac{1}{2}\tilde{\beta} \sum_{X_{d-p-1}(\hat{\Lambda})} (d\tilde{a} + \star \tilde{m})^2 \\ &+ i \sum_{X_{p+1}(\Lambda)} \tilde{m}(\star da). \end{aligned} \quad (59)$$

Consequently, the dual model describes a $U(1)$ -valued $(d - p - 2)$ -form gauge field in Villain form, but with the Lagrange multiplier a constraining $d\tilde{m}$. But a is quantized in units of $N^{-1}\mathbb{Z}$, which means that $d\tilde{m}$ only vanishes modulo N . So, we get a $U(1)$ gauge theory coupled to magnetic monopoles of charge N . Now, when $\beta \rightarrow 0$ (high temperature limit) in the original S_{mV} , so that center vortices proliferate but monopoles are still suppressed, then we see that we get a light $U(1)$ gauge field, so this is a Coulomb phase.

In the above, we have simply completely suppressed the magnetic monopoles using a Lagrange multiplier. If instead one simply gives the magnetic monopoles a very large but finite mass $M \gg 1$ using a term like

$$\frac{1}{2}M^2 \sum_{X_{p+2}(\Lambda)} (dm)^2, \quad (60)$$

one can expect the qualitative picture to be the same, following symmetry-based arguments as in [1].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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- [49] Here $V[k]$ denotes the degree-shift of a graded vector space $V = \bigoplus_i V^i$ by $k \in \mathbb{Z}$, where $V[k]^i := V^{i+k}$. Note that $\mathfrak{g}$ and $\mathfrak{h}$ are regarded as graded vector spaces concentrated in degree zero, i.e. $\mathfrak{g} = \mathfrak{g}^0$, so $\mathfrak{h}[1] \oplus \mathfrak{g}[0]$ has non-trivial degrees -1 and 0 .
- [50] Note that, by abuse of notation, we overload the symbols s , ρ and κ for both the finite (crossed module of Lie groups) and infinitesimal (crossed module of Lie algebras) cases; the intended meaning should be clear from the context.

- [51] If, however, κ is antisymmetric, then $\delta H = 0$ and $dH = 0$, even if $\mathfrak{h}$ is non-Abelian; in that case, H is the Noether current for a $(d - 4)$ -form symmetry.
- [52] In fact, since γ is valued in the group $Z(G) \subset G$ whereas λ is valued in $Z(\mathfrak{h}) \subset \mathfrak{h}$, this is better regarded as the centre of a differential graded Lie group with body G and with nonzero-degree coordinates corresponding to $\mathfrak{h}$ obtained as a partial linearization of the 2-group $\mathcal{G}$.
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