

Nonradial pulsations of gravitationally coupled two-fluid neutron stars in general relativity

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(Received 5 May 2026; accepted 4 August 2026; published 20 August 2026)

Nonradial oscillations of dark matter admixed neutron stars offer a powerful probe of stellar structure and relativistic gravity, but when the ordinary and dark matter components are modeled as dynamically distinct fluids, their consistent description requires a two-fluid treatment that retains the dynamical response of both components and their coupling to spacetime. The system considered here corresponds to the noninteracting limit of the established general relativistic formalism for polar perturbations of two-fluid stars, but is formulated directly for independently conserved perfect-fluid components that interact only through gravity, without entrainment or direct microphysical coupling. Within this formulation, we present the coupled linear perturbation equations for the spacetime metric and both fluid components, together with the central regularity conditions, the boundary conditions at the individual fluid surfaces, and the matching to the exterior vacuum spacetime required to determine the polar mode spectrum. We implement this system numerically for mirror dark matter admixed neutron stars, describing both components with the QHC21-BT equation of state, and compute representative polar mode spectra. The resulting spectra contain distinct fundamental (f) and pressure (p) mode branches with dominant inner- or outer-fluid character, which we identify from the associated eigenfunctions and their radial node structure. We find that the additional gravitationally coupled fluid shifts the ordinary-matter-led f-mode branch and that the equation-of-state-insensitive universal relation between the mass-scaled f-mode frequency and compactness known for single-fluid neutron stars is not preserved in the same form. These results provide a fully general relativistic characterization of the polar mode spectrum of mirror dark matter admixed neutron stars and illustrate the asteroseismic signatures introduced by an additional gravitationally coupled fluid component.

DOI: [10.1103/642h-xhjf](https://doi.org/10.1103/642h-xhjf)

I. INTRODUCTION

Neutron stars are now firmly established as sources of gravitational waves across a wide range of dynamical

scenarios, extending well beyond the inspiral phase of compact binaries. Whenever the stellar interior or the surrounding spacetime is driven out of equilibrium—by violent events such as gravitational collapse, the formation and relaxation of postmerger remnants, magnetar flares, crustal failures, or sudden angular-momentum rearrangements—the star responds through global stellar oscillations that can imprint characteristic spectral features on the gravitational-wave emission. Unlike chirp signals, which primarily reflect orbital dynamics, oscillation-driven radiation is directly linked to the internal structure of the star and therefore provides a potential observational channel for probing dense-matter physics through mode frequencies and damping times [1,2]. Although the detectability of such signals

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depends sensitively on excitation efficiency, dissipation mechanisms, and instrumental sensitivity—particularly in the kilohertz regime—the improving sensitivity of gravitational-wave interferometers and the development of dedicated analysis strategies have brought stellar asteroseismology into closer contact with observational science. In this context, a reliable theoretical description of neutron star oscillations is essential for establishing a robust connection between prospective gravitational-wave data and the microphysical properties of matter at supranuclear densities.

Nonradial oscillations of relativistic stars provide a direct window into the dynamical coupling between matter and spacetime, encoding detailed information about the internal structure and composition of neutron stars. In general relativity, these oscillations decompose into axial (odd-parity) and polar (even-parity) sectors, distinguished by their transformation properties under parity. The polar sector describes genuine pulsational degrees of freedom of the fluid and couples self-consistently to gravitational radiation, giving rise to families of modes, such as the fundamental (f), pressure (p), gravity (g), and spacetime (w) modes [3–5]. These polar oscillations are of particular interest for relativistic stellar asteroseismology, as they dominate the star’s gravitational-wave response to dynamical perturbations and carry imprints of the underlying microphysics through their characteristic frequencies and damping properties [6]. Axial perturbations, by contrast, do not involve compressional motion of the fluid. In nonrotating stars, the axial fluid degrees of freedom correspond only to stationary, toroidal motions without a restoring force, while the associated gravitational-wave solutions describe spacetime oscillations with only very weak coupling to the stellar fluid [7,8]. For this reason, polar nonradial oscillations constitute the physically relevant channel for studying neutron-star dynamics and gravitational-wave emission [1,9], and form the focus of the present work.

The theoretical description of polar nonradial oscillations in single-fluid relativistic stars is by now well established and has been developed systematically within general relativity [5–7,9,10]. In this framework, the coupled linear perturbations of the spacetime metric and the stellar fluid can be formulated as a closed system of linearized Einstein-matter equations, yielding a discrete spectrum of oscillation modes with well-defined physical interpretation. Extensive studies have clarified the properties of the fluid-dominated mode families (including the f, p, and, when stratification is present, g-modes) as well as spacetime-led w-modes; these studies have demonstrated how the modes’ frequencies and damping times depend sensitively on the star’s equation of state, stellar compactness, and internal composition [5,11–19]. The fundamental (f-) mode corresponds to a global oscillation of the stellar fluid coupled strongly to spacetime curvature, while the pressure (p-) modes are predominantly governed by

the fluid’s compressibility and sound speed. Gravity (g-) modes arise when stable stratification or composition gradients are present [20,21], reflecting buoyancy-driven restoring forces, whereas spacetime (w-) modes are dominated by oscillations of the gravitational field itself with only weak participation of the fluid [22,23]. For completeness, rotating neutron stars also support axial-led inertial oscillations, such as r-modes, which belong to the odd-parity sector and are restored primarily by the Coriolis force [24,25]. Together, these mode families provide complementary probes of neutron-star structure and have made nonradial asteroseismology a powerful theoretical tool for connecting microscopic physics to observable spectral properties. This standard construction, however, is rooted in a physical description where the stellar interior is treated as a single effective fluid, characterized by a unique four-velocity and thermodynamic response. When multiple fluids coexist and evolve dynamically within the same spacetime, the perturbation problem must instead accommodate additional fluid degrees of freedom and their coupling through the metric.

Neutron star interiors are not generically restricted to a single comoving fluid, and multifluid dynamics arises naturally in several physically motivated contexts. A well-known example is provided by superfluid neutron stars [26–28], where superfluid neutrons can move relative to a charged component, leading to a genuine multi-velocity description of the stellar interior. Dark matter admixed neutron stars provide another physically distinct realization of multifluid dynamics, since the two fluids represent separate matter sectors and may possess independently specified central densities and radial supports, as well as distinct equations of state in more general dark matter models. Such configurations cannot generally be reduced to an effective single-fluid description, because each component retains its own displacement field and thermodynamic response; the resulting additional degrees of freedom, coupled through the metric perturbations, can introduce a qualitatively distinct structure into the nonradial mode spectrum. Relativistic multifluid frameworks have been developed to describe such systems, including formulations that incorporate entrainment, and nonradial oscillations of superfluid stars have been studied within these approaches [29–31]. In these theories, the dynamics is derived from a variational principle built on a master function of the scalar invariants constructed from the constituent currents, with the derivatives of the master function encoding the thermodynamic properties and any microphysical coupling between the constituents and determining the structure of the perturbation equations and mode spectrum [31–34]. The same framework also includes the limit in which the relevant interaction and entrainment coefficients vanish, so that the constituents remain dynamically independent at the microphysical level while being coupled indirectly through the common spacetime. Dark matter admixed neutron stars provide a particularly

well-motivated realization of this regime, as the dark component possesses its own conserved current and four-velocity and is assumed here to interact with ordinary matter only through gravity [35–40]. The gravity-only system studied in the present work should therefore be understood as the noninteracting limit of the general relativistic two-fluid perturbation formalism developed in Ref. [31].

Despite substantial progress in these related directions, the nonradial dynamics of dark matter admixed neutron stars has not yet been systematically explored in full general relativity. Nonradial oscillations of neutron stars have been extensively studied in the single-fluid framework, while general relativistic two-fluid formulations have enabled detailed investigations of the oscillation spectra of superfluid stars. [15,29,41–53]. In parallel, gravitationally coupled two-fluid neutron stars—particularly in the context of dark matter admixture—have been explored primarily at the level of equilibrium structure, radial oscillations, and stability analyses [54–60]. Nonradial oscillations of dark matter admixed neutron stars have also been investigated within the relativistic Cowling approximation [61], providing an important step toward using nonradial mode spectra as probes of dark matter. In that study, although the equilibrium configuration was constructed as a two-fluid system, the perturbation analysis did not retain independent displacement fields for both components and therefore did not describe the full two-fluid dynamical response. A subsequent study developed a consistent two-fluid treatment within the Cowling approximation, retaining the independent dynamical degrees of freedom of both components [62]. These Cowling approximation studies demonstrated the importance of retaining the separate fluid degrees of freedom, but necessarily neglected the metric perturbations through which the two components are gravitationally coupled at the perturbative level. The general formalism of Ref. [31] was also applied in its noninteracting limit to a two-fluid stellar model in which the components obeyed distinct equations of state and their equilibrium densities were algebraically related.

The remaining gap is therefore not the absence of a general relativistic two-fluid perturbation framework, which was established in Ref. [31], but its systematic application to dark matter admixed neutron stars and the explicit numerical treatment of configurations in which the two components have distinct radial supports. For this application, we formulate the noninteracting limit directly in the conventional thermodynamic variables used in zero-temperature neutron star modeling, with each component specified by its pressure–energy–density relation. Although mathematically equivalent to the noninteracting limit of Ref. [31], the perturbation equations in that variational formulation retain explicit dependence on the constituent chemical potentials and on several thermodynamic coefficients constructed from derivatives of the master function, which must be evaluated for the adopted equations of state before the system can be

implemented numerically. By deriving the perturbation equations directly from the separately conserved stress-energy tensors of the two fluids and the linearized Einstein equations, we obtain a closed system whose matter sector coefficients are expressed directly in terms of the energy density, pressure, and adiabatic response of each component, without retaining the additional variational quantities appearing in the general interacting formulation. Within this representation, the nonradial problem retains the independent displacement fields of both fluids together with the metric perturbations that mediate their gravitational coupling, providing a direct basis for computing and interpreting the polar mode spectrum of dark matter admixed neutron stars in full general relativity.

In this work, we present a direct formulation and numerical implementation of polar nonradial perturbations for gravitationally coupled two-fluid neutron stars, specialized to the noninteracting limit of the general framework of Ref. [31] and tailored to dark matter admixed configurations. Starting from a two-fluid system with independently conserved stress–energy tensors and particle number currents, we derive the complete set of linearized perturbation equations governing the coupled dynamics of the spacetime metric and both fluid components in full general relativity, without invoking the Cowling approximation [63]. The analysis is restricted to the even-parity sector and assumes that the two fluids interact exclusively through the spacetime geometry, with no entrainment or direct microphysical coupling between them. The resulting perturbation equations form a closed, self-consistent system suitable for studying nonradial oscillations in a broad class of gravitationally coupled two-fluid configurations. We implement these perturbation equations numerically for mirror dark matter admixed neutron stars and compute representative polar mode spectra in full general relativity. This extends the earlier two-fluid analysis performed within the Cowling approximation [62] by retaining the dynamical spacetime perturbations that mediate the gravitational coupling between the two components. The representative mirror dark matter models considered in this work show that the additional fluid degree of freedom leaves a measurable imprint on the polar spectrum, producing distinct mode branches associated with the two fluid components. In particular, the ordinary-matter-led *f*-mode branch is shifted by the gravitational presence of the dark component, and the equation-of-state-insensitive mass-scaled *f*-mode universal relation with compactness, known for standard single-fluid neutron stars, is not preserved in the same form for gravitationally coupled two-fluid configurations.

The paper is organized as follows. In Sec. II, we formulate the fully relativistic theory of polar nonradial perturbations for gravitationally coupled two-fluid neutron stars. We begin by specifying the static, spherically symmetric background configuration of independently conserved fluids interacting solely through the spacetime geometry (Sec. II A), and then

derive the coupled linear perturbation equations for the metric and both fluid components in the even-parity sector (Sec. II C). The formulation is completed by imposing the appropriate regularity conditions at the center, boundary conditions at the stellar surfaces of each fluid (Sec. II D), and matching conditions to the exterior vacuum spacetime (Sec. II E). In Sec. III, we describe the numerical implementation of the perturbation formalism and the procedure used to compute the nonradial mode spectrum. In Sec. IV, we apply the framework to mirror dark matter admixed neutron star models, illustrating the practical application of the framework and discussing the resulting mode structure and representative asteroseismic features. We conclude in Sec. V with a summary and comment on future directions.

Throughout this paper, we will use the following conventions. Greek indices denote spacetime coordinates, and uppercase Latin indices, such as “ $X, Y = 1, 2, \dots, N$,” label the different fluid components. Summation over fluid labels is written explicitly and is not implied by repetition. We adopt the metric signature $(-, +, +, +)$ and work in geometric units with $G = c = 1$. The stellar background is assumed to be static and spherically symmetric, and the matter content is described by multiple perfect fluids with independently conserved currents, interacting solely through the spacetime geometry. No entrainment or direct microphysical coupling between the fluids is included. We restrict attention to polar (even-parity) perturbations, which describe genuine stellar pulsations and couple self-consistently to gravitational radiation.

II. NONRADIAL PERTURBATIONS OF GRAVITATIONALLY COUPLED TWO-FLUID NEUTRON STARS

The general relativistic polar perturbations of a two-fluid star were formulated within the variational multifluid framework in Ref. [31]. The system considered here corresponds to the noninteracting limit of that framework, in which entrainment and direct microphysical coupling are absent, while the fluids remain gravitationally coupled through the common background spacetime and its perturbations. We derive the corresponding perturbation system directly from the separately conserved perfect-fluid stress-energy tensors and particle-number currents, expressing the thermodynamic sector in terms of the energy density, pressure, and adiabatic response of each component.

A. Background two-fluid stellar configuration

We consider a static, spherically symmetric spacetime describing a compact star composed of N perfect-fluid components that interact only through gravity. Each component $X = 1, 2, \dots, N$ is characterized by its own energy density $\mathcal{E}_X$, pressure p_X , and four-velocity u_X^μ , and is assumed to possess an independently conserved stress-energy tensor. In equilibrium, the configuration is static,

and each fluid is at rest with respect to the chosen coordinate system, so that the background four-velocity of every component is purely timelike and aligned with the timelike Killing vector of the spacetime. Writing the metric in the standard form,

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

the background four-velocity of each component is therefore

$$u_X^\mu = e^{-\Phi(r)} \delta_t^\mu, \quad u_X^\mu u_{\mu X} = -1, \quad (2)$$

so that the equilibrium state is fully specified by the radial profiles $\Phi(r)$ and $\Lambda(r)$, and the thermodynamic variables $\mathcal{E}_X$ and p_X for each fluid. The stress-energy tensor of each component is taken to be that of a perfect fluid,

$$T_X^{\mu\nu} = (\mathcal{E}_X + p_X) u_X^\mu u_X^\nu + p_X g^{\mu\nu}. \quad (3)$$

We further assume that each component obeys its own barotropic equation of state $p_X = p_X(\mathcal{E}_X)$, with no entrainment or direct microphysical coupling between fluids. Since the interaction is purely gravitational, each component is separately conserved,

$$\nabla_\mu T_X^{\mu\nu} = 0, \quad \text{for each } X = 1, 2, \dots, N, \quad (4)$$

which, upon projection along u_X^μ and orthogonally to it, yields, respectively, the energy-conservation equation and the relativistic Euler equation for each component,

$$u_X^\nu \nabla_\nu \mathcal{E}_X + (\mathcal{E}_X + p_X) \nabla_\nu u_X^\nu = 0, \quad (5)$$

$$(\mathcal{E}_X + p_X) u_X^\nu \nabla_\nu u_X^\mu + (g^{\mu\nu} + u_X^\mu u_X^\nu) \nabla_\nu p_X = 0. \quad (6)$$

Moreover, we introduce a conserved particle-number current $n_X u_X^\mu$ for each component, where n_X denotes the particle number density of fluid X . The conservation of particle number is expressed as [55]

$$\nabla_\mu (n_X u_X^\mu) = 0, \quad (7)$$

which provides an independent conservation law for each fluid. For a barotropic component whose energy density can be regarded as a function of the number density, $\mathcal{E}_X = \mathcal{E}_X(n_X)$, the particle-number conservation law together with the energy-balance equation [Eq. (5)] implies the differential thermodynamic relation,

$$d\mathcal{E}_X = \frac{\mathcal{E}_X + p_X}{n_X} dn_X, \quad (8)$$

which closes the thermodynamic sector of the background and perturbed equations for each fluid.

The different fluids are coupled exclusively through the spacetime geometry via the Einstein field equations,

$$G^\mu_\nu = 8\pi \sum_X T^\mu_\nu. \quad (9)$$

For a static, spherically symmetric configuration, the Einstein equations together with the Euler equations for each component yield the multifluid generalization of the Tolman-Oppenheimer-Volkoff (TOV) equations [36,54,55,64],

$$\frac{d\Phi}{dr} = \frac{1}{r^2} \frac{1}{1 - \frac{2m}{r}} \left(m + \sum_X 4\pi r^3 p_X \right), \quad (10)$$

$$\frac{dm}{dr} = 4\pi r^2 \sum_X \mathcal{E}_X, \quad (11)$$

$$\frac{dp_X}{dr} = -(\mathcal{E}_X + p_X) \frac{d\Phi}{dr}, \quad (12)$$

where the mass function $m(r)$ is defined through the metric relation $e^{-2\Lambda(r)} = 1 - 2m/r$.

Together with the equations of state for the individual components, the multifluid TOV system defines a well-posed boundary-value problem that can be integrated from the center of the star outward. The integration for each component terminates at a radius R_X determined by the condition $p_X(R_X) = 0$, beyond which its pressure and energy density vanish identically. The outer boundary of the matter distribution is therefore located at

$$R_{\text{out}} \equiv \max_X \{R_X\}, \quad (13)$$

and the total gravitational mass of the configuration is given by $M = m(R_{\text{out}})$. For $r > R_{\text{out}}$, the spacetime is described by the exterior Schwarzschild solution with mass parameter M . This construction fixes the radial domain of the interior problem and specifies the matching surface to the exterior vacuum region that will be used in the formulation of the nonradial perturbation equations.

B. Perturbation variables and kinematics

We now consider linear perturbations of the spacetime geometry and of each fluid about the static, spherically symmetric background described in Sec. II A. We distinguish between Eulerian perturbations, denoted by δ , and Lagrangian perturbations following the motion of a given fluid X , denoted by Δ_X . The two are related through the Lagrangian displacement vector ξ^μ_X , which describes the displacement of fluid elements away from their equilibrium positions under the perturbation. For any tensor quantity T defined on the background spacetime, the Lagrangian and Eulerian variations are related by

$$\Delta_X T = \delta T + \mathcal{L}_{\xi_X} T, \quad (14)$$

where $\mathcal{L}_{\xi_X}$ denotes the Lie derivative along ξ^μ_X . In particular, for a background scalar function $f(r)$, one has $\Delta_X f = \delta f + \xi^\mu_X \nabla_\mu f$.

The Lagrangian displacement ξ^μ_X is not unique, since adding any component parallel to the background four-velocity u^μ_X corresponds to a relabeling of fluid elements along the unperturbed flow and does not change the physical perturbation. This reflects the invariance of the Lagrangian description under shifts of the displacement along the background fluid flow. We fix this gauge freedom by imposing the orthogonality condition $u^\mu_X \xi^\mu_X = 0$, so that the displacement vector is purely spatial in the local rest frame of fluid X . This choice provides a unique and physically transparent definition of the displacement associated with each fluid perturbation.

Using these definitions, together with the normalization $u^\mu_X u^\mu_X = -1$, which fixes the perturbation of the four-velocity in terms of the metric perturbation, and the thermodynamic relations of Sec. II A, the Lagrangian perturbations of the fluid variables can be written in terms of the metric perturbations and the displacement fields (see, e.g., Ref. [65]) as

$$\frac{\Delta_X n_X}{n_X} = -\frac{1}{2} (u^\alpha_X u^\beta_X + g^{\alpha\beta}) \Delta_X g_{\alpha\beta}, \quad (15a)$$

$$\Delta_X p_X = \gamma_X p_X \frac{\Delta_X n_X}{n_X}, \quad (15b)$$

$$\Delta_X \mathcal{E}_X = (\mathcal{E}_X + p_X) \frac{\Delta_X n_X}{n_X}, \quad (15c)$$

$$\Delta_X u^\mu_X = \frac{1}{2} u^\mu_X u^\alpha_X u^\beta_X \Delta_X g_{\alpha\beta}. \quad (15d)$$

In Eqs. (15) and in what follows, any quantity that does not carry either a Eulerian variation symbol δ or a Lagrangian variation symbol Δ_X refers to a *background* (equilibrium) quantity. Here, the adiabatic index of fluid X is defined by

$$\gamma_X \equiv \frac{\partial \ln p_X}{\partial \ln n_X}. \quad (16)$$

For a barotropic equation of state, this can be written equivalently in terms of the sound speed as

$$\gamma_X = c_X^2 \frac{\mathcal{E}_X + p_X}{p_X}, \quad (17)$$

where $c_X^2 \equiv \partial p_X / \partial \mathcal{E}_X$ is the squared sound speed of fluid X .

The perturbed Euler equation for each fluid is obtained most easily by taking the Eulerian perturbation of Eq. (6) and retaining only terms linear in the perturbations. Introducing the four-acceleration $a_X^\mu \equiv u_X^\nu \nabla_\nu u_X^\mu$, the Eulerian variation of Eq. (6) yields

$$(\mathcal{E}_X + p_X) \delta a_X^\mu + (\delta \mathcal{E}_X + \delta p_X) a_X^\mu + (g^{\mu\nu} + u_X^\mu u_X^\nu) \nabla_\nu \delta p_X - [h_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu} - \delta(u_X^\mu u_X^\nu)] \nabla_\nu p_X = 0. \quad (18)$$

Here $h_{\mu\nu} \equiv \delta g_{\mu\nu}$ denotes the Eulerian metric perturbation, and indices on $h_{\mu\nu}$ are raised/lowered with the background metric.

Together, the relations above provide a complete kinematical framework for describing linear perturbations of the fluid variables in terms of the metric perturbations and the displacement fields of each fluid component. These expressions will be combined with the perturbed Einstein equations in the following subsection, where we specialize to polar perturbations of a static, spherically symmetric background and introduce a harmonic decomposition of both the spacetime and fluid perturbations.

C. Even-parity (polar) perturbations and perturbation equations

As a consequence of Eq. (15), the perturbations of the fluid variables can be expressed in terms of the Lagrangian displacement vector ξ_X^μ and the Eulerian perturbation of the spacetime metric, $h_{\mu\nu} \equiv \delta g_{\mu\nu}$. The remaining freedom in the metric perturbation can be fixed by an appropriate choice of gauge for the problem. In what follows, the perturbations are decomposed into spherical harmonics with a harmonic time dependence, and we restrict attention to even-parity modes with $l \geq 2$. Adopting the Regge–Wheeler gauge for the metric decomposition, the non-vanishing components of the metric perturbation are written as [6,10]

$$h_{00} = \sum_{l,m} e^{2\Phi} r^l H_0^{(lm)} Y^{lm} e^{i\omega t}, \quad (19a)$$

$$h_{01} = \sum_{l,m} i\omega r^{l+1} H_1^{(lm)} Y^{lm} e^{i\omega t} = h_{10}, \quad (19b)$$

$$h_{11} = \sum_{l,m} e^{2\Lambda} r^l H_2^{(lm)} Y^{lm} e^{i\omega t}, \quad (19c)$$

$$h_{22} = \sum_{l,m} r^{l+2} K^{(lm)} Y^{lm} e^{i\omega t} = \csc^2 \theta h_{33}. \quad (19d)$$

The functions $H_0^{(lm)}(r)$, $H_1^{(lm)}(r)$, $H_2^{(lm)}(r)$ and $K^{(lm)}(r)$ represent the radial amplitudes of the metric perturbation functions.

For the Lagrangian displacement vector, we impose the spatial gauge condition $\xi_X^\mu u_\mu^X = 0$ (introduced in Sec. II B),

which ensures that the displacement is purely spatial in the rest frame of each fluid. Restricting to the even-parity sector, the displacement vector can then be decomposed into spherical harmonics as

$$\xi_X^r = \sum_{l,m} r^{l-1} e^{-\Lambda} W_X^{(lm)} Y^{lm} e^{i\omega t}, \quad (20a)$$

$$\xi_X^\theta = -\sum_{l,m} r^{l-2} V_X^{(lm)} \partial_\theta Y^{lm} e^{i\omega t}, \quad (20b)$$

$$\xi_X^\phi = -\sum_{l,m} r^{l-2} \frac{V_X^{(lm)}}{\sin^2 \theta} \partial_\phi Y^{lm} e^{i\omega t}. \quad (20c)$$

The fluid perturbations are described by the functions $W_X^{(lm)}(r)$ and $V_X^{(lm)}(r)$, which correspond to the radial and angular components of the Lagrangian displacement of each fluid. In what follows, we suppress the spherical harmonic indices (l, m), and all perturbation variables are understood to correspond to a fixed harmonic mode, with the common time dependence $e^{i\omega t}$ implicitly factored out.

Substituting the above decompositions into the linearized Einstein equations yields the coupled equations governing the polar perturbations. The geometric sector, represented by the perturbed Einstein tensor, retains the same form as in the single-fluid problem, since the introduction of additional fluids affects only the matter source through the perturbed stress-energy tensor. Thus, the multifluid structure enters the perturbed field equations explicitly through the source term $\delta T_\nu^\mu = \sum_X \delta T_{\nu X}^\mu$, while the differential operator acting on the metric perturbations is the same as in the single-fluid case evaluated on the multifluid background. As in the single-fluid case, the angular combination $\delta G_{\theta\theta} - \csc^2 \theta \delta G_{\phi\phi} = 0$ implies, for $l \geq 2$, that $H_2 = H_0$ [5,6]. We use this relation throughout the remainder of the derivation and eliminate H_2 in favor of H_0 . The relevant components of the perturbed Einstein tensor take the form

$$\delta G_t^r = i\omega r^l e^{-2\Lambda} \left[\frac{l(l+1)}{2r} H_1 + \frac{1}{r} H_0 - \left(\frac{l+1}{r} - \Phi' \right) K - \frac{dK}{dr} \right] Y^{lm}, \quad (21a)$$

$$\delta G_t^\theta = -\frac{i\omega}{2} r^{l-2} e^{-2\Lambda} \left\{ e^{2\Lambda} K + [(\Lambda' - \Phi')r - (l+1)] H_1 + e^{2\Lambda} H_0 - r \frac{dH_1}{dr} \right\} \partial_\theta Y^{lm}. \quad (21b)$$

The corresponding components of the perturbed stress-energy tensor take the form

$$\delta T_t^r = -i\omega e^{-\Lambda} r^{l-1} \sum_X (\mathcal{E}_X + p_X) W_X Y^{lm}, \quad (22a)$$

$$\delta T_t^\theta = i\omega r^{l-2} \sum_X (\mathcal{E}_X + p_X) V_X \partial_\theta Y^{lm}. \quad (22b)$$

Equating these expressions through the linearized Einstein equations, $\delta G^\mu_\nu = 8\pi \delta T^\mu_\nu$, yields

$$\begin{aligned} \frac{dK}{dr} = & -\left(\frac{l+1}{r} - \Phi'\right) K + \frac{l(l+1)}{2r} H_1 + \frac{1}{r} H_0 \\ & + \frac{8\pi e^\Lambda}{r} \sum_X (\mathcal{E}_X + p_X) W_X, \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{dH_1}{dr} = & H_1 \left[(\Lambda' - \Phi') - \frac{l+1}{r} \right] + \frac{e^{2\Lambda}}{r} (H_0 + K) \\ & + \frac{16\pi e^{2\Lambda}}{r} \sum_X (\mathcal{E}_X + p_X) V_X. \end{aligned} \quad (24)$$

These two relations, corresponding to the tr and $t\theta$ components of the linearized Einstein equations, determine the metric amplitudes K and H_1 in terms of the fluid displacements and provide the first part of the coupled perturbation system.

With the metric and displacement decompositions specified, the fluid perturbations can now be written explicitly. Substituting the harmonic expansions into Eq. (15), one obtains

$$\begin{aligned} \frac{\Delta_X n_X}{n_X} = & -\left(\frac{1}{2} H_0 + K\right) - \frac{l(l+1)}{r^2} V_X - e^{-\Lambda} \frac{l+1}{r^2} W_X \\ & - e^{-\Lambda} \frac{1}{r} \frac{dW_X}{dr}. \end{aligned} \quad (25)$$

Through the thermodynamic relation in Eq. (15b), this expression also determines the Lagrangian pressure perturbation of each fluid. It is therefore convenient to define the variable¹ $\mathcal{P}_X \equiv -\Delta_X p_X e^\Phi$, since the surface condition for each fluid is simply $\Delta_X p_X(R_X) = 0$. Using Eq. (15b), Eq. (25) can then be rewritten in terms of $\mathcal{P}_X$, which yields

$$\begin{aligned} \frac{dW_X}{dr} = & -\frac{l+1}{r} W_X - \frac{l(l+1)}{r} e^\Lambda V_X - r e^\Lambda \left(\frac{H_0}{2} + K \right) \\ & + \frac{r e^{\Lambda-\Phi}}{\gamma_X p_X} \mathcal{P}_X. \end{aligned} \quad (26)$$

The remaining equation for the fluid sector follows from the radial component of the perturbed Euler equation, Eq. (18). After substituting the harmonic expansions and eliminating the remaining perturbation variables in favor of W_X , V_X , H_0 , H_1 , and K , one obtains the following

¹Here and in what follows, the harmonic indices are suppressed. More explicitly, $\Delta_X p_X = -e^{-\Phi} \sum_{l,m} r^l \mathcal{P}_X^{(lm)} Y^{lm} e^{i\omega t}$. A similar suppression is understood in Eq. (25).

first-order equation² for $\mathcal{P}_X$:

$$\begin{aligned} \frac{d\mathcal{P}_X}{dr} = & -\frac{l}{r} \mathcal{P}_X + (\mathcal{E}_X + p_X) e^\Phi \left\{ \frac{1}{2} \left(\Phi' - \frac{1}{r} \right) H_0 \right. \\ & - \frac{1}{2} \left(\omega^2 r e^{-2\Phi} + \frac{l(l+1)}{2r} \right) H_1 \\ & + \left(\frac{1}{2r} - \frac{3}{2} \Phi' \right) K - \frac{l(l+1)}{r^2} \Phi' V_X \\ & - \frac{1}{r} \left[\omega^2 e^{\Lambda-2\Phi} - r^2 \frac{d}{dr} \left(\frac{1}{r^2} e^{-\Lambda} \Phi' \right) \right] W_X \Big\} \\ & - \frac{4\pi}{r} e^{\Lambda+\Phi} (\mathcal{E}_X + p_X) \sum_Y (\mathcal{E}_Y + p_Y) W_Y. \end{aligned} \quad (27)$$

The system of four first-order differential equations [Eqs. (23), (24), (26) and (27)] derived above is supplemented by two algebraic relations, which determine the variables H_0 and V_X . The algebraic relation for V_X follows from the θ -component of the perturbed Euler equation, Eq. (18), and takes the form³

$$\begin{aligned} (\mathcal{E}_X + p_X) \omega^2 V_X = & e^\Phi \mathcal{P}_X - e^{2\Phi-\Lambda} \frac{\mathcal{E}_X + p_X}{r} \Phi' W_X \\ & + \frac{1}{2} (\mathcal{E}_X + p_X) e^{2\Phi} H_0. \end{aligned} \quad (28)$$

The second algebraic relation, which determines H_0 , follows from an appropriate combination of the rr and $r\theta$ components of the linearized Einstein equations. The resulting expression is given by (see Appendix A for algebraic details)

$$\begin{aligned} & -8\pi r e^{2\Lambda-\Phi} \sum_X \mathcal{P}_X + H_0 \left[\left(\frac{1}{r} - \Phi' \right) - e^{2\Lambda} \frac{l(l+1)}{2r} \right] \\ & - rK \left[\Phi' \left(\Phi' - \frac{1}{r} \right) + e^{2\Lambda-2\Phi} \omega^2 - e^{2\Lambda} \frac{(l+2)(l-1)}{2r^2} \right] \\ & + H_1 r \left[e^{-2\Phi} \omega^2 - \Phi' \frac{l(l+1)}{2r} \right] = 0. \end{aligned} \quad (29)$$

Equations (23), (24), (26), and (27) define a system of $2N + 2$ coupled first-order differential equations for the variables K , H_1 , W_X , and $\mathcal{P}_X$, where N is the number of fluid components. These are supplemented by the algebraic relations (28) and (29), which determine V_X and H_0 , respectively. Together, these equations form a closed system that can be integrated numerically once appropriate boundary conditions are specified.

Within the present perfect-fluid representation, this closed system expresses the noninteracting two-fluid problem in variables directly analogous to those of the single-fluid polar

²Intermediate algebraic steps are outlined in the Appendix A.

³The full algebraic derivation is presented in the Appendix A.

perturbation equations derived in Ref. [10], while remaining mathematically equivalent to the corresponding limit of the more general variational two-fluid formalism developed in Ref. [31]. More explicitly, in the notation of Ref. [31],⁴ the noninteracting limit is obtained by setting the cross-fluid coefficients $\mathcal{A}$ and $\mathcal{A}_0^0$ to zero, which removes the explicit matter-sector coupling between the fluids while leaving their perturbations indirectly coupled through the common space-time variables. Upon identifying the background metric conventions through $\nu = 2\Phi$ and $\lambda = 2\Lambda$, accounting for the opposite overall sign convention in the metric-perturbation decomposition, and using the thermodynamic identity $\mu_X n_X = \mathcal{E}_X + p_X$, the first-order metric perturbation equations for H_1 and K , given by Eqs. (49) and (50) of Ref. [31] reduce directly to Eqs. (24) and (23), respectively. The correspondence in the fluid sector requires a further thermodynamic reduction, because Eqs. (51)–(56) of Ref. [31] retain the constituent chemical potentials and the master function response coefficients $\mathcal{B}$, $\mathcal{C}$, $\mathcal{B}_0^0$, and $\mathcal{C}_0^0$ even after the cross-fluid coefficients $\mathcal{A}$ and $\mathcal{A}_0^0$ are set to zero. In the noninteracting limit, the relevant thermodynamic quantities satisfy, for each constituent X ,

$$\mu_X n_X = \mathcal{E}_X + p_X, \quad n_X^2 (\mathcal{K}_0^0)_X = \gamma_X p_X, \quad (30)$$

where $(\mathcal{K}_0^0)_X$ denotes $\mathcal{B}_0^0$ or $\mathcal{C}_0^0$ for the two fluids, respectively, in the notation of Ref. [31]. Using these identities together with the background hydrostatic-equilibrium relation, the variables X_n and X_p defined in Eqs. (55) and (56) of Ref. [31] are identified with the radial amplitudes of the constituent Lagrangian pressure perturbations and therefore correspond to $\mathcal{P}_X$ in the present notation, after accounting for the differing metric perturbation convention. With this identification, Eqs. (55) and (56) of Ref. [31], when rearranged for angular displacement amplitudes V_n and V_p , yield Eq. (28), while, after substituting the above thermodynamic relations, employing the background hydrostatic and Einstein equations, and carrying out the required nontrivial algebraic rearrangements, the radial displacement equations (51) and (52), the Lagrangian pressure perturbation equations (53) and (54), and the algebraic constraint (48) of Ref. [31] reduce, respectively, to the $N = 2$ specialization of Eqs. (26), (27), and (29). Thus, although the

⁴In Ref. [31], n^μ and p^μ denote the two conserved constituent currents, with scalar number densities n and p , respectively. These symbols are specific to the two constituents considered in that reference and should be distinguished from the notation adopted here, in which n_X denotes the number density of a generic constituent X , whereas p_X denotes its pressure. The quantities $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, together with $\mathcal{A}_0^0$, $\mathcal{B}_0^0$, and $\mathcal{C}_0^0$, are coefficients constructed from derivatives of the master function. The pairs (W_n, V_n) and (W_p, V_p) denote the radial and angular displacement amplitudes of the two constituents, respectively, while X_n and X_p represent the corresponding Lagrangian pressure perturbation amplitudes.

variational master function framework of Ref. [31] contains the present system as its noninteracting limit, implementing this limit using conventional barotropic equation of state data requires translating the constituent chemical potentials and derivatives of the master function into the corresponding perfect fluid variables. The representation adopted here performs this reduction at the outset and expresses the dynamics directly in terms of $\mathcal{E}_X$, p_X , and γ_X , thereby providing a more transparent and practical basis for the gravitationally coupled dark matter admixed neutron star configurations considered in this work, including models with independently specified equations of state and potentially distinct radial supports.

D. Regularity and boundary conditions

The system of perturbation equations must be supplemented by imposing regularity at the center and boundary conditions at the surfaces of the fluid components. In a multifluid configuration, these conditions must be imposed at $r = 0$, at each radius R_X where a given fluid component vanishes, including the outer stellar surface. Since the system consists of $2N + 2$ first-order differential equations, a corresponding set of $2N + 2$ boundary conditions is required to obtain a unique solution. We first consider the conditions at the surfaces of the individual fluid components.

At the surface of a given fluid X , defined by $p_X(R_X) = 0$, the appropriate boundary condition is that the Lagrangian pressure perturbation vanishes, i.e. $\Delta_X p_X(R_X) = 0$, or equivalently $\mathcal{P}_X(R_X) = 0$. For a two-fluid star in which both constituents terminate at the same outer surface, these conditions reproduce the vanishing of the two constituent Lagrangian pressure perturbation amplitudes, denoted X_n and X_p in Ref. [31], at that common surface. In the present treatment, however, the Lagrangian pressure perturbation condition is imposed independently at the surface of each constituent, thereby allowing one fluid component to terminate within the radial domain occupied by the other. The remaining perturbation variables, namely H_1 and K , and the fluid variables associated with the other components (W_Y , $\mathcal{P}_Y$ with $Y \neq X$), must remain continuous across the surface. For radii $r > R_X$, the fluid X is absent, so that $\mathcal{E}_X = p_X = 0$, and the corresponding variables W_X and $\mathcal{P}_X$ no longer enter the system of equations. The perturbation problem, therefore, reduces to that of the remaining fluid components in this outer region.

At the stellar center, regularity requires that both the background geometry and the perturbation variables remain finite. To derive the corresponding boundary conditions, we first record the leading behavior of the background quantities near $r = 0$. Using Eqs. (10) and (11), one finds

$$m(r) \sim \frac{4}{3} \pi r^3 \sum_X \mathcal{E}_X(0) + \mathcal{O}(r^5), \quad (31a)$$

$$e^{2\Lambda}(r) \sim 1 + \mathcal{O}(r^2), \quad (31b)$$

$$\Phi'(r) \sim 4\pi r \sum_X \left[\frac{1}{3} \mathcal{E}_X(0) + p_X(0) \right] + \mathcal{O}(r^3), \quad (31c)$$

so that, in particular, $\Lambda'(r) = \mathcal{O}(r)$ near the center.

With these background scalings, regularity of the perturbation equations implies that the variables entering the first-order system admit regular Taylor expansions about $r = 0$. In particular, one finds

$$W_X(r) \sim W_X(0) + \mathcal{O}(r^2), \quad (32a)$$

$$\mathcal{P}_X(r) \sim \mathcal{P}_X(0) + \mathcal{O}(r^2), \quad (32b)$$

$$H_1(r) \sim H_1(0) + \mathcal{O}(r^2), \quad (32c)$$

$$K(r) \sim K(0) + \mathcal{O}(r^2). \quad (32d)$$

The explicit regularity conditions at the origin are obtained by substituting the expansions of Eq. (32) into the perturbation system and requiring that the leading-order terms remain finite. We have checked that the next order terms in this local expansion about the center of the star are negligible for the starting radii chosen in this work.

We begin with the algebraic relation of Eq. (29). Its leading contribution near $r = 0$ is

$$\frac{2 - l(l+1)}{2r} H_0(0) + \frac{(l+2)(l-1)}{2r} K(0) + \mathcal{O}(r^0) = 0.$$

Since $2 - l(l+1) = -(l+2)(l-1)$, regularity for $l \geq 2$ requires

$$H_0(0) = K(0). \quad (33)$$

Next, we evaluate Eq. (26) near the center. Because the expansion (32) implies $dW/dr = \mathcal{O}(r)$, the singular terms at order r^{-1} must cancel, giving

$$-\frac{(l+1)}{r} W_X(0) - \frac{l(l+1)}{r} V_X(0) + \mathcal{O}(r) = 0.$$

It follows that regularity at the origin demands

$$W_X(0) = -lV_X(0). \quad (34)$$

The remaining central relations are obtained from Eqs. (23) and (27). Using the near-center behavior of the background and perturbation variables and retaining the leading-order terms at $\mathcal{O}(r^{-1})$, Eq. (23) gives

$$H_1(0) = \frac{2}{(l+1)} K(0) - \frac{16\pi}{l(l+1)} \sum_X [\mathcal{E}_X(0) + p_X(0)] W_X(0). \quad (35)$$

The same relation [Eq. (35)] is obtained by imposing regularity on Eq. (24), providing a useful consistency check of the expansion. Finally, imposing regularity at order $\mathcal{O}(r^{-1})$ in Eq. (27) yields the central value of $\mathcal{P}_X$ as

$$\mathcal{P}_X(0) = \left[\left\{ \frac{4\pi}{3} \sum_Y (\mathcal{E}_Y(0) + 3p_Y(0)) - \frac{\omega^2}{l} e^{-2\Phi(0)} \right\} \times W_X(0) - \frac{1}{2} K(0) \right] e^{\Phi(0)} [\mathcal{E}_X(0) + p_X(0)]. \quad (36)$$

An equivalent expression follows by imposing regularity on Eq. (28) at leading order, confirming the consistency of the system.

Together, these relations determine the regular central values of all perturbation variables in terms of the amplitudes $K(0)$ and $W_X(0)$, which represent the $N+1$ independent degrees of freedom of the solution at the origin and serve as the starting point for the numerical integration of the system. For two constituents, this central construction agrees with that of Ref. [31]: the metric amplitude $K(0)$ and the two constituent radial displacement amplitudes are independent, while the remaining central perturbation amplitudes are fixed by regularity. Imposing the surface conditions $\mathcal{P}_X(R_X) = 0$ for each fluid component provides N additional constraints. The system is therefore left with one final condition, which determines the allowed eigenfrequencies. This condition is supplied by the behavior of the perturbations in the exterior vacuum region, where the solution must satisfy a purely outgoing-wave boundary condition. The implementation of this exterior matching is described in the next subsection.

E. Exterior spacetime and matching conditions

Outside the outermost fluid surface, $r > R_{\text{out}}$, the matter fields vanish and the background spacetime reduces to the Schwarzschild solution with mass $M = m(R_{\text{out}})$. In this exterior region, the perturbation problem reduces to that of even-parity vacuum metric perturbations of a Schwarzschild spacetime. The interior solution obtained by integrating the coupled two-fluid perturbation equations must therefore be matched at $r = R_{\text{out}}$ to a corresponding exterior vacuum solution. Once the perturbation variables are specified at the matching surface, the subsequent determination of the mode spectrum follows the standard procedure for vacuum perturbations of Schwarzschild spacetime. For completeness, we outline this procedure below, closely following the established treatments in the literature [21,31,66]. The remaining condition required to determine the

eigenfrequencies is then provided by the asymptotic behavior of the perturbations at large distances, where physically admissible solutions must satisfy a purely outgoing-wave condition.

In the exterior vacuum region, the even-parity metric perturbations can be expressed in terms of a single gauge-invariant master function, the Zerilli function $Z(r)$, which satisfies a second-order wave equation on the Schwarzschild background. Introducing the tortoise coordinate $r_* = r + 2M \ln(r/2M - 1)$, the perturbation equations reduce to a Schrödinger-like equation of the form [67]

$$\frac{d^2 Z}{dr_*^2} + (\omega^2 - V_Z)Z = 0, \quad (37)$$

where $V_Z(r)$ is the Zerilli potential, given by

$$V_Z = \left(1 - \frac{2M}{r}\right) \times \left(\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 M r^2 + 18\lambda M^2 r + 18M^3}{r^3(\lambda r + 3M)^2}\right), \quad (38)$$

with $\lambda \equiv [(l-1)(l+2)]/2$. The exterior even-parity solution is therefore fully determined once Z and its radial derivative are specified at the matching surface.

At the matching surface, i.e., $r = R_{\text{out}}$, the interior solution provides the metric perturbation functions $K(R_{\text{out}})$ and $H_1(R_{\text{out}})$. To connect this interior solution to the exterior Zerilli formulation, these variables must be expressed in terms of the Zerilli function Z and its derivative. In the exterior region, the metric perturbations can be written in terms of Z through the standard relations [5,68], which allow one to determine Z and dZ/dr_* at the stellar surface (R_{out}) from the interior solution. These relations take the form

$$r^l K = \frac{\lambda(\lambda+1)r^2 + 3\lambda r M + 6M^2}{r^2(\lambda r + 3M)} Z + \frac{dZ}{dr_*} \quad (39a)$$

$$r^{l+1} H_1 = \frac{\lambda r^2 - 3\lambda r M - 3M^2}{(r-2M)(\lambda r + 3M)} Z + \frac{r^2}{r-2M} \frac{dZ}{dr_*}, \quad (39b)$$

which are to be evaluated at the outer stellar surface, $r = R_{\text{out}}$. In this way, the metric perturbations at the stellar surface, $K(R_{\text{out}})$ and $H_1(R_{\text{out}})$, determine the corresponding exterior values $Z(R_{\text{out}})$ and $dZ/dr_*(R_{\text{out}})$, thereby providing the necessary initial data for the exterior problem.

Since we are considering even-parity perturbations, the exterior spacetime is naturally described by the Zerilli function Z . For the quasi-normal-mode calculation, however, we transform the Zerilli equation to the Regge-Wheeler form using the Chandrasekhar transformation [68]. We adopt this representation because the Regge-Wheeler potential is

simpler, which makes the asymptotic analysis and the continued-fraction implementation more transparent, while leaving the resulting eigenfrequencies unchanged (since isospectrality is preserved here). In terms of the Regge-Wheeler function $Q(r)$, the exterior perturbations satisfy [69]

$$\frac{d^2 Q}{dr_*^2} + (\omega^2 - V_{\text{RW}})Q = 0, \quad (40)$$

where the Regge-Wheeler potential V_{RW} is given by

$$V_{\text{RW}} = \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right). \quad (41)$$

The Zerilli and Regge-Wheeler functions are related through the Chandrasekhar transformation [23,68], which takes the form

$$(\kappa + 2i\omega\beta)Z = \left(\kappa + \beta^2 \frac{r-2M}{r^2(\lambda r + 3M)}\right)Q + 2\beta \frac{dQ}{dr_*} \quad (42a)$$

$$(\kappa - 2i\omega\beta)Q = \left(\kappa + \beta^2 \frac{r-2M}{r^2(\lambda r + 3M)}\right)Z - 2\beta \frac{dZ}{dr_*}, \quad (42b)$$

with $\beta = 6M$ and $\kappa = 4\lambda(\lambda+1)$. This representation is particularly convenient for imposing the outgoing-wave boundary condition.

To determine the allowed eigenfrequencies, one must impose the appropriate asymptotic behavior on the exterior solution. In the limit $r \rightarrow \infty$, the Regge-Wheeler potential vanishes, and the exterior solution approaches a superposition of ingoing and outgoing waves. With the time dependence $e^{i\omega t}$ adopted throughout this work, the physically admissible quasinormal mode solution is purely outgoing and behaves as $e^{-i\omega r_*}$ at spatial infinity, corresponding to $e^{i\omega(t-r_*)}$. To implement this condition, we expand the Regge-Wheeler function in a form that explicitly factors out the required asymptotic behavior.

Following the standard continued-fraction construction, the exterior solution is expanded about a finite radius r_a in the exterior vacuum region, chosen so that the series converges [14]. In practice, we take $r_a = R_{\text{out}}$ when $R_{\text{out}} \geq 4M$, since the stellar surface itself then provides a suitable expansion point for the continued-fraction series. For more compact configurations with $R_{\text{out}} < 4M$, we instead set $r_a = 8M$, so that the expansion point is moved farther out in the vacuum exterior and the series remains convergent up to spatial infinity. When $r_a \neq R_{\text{out}}$, the exterior solution is first propagated from the stellar surface to the expansion radius. The interior solution determines K and H_1 at R_{out} , from which Z and dZ/dr_* are obtained using Eq. (39). The Chandrasekhar transformation [Eq. (42)] is

then used to construct Q and dQ/dr_* at R_{out} , and the Regge-Wheeler equation is integrated outward through the vacuum region up to $r = r_a$. The resulting $Q(r_a)$ and $(dQ/dr)|_{r_a}$ are used to initialize the continued-fraction expansion. We verified that shifting r_a within the exterior region does not change the extracted real mode frequencies to the numerical precision of the present calculation.

The exterior solution is then written as

$$Q(r) = e^{-i\omega r_*} \sum_{n=0}^{\infty} a_n \left(1 - \frac{r_a}{r}\right)^n. \quad (43)$$

Substituting the series representation above into the Regge-Wheeler equation and collecting terms order by order in $(1 - r_a/r)$ yields a recurrence relation for the expansion coefficients a_n . For $n \geq 2$, the coefficients satisfy the four-term relation (one could also obtain a three-term recursion relation directly with a different form of the continued fraction ansatz [70])

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} + \delta_n a_{n-2} = 0, \quad (44)$$

where the recurrence coefficients are given by

$$\alpha_n = \left(1 - \frac{2M}{r_a}\right) n(n+1), \quad (45a)$$

$$\beta_n = -2 \left[i\omega r_a + \left(1 - \frac{3M}{r_a}\right) n \right], \quad (45b)$$

$$\gamma_n = \left(1 - \frac{6M}{r_a}\right) n(n-1) + \frac{6M}{r_a} - l(l+1), \quad (45c)$$

$$\delta_n = \frac{2M}{r_a} (n-3)(n+1). \quad (45d)$$

To initialize the recurrence, the first coefficients must be specified separately. The structure of the series expansion implies $a_{-1} = 0$, while the coefficients a_0 and a_1 are determined by the values of Q and its radial derivative with respect to r at $r = r_a$. These coefficients are given by

$$a_0 = \frac{Q(r_a)}{e^{-i\omega r_*}|_{r=r_a}}, \quad (46a)$$

$$a_1 = \frac{r_a}{e^{-i\omega r_*}|_{r=r_a}} \left[\frac{dQ}{dr} \Big|_{r_a} + \frac{i\omega r_a}{r_a - 2M} Q(r_a) \right]. \quad (46b)$$

For the continued-fraction construction, it is convenient to reduce the four-term recurrence relation to a three-term form. This can be achieved through a standard Gaussian-elimination procedure applied to the recurrence relation. After Gaussian elimination, the resulting three-term recurrence involves coefficients $\hat{\alpha}_n$, $\hat{\beta}_n$ and $\hat{\gamma}_n$, which we define

recursively by

$$\hat{\alpha}_1 = \alpha_1, \quad \hat{\beta}_1 = \beta_1, \quad \hat{\gamma}_1 = \gamma_1, \quad (47)$$

and, for $n \geq 2$,

$$\hat{\alpha}_n = \alpha_n, \quad (48a)$$

$$\hat{\beta}_n = \beta_n - \frac{\hat{\alpha}_{n-1} \delta_n}{\hat{\gamma}_{n-1}}, \quad (48b)$$

$$\hat{\gamma}_n = \gamma_n - \frac{\hat{\beta}_{n-1} \delta_n}{\hat{\gamma}_{n-1}}. \quad (48c)$$

The four-term recurrence relation can be recast into the three-term form [71],

$$\hat{\alpha}_n a_{n+1} + \hat{\beta}_n a_n + \hat{\gamma}_n a_{n-1} = 0. \quad (49)$$

After reducing the recurrence relation to the three-term form, the quasinormal mode condition can be imposed through the associated continued fraction. In particular, requiring the sequence of coefficients a_n to define the minimal solution of the recurrence leads to a continued-fraction equation for the frequency ω (Leaver's method) [23], given by

$$\mathcal{F}(\omega) \equiv \frac{a_1}{a_0} + \frac{\hat{\gamma}_1}{\hat{\beta}_1 - \frac{\hat{\alpha}_1 \hat{\gamma}_2}{\hat{\beta}_2 - \frac{\hat{\alpha}_2 \hat{\gamma}_3}{\hat{\beta}_3 \dots}}} = 0. \quad (50)$$

Here, the coefficients $\hat{\alpha}_n$, $\hat{\beta}_n$, and $\hat{\gamma}_n$ of the three-term recurrence relation are determined entirely by the exterior Regge-Wheeler problem, whereas the dependence on the stellar interior enters through the ratio a_1/a_0 , fixed by the values of Q and dQ/dr at the expansion radius r_a . The allowed eigenfrequencies are determined by the roots of $\mathcal{F}(\omega) = 0$.

III. NUMERICAL IMPLEMENTATION

In this section, we describe the numerical procedure used to solve the perturbation equations derived in Sec. II. Although the formalism was developed for a general multi-fluid system, the numerical construction presented here is restricted to the case of two gravitationally coupled fluids. In what follows, we label these components as I and O , denoting the inner and outer fluids, respectively. For a given choice of equations of state, equilibrium configurations are first constructed by solving the background equations outlined in Sec. II A. Each fluid has its own surface, determined by the vanishing of its pressure, and the corresponding radii are denoted by R_I and R_O . In general, the two radii need not coincide, although they may do so in special cases. We choose the labels such that $R_I \leq R_O$, so that the surface of the outer fluid component defines the outer stellar boundary, $R_{\text{out}} = R_O$. The numerical matching

construction described below therefore applies to the configurations with $R_I < R_O$ considered here. The resulting background profiles, including the metric functions and thermodynamic variables of both fluids, are then used as input for the numerical integration of the perturbation equations.

The perturbation equations define a linear boundary-value problem for each trial frequency ω . For two fluids, the first-order system contains six dynamical variables, and hence six boundary conditions are required. Regularity at the center provides three relations among the central values, leaving three independent central amplitudes. The two fluid-surface conditions $\mathcal{P}_I(R_I) = 0$ and $\mathcal{P}_O(R_O) = 0$ provide two additional conditions, and the final condition is the purely outgoing-wave condition imposed on the exterior solution. The regularity conditions at the center do not select a unique solution, because the central expansion still contains free amplitudes. Similarly, imposing the surface boundary conditions does not by itself determine a unique global solution, since additional free amplitudes remain in the solutions constructed from the outer surface. We therefore solve the problem by constructing independent basis solutions that satisfy the admissible local conditions at the center and at the relevant fluid surfaces, and then determine the physical solution by matching these basis solutions at an intermediate radius. The general basis-solution matching strategy is analogous to that employed in Ref. [31] for two fluids terminating at a common stellar surface. For the configurations considered here, the distinct constituent radii require a piecewise implementation: the inner-fluid boundary condition is imposed at R_I , and the matching is performed in the outer fluid region $R_I < r_m < R_O$, where only the outer fluid degrees of freedom remain in the matter sector.

For a given trial frequency ω , the perturbation equations [Eqs. (23), (24), (26) and (27)] are first integrated outward from the stellar center. The regularity conditions derived in Sec. II D determine the central values of the perturbation variables in terms of the free amplitudes $K(0)$, $W_I(0)$, and $W_O(0)$ [Eqs. (33)–(36)], where we recall that I and O denote the inner and outer fluids, respectively. To construct a convenient pair of independent regular interior basis solutions, we choose the central data $(K(0), W_I(0)) = (\pm[\mathcal{E}(0) + p(0)], 1)$,⁵ where $\mathcal{E}(0) \equiv \mathcal{E}_I(0) + \mathcal{E}_O(0)$ and $p(0) \equiv p_I(0) + p_O(0)$ denote the total central energy density and pressure of the two-fluid background configuration. For each of these two choices, the remaining central amplitude $W_O(0)$ is treated as a shooting parameter and adjusted until the inner-fluid surface condition $\mathcal{P}_I(R_I) = 0$

is satisfied. This procedure yields two linearly independent interior solutions that are regular at the center and satisfy the boundary condition at the inner-fluid surface, and these solutions are then integrated outward from the center to the intermediate matching radius.

The matching radius r_m is an auxiliary point at which the independently constructed outward- and inward-integrated interior basis solutions are matched. In the implementation described here, r_m is chosen in the outer-fluid region, $R_I < r_m < R_O$, where the inner fluid is absent and the reduced perturbation vector $\Psi(r) = (H_1(r), K(r), W_O(r), \mathcal{P}_O(r))^T$ is appropriate.⁶ The computed mode frequencies are independent of the precise location of r_m , provided it is varied within this regular matching region. For the calculations presented here, we choose $r_m = (R_I + R_O)/2$. In this region, the variables associated with fluid I no longer enter the perturbation equations, and the system reduces to the metric variables together with the perturbation variables of the outer fluid O . For the same trial frequency ω , we then construct a second set of independent solutions by integrating the reduced perturbation system inward from the outer stellar surface $r = R_O$ to the matching radius $r = r_m$. At $r = R_O$, we impose the outer-fluid surface condition $\mathcal{P}_O(R_O) = 0$, while the remaining variables $H_1(R_O)$, $K(R_O)$, and $W_O(R_O)$ are chosen freely to generate a basis of three independent inward-integrated solutions. In practice, we use the three basis choices $(H_1, K, W_O) = (1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ at $r = R_O$, with $\mathcal{P}_O(R_O) = 0$ in each case. These three solutions are integrated inward to r_m , where they are matched to the two outward-integrated interior basis solutions constructed from the center.

To make this matching construction explicit, we introduce the relevant perturbation vector to be evaluated at $r = r_m$,

$$\Psi(r) = (H_1(r), K(r), W_O(r), \mathcal{P}_O(r))^T, \quad (51)$$

which contains the independent metric and outer-fluid variables of the reduced perturbation system in the region $R_I < r < R_O$. For a fixed trial frequency ω , the general outward-integrated solution in this region can be written as a linear combination of the two regular interior basis solutions,

$$\Psi_{\text{in}}(r) = a_1 \Psi_{\text{in}}^{(1)}(r) + a_2 \Psi_{\text{in}}^{(2)}(r). \quad (52)$$

Here a_1 and a_2 are matching coefficients to be determined, while the superscripts (1) and (2) label the two center-regular

⁵The choice $W_I(0) = 1$ is a normalization used to construct a convenient pair of independent regular solutions satisfying $\mathcal{P}_I(R_I) = 0$. Since the perturbation equations are linear, any solution with $W_I(0) \neq 0$ can be rescaled to this form. An exactly inner-displacement-free solution with $W_I(0) = 0$, if present, would require constructing the full three-dimensional regular center basis and imposing $\mathcal{P}_I(R_I) = 0$ by linear algebra.

⁶If the matching point were chosen in the two-fluid region, $0 < r_m < R_I$, the matching vector would have to be enlarged to include the inner-fluid variables W_I and $\mathcal{P}_I$. The reduced four-dimensional vector used in the present implementation is therefore tied to the choice $R_I < r_m < R_O$.

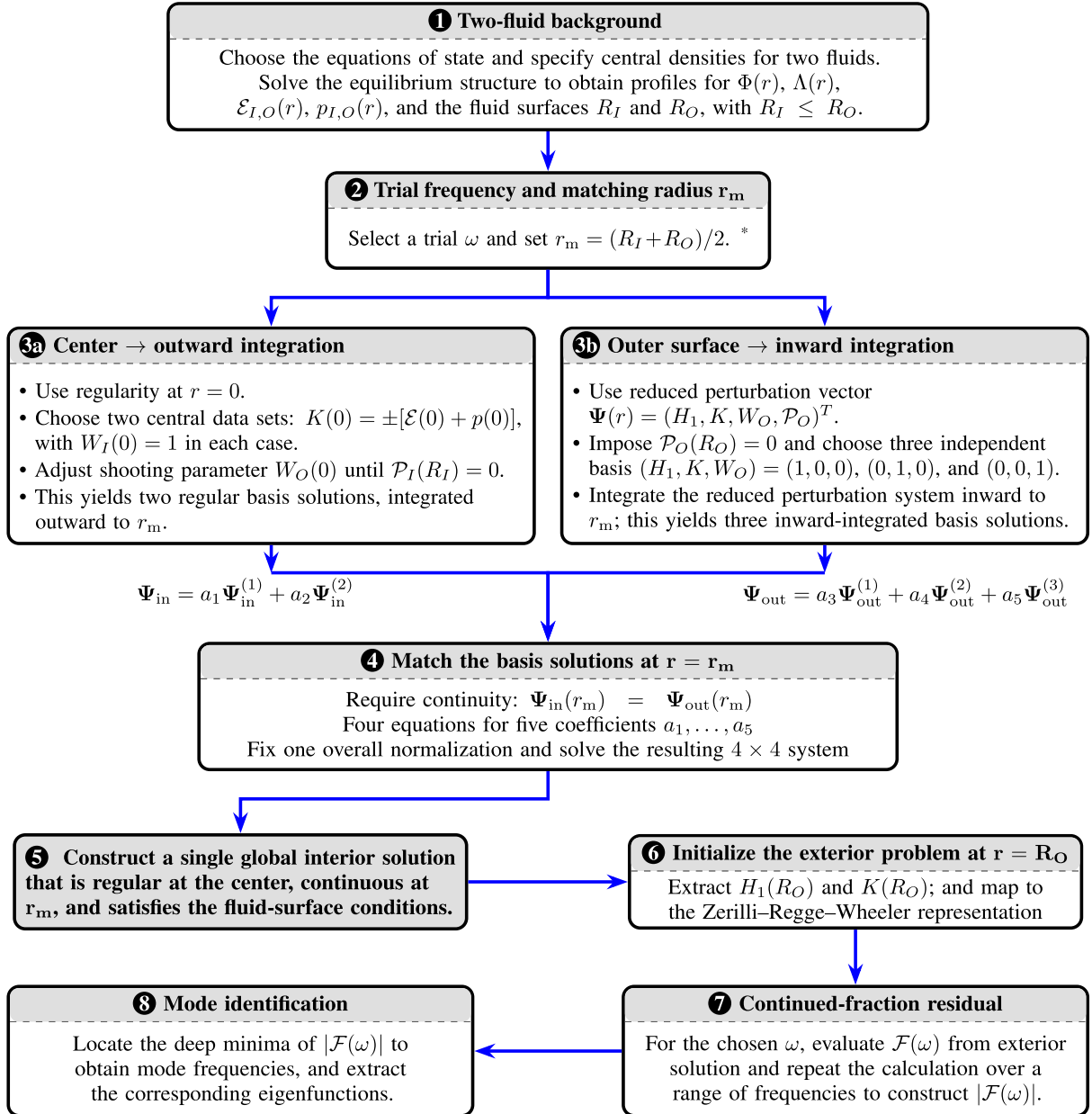


FIG. 1. Flowchart of the numerical procedure used to construct the nonradial mode spectrum of a gravitationally coupled two-fluid star. The notations I and O denote the inner and outer fluid components, respectively. *The matching point is chosen in the outer-fluid region, $R_I < r_m < R_O$, where the reduced matching vector $\Psi = (H_1, K, W_O, \mathcal{P}_O)^T$ is appropriate. The extracted frequencies are independent of the precise location of r_m within this region.

basis solutions constructed from the central choices $K(0) = +[\mathcal{E}(0) + p(0)]$ and $K(0) = -[\mathcal{E}(0) + p(0)]$, respectively, with $W_I(0) = 1$ in both cases and $W_O(0)$ fixed by $\mathcal{P}_I(R_I) = 0$. For the inward-integrated solutions, the general solution in the same outer-fluid region is written as a linear combination of the three basis solutions constructed from the outer surface,

$$\Psi_{\text{out}}(r) = a_3 \Psi_{\text{out}}^{(1)}(r) + a_4 \Psi_{\text{out}}^{(2)}(r) + a_5 \Psi_{\text{out}}^{(3)}(r). \quad (53)$$

Here a_3 , a_4 , and a_5 are additional matching coefficients, and the three superscripts label the basis solutions generated by the choices $(H_1, K, W_O) = (1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ at $r = R_O$, with $\mathcal{P}_O(R_O) = 0$. Continuity of the physical solution at the matching radius requires the outward- and inward-integrated linear combinations to agree, so that

$$a_1 \Psi_{\text{in}}^{(1)}(r_m) + a_2 \Psi_{\text{in}}^{(2)}(r_m) = a_3 \Psi_{\text{out}}^{(1)}(r_m) + a_4 \Psi_{\text{out}}^{(2)}(r_m) + a_5 \Psi_{\text{out}}^{(3)}(r_m). \quad (54)$$

The matching condition above represents four scalar equations, one for each component of Ψ , for the five coefficients a_1, a_2, a_3, a_4 , and a_5 . Since the overall normalization of the perturbation is arbitrary, one coefficient may be fixed, and the remaining four coefficients are obtained by solving the resulting 4×4 linear system. This determines the relative weights of the inward- and outward-integrated basis solutions and therefore constructs, for the chosen trial frequency ω , a single global interior solution that is regular at the center, satisfies the fluid-surface conditions, and is continuous at the matching radius.

Once this global interior solution has been constructed, the values of H_1 and K at the outer stellar surface R_O are obtained from it and used to initialize the exterior Zerilli–Regge–Wheeler construction described in Sec. II E. This completes the evaluation of the continued-fraction function $\mathcal{F}(\omega)$ for the chosen trial frequency, which will generically not be zero, and we thus, hereafter, refer to as the residual. By repeating the same procedure over a range of trial frequencies, we obtain $|\mathcal{F}(\omega)|$ as a function of ω . The oscillation frequencies are identified from the deep minima of this function, and the associated eigenfunctions are obtained from the corresponding global interior solutions. The numerical workflow is summarized schematically in Fig. 1. In the present implementation, the scan is performed along the real-frequency axis; the imaginary part of the quasinormal mode frequency, which determines the gravitational-wave damping time, is not extracted here.

IV. APPLICATION TO MIRROR DARK MATTER ADMIXED NEUTRON STARS

A. Residual spectra and mode identification

In this section, we apply the two-fluid perturbation formulation developed above for dynamically independent constituents coupled through the spacetime geometry to neutron stars admixed with mirror dark matter. In the present work, by “mirror dark matter” we mean an *idealized* realization in which the ordinary and dark components are assigned the *same* QHC21-BT equation of state [72,73], while being treated as dynamically distinct fluids coupled only through gravity. The QHC21-BT model is a quark-hadron crossover equation of state in which the low-density hadronic sector is described by the Togashi equation of state, obtained through variational many-body calculations based on realistic two- and three-nucleon interactions, while the high-density sector is modeled using the Nambu–Jona-Lasinio description of strongly interacting quark matter. The two regimes are connected by a smooth interpolation over the baryon-density interval $1.5n_0 \leq n \leq 3.5n_0$, where $n_0 \simeq 0.16 \text{ fm}^{-3}$ denotes the nuclear saturation density, with the interpolation constrained by thermodynamic stability and causality. The resulting equation of state supports a maximum nonrotating single-fluid neutron star mass of approximately $2.20M_\odot$, consistent with the existence of massive

neutron stars. This idealized assignment of the same equation of state to both sectors is not intended to represent the full range of possible mirror-sector microphysics; rather, it allows us to isolate the effects of the two-fluid structure and gravitational coupling without simultaneously varying an independent dark-sector equation of state. The formalism employed here, however, is not restricted to this specific case and applies more generally to any two-component neutron star model in which the constituents interact only gravitationally and can be treated as dynamically independent fluids. The results presented below therefore provide a concrete demonstration of the numerical implementation and a fully general relativistic characterization of the polar nonradial mode structure in gravitationally coupled two-fluid stars with independently specified central densities and potentially distinct radial supports.

Figure 2 shows the equilibrium configuration space of mirror dark matter admixed two-fluid neutron stars constructed with the QHC21-BT equation of state in the plane of the central energy densities of the nuclear matter and dark matter components, $(\mathcal{E}_c^{\text{NM}}, \mathcal{E}_c^{\text{DM}})$. These equilibrium configurations are obtained by solving the two-fluid TOV equations, Eqs. (10)–(12), for each pair of central densities. The color scale represents the total gravitational mass M of

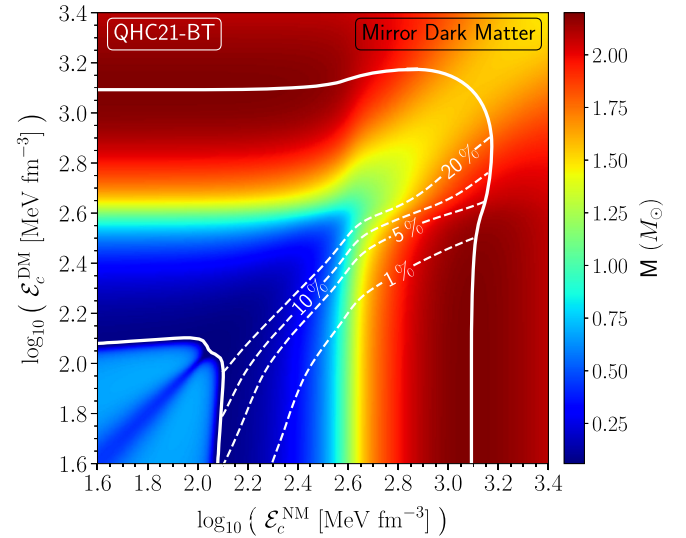


FIG. 2. Gravitational mass contours of two-fluid neutron stars with mirror dark matter constructed using the QHC21-BT equation of state. The color map shows the total stellar mass M (in units of $M_\odot$) in the plane spanned by the central energy densities of the nuclear matter ($\mathcal{E}_c^{\text{NM}}$) and dark matter ($\mathcal{E}_c^{\text{DM}}$) components. Solid white curves delineate the stability boundary of two-fluid equilibrium configurations; dynamically stable equilibria exist only within the region enclosed by these curves, while configurations outside this domain are unstable [56]. Dashed white curves indicate contours of fixed dark matter mass fraction $F_{\text{DM}} \equiv (M_{\text{DM}}/M) \times 100 = 1\%, 5\%, 10\%$, and 20% . The mirror dark matter adopted here assigns the dark and nuclear components the same QHC21-BT equation of state.

the star, expressed in units of $M_\odot$. Owing to the mirror dark matter setup adopted here, where the nuclear matter and dark matter components are assigned the same equation of state, the mass contour structure is symmetric about the line $\mathcal{E}_c^{\text{NM}} = \mathcal{E}_c^{\text{DM}}$. The solid white curves mark the stability boundary of the two-fluid equilibrium configurations, as determined from the radial oscillation analysis of Refs. [55,56]. Stable equilibrium configurations are confined within the region enclosed by these curves, whereas configurations outside this region are unstable. In particular, the stable region occupies a finite and nontrivial domain in the $(\mathcal{E}_c^{\text{NM}}, \mathcal{E}_c^{\text{DM}})$ plane, rather than a simple rectangular area, reflecting the genuinely two-dimensional nature of the stability problem in gravitationally coupled two-fluid stars. The slight rightward shift of the upper stability branch at intermediate $\mathcal{E}_c^{\text{DM}}$ is another manifestation of this coupled response: the dark component modifies the mass distribution, pressure support, and gravitational potential, allowing stable models with somewhat larger $\mathcal{E}_c^{\text{NM}}$ than in the nearly dark-matter-free limit; see also Ref. [56]. The dashed white curves trace sequences of fixed dark matter mass fraction, $F_{\text{DM}} \equiv (M_{\text{DM}}/M) \times 100$, where M_{DM} denotes the gravitational mass of the dark matter component and M is the total stellar mass; the curves shown here correspond to $F_{\text{DM}} = 1\%, 5\%, 10\%$, and 20% , and illustrate how configurations with the same global dark matter content are distributed across the two-dimensional central energy density plane. These fixed- F_{DM} sequences will be

particularly useful in the following analysis, since they allow the oscillation spectra to be compared across equilibrium models with systematically varying dark matter content. The corresponding evolution of the constituent radii and the outer stellar surface across this configuration space is shown in Fig. 3.

Figure 3 maps the radial extents of the nuclear matter and dark matter components across the same two-dimensional equilibrium configuration space considered in Fig. 2. Over the common color map of the total gravitational mass M , the left panel displays black dashed contours of the nuclear matter radius R_{NM} , defined by the radial position at which the nuclear matter pressure vanishes, and white dotted contours of the dark matter radius R_{DM} , defined analogously by the vanishing of the dark matter pressure. The right panel then identifies the physical stellar radius $R \equiv \max(R_{\text{NM}}, R_{\text{DM}})$ through black dashed contours and superposes the white dashed fixed- F_{DM} sequences, so that the variation of the outer surface can be followed directly with stellar mass and dark matter content. The numerical labels placed along the radius contours give the corresponding values in kilometers, whereas those attached to the white dashed sequences in the right panel specify the dark matter mass fraction in percentage. Because the two constituents are governed by the same equation of state, the component-radius contours exhibit the expected reflection symmetry about the line $\mathcal{E}_c^{\text{NM}} = \mathcal{E}_c^{\text{DM}}$, along which $R_{\text{NM}} = R_{\text{DM}} = R$, while the component defining the outer stellar surface is

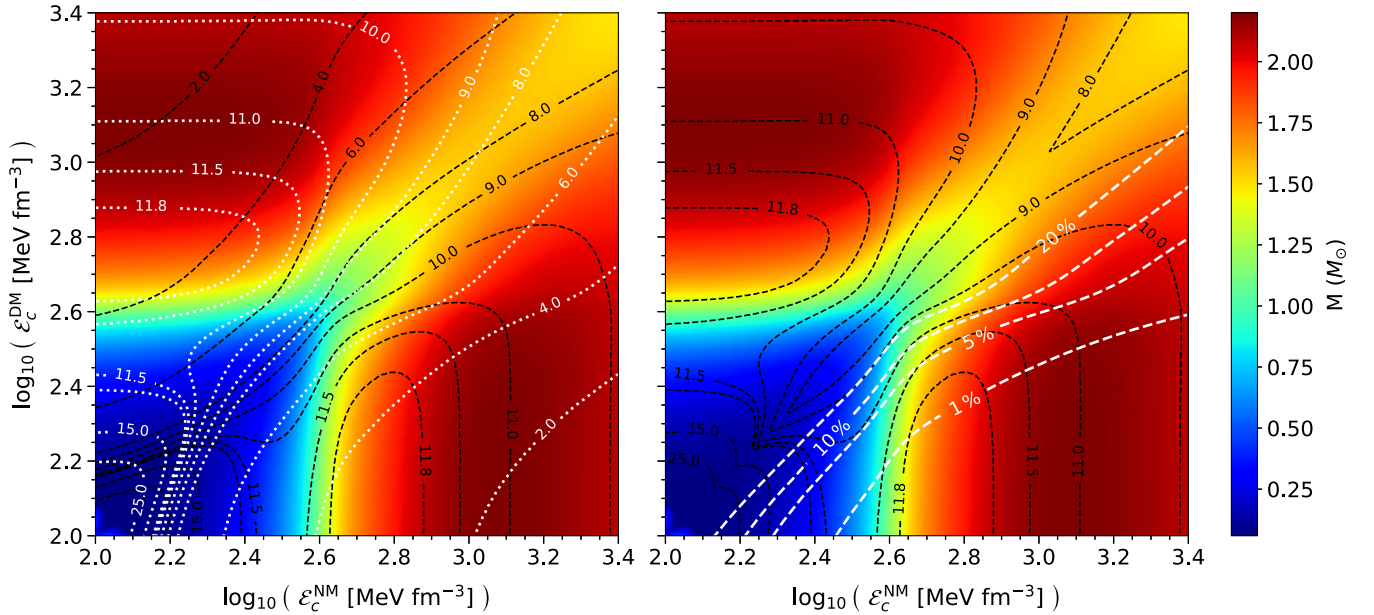


FIG. 3. Component and total radii of mirror dark matter admixed two-fluid neutron stars constructed with the QHC21-BT equation of state in the plane of nuclear matter and dark matter central energy densities, $(\mathcal{E}_c^{\text{NM}}, \mathcal{E}_c^{\text{DM}})$. In both panels, the color map represents the total gravitational mass M in units of $M_\odot$. The left panel shows contours of the nuclear matter radius R_{NM} (black dashed curves) and dark matter radius R_{DM} (white dotted curves), with all radius values given in km. The right panel shows contours of the total stellar radius $R \equiv \max(R_{\text{NM}}, R_{\text{DM}})$ (black dashed curves), together with sequences of fixed dark matter mass fraction $F_{\text{DM}} \equiv (M_{\text{DM}}/M) \times 100$ (white dashed curves) for $F_{\text{DM}} = 1\%, 5\%, 10\%$, and 20% . The combined representation illustrates how the radial extents of the two constituents and the outer stellar surface vary with the total mass and dark matter content of the configurations.

interchanged across this line. Away from this symmetric line, the constituent with the larger central energy density acquires the greater radial extent and therefore determines R , whereas the other component is confined to an increasingly smaller inner region as the contrast between the two central densities grows. The fixed- F_{DM} sequences considered here remain on the $\mathcal{E}_c^{\text{NM}} > \mathcal{E}_c^{\text{DM}}$ side of the configuration space, so that the nuclear matter component defines the outer stellar surface, $R = R_{\text{NM}}$, while the dark matter component occupies an inner region with $R_{\text{DM}} < R_{\text{NM}}$. This constituent-dependent radial structure, with the dark matter component embedded within the nuclear matter envelope for the sequences examined here, provides the physical basis for the distinct inner- and outer-fluid character of the oscillation modes identified below.

Figure 4 shows the boundary-condition residual, $|\mathcal{F}(\omega)|$, as a function of the real trial frequency $f = \omega/2\pi$ for representative s.f. and mirror dark matter two-fluid neutron star models. The three columns correspond to fixed total masses $M = 0.25M_\odot$, $1.00M_\odot$, and $1.75M_\odot$. The low-mass $0.25M_\odot$ model is included only as a diagnostic configuration: it makes several mode branches and their eigenfunction node structure visible within the frequency range shown, and is not intended to represent an observationally typical neutron star. The rows compare the standard single-fluid configurations constructed with the QHC21-BT equation of state (top) with gravitationally coupled two-fluid configurations having dark matter mass fractions $F_{\text{DM}} = 1\%$ (middle) and 5% (bottom). In the single-fluid case, the residual is obtained from the usual polar perturbation problem built on the corresponding single-fluid TOV background, which is recovered in the present framework when the second fluid is absent. For each stellar model, the eigenfrequencies are identified by the deep minima of the residual, which represent the frequencies at which the interior solution can be matched consistently to a purely outgoing exterior spacetime solution. In the standard single-fluid case, the residual exhibits the expected sequence of minima associated with the usual f- and p-modes. Once a second gravitationally coupled fluid is introduced, however, the structure of the spectrum becomes richer: additional minima appear, the original single-fluid sequence is supplemented by additional branches associated with the inner and outer fluid components, and the ordering of the pressure modes changes with both stellar mass and dark matter fraction. Even at the level of the residual function, the emergence of these extra deep minima already makes clear that the two-fluid system supports a substantially more intricate polar oscillation spectrum than its single-fluid counterpart.

In principle, the quasinormal mode frequencies of relativistic stars are complex, and may be written as $\omega = \omega_R + i\omega_I$, where the real part ω_R determines the oscillation frequency and the imaginary part ω_I encodes the damping due to gravitational-wave emission. In the present analysis,

however, we restrict the scan in Fig. 4 to real trial frequencies. This is adequate for the purpose of identifying and comparing the mode spectrum, since for the fluid modes considered here, the imaginary part is typically much smaller than the real part ($|\omega_I| \ll |\omega_R|$), so that the locations of the deep minima of $|\mathcal{F}(\omega)|$ track the oscillation frequencies quite accurately, e.g., [74]. The residual plots in Fig. 4 should therefore be understood as a practical and physically transparent way of resolving the real-frequency structure of the polar spectrum before turning to the detailed mode identification. Viewed in this way, the top row of Fig. 4 provides the standard single-fluid reference spectrum at the three representative masses considered here. For the lowest-mass model, $M = 0.25M_\odot$, one clearly resolves the fundamental mode together with a sequence of higher-order pressure modes (up to the p_4 -mode) within the scanned frequency range shown here, which extends up to 5 kHz. As the stellar mass is increased to $1.00M_\odot$ and $1.75M_\odot$, the scan is extended to 8 and 10 kHz, respectively, and the number and spacing of the visible minima change, but the residual still exhibits the familiar single-fluid polar mode structure, with a single f-mode followed by the ordered sequence of p-modes. Once mirror dark matter is introduced, however, the spectral structure changes both qualitatively and quantitatively. The two-fluid panels no longer display a single ordered branch of minima, but instead show multiple families associated with the outer nuclear matter fluid and the inner dark matter fluid, labeled here by the superscripts O and I , respectively. For the configurations shown in Fig. 4, the dark matter component remains the inner fluid, while the nuclear matter extends to the outer stellar surface, so this identification is unambiguous throughout the panels. The presence of these additional minima, however, does not by itself determine their fluid character, and a reliable assignment of the corresponding modes to the inner dark matter or outer nuclear matter component requires examination of the associated eigenfunctions.

To make this assignment explicit, Fig. 5 shows the normalized radial-displacement eigenfunctions⁷ associated with the minima identified in Fig. 4 for the $M = 0.25M_\odot$, $F_{\text{DM}} = 1\%$ configuration, together with the corresponding single-fluid reference spectrum at the same mass. The top-left panel of Fig. 5 displays the normalized displacement eigenfunctions of the single-fluid star for the f- and p_{1-4} -modes, while the remaining panels show the corresponding two-fluid eigenfunctions W_{norm}^I and W_{norm}^O for the eight deep minima appearing in the $F_{\text{DM}} = 1\%$ residual spectrum. In the two-fluid panels, the vertical dashed line marks

⁷Throughout this section, the plotted displacement eigenfunctions are normalized by their value at the relevant surface: $W_{\text{norm}} \equiv W/W(R)$ in the single-fluid case, $W_{\text{norm}}^I \equiv W_I/W_I(R_I)$ for the inner component, and $W_{\text{norm}}^O \equiv W_O/W_O(R_O)$ for the outer component.

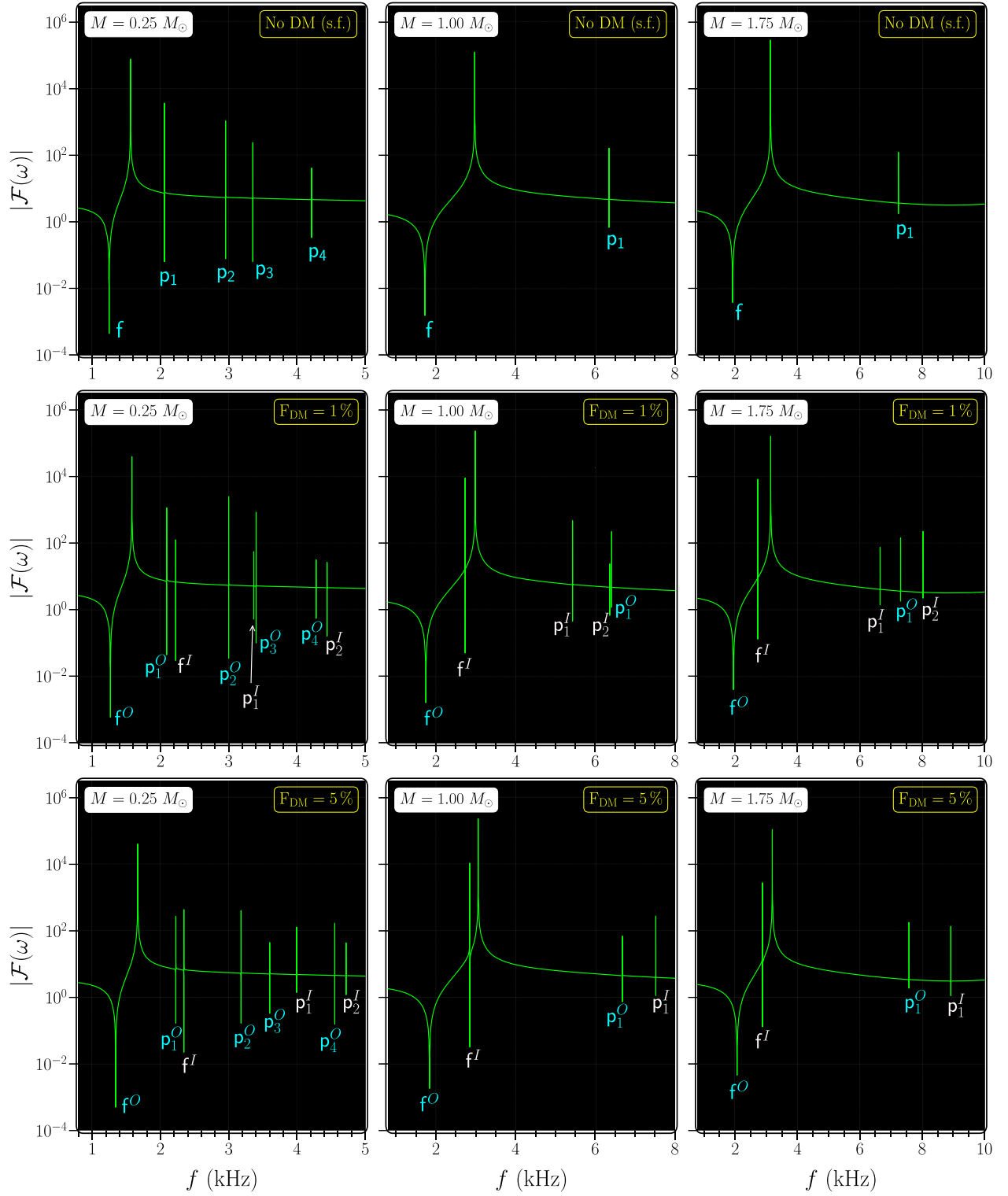


FIG. 4. Boundary-condition residual (error function) $|\mathcal{F}(\omega)|$ as a function of the real trial frequency $f = \omega/2\pi$ for polar nonradial oscillations. The top row corresponds to single-fluid (s.f.) neutron stars constructed using the QHC21-BT equation of state, while the middle and bottom rows show gravitationally coupled two-fluid configurations with mirror dark matter based on the same microphysics, for dark matter mass fractions $F_{\text{DM}} = 1\%$ and 5% , respectively. Each panel shows the scanning result for a fixed total mass $M = 0.25M_\odot$ (left column), $M = 1.00M_\odot$ (middle column), and $M = 1.75M_\odot$ (right column). The eigenfrequencies are identified by the deep minima of $|\mathcal{F}(\omega)|$ along the real-frequency axis, which locate the real parts of weakly damped modes; the imaginary parts associated with gravitational-wave damping are not extracted in this scan. Labels indicate the dominant outer/nuclear- and inner/dark-fluid character of the fundamental and pressure modes, highlighting the hybridization of modes induced by the presence of a second gravitationally coupled fluid.

the radius of the inner fluid, so that W_{norm}^I is defined only inside this boundary, whereas W_{norm}^O extends across the full stellar radius. In the single-fluid panel, the mode identification is straightforward: the f-mode is the node-free solution, while the successive p-modes are distinguished by the increasing number of radial nodes in the displacement eigenfunction, made particularly clear by the inset around $W_{\text{norm}} \simeq 0$. This single-fluid pattern provides the reference against which the two-fluid eigenfunctions can be interpreted, particularly for identifying those modes whose outer-fluid displacement W_{norm}^O retains the characteristic node structure of the standard single-fluid spectrum.

For the two-fluid configurations, the outer-fluid-led modes (*O*-type) are identified by matching the node structure of W_{norm}^O to the standard single-fluid sequence, whereas the additional modes that do not follow this outer-fluid pattern are classified through the corresponding node structure of W_{norm}^I within the inner-fluid region. The lowest-frequency deep minimum is identified as the outer-fluid-led fundamental mode (f^{*O*}-mode), since the corresponding outer-fluid eigenfunction W_{norm}^O remains node free and follows the same qualitative pattern as the single-fluid fundamental mode. This identification is not based on node counting of W_{norm}^O alone, because the inner-fluid eigenfunction W_{norm}^I is also node free within its domain; rather, the distinction becomes clear only after comparison with the eigenfunctions associated with the subsequent minima. The next minimum, whose associated eigenfunctions are shown in the upper-rightmost panel of Fig. 5, is identified as the first pressure mode of the *O*-led fluid (p₁^{*O*}-mode), because the corresponding outer-fluid eigenfunction W_{norm}^O develops a single radial node, matching the characteristic p₁ mode pattern of the single-fluid reference, even though the inner-fluid eigenfunction W_{norm}^I remains node free within the inner-fluid domain in this case too. The following minimum, whose associated eigenfunctions are shown in the leftmost panel of the middle row of Fig. 5, is then identified as the inner-fluid-led fundamental mode (f^{*I*}-mode): in this case, W_{norm}^I remains node free within the inner-fluid region, while the accompanying outer-fluid eigenfunction W_{norm}^O no longer matches the characteristic pattern of the single-fluid f- or p-mode sequence, and instead exhibits a distinctly distorted behavior near $W_{\text{norm}} \simeq 0$, indicating that this mode does not belong to the outer-fluid-led branch.

This also clarifies why the first two minima are not assigned to the inner-fluid-led (*I*-) branch: although W_{norm}^I remains node free in both cases, the corresponding outer-fluid eigenfunctions W_{norm}^O still retain the standard single-fluid f- and p₁-mode patterns, and are therefore naturally classified as *O*-led modes. Taken together, these two node free minima therefore admit a natural interpretation as the fundamental modes of the outer and inner fluids, respectively: the lower-frequency one (at 1.2665 kHz) preserves

the standard outer-fluid f-mode pattern and is identified as f^{*O*}, whereas the higher-frequency one (at 2.2225 kHz) is assigned to f^{*I*}, since its outer-fluid eigenfunction no longer follows any of the standard single-fluid mode patterns.

The same comparison can be extended to the higher-frequency minima of the $M = 0.25M_{\odot}$, $F_{\text{DM}} = 1\%$ configuration shown in Fig. 4, using the corresponding eigenfunctions plotted in the remaining panels of Fig. 5. The eigenmodes at 2.9987, 3.4017, and 4.2805 kHz are identified as the p₂^{*O*}-, p₃^{*O*}- and p₄^{*O*}-modes, respectively, because the corresponding outer-fluid eigenfunctions W_{norm}^O reproduce the expected two-node, three-node, and four-node structure of the standard single-fluid p-mode sequence. By contrast, the interleaved minima at 3.3662 and 4.4408 kHz are assigned to the *I*-led branch as the p₁^{*I*}- and p₂^{*I*}-modes, since the corresponding inner-fluid eigenfunctions W_{norm}^I exhibit one and two radial nodes, respectively, within the inner-fluid region, while the accompanying outer-fluid eigenfunctions W_{norm}^O no longer follow the characteristic pattern of the single-fluid p-mode sequence, and instead exhibit a noticeably distorted behavior near $W_{\text{norm}} \simeq 0$. The set of eight minima shown in Fig. 5 therefore separates naturally into two interwoven mode sequences: one associated with the outer nuclear matter fluid, which preserves the ordering pattern of the standard single-fluid spectrum, and the other associated with the inner dark matter fluid, which appears through additional modes interleaved with the outer-fluid-led branches. The same identification procedure is used for the corresponding minima in the $F_{\text{DM}} = 1\%$ models at $M = 1.00M_{\odot}$ and $1.75M_{\odot}$, and likewise for the $F_{\text{DM}} = 5\%$ configurations shown in Fig. 4.

B. Mode frequencies and compactness universality

With the mode identification established, we now examine how the identified f- and p₁-mode branches evolve with stellar mass and dark matter content. Figures 6 and 7 summarize, respectively, the mass dependence of the f-mode and p₁-mode frequencies for the single-fluid sequence and for gravitationally coupled two-fluid mirror dark matter configurations constructed with the QHC21-BT equation of state. In both figures, the black ringed square markers denote the standard single-fluid reference sequence, while the colored markers correspond to two-fluid configurations with fixed dark matter mass fractions $F_{\text{DM}} = 1\%, 5\%, 10\%$, and 20% . The filled circular markers represent the inner-fluid branch, which in the present models corresponds to the dark matter component, whereas the ringed circular markers denote the outer-fluid branch associated with the nuclear matter component. Taken together, these two figures provide a direct summary of how the standard single-fluid f- and p₁-mode sequences are reorganized into separate inner- and outer-fluid branches once a second gravitationally coupled fluid is present, with

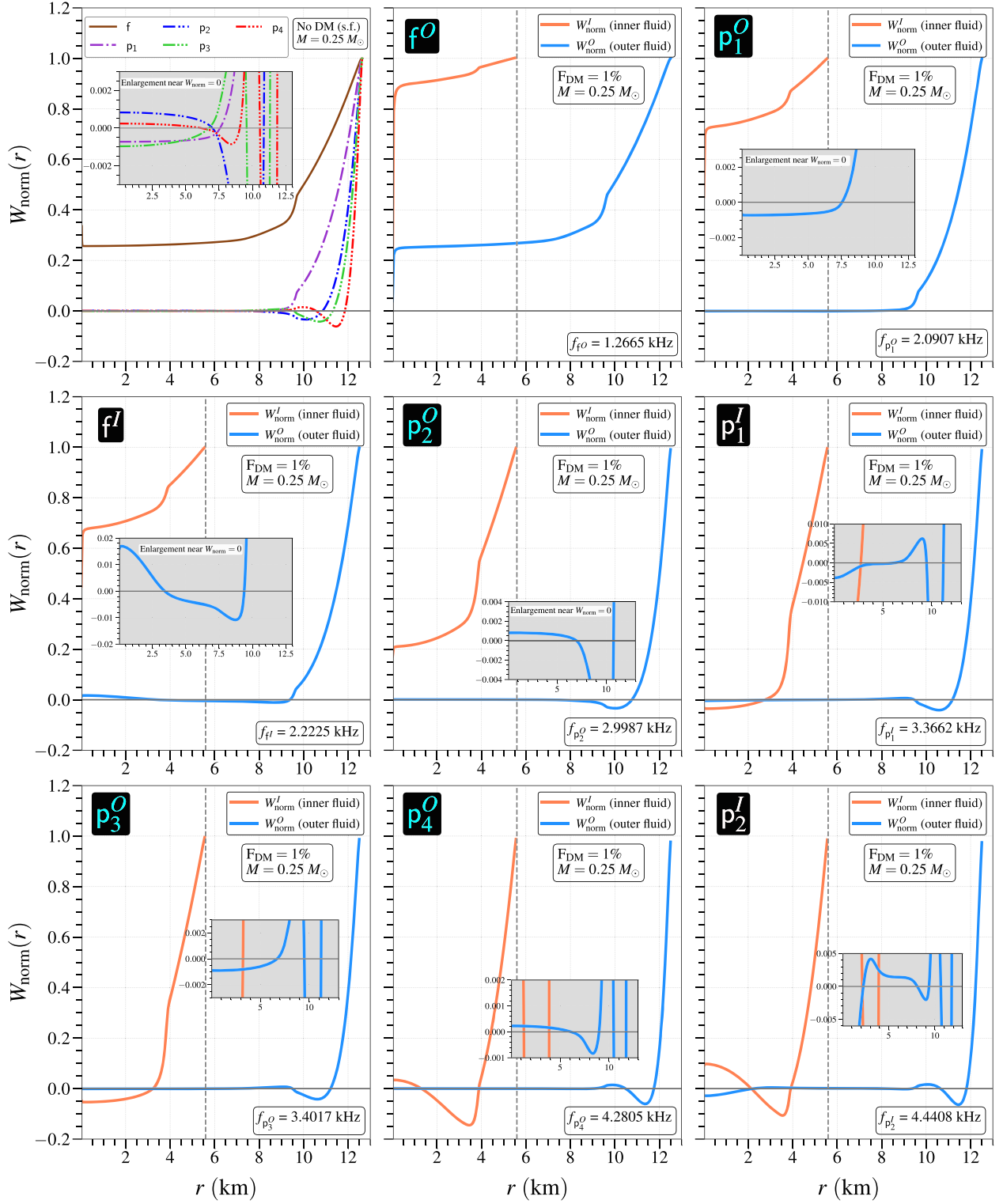


FIG. 5. Normalized radial-displacement eigenfunctions for polar nonradial modes in the QHC21-BT model at fixed gravitational mass $M = 0.25M_{\odot}$. The top-left panel shows the reference single-fluid neutron star (no dark matter), with the normalized displacement W_{norm} for the f and p_{1-4} modes. The remaining eight panels correspond to a gravitationally coupled two-fluid configuration with mirror dark matter and dark matter mass fraction $F_{\text{DM}} = 1\%$, showing the normalized displacements of the inner (dark) and outer (nuclear) components, $W_{\text{norm}}^I(r)$ and $W_{\text{norm}}^O(r)$, for the identified mode families. Each panel is labeled by the mode assignment and the corresponding eigenfrequency (in kHz), as extracted from the deep minima of $|\mathcal{F}(\omega)|$: f^O , p_1^O , f^I , p_2^O , p_1^I , p_3^O , p_4^O , p_2^I . Vertical dashed lines mark the radius of the inner-fluid component. Insets enlarge the neighborhood of $W_{\text{norm}} \simeq 0$ to make node structure and sign changes visible, enabling an unambiguous identification of which spectral feature corresponds to an outer- or inner-led mode.

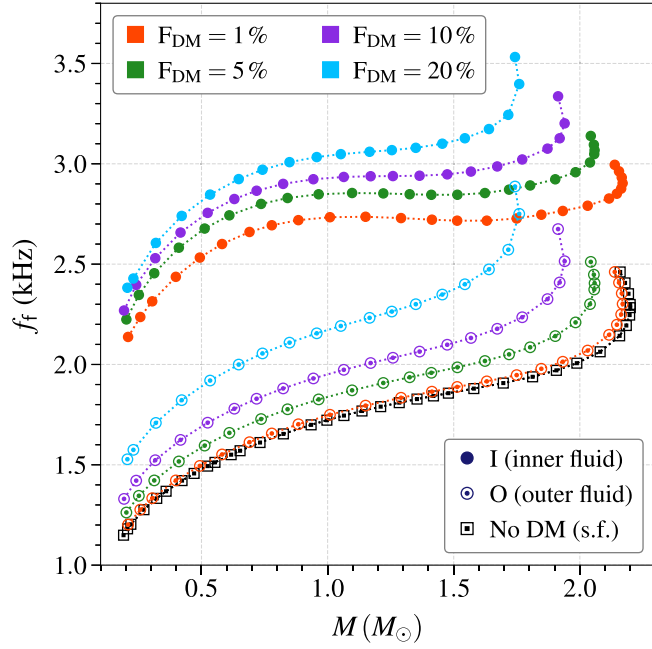


FIG. 6. Fundamental mode (f-mode) frequency f_f as a function of the gravitational mass M for single-fluid neutron stars and gravitationally coupled two-fluid configurations with mirror dark matter constructed using the QHC21-BT equation of state. Filled colored circular markers show the f-mode frequencies of the inner (dark matter) fluid led component, while ringed (bulls-eye) circular markers correspond to the outer (nuclear matter) fluid component. Different colors indicate dark matter mass fractions $F_{\text{DM}} = 1\%$, 5% , 10% and 20% , as shown in the legend. Black ringed square markers indicate the single-fluid (no dark matter) sequence. The two-fluid sequences exhibit a systematic separation of the fundamental mode into inner- and outer-fluid branches.

the branch assignment made using the eigenfunction-based identification described above.

Figure 6 shows that, in the single-fluid case, the fundamental-mode frequency increases smoothly with stellar mass for neutron stars constructed with the QHC21-BT equation of state, thereby providing a useful reference against which the two-fluid trends can be assessed. A clear feature of Fig. 6 is that the outer-fluid branch, associated with the nuclear matter component, remains relatively closer to this single-fluid reference, whereas the inner-fluid branch, associated with the dark matter component, is shifted to systematically higher f-mode frequencies over the full mass range considered here. At fixed mass, the inner-fluid f-mode is always found at a higher frequency than the corresponding outer-fluid mode, while the latter continues to track the behavior of the standard single-fluid sequence more closely. This trend is expected because the outer-fluid branch is associated with the nuclear matter component that still extends across the stellar radius, so that its displacement samples the full radial extent of the configuration and therefore continues to govern the global bulk oscillation

of the star in a way that closely resembles the standard one-fluid case. By contrast, the inner-fluid branch is associated with oscillations of the more compact dark matter core. Since this motion is confined to a smaller radial region, it is characterized by a shorter effective length scale, and the corresponding restoring response operates over that shorter scale, leading naturally to a higher characteristic frequency. As the dark matter fraction increases, the equilibrium configuration itself is modified more strongly by the presence of the inner dark matter component, so that the common gravitational field and the associated restoring dynamics of the two-fluid system depart progressively farther from the single-fluid case. This is reflected in the progressively larger displacement of both two-fluid f-mode branches away from the single-fluid reference as F_{DM} increases and the two-fluid character of the star becomes more pronounced.

Figure 7 depicts the p_1 -mode frequency as a function of gravitational mass for the same single-fluid and two-fluid mirror dark matter sequences considered in Fig. 6. In contrast to the f-mode case, the behavior of the p_1 -mode branches for these two-fluid stars is qualitatively different, especially for the inner-fluid sequence. The outer-fluid p_1 -branch, associated with the nuclear matter component, still follows the overall trend of the standard single-fluid sequence closely, but the inner-fluid branch behaves in a qualitatively different manner. For $F_{\text{DM}} = 1\%$, the inner-fluid p_1 -branch lies below both the outer-fluid branch and the single-fluid reference over a broad intermediate-mass range, before turning upward again toward the high-mass

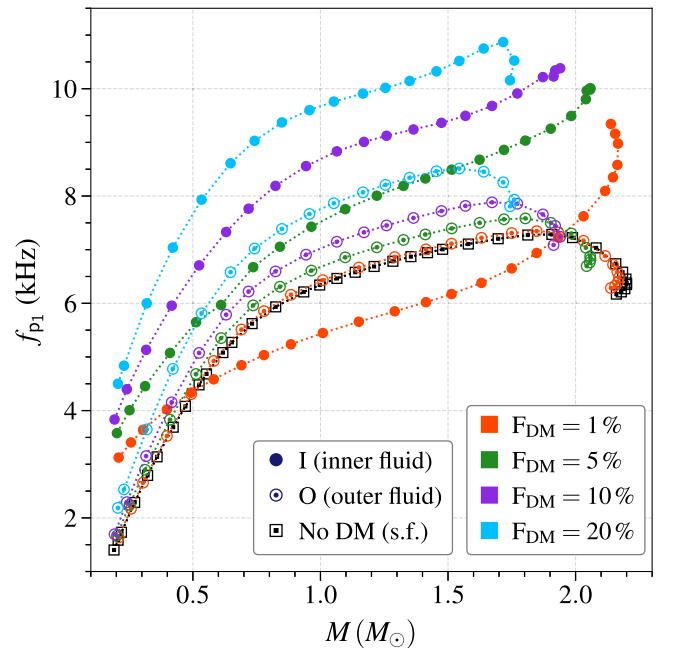


FIG. 7. Same as Fig. 6, but showing the first pressure mode (p_1) frequency as a function of M with the same marker and color conventions.

end. This more complicated behavior reflects the stronger sensitivity of pressure modes to where the compressional energy is localized between the extended nuclear layer and the inner dark matter region, so small changes in the two-fluid stratification can reorganize the p_1 branch more strongly than the f branch. For larger dark matter fractions, by contrast, the inner-fluid p_1 -branch lies above both the outer-fluid branch and the single-fluid reference over the full mass range considered here, with its frequency generally increasing as F_{DM} is raised from 5% to 20%. Overall, the p_1 -mode shows a stronger sensitivity than the f -mode to the detailed two-fluid structure of the star, rather than primarily to its overall bulk motion.

Figure 8 presents the fundamental-mode results of Fig. 6 in the form commonly used to examine the compactness-based universal relation, by plotting the mass-scaled frequency $f_t M_{1.4}$, with $M_{1.4} \equiv M/(1.4M_\odot)$, against the stellar compactness $C \equiv M/R$. For standard single-fluid neutron stars, this representation is known to yield a relation that is nearly insensitive to the equation of state between the f -mode frequency and compactness [46]. Here, it provides a useful way to assess how the standard single-fluid behavior is altered in the two-fluid configurations, where a second gravitationally coupled fluid is present. The black ringed squares denote the single-fluid QHC21-BT sequence, while the colored points correspond to mirror dark matter two-fluid configurations at fixed F_{DM} , with filled and ringed circles representing the inner- and outer-fluid branches, respectively.

For the single-fluid sequence, the mass-scaled f -mode frequency follows the expected monotonic increase with compactness, consistent with the standard compactness-based f -mode relation for ordinary neutron stars. The mirror dark matter two-fluid configurations, however, no longer collapse onto a single compactness sequence, but instead separate into two distinct branches: an outer-fluid branch that remains close to the single-fluid trend, and an inner-fluid branch that is shifted systematically to higher values of $f_t M_{1.4}$ at the same compactness. This behavior is a direct consequence of the two-fluid character of the oscillation spectrum: even in the adopted mirror dark matter case, where both fluids are assigned the same equation of state, the two branches correspond to perturbations that are preferentially supported in different radial regions and are coupled through the common spacetime. As a result, the two-fluid configurations do not collapse onto the single-fluid compactness trend: configurations with comparable total compactness can exhibit different mass-scaled f -mode frequencies depending on whether the mode belongs to the inner- or outer-fluid-led branch.

Another notable feature is the systematic evolution of this branch structure with increasing F_{DM} . In the total compactness representation used in Fig. 8, this evolution is most clearly visible in the outer-fluid branch, which shifts progressively upward away from the single-fluid sequence

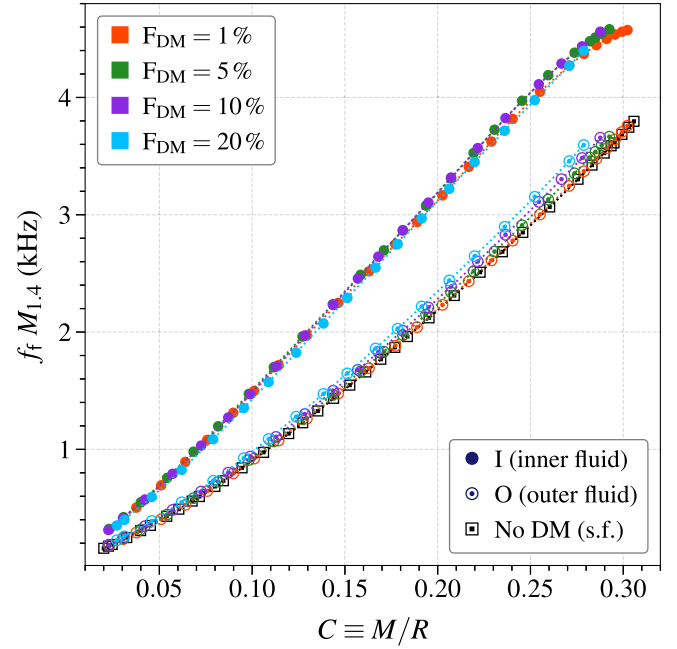


FIG. 8. Mass-scaled fundamental-mode relation for mirror dark matter two-fluid stars constructed with the QHC21-BT equation of state. The f -mode frequency is rescaled as $f_t M_{1.4}$ with $M_{1.4} \equiv M/(1.4M_\odot)$, and plotted against the stellar compactness $C \equiv M/R$. Colored points correspond to gravitationally coupled two-fluid configurations with different colors indicating the dark matter mass fraction $F_{\text{DM}} = 1\%$, 5% , 10% and 20% (see legend). For each F_{DM} sequence, filled (solid) circular markers show the inner-fluid branch, while ringed circular markers show the outer-fluid branch, as indicated in the marker-style legend box. Black ringed square markers denote the standard single-fluid (no dark matter) sequence.

as the dark component becomes more important. The corresponding change in the inner-fluid branch is less transparent in this projection, since the horizontal axis is the total stellar compactness $C \equiv M/R$, whereas the two mode branches are more naturally interpreted using the compactness scale of the fluid component that predominantly supports each branch. This can be seen more clearly in Fig. 9 of Appendix B, where the same mass-scaled frequencies are plotted against the component wise diagnostic compactnesses, $C_I \equiv M_I/R_I$ and $C_O \equiv M_O/R_O$, with M_I and M_O denoting the mass-function contributions of the inner and outer fluids, respectively. In that representation, increasing F_{DM} drives both the inner- and outer-fluid sequences toward one another. This is the expected behavior as the mirror configuration approaches the symmetric case, $F_{\text{DM}} = 50\%$, where the two fluids have the same equation of state and contribute equally to the equilibrium stellar structure. In this limit, the system is invariant under interchange of the two components, so the distinction between inner- and outer-fluid-dominated oscillations no longer has a unique physical meaning. The fundamental mode is then more naturally understood as a

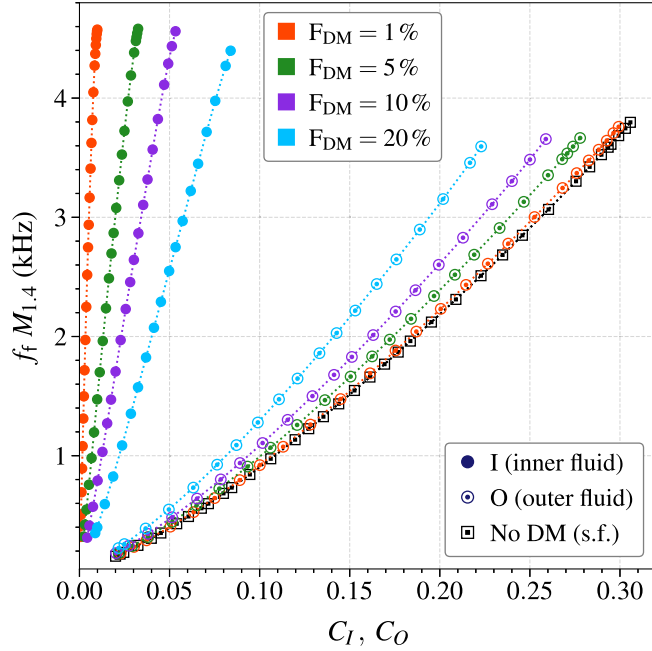


FIG. 9. Mass-scaled fundamental-mode relation plotted against component-wise diagnostic compactnesses of the inner and outer fluid components for mirror dark matter two-fluid stars constructed with the QHC21-BT equation of state. The same quantity $f_f M_{1.4}$ shown in Fig. 8 is plotted here against the componentwise diagnostic compactness of each fluid component, $C_I \equiv M_I/R_I$ for the inner component and $C_O \equiv M_O/R_O$ for the outer component, where M_I and M_O denote the additive mass-function contributions of the corresponding fluids, as defined in the text, and R_I and R_O are the associated fluid surfaces. Filled circular markers show the inner-fluid branch, while ringed circular markers show the outer-fluid branch. Different colors correspond to fixed dark matter mass fractions $F_{\text{DM}} = 1\%, 5\%, 10\%$ and 20% , and black ringed square markers denote the single-fluid QHC21-BT reference sequence. This representation makes clear that the evolution of the two branches with increasing F_{DM} is more naturally interpreted in terms of the compactness scale of the fluid component that predominantly supports each mode branch.

collective oscillation of two identical gravitationally coupled fluids, rather than as two separately identifiable branches.

Taken together, these results show that, in the mirror dark matter configurations considered here, the presence of a second fluid, even when it is assigned the same equation of state as the ordinary matter component and interacts only through gravity, qualitatively changes the nonradial oscillation spectrum of the star. In particular, the emergence of distinct inner- and outer-fluid-led branches and their departure from the standard mass-scaled f-mode universal relation with compactness demonstrate that the usual single-fluid asteroseismic intuition cannot be carried over unchanged to such gravitationally coupled two-fluid configurations. The mirror dark matter setup adopted here therefore provides a particularly clean example of how additional fluid degrees

of freedom can leave identifiable signatures in the mode spectrum, even in the absence of any nongravitational coupling.

V. SUMMARY AND OUTLOOK

In this work, we formulated the polar nonradial perturbations of gravitationally coupled multifluid neutron stars in full general relativity directly from the independently conserved perfect fluid stress-energy tensors of the constituent fluids. This construction corresponds to the non-interacting limit of the general relativistic variational multifluid formalism developed in Ref. [31], in which the coefficients governing direct cross-fluid coupling and entrainment vanish while the constituents remain dynamically coupled through the metric perturbations of the common spacetime. By expressing the constituent chemical potentials and master-function response coefficients in terms of the energy density, pressure, and adiabatic response of each fluid, we verified that the resulting metric and fluid perturbation equations are mathematically equivalent to this established noninteracting limit. Within this perfect-fluid representation, we derived the coupled linear perturbation equations for the metric and the constituent fluid components, and completed the formulation by imposing regularity at the stellar center, boundary conditions at the individual fluid surfaces, and the matching conditions at the outer surface needed to connect the interior solution to the exterior vacuum spacetime and define the eigenvalue problem. Taken together, these ingredients provide a self-consistent general relativistic basis for studying nonradial oscillations in a broad class of gravitationally coupled multicomponent compact stars.

To demonstrate the practical use of the formalism, we implemented the coupled perturbation system numerically and computed representative polar mode spectra for gravitationally coupled two-fluid stellar models. This implementation made it possible to solve the full two-fluid eigenvalue problem in the presence of spacetime coupling and to identify how the additional fluid degree of freedom reorganizes the structure of the nonradial mode spectrum. A central outcome of this implementation is that it provides a practical way to address mode identification in gravitationally coupled two-fluid stars, allowing the f- and p-mode branches of the spectrum to be classified by their dominant inner- or outer-fluid character through the associated eigenfunctions and their node structure. In this way, the present implementation establishes a workable strategy for computing and interpreting the polar oscillation spectrum of gravitationally coupled two-fluid stars with distinct constituent surfaces.

We applied this formulation to mirror dark matter admixed neutron stars constructed with the QHC21-BT equation of state. This choice isolates the effect of the two-fluid structure, since the nuclear and dark components obey the same equation of state, while remaining dynamically

independent and coupled only through the spacetime geometry. The resulting nonradial spectrum differs qualitatively from the standard single-fluid one. The residual function develops additional minima associated with the extra fluid degree of freedom, and the corresponding eigenfunctions separate the spectrum into inner- and outer-fluid branches. For the fundamental mode, the outer-fluid branch remains closest to the single-fluid sequence, but is shifted systematically as the dark matter fraction increases, showing that the ordinary-matter-led oscillation is modified by the gravitational presence of the dark component. The inner-fluid f-mode branch appears at higher frequencies over the range of dark matter fractions considered, indicating that the second fluid introduces a distinct oscillation channel rather than merely perturbing the usual single-fluid mode. Consequently, the usual mass-scaled f-mode relation with compactness does not remain a single universal sequence: configurations with similar total compactness can yield different values of $f_t M_{1.4}$, depending on whether the mode belongs to the inner- or outer-fluid branch. This branch-dependent departure from the single-fluid relation provides one of the clearest signatures of the two-fluid character of the oscillation spectrum.

The direct multifluid formulation based on independently conserved perfect fluid stress-energy tensors, together with its numerical implementation, presented here also opens several natural directions for future work. Although the present study has focused on an idealized mirror dark matter realization as a clean baseline application, the same noninteracting formulation can be applied to more general gravitationally coupled two-fluid scenarios involving distinct equations of state and nontrivial dark-sector microphysics, including fermionic and self-interacting dark matter models. Such extensions will make it possible to explore how the detailed properties of the second fluid modify the nonradial spectrum, the organization of the mode branches, and the associated asteroseismic signatures in a quantitatively richer setting. Models incorporating entrainment or direct microphysical coupling between the constituents would instead require the corresponding interacting sector of the general variational multifluid formalism. In addition, while the present study has concentrated on representative polar fluid modes, the present implementation can be extended to other important sectors of the oscillation spectrum, including buoyancy-driven g-modes and spacetime-dominated w-modes. The latter are especially interesting in the present setting because both fluids couple to the same dynamical spacetime and may therefore modify the spacetime-led spectrum in a nontrivial manner. More broadly, extending relativistic asteroseismology to genuinely multicomponent compact stars may help clarify how additional internal degrees of freedom are encoded in oscillation spectra and, ultimately, how such effects could be probed through future gravitational-wave observations.

ACKNOWLEDGMENTS

This work is supported in part by the Japan Society for the Promotion of Science (JSPS) KAKENHI Grants No. JP23K20848 and No. JP24KF0090, and by the Mitsubishi Foundation through Grant No. 202510029. D. A. C. is grateful for support from UIUC Graduate College and the Grainger College of Engineering, and from the Sloan Foundation. N. Y. acknowledges the support of the Simons Foundation through Award No. 896696, the Simons Foundation International through Award No. SFI-MPS-black hole-00012593-01 and the National Science Foundation (NSF) through Grant No. PHY-25-12423.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: ALGEBRAIC DETAILS OF THE PERTURBATION EQUATIONS

In this Appendix, we complement Sec. II C by providing the intermediate algebraic steps leading to the derivation of Eqs. (27), (28) and (29). These relations are used in Sec. II C to close the first-order perturbation system for the variables K , H_1 , W_X , and $\mathcal{P}_X$. Throughout this Appendix, the harmonic time dependence and the summation over spherical-harmonic indices are suppressed, while the explicit r^l scaling and the angular dependence through Y^{lm} are retained.

1. Derivation of the algebraic relation for V_X

We first derive Eq. (28) from the angular component of the perturbed Euler equation, Eq. (18). We evaluate the angular component in index-lowered form. For polar perturbations, the θ -component reduces to

$$(\mathcal{E}_X + p_X)\delta a_\theta^X + \partial_\theta \delta p_X = 0, \quad (\text{A1})$$

where δa_θ^X is the Eulerian perturbation of the four-acceleration of fluid X . The corresponding Eulerian perturbation of the four-acceleration is

$$\delta a_\theta^X = -r^l \left(\frac{1}{2} H_0 - e^{-2\Phi} \omega^2 V_X \right) \partial_\theta Y^{lm}. \quad (\text{A2})$$

The pressure term (i.e., $\partial_\theta \delta p$) follows from the relation between Eulerian and Lagrangian perturbations,

$$\begin{aligned}\partial_\theta \delta p_X &= \partial_\theta \left[\Delta_X p_X - \xi_X^r \frac{d}{dr} p_X \right] \\ &= r^l \left[-e^{-\Phi} \mathcal{P}_X + e^{-\Lambda} \frac{\mathcal{E}_X + p_X}{r} \Phi' W_X \right] \partial_\theta Y^{lm}. \quad (\text{A3})\end{aligned}$$

Substituting Eqs. (A2) and (A3) into Eq. (A1), and using the fact that the angular dependence factors out through $\partial_\theta Y^{lm}$, gives the algebraic relation for V_X ,

$$\begin{aligned}(\mathcal{E}_X + p_X) \omega^2 V_X &= e^\Phi \mathcal{P}_X - e^{2\Phi-\Lambda} \frac{\mathcal{E}_X + p_X}{r} \Phi' W_X \\ &\quad + \frac{1}{2} (\mathcal{E}_X + p_X) e^{2\Phi} H_0, \quad (\text{A4})\end{aligned}$$

which is Eq. (28).

2. Derivation of the constraint for H_0

We next derive the algebraic relation (29), which determines H_0 in terms of the metric amplitudes K and H_1 , together with the fluid pressure variables $\mathcal{P}_X$. The derivation

uses the $r\theta$ and rr components of the linearized Einstein equations, since these components contain dH_0/dr and dK/dr and can be combined to obtain an independent expression for dK/dr . The $r\theta$ component of the perturbed Einstein tensor is

$$\begin{aligned}\delta G_r^\theta &= \frac{r^{l-2}}{2} \left[\frac{dH_0}{dr} - \frac{dK}{dr} + H_0 \left(\frac{l}{r} + 2\Phi' \right) - \frac{l}{r} K \right. \\ &\quad \left. - r e^{-2\Phi} \omega^2 H_1 \right] \partial_\theta Y_{lm}. \quad (\text{A5})\end{aligned}$$

For the perfect-fluid stress-energy tensor considered here, there is no $r\theta$ component of the matter perturbation, so that $\delta T_r^\theta = 0$. The $r\theta$ component of the linearized Einstein equation therefore gives $\delta G_r^\theta = 0$, which provides one relation between dH_0/dr and dK/dr .

The second relation is obtained from the rr component of the linearized Einstein equations. The corresponding perturbation of the Einstein tensor is

$$\begin{aligned}\delta G_r^r &= r^l \left\{ e^{-2\Lambda} \left[\left(\frac{1}{r} + \Phi' \right) \frac{dK}{dr} - \frac{1}{r} \frac{dH_0}{dr} \right] - K \left[\frac{(l-1)(l+2)}{2r^2} - e^{-2\Phi} \omega^2 - e^{-2\Lambda} \frac{l}{r} \left(\frac{1}{r} + \Phi' \right) \right] \right. \\ &\quad \left. + \frac{H_0}{r} \left[\frac{l(l+1)}{2r} - e^{-2\Lambda} \left(\frac{l+1}{r} + 2\Phi' \right) \right] - 2e^{-2\Lambda-2\Phi} \omega^2 H_1 \right\} Y_{lm}. \quad (\text{A6})\end{aligned}$$

The matter source in this component is given by the Eulerian pressure perturbation summed over all fluids,

$$\delta T_r^r = \sum_X \delta p_X = r^l \sum_X \left[-e^{-\Phi} \mathcal{P}_X + e^{-\Lambda} \frac{\mathcal{E}_X + p_X}{r} \Phi' W_X \right] Y^{lm}. \quad (\text{A7})$$

The rr component of the linearized Einstein equations is therefore obtained by imposing $\delta G_r^r = 8\pi \delta T_r^r$. This gives the second relation involving dH_0/dr and dK/dr , together with algebraic terms in H_0 , H_1 , K , W_X , and $\mathcal{P}_X$.

Together, the $r\theta$ and rr components provide two linear algebraic equations for the derivative pair dH_0/dr and dK/dr . Solving this pair for dK/dr , and using the background Einstein equations to simplify the result, gives an expression independent of Eq. (23):

$$\begin{aligned}\Phi' \frac{dK}{dr} &= K \left[e^{2\Lambda} \frac{(l-1)(l+2)}{2r^2} - e^{2\Lambda-2\Phi} \omega^2 - \frac{l}{r} \Phi' \right] + e^{-2\Phi} \omega^2 H_1 - H_0 \left[e^{2\Lambda} \frac{l(l+1)}{2r^2} - \frac{1}{r^2} \right] \\ &\quad + 8\pi e^\Lambda \sum_X \Phi' \frac{\mathcal{E}_X + p_X}{r} W_X - 8\pi e^{2\Lambda-\Phi} \sum_X \mathcal{P}_X. \quad (\text{A8})\end{aligned}$$

Equation (A8) is obtained solely from the $r\theta$ and rr components of the linearized Einstein equations. It therefore provides an independent expression for dK/dr , whereas Eq. (23) was obtained from the tr component. Multiplying Eq. (23) by Φ' and equating the result to Eq. (A8) eliminates dK/dr . After collecting terms, one obtains the algebraic constraint equation for H_0 , Eq. (29).

3. Derivation of the differential equation for $\mathcal{P}_X$

We finally derive Eq. (27), which gives the first-order evolution equation for the pressure variable $\mathcal{P}_X$. This equation follows from the radial component of the perturbed Euler equation, Eq. (18). Evaluating the radial component in index-lowered form, the relevant terms reduce to

$$(\mathcal{E}_X + p_X)\delta a_r^X + (\delta\mathcal{E}_X + \delta p_X)a_r^X + \partial_r\delta p_X = 0. \quad (\text{A9})$$

The Eulerian perturbation of the radial component of the four-acceleration is

$$\delta a_r^X = r^l \left[-\omega^2 \frac{e^{\Lambda-2\Phi}}{r} W_X - \frac{l}{2r} H_0 - \frac{1}{2} \frac{dH_0}{dr} - e^{-2\Phi} \omega^2 r H_1 \right] Y^{lm}. \quad (\text{A10})$$

The radial derivative of the Eulerian pressure perturbation follows from $\delta p_X = \Delta_X p_X - \xi_X^r (dp_X/dr)$, together with

$\mathcal{P}_X = -e^\Phi \Delta_X p_X$ and the background Euler equation $dp_X/dr = -(\mathcal{E}_X + p_X)\Phi'$. This gives

$$\partial_r \delta p_X = \frac{d}{dr} [-r^l \mathcal{P}_X e^{-\Phi} + e^{-\Lambda} r^{l-1} W_X (\mathcal{E}_X + p_X) \Phi'] Y^{lm}. \quad (\text{A11})$$

Substituting these expressions into Eq. (A9), expanding the radial derivative of δp_X , and removing the common factor $r^l Y^{lm}$, one obtains

$$0 = -e^{-\Phi} \frac{d\mathcal{P}_X}{dr} + e^{-\Lambda} \frac{\mathcal{E}_X + p_X}{r} \Phi' \frac{dW_X}{dr} - \frac{1}{2} (\mathcal{E}_X + p_X) \frac{dH_0}{dr} - e^{-\Phi} \left[\frac{l}{r} + \frac{\Phi'}{c_X^2} \right] \mathcal{P}_X - (\mathcal{E}_X + p_X) \left\{ \frac{l}{2r} H_0 + r e^{-2\Phi} \omega^2 H_1 + \frac{e^\Lambda}{r} \left[e^{-2\Phi} \omega^2 - e^{-2\Lambda} \Phi'' + e^{-2\Lambda} \left(\Lambda' - \frac{l-1}{r} \right) \Phi' \right] W_X \right\}. \quad (\text{A12})$$

Here $c_X^2 \equiv dp_X/d\mathcal{E}_X$ has been used to write $d\mathcal{E}_X/dr = (d\mathcal{E}_X/dp_X)(dp_X/dr) = -(\mathcal{E}_X + p_X)\Phi'/c_X^2$. The intermediate equation still contains the derivatives dW_X/dr and dH_0/dr . The first of these is eliminated using Eq. (26). To remove dH_0/dr , we use the rr component of the linearized Einstein equations together with Eq. (23). Specifically, substituting Eq. (23) into Eq. (A6) and imposing $\delta G_r{}^r = 8\pi\delta T_r{}^r$ gives

$$\frac{dH_0}{dr} = \left[e^{2\Lambda} \frac{l(l+1)}{2r} - \left(\frac{l}{r} + \Phi' \right) \right] H_0 + \left[\frac{1}{2} l(l+1) \left(\frac{1}{r} + \Phi' \right) - 2e^{-2\Phi} r \omega^2 \right] H_1 - r \left[e^{2\Lambda} \frac{(l-1)(l+2)}{2r^2} + \frac{1}{r^2} - (\Phi')^2 - e^{2\Lambda-2\Phi} \omega^2 \right] K + 8\pi \sum_X \left[r e^{2\Lambda-\Phi} \mathcal{P}_X + e^\Lambda \frac{\mathcal{E}_X + p_X}{r} W_X \right]. \quad (\text{A13})$$

Substituting Eqs. (26) and (A13) into the intermediate radial Euler equation eliminates the remaining derivatives dW_X/dr and dH_0/dr . After collecting terms, one obtains

$$\begin{aligned} \frac{d\mathcal{P}_X}{dr} = & -\frac{l}{r} \mathcal{P}_X - (\mathcal{E}_X + p_X) e^\Phi \left\{ \Phi' \frac{l(l+1)}{r^2} V_X + \frac{l(l+1)}{4} \left(\frac{1}{r} + \Phi' \right) H_1 + e^{2\Lambda} \frac{l(l+1)}{4r} H_0 \right\} \\ & + (\mathcal{E}_X + p_X) e^\Phi \left\{ r K \left[\frac{1}{2r^2} + e^{2\Lambda} \frac{(l+2)(l-1)}{4r^2} - \frac{1}{2} e^{2\Lambda-2\Phi} \omega^2 - \Phi' \left(\frac{1}{r} + \frac{1}{2} \Phi' \right) \right] \right. \\ & \left. - \frac{W_X}{r} \left[e^{\Lambda-2\Phi} \omega^2 - r^2 \frac{d}{dr} \left(\frac{1}{r^2} e^{-\Lambda} \Phi' \right) \right] \right\} - 4\pi (\mathcal{E}_X + p_X) \left[\sum_Y e^{\Lambda+\Phi} \frac{(\mathcal{E}_Y + p_Y)}{r} W_Y + r e^{2\Lambda} \mathcal{P}_Y \right]. \quad (\text{A14}) \end{aligned}$$

The last line contains the summed pressure perturbation $\sum_Y \mathcal{P}_Y$. This term is removed using the algebraic constraint (29), which expresses the same pressure sum in terms of H_0 , H_1 , and K . Substituting Eq. (29) into the expression above and simplifying gives the final first-order equation for $\mathcal{P}_X$, Eq. (27).

APPENDIX B: MASS-SCALED f-MODE RELATION USING INDIVIDUAL FLUID COMPACTNESS

The compactness-based relation shown in Fig. 8 uses the total stellar compactness $C \equiv M/R$, where M is the

total gravitational mass and $R = R_O$ is the outer stellar radius. This choice is appropriate for testing the usual mass-scaled f-mode universal relation with compactness, but it does not separately display the mass and length scales associated with each fluid component in a two-fluid configuration. Since the two branches in Fig. 8 are identified through the inner- and outer-fluid perturbation variables, it is useful to examine the same mass-scaled frequencies using a componentwise compactness associated with the corresponding fluid sector. To construct this diagnostic, we define the additive mass-function contribution of each fluid as

$$M_j \equiv 4\pi \int_0^{R_j} \mathcal{E}_j(r) r^2 dr, \quad j = I, O, \quad (\text{B1})$$

and introduce $C_j \equiv M_j/R_j$. These quantities should be understood as bookkeeping compactnesses associated with the individual fluid sectors, rather than as separately measurable gravitational compactnesses. In particular, the gravitational field at R_I is determined by the total enclosed mass $M(R_I)$, not only by M_I . Nevertheless, C_I and C_O provide a useful way to display how the two mode branches evolve with the characteristic mass and radius scales of the fluid component used to identify each branch.

Figure 9 shows that the separation of the two branches in the mass-scaled f-mode relation depends on the compactness variable used to represent the sequence. When plotted against the total compactness C , as in Fig. 8, increasing F_{DM} produces a clear upward shift of the outer-fluid branch, while the corresponding evolution of the inner-fluid branch

is less directly visible. The present representation separates these two componentwise scales by using C_I for the inner-fluid branch and C_O for the outer-fluid branch, thereby tracking the compactness scale associated with the fluid degree of freedom used to identify each branch. In this representation, increasing F_{DM} moves the inner- and outer-fluid sequences toward one another, making the approach to the symmetric mirror configuration more transparent.

This comparison does not replace the total-compactness relation used in the main text, which remains the appropriate quantity for testing the standard single-fluid $f_1 M_{1.4}$ -compactness universality. Rather, it provides a complementary diagnostic for two-fluid configurations by separating the mass-radius scales associated with the two fluid sectors. The result supports the interpretation given in Sec. IV: the split branches in Fig. 8 arise from the two-fluid nature of the oscillation spectrum, while their evolution with increasing F_{DM} reflects the gradual approach toward the symmetric mirror limit.

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