Particle decay as asymptotic narrow parametric resonance

Shao-Ping Li*

Department of Physics, The University of Osaka, Toyonaka, Osaka 560-0043, Japan

(Received 7 May 2025; accepted 16 June 2025; published 7 July 2025)

Parametric resonance can produce particles from an oscillating scalar field, where an exponential growth of the particle number density can be developed. While it has been noticed that stimulated decay in Boltzmann equations exhibits similar parametric dependence of the exponential growth, it is not yet quantitatively clear under what circumstances the two phenomena can reconcile. We demonstrate that trilinear particle interaction in the Boltzmann equation can provide a good approximation to describe the distribution function and the number density in the asymptotic regime of narrow parametric resonance. We find that the crucial treatment leading to the quantitative agreement is a proper Gaussian simulation of the momentum spread inherited from nonrelativistic particle decay. With the simple particle picture, the analytic Boltzmann approximation can be applied to explosive photon production from axions/axionlike particles and to dark matter production from oscillating fields.

DOI: 10.1103/6849-rxjs

I. INTRODUCTION

It has been known for decades that particles can be produced abundantly in a background of oscillating scalar/pseudoscalar fields, previously introduced as a short period of preheating right after inflation but prior to reheating [1,2]. Over the years, this production mechanism has been widely considered as an efficient particle process for ultralight dark matter production [3–5], photon production from axion clump condensates [6–8], primordial black hole formation [9,10], gravitational wave generation [11,12], and axion production and fragmentation [13,14], among others.

Parametric resonance may be understood as an induced process [15], where particles produced at late times depend on those at early times. This pattern was qualitatively confirmed in the so-called narrow parametric resonance (NPR) [1,2,16,17], where the initial amplitude of the oscillating field is small. In the NPR mechanism, a small initial amplitude allows a perturbative treatment to evaluate the particle growth rate using either the Floquet theorem [18] or the Mathieu equation [19]. The growth rate can feature an exponential behavior in time, $\Gamma \propto e^{qmt}$, where m denotes the mass of the oscillating field and $q \ll 1$ corresponds to the typical NPR regime.

*Contact author: lisp@het.phys.sci.osaka-u.ac.jp

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

In recent years, there has been increasing attention paid to the equivalence between NPR and stimulated decay based on the Boltzmann equation. Typically, NPR is formulated by equations of motion derived in classical field theory, while the semiclassical Boltzmann equations feature particle collisions derived in the S-matrix formalism of quantum field theory (QFT). Since oscillating field condensates reflect a collection of numerous nonrelativistic particles, it was previously argued that these condensates can be formulated in QFT, leading to a decay rate equivalent to the S-matrix formalism [20], which was confirmed earlier by using a Schrödinger picture of QFT [17,21]. For bosonic particle production, Boltzmann equations exhibit an exponential growth due to final-state Bose enhancement. This was noticed in photon production from axion/axionlike fields using the uncertainty principle [22]; however, the quantitative dependence of the growth rate, by default, cannot be precisely determined.

Indeed, if one follows the Boltzmann equation of axion decay to photons $a \to 2\gamma$, the stimulated decay rate reads

$$\Gamma_{\rm stim} = \Gamma_0(1 + 2f_\gamma), \tag{1}$$

where f_γ is the final-state photon distribution function from axion production, with the spontaneous decay rate Γ_0. From the photon number Boltzmann equation

$$\frac{dn_\gamma}{dt} = 2\Gamma_{\rm stim} n_a, \tag{2}$$

one can derive an exponential solution in the limit $f_\gamma \gg 1$,

$$n_\gamma \propto e^{\tilde{\mu}t}, \tag{3}$$

as, Eq. (3), by substituting the relation $f_\gamma = n_\gamma/(2\pi^2 k^2 \delta k)$ into Eq. (1). The exponential growth rate $\tilde{\mu}$ depends crucially on the momentum spread δk that is determined by nonrelativistic axion decay. Following this approach, the growth rate was estimated in Ref. [23] to be $\tilde{\mu} = q m_a \pi/4$, which differs from the $\tilde{\mu} = q m_a/2$ growth rate predicted by NPR [2]. While the numerical difference in the exponent is small, the difference will become increasingly large after several periods of axion oscillations, namely, when $m_a t \gg 1$. It was also recently shown that, starting with a distribution function constructed in QFT, the correct exponential growth rate can be matched to NPR [24]. However, this QFT-constructed distribution function was later inserted into the Boltzmann equation to calculate the number density, leading to a conclusion that the Boltzmann equation can yield a number density that is orders of magnitude larger than NPR in the asymptotic limit $m_a t \gg 1$.

In this work, we demonstrate that the distribution function directly obtained from the Boltzmann equation can provide a good approximation to describe the exponential growth in the asymptotic NPR regime. The resulting number density can also approximately work to predict the explosively increasing number density before backreaction comes into play. It provides a simple analytic approach to determine the final particle number density that can have impacts on observables in the early Universe. In particular, it allows one to analyze late-time thermalization processes between the produced particles and the background plasma, where Boltzmann equations have been well established e.g., in big-bang nucleosynthesis (BBN) and the cosmic microwave background (CMB).

We begin in Sec. II to calculate the distribution function, or the occupation number, within the field theory by taking the trilinear interaction as a prototype scenario. Then we show in Sec. III how the evolution of the distribution function obtained from the Boltzmann equation can be matched to that predicted in the asymptotic NPR regime. We discuss in Sec. IV the application of the simple Boltzmann equation and its validity in particle production from oscillating fields, especially in regimes where cosmic expansion, backreaction, and scattering with a background thermal plasma can be neglected. Finally, we draw conclusions in Sec. V.

II. TRILINEAR PARAMETRIC RESONANCE

Let us consider the typical Yukawa or trilinear interaction

$$\mathcal{L} = -\frac{1}{2}\mu\phi\chi^2, \tag{4}$$

where ϕ denotes the background oscillating scalar field and χ denotes a bosonic particle. We parametrize the ϕ evolution after the oscillation begins at some initial time t_i with the field amplitude

$$\phi = \phi_i \left(\frac{a(t_i)}{a(t)}\right)^{3/2} \cos(m_\phi t) \equiv \bar{\phi}\cos(m_\phi t), \tag{5}$$

where $a(t)$ is the scale factor as a function of the cosmic time t and m_ϕ is the mass of ϕ. For definiteness, we focus on the regime where $m_\chi \ll m_\phi$ and hence neglect the mass of χ. We assume that right after the oscillation begins, the kinetic energy of ϕ is negligible such that the energy density of nonrelativistic ϕ gives

$$\rho_\phi \approx \frac{1}{2} m_\phi^2 \bar{\phi}^2, \tag{6}$$

and the number density scales like nonrelativistic matter in the radiation-dominated Universe,

$$n_\phi = \frac{1}{2} m_\phi \phi_i^2 \left(\frac{T}{T_i}\right)^3 \propto a(t)^{-3}, \tag{7}$$

with T the cosmic temperature.

In Fourier space, the mode function of χ, denoted as $u_k(t)$ with the wave number $k \equiv |\vec{k}|$ and a mass dimension of $-1/2$, obeys the Klein-Gordon equation

$$\frac{d^2 u_k}{dt^2} + \omega_k^2(t) u_k = 0, \tag{8}$$

with the time varying frequency $\omega_k(t) = \sqrt{k^2 - \mu\phi(t)}$. If we can neglect the Hubble expansion, backreaction to ϕ from accumulated χ, and any interaction between the produced χ with a background thermal plasma, the above equation can be converted into the Mathieu equation [19]

$$\frac{d^2 u_k}{dz^2} + (A_k - 2q\cos(2z)) u_k = 0, \tag{9}$$

where

$$A_k \equiv \frac{4k^2}{m_\phi^2}, \qquad q \equiv \frac{2\mu\bar{\phi}}{m_\phi^2}, \qquad z \equiv \frac{m_\phi t}{2}. \tag{10}$$

The parameter q determines whether the resonance is in the narrow ($q \ll 1$) or broad regime ($q \gg 1$) [2,25–28]. When $A_k - 2q < 0$, χ carries a negative effective mass. This regime can lead to tachyonic instability, though it generally occurs for a large q [28]. The Mathieu equation can be solved numerically with the following initial conditions

$$u_k(0) = \frac{1}{\sqrt{2k}}, \tag{11}$$

$$\left.\frac{du_k}{dt}\right|_{t=0} = \frac{m_\phi}{2}\left.\frac{du_k}{dz}\right|_{z=0} = -i\sqrt{\frac{k}{2}}, \tag{12}$$

where we have set $t_i = 0$ for simplicity. Under this setup, the above conditions correspond to a plane-wave state, where no χ particle is produced initially. For clarity, we will set $m_\phi = 1$ eV throughout, but the conclusions drawn below do not depend on this scale choice.

The occupation number is constructed by the mode function via [2]

$$n_k = k\left(\frac{|du_k/dt|^2}{k^2} + |u_k|^2\right) - 1 \tag{13}$$

$$= \frac{m_\phi^2}{4k}\left|\frac{du_k}{dz}\right|^2 + k|u_k|^2 - 1, \tag{14}$$

where we have multiplied the occupation number by a factor of 2 to account for the pair production of χ. This is more clear when χ is identified as photons, where two helicity states are produced [3]. The particle number and energy densities are obtained by integrating the occupation number in the entire momentum space,

$$n_\chi^{\rm M}(z) = \int \frac{d^3k}{(2\pi)^3} n_k(z), \tag{15}$$

$$\rho_\chi^{\rm M}(z) = \int \frac{d^3k}{(2\pi)^3} k n_k(z). \tag{16}$$

with the superscript M for results obtained from the Mathieu equation.

It is worthwhile to mention that the trilinear interaction can also be specified as photon production from ultralight dark matter, axions, or axionlike particles, where the generic Lagrangian reads

$$\mathcal{L} = -\frac{g_{\phi\gamma}}{4}\phi F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{17}$$

with $F_{\mu\nu}$ the photon field strength tensor and its dual $\tilde{F}_{\mu\nu}$. The resulting Mathieu equations read

$$\frac{d^2u_k^\pm}{dz^2} + (A_k \mp 2q\cos(2z))u_k^\pm = 0 \tag{18}$$

for two helicity states of photons $u_k^\pm$, with

$$A_k = \frac{4k^2}{m_\phi^2}, \qquad q \equiv \frac{2g_{\phi\gamma}\bar{\phi}k}{m_\phi}, \qquad z \equiv \frac{m_\phi t}{2} - \frac{\pi}{4}. \tag{19}$$

The number and energy densities in this case sum over the two helicity states $u_k^\pm$ [3].

Note that the χ-particle production can also be formulated in terms of the Bogoliubov approach, as widely applied to particle creation in curved spacetime [29]. Potential relations between the Bogoliubov and Boltzmann approaches were discussed in different regimes [30,31]. Here, we do not delve into the identification of potential equivalence between the two approaches but instead focus on how n_k and $n_\chi^{\rm M}$, numerically obtained from Eq. (9), can be well described by the Boltzmann equation. The calculation performed below can be straightforwardly applied to Eq. (17), though it will not be detailed in this work.

III. GAUSSIAN BOLTZMANN EQUATION

When neglecting Hubble expansion and scattering with background particles, we can write the Boltzmann equation for distribution function f_χ that is determined by a collection of ϕ particles depleting into χ particles, $j\phi \to 2\chi$,

$$\frac{\partial f_\chi}{\partial t} = \sum_{j=1}^{\infty}\frac{1}{k_j}\int dP_\phi dP'_\chi (2\pi)^4\delta^4(k)|\mathcal{M}_j|^2\mathcal{F}, \tag{20}$$

where we have multiplied the right-hand side by a factor of 2 to account for pair production of χ. In Eq. (20), the phase-space differential is defined by

$$dP_\phi \equiv \prod_{i=1}^{j}\frac{d^3k_i}{(2\pi)^2 2E_i}, \qquad dP'_\chi \equiv \frac{d^3k'_\chi}{(2\pi)^3 2E'_\chi}, \tag{21}$$

and the statistics function reads

$$\mathcal{F} \equiv \left(\prod_{i=1}^{j} f_i\right)(1+f_\chi)(1+f'_\chi) - f_\chi f'_\chi\prod_{i=1}^{j}(1+f_i) \tag{22}$$

$$\approx \left(\prod_{i=1}^{j} f_i\right)(1+2f_\chi), \tag{23}$$

where f_i is the distribution function of ϕ and the second approximation holds in the limit $f_i \gg 1$ and $f_\chi \approx f'_\chi$.

The general formula for the amplitude of $j\phi \to 2\chi$ can be obtained from Feynman diagrams. In the limit of $p_i = 0$, we arrive at

$$\mathcal{M}_j = j!\mu^j\prod_{i=1}^{j-1}\frac{1}{(i^2-ij)m_\phi^2} = \frac{j!(-1)^{j-1}}{[(j-1)!]^2}\frac{\mu^j}{m_\phi^{2(j-1)}}, \tag{24}$$

where the factor of $j!$ appears owing to the degeneracy of the Feynman diagrams, namely, the number of Feynman diagrams due to permutation of momenta [32]. The squared amplitude after taking into account a symmetry factor of $1/(2j!)$ yields

$$|\mathcal{M}_j|^2 = \frac{j!}{2[(j-1)!]^4}\frac{\mu^{2j}}{m_\phi^{4(j-1)}}. \tag{25}$$

We have checked this result by building a model file in FEYNARTS [33] and calculate the amplitude in FEYNCALC [34].

Integrating dP'_χ via $\delta^3(\sum \vec{k}_i - \vec{k}_\chi - \vec{k}'_\chi)$ in Eq. (20) gives rise to

$$\frac{\partial f_\chi}{\partial t} = \sum_{j=1}^{\infty}\frac{\pi|\mathcal{M}_j|^2}{k_j^2}\left(\frac{\bar{\phi}}{2}\right)^{2j}\delta(jm_\phi - 2k_j)(1+2f_\chi) \tag{26}$$

$$\approx \frac{\pi m_\phi^4}{2k_1^2}\left(\frac{q}{4}\right)^2\delta(m_\phi - 2k_1)(1+2f_\chi), \tag{27}$$

where we have used the approximation of nonrelativistic ϕ,

$$\int\prod_{i=1}^{j}\frac{d^3k_i}{(2\pi)^2 2E_i}f_i = \left(\frac{n_\phi}{2m_\phi}\right)^j = \left(\frac{\bar{\phi}}{2}\right)^{2j}, \tag{28}$$

with Eq. (7) used. The result given by Eq. (27) is obtained by taking only the decay channel $j=1$ with $k_1 = m_\phi/2$. This approximation results from the fact that the product of $|\mathcal{M}_j|^2$ and $\bar{\phi}^{2j}$ yields a factor of $(q/4)^{2j}$, which is dominated by the $j=1$ channel for $q \ll 1$. This coincides with the common lore in NPR that the resonance production dominantly occurs in the first bandwidth where $A_k = 1$.

To proceed with Eq. (27), we simulate the Dirac δ function with a Gaussian distribution

$$\delta(m_\phi - 2k_1) \approx \frac{a}{\sigma}e^{-\frac{(k_1 - m_\phi/2)^2}{b\sigma^2}}, \tag{29}$$

where the simulation coefficients a and b are determined by the normalization condition

$$\int_0^\infty \frac{a}{\sigma}e^{-\frac{(k_1-m_\phi/2)^2}{b\sigma^2}}dk_1 = \frac{1}{2}, \tag{30}$$

leading to

$$a\sqrt{b} = \frac{1}{2\sqrt{\pi}}. \tag{31}$$

The width σ describes the momentum spread inherited from nonrelativistic ϕ decay, which can be perturbatively calculated from the effective frequency with $k^2 \gg \mu\phi(t)$. At leading order, it reads

$$k_1\left(1 - \frac{\mu\bar{\phi}}{2k_1^2}\right) \le \omega_k \le k_1\left(1 + \frac{\mu\bar{\phi}}{2k_1^2}\right). \tag{32}$$

Therefore, the momentum has a spread

$$\sigma \equiv \delta k_1 = \frac{\mu\bar{\phi}}{k_1} = qm_\phi, \tag{33}$$

which is the same as that derived from Eq. (9) by using, e.g., the *harmonic balancing* [19]. Substituting Eq. (29) into Eq. (27) with the following simulation coefficients,

$$a = \frac{2}{\pi}, \qquad b = \frac{\pi}{16}, \tag{34}$$

we arrive at

$$f_\chi(z) = \frac{1}{2}e^{\mathcal{G}qz} - \frac{1}{2}, \tag{35}$$

which has the same growth rate $qm_\phi/2$ as in the NPR regime, except for the Gaussian modification

$$\mathcal{G} \equiv \exp\left(-\frac{16(k_1 - m_\phi/2)^2}{\pi q^2 m_\phi^2}\right) \tag{36}$$

having the peak at $k_1 = m_\phi/2$.

The resulting number density of χ in the Boltzmann approach is then obtained by integrating Eq. (35) over momentum space,

$$n_\chi^{\mathrm{B}}(z) = \int\frac{d^3k}{(2\pi)^2}f_\chi(z) \tag{37}$$

$$\approx \frac{1}{2\pi^2}\int_{k_1-n\sigma}^{k_1+n\sigma}dkk^2 f_\chi(z), \tag{38}$$

where the second approximation works for momentum expanding a few σ, i.e., $n = \mathcal{O}(1)$, owing to the Gaussian peaked distribution function.

We show in Fig. 1 the comparisons between the distribution functions obtained from the Mathieu and Boltzmann equations, respectively. In the top panel, the peak structures of n_k and f_χ are shown for $q = 0.001$ and $q = 0.1$. In general, n_k and f_χ are not identical, but there is a large overlap between n_k and f_χ near the peak $k_1 = m_\phi/2$. This points out that the two distribution functions can reach approximate consistency between particle production from oscillating ϕ in the NPR regime and that from nonrelativistic ϕ decay, basically independent of q as long as $q \ll 1$ holds.

The middle panel of Fig. 1 shows the evolution of distribution functions at the peak $k_1 = m_\phi/2$. We see that f_χ will reach the asymptotic behavior of n_k after the occupation number becomes larger than 1. Such consistency confirms the qualitative interpretation of particle production in the NPR regime. Furthermore, the exponential growth of bosonic particles from oscillating fields can

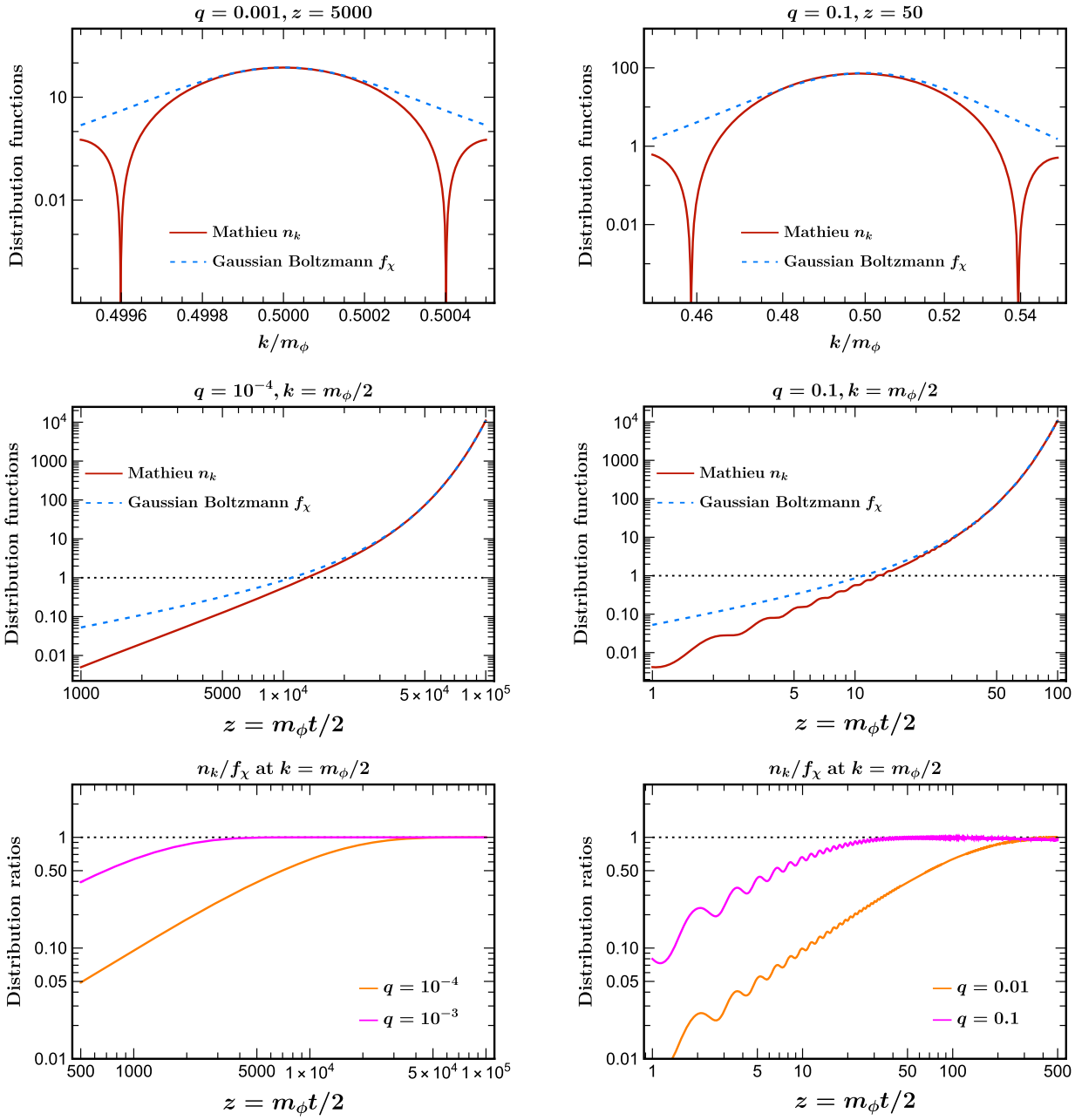


FIG. 1. Top panels: peak structures of the distribution function n_k obtained from the Mathieu equation, Eq. (13), and f_χ from the Boltzmann equation, Eq. (35). Middle panels: time evolution of distribution functions at the resonance peak $k = m_\phi/2$. Bottom panels: time evolution of the ratio n_k/f_χ at $k = m_\phi/2$.

be quantitatively described by stimulated decay of nonrelativistic particles.

In the bottom panel of Fig. 1, we show the evolution of the ratio n_k/f_χ at the peak $k_1 = m_\phi/2$ for q ranging from 10^{-4} to 0.1, confirming that the two functions coincide in the asymptotic limit $z \gg 1$. For larger q, the timescale for reaching the consistency is shorter since a larger q will have a larger production rate to create χ particles, consequently allowing for faster development of the exponential growth.

We have shown the consistency between the distribution functions obtained from the Mathieu and Boltzmann equations only at the peak. As can be inferred from the top panel of Fig. 1, there are still regions where the two approaches are not identical. However, these regions correspond to the distribution of produced χ particles away from the peak and hence exhibit suppressed occupation numbers. When integrating over the full momentum space, we may interpret these nonidentical regions as the theoretical errors in predicting particle production from the Boltzmann equation.

In Fig. 2, we show the discrepancy via the evolution of the ratio $n_\chi^{\mathrm{M}}/n_\chi^{\mathrm{B}}$, where n_χ^{M} denotes the χ particle number

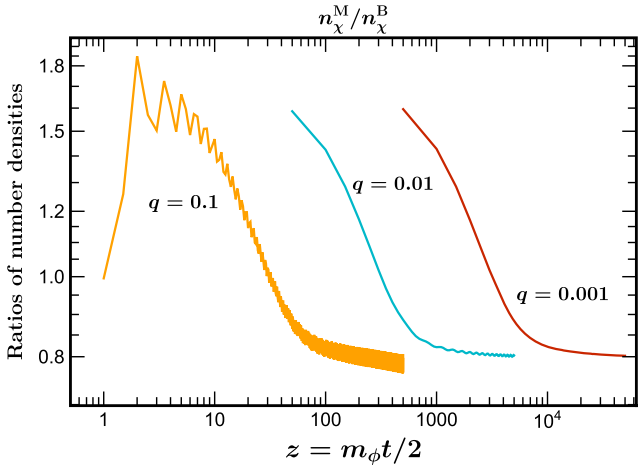


FIG. 2. The time evolution of the ratios $n_\chi^{\rm M}/n_\chi^{\rm B}$, where $n_\chi^{\rm M}$ is given by Eq. (15) and $n_\chi^{\rm B}$ is from Eq. (37).

density obtained from the Mathieu equation given in Eq. (15) and $n_\chi^{\rm B}$ is obtained in the Boltzmann equation based on Eq. (37). We see that after the ratio n_k/f_χ reaches 1, the ratio $n_\chi^{\rm M}/n_\chi^{\rm B}$ will reach a constant at 0.8, basically independent of q. It implies that there is a 20% relative deviation of the particle number density between the Mathieu and Boltzmann approaches. Nevertheless, such a deviation, which is also the typical scale between using the Boltzmann and Bose-Einstein distributions, is usually durable when determining physical impacts on observables.

IV. VALIDITY AND APPLICATION

The discussions thus far have not taken into account effects from cosmic expansion, backreaction, and interactions between the produced χ particles and a potential thermal plasma. In fact, it has been widely recognized that these effects can drastically change the exponential growth of the particle distribution function and the number density [2,15,22,23,35,36]. In particular, the NPR regime is quite sensitive to momentum redshift and any scattering process that would drive the generated χ momentum away from the bandwidth $\sigma = qm_\phi$, such that successful accumulation of produced particles cannot be prepared to trigger the late-time exponential growth after several periods of oscillations. Once this situation occurs, we would not expect NPR to generate abundant particles, and the associated observational consequences would hardly arise. This case renders applications of the NPR theory limited and tests of the NPR theory challenging. In addition, it becomes less meaningful to discuss the consistency between the NPR and Boltzmann approaches.

Therefore, applying the Boltzmann equation to study the exponential growth of particles in an oscillating field background is beneficial for scenarios where cosmic expansion, backreaction, and scattering do not prevent the explosive particle production. In the following, we discuss these potential scenarios.

Cosmic expansion leads to momentum redshift after particles are produced. The timescale for redshifting the momentum spread out of the bandwidth can be estimated via the conserved comoving momentum $k_i a(t_i) = k_f a(t_f)$, with the Hubble parameter $H = a^{-1}da/dt$. Let us focus on the radiation-dominated Universe, where $a(t) \propto \sqrt{t}$ and $H = 1/(2t)$. Then the timescale of momentum redshift from the one boundary to the other boundary [see Eq. (32)] yields

$$\Delta t = t_f - t_i = \frac{\sigma}{H_{\rm osc} k}, \tag{39}$$

where we have denoted the Hubble scale as $H_{\rm osc}$ at the initial time t_i or during the scalar oscillation epoch. If this timescale is much shorter than that of the exponential development $\tau_{\rm NPR} \equiv (qm_\phi/2)^{-1}$ [see Eq. (35)], then we should expect no abundant particle production from NPR. Therefore, for working NPR, we have the condition

$$q^2 \gg \frac{H_{\rm osc}}{m_\phi}. \tag{40}$$

Clearly, for $q \ll 1$, a large-scale hierarchy $H_{\rm osc} \ll m_\phi$ should be met once the oscillation of ϕ begins.

The oscillation time for ϕ is determined by the equation of motion

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + \frac{dV_\phi}{d\phi} = 0. \tag{41}$$

For scalar potential V_ϕ with suppressed higher-order ϕ terms, we can approximate $V_\phi \approx m_\phi^2\phi^2/2$ such that $dV_\phi/d\phi = m_\phi^2\phi$. This situation occurs in standard QCD axion theories [37,38] and in ultralight dark matter production via the conventional misalignment mechanism [39–41], where the oscillation time is determined by

$$m_\phi = 3H(T_{\rm osc}), \tag{42}$$

with $T_{\rm osc}$ denoting the cosmic temperature when the oscillation begins. It immediately suggests that it is not easy for these scenarios to feature explosive particle production in the NPR regime, since a sufficiently large q is required to overcome condition (40) right after the oscillation begins.

In several realistic applications, cosmic expansion may not be the dominant effect in changing the momentum of the produced particles. In particular, when the produced particles are photons, the thermalization timescale in the radiation-dominated Universe is much shorter than the Hubble expansion. The scattering effect may be estimated using a frequency-dependent medium-induced mass, or the

thermal mass, which also prevents the development of the exponential growth [22,39]. Note that inserting the thermal mass into the Mathieu equation is not well justified [42–44], since the timescale for developing such a thermal mass is not instantaneous, depending on the interaction rate with the thermal plasma. Nevertheless, for scattering effects that drive the created momentum away from the bandwidth at a speed faster than Hubble redshift, condition (40) would be strengthened,

$$q^2 \gg \xi \frac{H_{\rm osc}}{m_\phi}, \tag{43}$$

with $\xi \gg 1$ depending on the relative size of the Hubble expansion rate and the scattering rate. Again in this case, a large-scale hierarchy $H_{\rm osc} \ll m_\phi$ is required in the NPR regime when the oscillation of ϕ begins.

The mentioned scale hierarchy can be achieved by scenarios of delayed oscillations. In particular, when ϕ has a large initial velocity, the second term in Eq. (41) cannot be neglected even in the regime $H \ll m_\phi$, such that the oscillation time is not determined by Eq. (42). This situation occurs in axion production from the kinetic [45,46] or trapped [47,48] misalignment mechanism, and dark matter production from axion level crossing [49,50].

Observational consequences are likely to arise if the oscillation starts after MeV temperatures, where NPR may leave imprints on sensitive observables inherited from, e.g., neutrino decoupling, BBN, CMB spectral distortions, and CMB anisotropies. In this regime, we have

$$\frac{H}{m_\phi} \lesssim 4.4 \times \left(\frac{10^{-16}\ \text{eV}}{m_\phi}\right), \tag{44}$$

implying that an eV-scale mass of ϕ can readily satisfy conditions (40) and (43) with $q \ll 1$ and hence feature explosive particle production. It is worthwhile to emphasize that for the aforementioned observables, the Boltzmann equation has been well established to make precise predictions. Therefore, applying the Boltzmann approach to the NPR mechanism merits easier evaluation for the impacts of explosive particle production on observables.

Once conditions (40) and (43) are satisfied, it is easier to prepare the exponential growth under several periods of oscillations. Nevertheless, when the generated energy density becomes comparable with the background field, the backreaction to the ϕ field cannot be neglected, which drives Eq. (41) into a more involved nonlinear regime [51,52]. The backreaction effect is often associated with the stability of ϕ, particularly when ϕ is identified with the dark matter candidate. When to consider the backreaction effect also depends on the products and on when NPR occurs.

If the products are photons or other relativistic bosonic species such as Majorons [53], the generated energy density should not exceed the energy budget of the Universe during which primordial observables are formed from BBN and CMB. In these regimes, the effective neutrino number $N_{\rm eff}$ plays a key observable [54,55]. Effectively, one can constrain the generated energy density by requiring

$$R_{\chi\gamma}(z) \equiv \frac{\rho_\chi}{\rho_\gamma} \lesssim \mathcal{O}(0.1), \tag{45}$$

where $\rho_\gamma \approx 0.66 T^4$ is the photon energy density and ρ_χ can be estimated via the Boltzmann approach,

$$\rho_\chi^{\rm B}(z) = \int \frac{d^3k}{(2\pi)^2} k f_\chi(z). \tag{46}$$

For relativistic particle production, condition (45) indicates a modification to $N_{\rm eff}$,

$$\Delta N_{\rm eff} = \frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\left(\frac{\rho_\chi}{\rho_\gamma}\right) \lesssim \mathcal{O}(0.1), \tag{47}$$

at the order that satisfies current bounds [55] and can be probed in upcoming CMB experiments.

Then one should check if the Boltzmann approach can be approximately applied for the entire interested timescale under condition (45). To see this, we show in Fig. 3 the evolution of the ratio $R_{\chi\gamma}$ at different temperatures. We see that while the distribution function f_χ obtained from the Boltzmann approach is not exactly equivalent to n_k obtained in NPR during the initial periods of oscillations, the generated energy density is initially too small to cause significant observational consequences. After ρ_χ becomes larger and approaches the magnitude of the photon energy density, the Boltzmann approach has already provided a

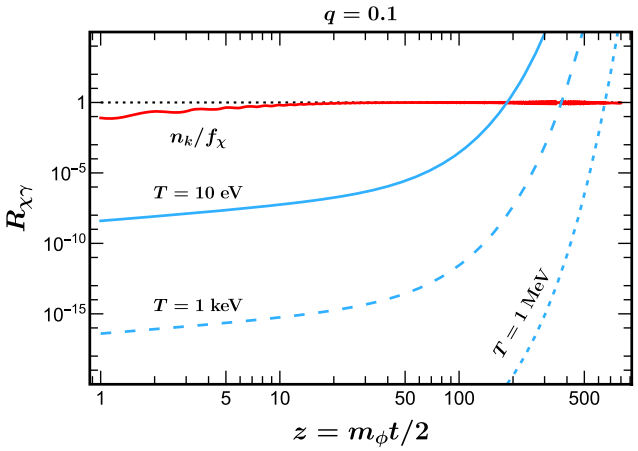


FIG. 3. The evolution of χ energy density normalized to the photon energy density in different radiation-dominated epochs, $T = 10$ eV, 1 keV, and 1 MeV. It serves to justify the Boltzmann approach with potential observational consequences but without significant backreaction effects.

consistent description of the distribution function predicted in NPR. Therefore, for observational consequences predicted from NPR at temperatures below a few MeV, it suffices to use the Boltzmann approach in describing the explosive particle production. Note that we have fixed $q = 0.1$ in Fig. 3 for illustrative purposes, but the conclusion is also valid for $q < 0.1$.

Finally, let us comment on another pattern that satisfies condition (40). This occurs if q is large ($q \gg 1$), corresponding to the broad parametric resonance. Particle production in this regime also features exponential growth, though in a quite different way [2,25–28], but the resonance width cannot be determined perturbatively. Near the maximum of the field amplitude in a single oscillation [$\phi(t) = \bar{\phi}$], the frequency $\omega_k(t)$ becomes purely imaginary since $k^2 < \mu\bar{\phi}$. In this regime, the calculation given in Eq. (32) fails to determine the momentum spread. It becomes problematic to calculate particle collision rates embedded in the Boltzmann equation. Nevertheless, near the minimum of the field amplitude where $k^2 \gg \mu\phi(t)$, the frequency $\omega_k(t)$ becomes real even for $q \gg 1$. In this regime, particle collision may work to describe the interaction between ϕ and χ, and in this case, the higher-order annihilation channels $j\phi \to 2\chi$ become more important than the decay mode, requiring proper summation over j [32].

V. CONCLUSION

We have demonstrated that by simulating the momentum spread with a Gaussian distribution, the Boltzmann equation can describe the exponential growth of the distribution function in bosonic particle production well, which is consistent with the prediction of the NPR mechanism in the asymptotic regime. As far as the particle number density is concerned, the prediction from the Boltzmann approach only shows a relative discrepancy at 20%, rather than orders of magnitude caused by the mismatch of the exponential growth rate or a different construction of the distribution function.

We have also discussed the potential regimes where the NPR mechanism may induce observational consequences at cosmic temperatures below a MeV. Under these circumstances, the cosmic expansion, backreaction, and scattering with the thermal plasma can be neglected such that explosive particle production is not prevented. The Boltzmann equation then provides a simple and analytic approach to study the evolution of distribution functions from oscillating scalar decay, where late-time thermalization processes can be incorporated in a straightforward way.

ACKNOWLEDGMENTS

The author would like to thank Shinya Kanemura, Kunio Kaneta, Kodai Sakurai, and Ryosuke Sato for the helpful discussions and feedback. This work is supported by JSPS Grant-in-Aid for JSPS Research Fellows No. 24KF0060.

DATA AVAILABILITY

No data were created or analyzed in this study.

[1] L. Kofman, A. D. Linde, and A. A. Starobinsky, Reheating after inflation, Phys. Rev. Lett. **73**, 3195 (1994).
[2] L. Kofman, A. D. Linde, and A. A. Starobinsky, Towards the theory of reheating after inflation, Phys. Rev. D **56**, 3258 (1997).
[3] P. Agrawal, G. Marques-Tavares, and W. Xue, Opening up the QCD axion window, J. High Energy Phys. 03 (2018) 049.
[4] R. T. Co, L. J. Hall, and K. Harigaya, QCD axion dark matter with a small decay constant, Phys. Rev. Lett. **120**, 211602 (2018).
[5] J. A. Dror, K. Harigaya, and V. Narayan, Parametric resonance production of ultralight vector dark matter, Phys. Rev. D **99**, 035036 (2019).
[6] I. I. Tkachev, Fast radio bursts and axion miniclusters, JETP Lett. **101**, 1 (2015).
[7] M. P. Hertzberg and E. D. Schiappacasse, Dark matter axion clump resonance of photons, J. Cosmol. Astropart. Phys. 11 (2018) 004.
[8] A. Arza, Photon enhancement in a homogeneous axion dark matter background, Eur. Phys. J. C **79**, 250 (2019).
[9] T. Suyama, T. Tanaka, B. Bassett, and H. Kudoh, Black hole production in tachyonic preheating, J. Cosmol. Astropart. Phys. 04 (2006) 001.
[10] Y.-F. Cai, X. Tong, D.-G. Wang, and S.-F. Yan, Primordial black holes from sound speed resonance during inflation, Phys. Rev. Lett. **121**, 081306 (2018).
[11] S. Kawai and J. Kim, Gauss–Bonnet Chern–Simons gravitational wave leptogenesis, Phys. Lett. B **789**, 145 (2019).
[12] Y.-F. Cai, C. Lin, B. Wang, and S.-F. Yan, Sound speed resonance of the stochastic gravitational wave background, Phys. Rev. Lett. **126**, 071303 (2021).
[13] N. Fonseca, E. Morgante, R. Sato, and G. Servant, Axion fragmentation, J. High Energy Phys. 04 (2020) 010.
[14] R. T. Co, L. J. Hall, K. Harigaya, K. A. Olive, and S. Verner, Axion kinetic misalignment and parametric resonance from inflation, J. Cosmol. Astropart. Phys. 08 (2020) 036.
[15] S. Kasuya and M. Kawasaki, Restriction to parametric resonant decay after inflation, Phys. Lett. B **388**, 686 (1996).

[16] Y. Shtanov, J. H. Traschen, and R. H. Brandenberger, Universe reheating after inflation, Phys. Rev. D **51**, 5438 (1995).
[17] M. Yoshimura, Catastrophic particle production under periodic perturbation, Prog. Theor. Phys. **94**, 873 (1995).
[18] M. A. Amin, M. P. Hertzberg, D. I. Kaiser, and J. Karouby, Nonperturbative dynamics of reheating after inflation: A review, Int. J. Mod. Phys. D **24**, 1530003 (2014).
[19] R. R. Kovacic, I., and S. Mohamed Sah, Mathieu's equation and its generalizations: Overview of stability charts and their features, Appl. Mech. Rev. **70**, 020802 (2018).
[20] S. Matsumoto and T. Moroi, Decay of scalar condensation in quantum field theory, Phys. Rev. D **77**, 045014 (2008).
[21] M. Yoshimura, Decay rate of coherent field oscillation, in *Frontiers in Quantum Field Theory in Honor of the 60th Birthday of Prof. K. Kikkawa* (1996), pp. 394–400, arXiv: hep-ph/9603356.
[22] G. Alonso-Álvarez, R. S. Gupta, J. Jaeckel, and M. Spannowsky, On the wondrous stability of ALP dark matter, J. Cosmol. Astropart. Phys. 03 (2020) 052.
[23] P. Carenza, A. Mirizzi, and G. Sigl, Dynamical evolution of axion condensates under stimulated decays into photons, Phys. Rev. D **101**, 103016 (2020).
[24] T. Moroi and W. Yin, Particle production from oscillating scalar field and consistency of Boltzmann equation, J. High Energy Phys. 03 (2021) 296.
[25] H. Fujisaki, K. Kumekawa, M. Yamaguchi, and M. Yoshimura, Particle production and dissipative cosmic field, Phys. Rev. D **53**, 6805 (1996).
[26] H. Fujisaki, K. Kumekawa, M. Yamaguchi, and M. Yoshimura, Particle production and gravitino abundance after inflation, Phys. Rev. D **54**, 2494 (1996).
[27] S. Y. Khlebnikov and I. I. Tkachev, The Universe after inflation: The wide resonance case, Phys. Lett. B **390**, 80 (1997).
[28] J. F. Dufaux, G. N. Felder, L. Kofman, M. Peloso, and D. Podolsky, Preheating with trilinear interactions: Tachyonic resonance, J. Cosmol. Astropart. Phys. 07 (2006) 006.
[29] L. Parker, Quantized fields and particle creation in expanding universes. I, Phys. Rev. **183**, 1057 (1969).
[30] T. Asaka and H. Nagao, Non-perturbative corrections to particle production from coherent oscillation, Prog. Theor. Phys. **124**, 293 (2010).
[31] K. Kaneta, S. M. Lee, and K.-y. Oda, Boltzmann or Bogoliubov? Approaches compared in gravitational particle production, J. Cosmol. Astropart. Phys. 09 (2022) 018.
[32] S.-P. Li and K.-P. Xie, Photon proliferation from N-body dark matter annihilation, arXiv:2412.15749.
[33] T. Hahn, Generating Feynman diagrams and amplitudes with FEYNARTS3, Comput. Phys. Commun. **140**, 418 (2001).
[34] V. Shtabovenko, R. Mertig, and F. Orellana, FEYNCALC10: Do multiloop integrals dream of computer codes?, Comput. Phys. Commun. **306**, 109357 (2025).
[35] A. Arza, T. Schwetz, and E. Todarello, How to suppress exponential growth—On the parametric resonance of photons in an axion background, J. Cosmol. Astropart. Phys. 10 (2020) 013.
[36] J. Jaeckel and S. Schenk, Challenging the stability of light millicharged dark matter, Phys. Rev. D **103**, 103523 (2021).
[37] D. J. E. Marsh, Axion cosmology, Phys. Rep. **643**, 1 (2016).
[38] L. Di Luzio, M. Giannotti, E. Nardi, and L. Visinelli, The landscape of QCD axion models, Phys. Rep. **870**, 1 (2020).
[39] L. F. Abbott and P. Sikivie, A cosmological bound on the invisible axion, Phys. Lett. **120B**, 133 (1983).
[40] M. Dine and W. Fischler, The not so harmless axion, Phys. Lett. **120B**, 137 (1983).
[41] J. Preskill, M. B. Wise, and F. Wilczek, Cosmology of the invisible axion, Phys. Lett. **120B**, 127 (1983).
[42] D.-S. Lee and K.-W. Ng, Photon production of axionic cold dark matter, Phys. Rev. D **61**, 085003 (2000).
[43] J. Yokoyama, Fate of oscillating scalar fields in the thermal bath and their cosmological implications, Phys. Rev. D **70**, 103511 (2004).
[44] J. Yokoyama, Can oscillating scalar fields decay into particles with a large thermal mass?, Phys. Lett. B **635**, 66 (2006).
[45] R. T. Co, L. J. Hall, and K. Harigaya, Axion kinetic misalignment mechanism, Phys. Rev. Lett. **124**, 251802 (2020).
[46] R. T. Co and K. Harigaya, Axiogenesis, Phys. Rev. Lett. **124**, 111602 (2020).
[47] L. Di Luzio, B. Gavela, P. Quilez, and A. Ringwald, Dark matter from an even lighter QCD axion: Trapped misalignment, J. Cosmol. Astropart. Phys. 10 (2021) 001.
[48] L. Di Luzio, B. Gavela, P. Quilez, and A. Ringwald, An even lighter QCD axion, J. High Energy Phys. 05 (2021) 184.
[49] R. Daido, N. Kitajima, and F. Takahashi, Level crossing between the QCD axion and an axionlike particle, Phys. Rev. D **93**, 075027 (2016).
[50] R. Daido, N. Kitajima, and F. Takahashi, Domain wall formation from level crossing in the axiverse, Phys. Rev. D **92**, 063512 (2015).
[51] S. Y. Khlebnikov and I. I. Tkachev, Classical decay of inflaton, Phys. Rev. Lett. **77**, 219 (1996).
[52] S. Y. Khlebnikov and I. I. Tkachev, Resonant decay of Bose condensates, Phys. Rev. Lett. **79**, 1607 (1997).
[53] G. B. Gelmini and M. Roncadelli, Left-handed neutrino mass scale and spontaneously broken lepton number, Phys. Lett. **99B**, 411 (1981).
[54] C. Pitrou, A. Coc, J.-P. Uzan, and E. Vangioni, Precision big bang nucleosynthesis with improved helium-4 predictions, Phys. Rep. **754**, 1 (2018).
[55] N. Aghanim *et al.* (Planck Collaboration), Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. **641**, A6 (2020); **652**, C4(E) (2021).