

Anomalous scaling of linear power corrections

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Nonperturbative corrections to hadronic observables represent a critical obstacle to increasing accuracy at colliders. Long taken to scale simply as $1/Q$, where Q is the center-of-mass scattering energy, recent work has opened the path toward calculating the anomalous dimension that modifies that scaling. *A priori*, the problem is complex, requiring a resummation involving arbitrary numbers of large-angle and low-energy gluons. Within a specific framework for kinematic recoil, we show that it reduces to a simple exponential for key observables like the thrust, C -parameter and energy correlators. This simplicity holds for a specific hadron-mass scheme, and also even beyond the two-jet limit.

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The interpretation of ever more precise data from high-energy colliders requires corresponding improvements in the accuracy of theoretical predictions within the Standard Model of particle physics. One major avenue of progress involves advances [1,2] in calculations using perturbative quantum chromodynamics (QCD). Yet, as perturbative accuracy increases, a second problem comes to the fore, namely that experiments measure nonperturbative hadrons rather than the quarks and gluons of perturbative QCD, and one must understand the relation between the two.

One powerful and widely used approach to this problem is to use phenomenological models to describe the transition from partons to hadrons, i.e. “hadronization” [3–11]. However, such models bring little analytical insight into the problem and suffer from long-standing open questions about how to consistently combine them with the highest-accuracy perturbative calculations. An alternative approach exploits the renormalon breakdown of perturbation theory at high orders, a factorial growth in the perturbative coefficients [12], to gain insight into the potential structure of nonperturbative effects. This path

offers the potential for greater understanding of the analytical structure of hadronization.¹

The class of nonperturbative corrections that is largest numerically is that with a Λ/Q power scaling, where Λ is the nonperturbative scale of QCD and Q is the center-of-mass energy. Such linear power corrections affect essentially all infrared safe observables that are based on hadron momenta. They were first explored in the mid-1990s [13–24] and are among the key elements that are extensively debated in determinations of the strong coupling constant from event shapes [25–30], with tensions of up to 3 standard deviations [27] relative to the world average [31].

One of the key results of the 1990s is that the renormalon approach reduces to probing how the emission of a single gluon at low transverse momentum modifies a given observable. That single gluon is sometimes referred to as a “gluer” [14]. This is illustrated in Fig. 1(a), which shows the emission of a gluer (in red) in $e^+e^- \rightarrow q\bar{q}$ collisions.

Such an approach inevitably oversimplifies hadronization, which seldom happens from a bare $q\bar{q}$ system, but instead almost always from a system where the $q\bar{q}$ has radiated multiple perturbative gluons, as in Fig. 1(b). What long remained unclear, however, was how to extend the renormalon picture to systems with more than a $q\bar{q}$ pair. The past few years have started to see insight into this question [32–39] (for other recent hadronization studies, including operator approaches, see e.g. Refs. [26,27,30,40–42]).

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¹In hadron-hadron collisions, specifically the part not related to multiparton interactions.

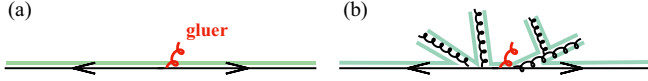


FIG. 1. (a) Emission of a nonperturbative gluer from a $q\bar{q}$ system, as in standard calculations; (b) a realistic set of perturbative partons from which a gluer would actually be emitted.

In the picture that is emerging—but that remains to be fully established—one can understand the effect of hadronization by evaluating how an observable changes when inserting a gluer into each and every color dipole of the event, using a well-chosen recoil scheme to account for momentum conservation.

Developing that picture for the thrust [43,44] and C -parameter [45], Dasgupta and Hounat [46] have considered the average effect of gluer insertion, after integrating over the kinematics of a single perturbative soft gluon. They find that this leads to a modification of the coefficient of Λ/Q for the average value of those observables, observing a correction factor that involves an anomalous dimension, $1 - (\alpha_s/2\pi) C_A \mathcal{S}_1 \ln Q/\Lambda$, with $\mathcal{S}_1 \simeq 2.455$ common to both observables. As discussed there, the all-order resummation of the anomalous scaling would involve considering gluer emission from systems with arbitrary numbers of large-angle soft gluons, Fig. 1(b). The goal of this Letter is to carry out that resummation. *A priori*, one expects [46] to find a structure similar to that of nonglobal logarithms [47], for which no analytic resummation result is known.

Starting from a $q\bar{q}$ system, the nonperturbative correction to an observable V can be written as

$$\langle \delta V_{\text{np}} \rangle_{q\bar{q}} = \int_0^{\mu_{\text{np}}} \frac{dk_{tn}}{k_{tn}} \int d\eta_n \frac{2C_F \alpha_s^{(\text{eff})}(k_{tn})}{\pi} \times [V(p_1, p_2, k_n) - V(\tilde{p}_1, \tilde{p}_2)], \quad (1)$$

where the p_i ($\tilde{p}_i$) are the hard momenta in the event after (before) emission of a soft nonperturbative gluer k_n . The integral is performed over its transverse momentum k_{tn} and rapidity η_n , weighted with $\alpha_s^{(\text{eff})}(k_{tn})$, an effective nonperturbative coupling for emitting the gluer at scales below an infrared matching scale $\mu_{\text{np}} \sim 1$ GeV.

We consider observables V that are *linear*, i.e. reduced to linear functions of any number of soft massless momenta k_i emitted from the hard $q\bar{q}$ system

$$V(p_1, p_2, k_1, \dots, k_m) - V(\tilde{p}_1, \tilde{p}_2) = \sum_i \frac{k_{ti}}{Q} f_V(\eta_i) + \mathcal{O}\left(\frac{k_{ti}^2}{Q^2}\right), \quad (2)$$

specifically, the thrust [T , $f_\tau(\eta) = e^{-|\eta|}$, with $\tau \equiv 1 - T$], the C -parameter [$f_C(\eta) = 3/\cosh \eta$], and energy-energy correlators $\text{EEC}(\theta) = \sum_{ij} E_i E_j / Q^2 \delta(\theta_{ij} - \theta)$ [48,49]

[$f_{\text{EEC}}(\eta) = 2 \cosh^2 \eta \delta(\eta - \log(\tan \theta/2))$]. Here k_{ti} and η_i are the transverse momentum and rapidity of i with respect to the $q\bar{q}$ pair. Equation (1) then reduces to

$$\langle \delta V_{\text{np}} \rangle_{q\bar{q}} = c_V \frac{T_{q\bar{q}}}{Q}, \quad T_{q\bar{q}} = \int_0^{\mu_{\text{np}}} dk_{tn} \frac{2C_F \alpha_s^{(\text{eff})}(k_{tn})}{\pi}, \quad (3)$$

where $T_{q\bar{q}}$ can be interpreted as the effective nonperturbative transverse momentum produced per unit rapidity from a $q\bar{q}$ system, and $c_V = \int d\eta f_V(\eta)$, e.g. $c_\tau = 2$, $c_C = 3\pi$ and $c_{\text{EEC}}(|\cos \theta| < 1/2) = 4/\sqrt{3}$. The linearity of the observables ensures that the details of how that transverse momentum is produced, e.g. correlations between particles, do not matter.

The key question we ask here is how the transverse momentum per unit rapidity changes when the nonperturbative gluer emission takes place not from a $q\bar{q}$ pair, but from a system with arbitrary numbers of additional soft gluons, $qg_1 \dots g_m \bar{q}$. We will work in the large- N_c limit, considering gluer emission from a sequence of color dipoles qg_1 , $g_1 g_2$, etc. The essential observation of Refs. [35,36] is that for linear observables, the result of the renormalon calculation can be obtained using kinematic maps where the recoil of the hard partons is a linear function of the soft emitted gluer's momentum. It is important to be aware that this is yet to be proved for final states with gluons, though the universality of soft emission suggests that the results should carry over.

We will consider two maps that satisfy the soft-gluon linearity condition: the 3-jet matched [50] PanGlobal [51,52] and PanLocal [51] maps. Both have the property that longitudinal recoil is local to the dipole. We have verified that our implementation reproduces the 3-jet nonperturbative shifts of Refs. [25,35,36] (Supplemental Material [53], Sec. I) and that the maps' predicted nonperturbative shifts for linear observables agree also for more complex dipole configurations, for sufficiently soft gluons.

We start by considering a gluer emission from a $qg\bar{q}$ system, where the gluon itself is soft, and evaluate the nonperturbative transverse momentum per unit rapidity as a function of rapidity η relative to the $q\bar{q}$ system. That can be written as

$$T_{qg\bar{q}}(\eta, \eta_g) = \frac{C_A}{\pi} \sum_{qg, g\bar{q}} \int \frac{dz_+}{z_+} \frac{dk_{\perp n}}{k_{\perp n}} \frac{d\phi_n}{2\pi} [k_{tn} \delta(\eta - \eta_n) + (p_{tg} - \tilde{p}_{tg}) \delta(\eta - \eta_g)] \alpha_s^{(\text{eff})}(k_{\perp n}). \quad (4)$$

The square brackets contain the change in scalar transverse momentum (subscript t , with respect to the $q\bar{q}$ direction) due to the gluer emission n and recoil of the perturbative gluon g . The gluer integration variables $k_{\perp n}$ and z_+ are the

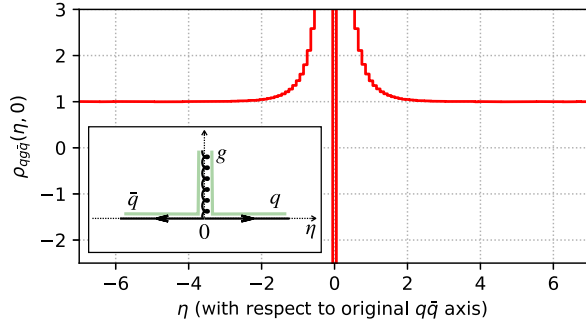


FIG. 2. The effect of gluon emission on transverse momentum per unit rapidity (η) in events consisting of a hard $q\bar{q}$ pair and a perturbative soft gluon at zero rapidity.

transverse component and plus light-cone fraction defined with respect to the emitting qg or $g\bar{q}$ dipole. We neglect the transverse momenta of the $q, \bar{q}$, because they are at large rapidities, where $f_V(\eta)$ in Eq. (2) vanishes. Finally, the gluon must be much softer than the gluon. Rather than using some specific form for $\alpha_s^{(\text{eff})}(\mu)$, we probe how the observable changes on insertion of the gluon at a specific transverse momentum $k_{\perp,n}$, then taking the limit of the gluon $k_{\perp,n} \rightarrow 0$ [53], Sec. I. Figure 2 shows a numerical evaluation of

$$\rho_{qg\bar{q}}(\eta, \eta_g) \equiv T_{qg\bar{q}}(\eta, \eta_g) / T_{q\bar{q}} \quad (5)$$

for $\eta_g = 0$. At large rapidities, $\rho_{qg\bar{q}}(\eta, \eta_g)$ tends to 1 as expected from angular ordering. For η near zero, i.e. close to the perturbative gluon, a small $k_{\perp,n}$ with respect to the qg or $g\bar{q}$ dipoles translates into a larger $k_{\perp,n}$ with respect to the $q\bar{q}$ direction, hence $\rho_{qg\bar{q}}(\eta) \gg 1$. Finally there is a negative δ -function at $\eta = \eta_g = 0$ associated with the $(p_{tg} - \tilde{p}_{tg})$ recoil of the perturbative gluon.

To obtain the logarithmically dominant contribution to Eq. (3) at next-to-leading order (NLO) accuracy, we need to integrate $\rho_{qg\bar{q}}(\eta, \eta_g)$ over the perturbative soft-gluon kinematics and account for virtual corrections,

$$\mathcal{R}^{\text{NLO}} = 1 + \int d\eta_g \int_{\mu_{\text{np}}}^Q \frac{d\tilde{p}_{tg}}{\tilde{p}_{tg}} \frac{2C_F\alpha_s(\tilde{p}_{tg})}{\pi} [\rho_{qg\bar{q}}(\eta, \eta_g) - 1]. \quad (6)$$

This is the NLO factor that multiplies the average transverse momentum density. It is straightforward to show (Supplemental Material [53], Sec. II) that

$$\int_{-\infty}^{+\infty} d\eta_g [\rho_{qg\bar{q}}(\eta, \eta_g) - 1] = -4(1 - \ln 2). \quad (7)$$

Replacing $2C_F \rightarrow C_A$ (large- N_c limit) yields

$$\mathcal{R}^{\text{NLO}} = 1 - \lambda(Q, \mu_{\text{np}}) \frac{C_A \mathcal{S}_1}{2\pi}, \quad (8)$$

with

$$\mathcal{S}_1 = 8(1 - \ln 2), \quad \lambda(Q, \mu_{\text{np}}) = \int_{\mu_{\text{np}}}^Q \frac{dp_t}{p_t} \alpha_s(p_t), \quad (9)$$

in agreement with the numerical result of Ref. [46], but obtained generically for observables that are linear in the soft momenta and with $f(\eta) \rightarrow 0$ for large $|\eta|$.

Our approach can be straightforwardly extended to all orders. Specifically, let $\mathcal{R}$ be a function of the hard scale Q , $\mathcal{R}(Q)$, such that $\mathcal{R}(\mu_{\text{np}}) = 1$. Given a certain $\mathcal{R}(Q)$, if we increase the hard scale by an infinitesimal factor $1 + \epsilon$, we allow for extra configurations with a soft gluon between scales Q and $(1 + \epsilon)Q$. Those extra configurations have a radiation pattern corresponding to dressed qg and $g\bar{q}$ dipoles, where the nonperturbative transverse momentum density with respect to the qg or $g\bar{q}$ dipole is $\mathcal{R}(Q)T_{q\bar{q}}$. Recall that we are using the large- N_c limit, thus the qg and $g\bar{q}$ dipoles have the same nonperturbative emission density as a $q\bar{q}$ dipole. This yields the following differential equation:

$$\frac{d\mathcal{R}(Q)}{d \ln Q} = \frac{2C_F\alpha_s(Q)}{\pi} \int d\eta_g [\rho_{qg\bar{q}}(\eta, \eta_g) - 1] \mathcal{R}(Q), \quad (10)$$

where we make use of the property that the nonperturbative all-order longitudinal recoil from the qg or $g\bar{q}$ dipoles gets the same $\mathcal{R}(Q)$ factor as the transverse momentum density ([53], Sec. III). Once again using the large- N_c limit, $2C_F \rightarrow C_A$, the solution to Eq. (10) is

$$\mathcal{R}(Q) = \exp \left[-\lambda(Q, \mu_{\text{np}}) \frac{C_A \mathcal{S}_1}{2\pi} \right] = \left(\frac{\alpha_s(Q)}{\alpha_s(\mu_{\text{np}})} \right)^{\frac{C_A \mathcal{S}_1}{\beta_0}}, \quad (11)$$

with $\beta_0 = (11C_A - 2n_f)/3$, such that Eq. (3) becomes

$$\langle \delta V_{\text{np}} \rangle = c_V \frac{T_{\text{all-order}}(Q)}{Q}, \quad T_{\text{all-order}}(Q) = T_{q\bar{q}} \mathcal{R}(Q). \quad (12)$$

This is much simpler than the structure one would expect from a nonglobal resummation, and part of the reason why such a structure emerges is the requirement of linearity of the observable in the soft limit, Eq. (2). The exponentiated structure is important because there is a fundamental ambiguity in what scale one should take for μ_{np} , and Eq. (11) tells us that that ambiguity can simply be absorbed into the normalization of $T_{q\bar{q}}$.

We supplement our analytic derivation with numerical tests. We start with a $q\bar{q}$ system, and use a perturbative parton shower [52,54,55] to add a cloud of soft gluons

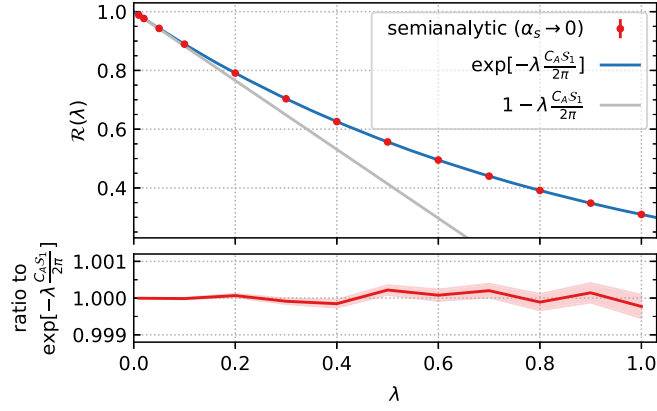


FIG. 3. All-order evaluation of $\mathcal{R}$, using a perturbative shower for the gluon cloud and a semianalytic evaluation of the gluer insertion. The result is shown in the limit $\alpha_s \rightarrow 0$ at any given fixed λ , plotted as a function of λ and compared to the all-order analytic result, Eq. (11) (blue line).

between scales $p_{t,\min}$ and $p_{t,\max} \ll Q$ (using fixed α_s for simplicity). We then have two ways of probing the gluer's effect. In one, we directly insert the gluer into the showered event. In the other, we use a semianalytic calculation of the gluer's impact on the transverse momentum per unit rapidity for an arbitrary dipole [53], Sec. 4 a. Figure 3 shows results with the latter. We have run the shower in an asymptotic regime, $\alpha_s \rightarrow 0$ for fixed $\lambda = \alpha_s \ln p_{t,\max}/p_{t,\min}$, so as to extract just the $(\alpha_s \ln p_{t,\max}/p_{t,\min})^n$ terms, free of higher-logarithmic contamination [51]. The results agree with Eq. (11) to within statistical errors, of order a few times 10^{-4} . The comparison to the pure first-order result (gray line) illustrates the size of the resummation effect.

Our analysis has so far been for a $q\bar{q}$ event with a cloud of soft gluons, and since we integrate over all p_t values for the soft gluons, it is in effect inclusive over the value of the observable. Insofar as the linearity property of an observable holds beyond $q\bar{q}$ events, we would expect the same modification factor as in Eq. (11) to hold for the Λ/Q power correction also in the 3-jet limit. We test this by starting from 3-parton events, adding a cloud of soft gluons as done for Fig. 3, and then inserting gluers into all dipoles and determining the change in the value of the observable. Figure 4 shows the resulting effective coefficient of the power correction to the C -parameter, $\delta C_{\text{np}}/[T_{q\bar{q}}/Q]$, as a function of the perturbative (3-parton) value of the C parameter. The red-dashed curve is without the cloud of soft gluons, the orange and cyan curves include the soft gluons. The lower panel shows the ratio to the plain 3-parton (no-cloud) result for two finite α_s values at fixed λ . The ratio coincides with the corresponding 2-jet result at the same λ and finite α_s values, slightly different from Eq. (11) because of subleading logarithmic effects present in the shower at finite α_s (“2-parton semi-anl.” [53], Sec. 4 b). This confirms that the universality of the anomalous dimension holds across the full spectrum. We find

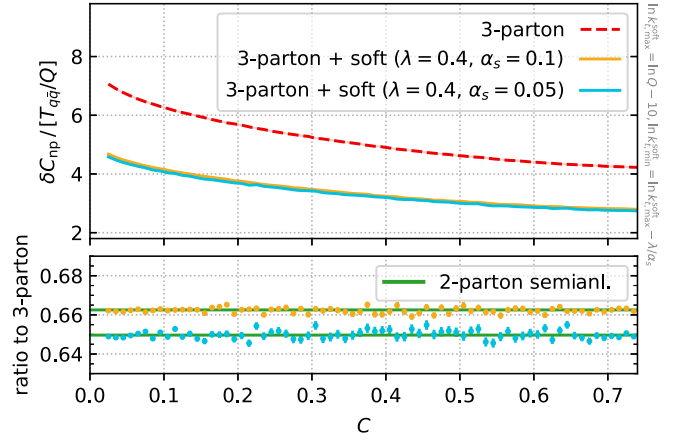


FIG. 4. The C -parameter nonperturbative correction in the 3-jet region, without (red) and with a cloud of soft gluons (orange, cyan). The lower panel shows that the ratio with and without soft gluons is independent of C and agrees with the corresponding 2-jet result at the same λ and α_s values (“2-parton semi-anl.”).

similar results for the thrust. Note that for small V , the 3-parton result itself goes as $c_V(1 - \frac{1}{2}S_1/\ln \frac{1}{V})$, as required for consistency with the anomalous dimension [53], Sec. I (see also Ref. [56]).

Figure 4 uses a soft-gluon cloud with $p_{t,\max} \ll Q$. In practice, when the observable has a value $V \ll 1$, the actual $p_{t,\max}$ is limited to be $\lesssim \xi V Q$, with ξ of order 1. Thus we expect the shift to involve a factor $\mathcal{R}(\xi V Q)$ rather than $\mathcal{R}(Q)$, bringing additional V dependence relative to the shape of the pure 3-parton result. Note that for $VQ \sim \Lambda$, higher-power corrections, including full shape functions [24], become relevant.

A further consideration is that hadrons have masses, while our analysis so far has been for massless gluer emission. There are various schemes for treating the hadron-mass dependence in the definitions of observables [57]. Differences between schemes involve an anomalous scaling [57,58] that differs from Eq. (11) and whose normalization involves nonperturbative parameter(s) beyond $T_{q\bar{q}}$. We conjecture that one can probe the effect of hadron masses by examining the same class of shower kinematic maps with a *massive* gluer ([53], Sec. V). The results, Fig. 5, suggest one hadron-mass scheme is favored, the D (or decay) scheme, where each hadron is isotropically decayed to two massless particles. In this scheme the anomalous dimension is universal, i.e. independent of the ratio of the gluer mass m_n to its transverse momentum $k_{\perp n}$. In other schemes, $\mathcal{R}$ depends strongly on $m_n/k_{\perp n}$, implying that the hadronization corrections depend on additional nonperturbative parameters beyond $T_{q\bar{q}}$, connected with the precise distribution of hadron masses and their transverse momenta.

We close with a comparison to hadronization effects employing PYTHIA's [59] widely used Lund string model [5,6]. Exploring a range of Q values, Fig. 6 (top)

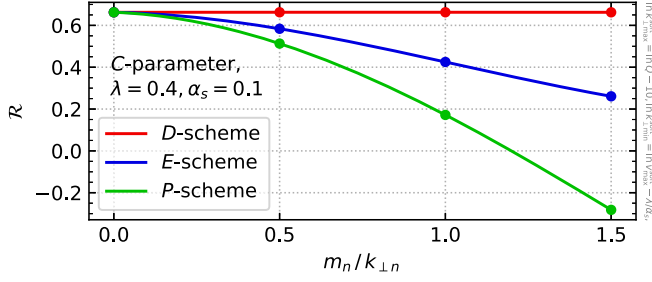


FIG. 5. Dependence of the scaling factor $\mathcal{R}$ on the ratio of gluon mass to transverse momentum, in three difference hadron-mass schemes. Only in the decay (D) scheme are the results consistent with Eq. (11) also at nonzero gluon masses.

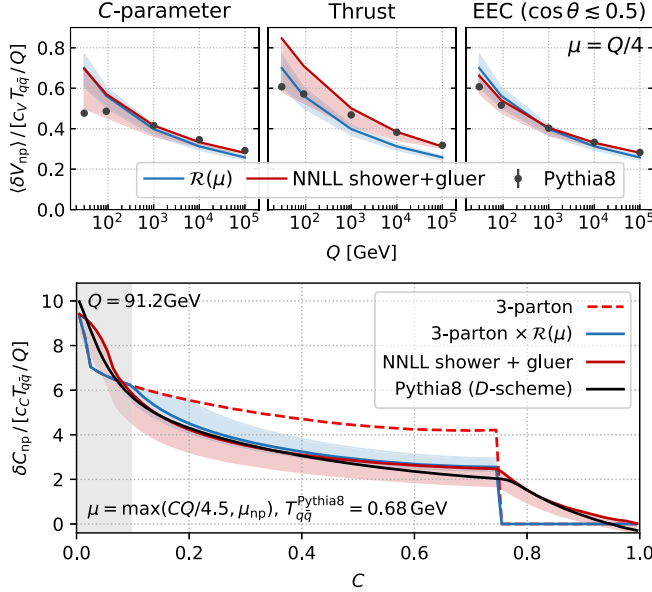


FIG. 6. Top: average C -parameter, τ and EEC, as a function of Q , comparing the $\mathcal{R}(\mu)$ calculation [Eq. (11)], gluon insertion into the NNLL PanGlobal shower, and PYTHIA8. For the EEC, ≤ 0.5 indicates a continuous bin boundary [53], Sec. VI. Bottom: the analogous set of results for the C -parameter distribution, with additionally the plain 3-parton result for comparison. In the gray area, we freeze μ in $\mathcal{R}(\mu)$ to $\mu_{np} = 2$ GeV.

shows the coefficient of the all-order nonperturbative correction to the average C -parameter, thrust and EEC results, specifically the ratio to the plain 2-parton expectation. The blue curve is Eq. (12) with $\mu_{np} = 2$ GeV, and the band covers $1/6 < \mu/Q < 1/2$. Our best prediction, the dark red curve, corresponds to the insertion of an asymptotically soft gluon, $k_{\perp n} \simeq 0.0015$ GeV, into actual showered events (NNLL (next-to-next-to-leading logarithmic) PanGlobal [60] with a cutoff at μ_{np}). The light red band shows the effect of increasing $k_{\perp n}$ to 0.7 GeV, which gives a sense of the magnitude of higher-power effects. These are surprisingly large even for $Q \sim 100$ GeV. The difference between the dark red and blue curves gives a measure of further higher-order perturbative effects.

The black points in Fig. 6 show the hadronization corrections from PYTHIA8.312 (Monash2013 tune [61]), using the D -scheme. To relate those to Eq. (12), we need to determine PYTHIA's effective $T_{q\bar{q}}$, which we do by requiring agreement between the gluon insertion result and PYTHIA for the average C -parameter at $Q = 10^3$ GeV. This yields $T_{q\bar{q}}^{\text{Pythia}} = 0.68$ GeV. Across observables, there is good agreement between PYTHIA and our predicted scaling, notably at high Q values.

Finally, the lower panel of Fig. 6 shows the full C -parameter distribution at $Q = 91.2$ GeV, with the same set of curves as for the average, and additionally the pure 3-parton result for comparison (dashed red). The agreement between our curves and PYTHIA is striking and one observes also a significant shape difference relative to the pure 3-parton result. This is connected with the dependence of the anomalous dimension's scale on the C -parameter. For other energies, and the thrust (subject to larger subleading effects), see Supplemental Material [53], Sec. VII.

To summarize: our main result is the determination of the all-order structure of the anomalous scaling for the non-perturbative Λ/Q power correction to widely studied linear observables like the thrust, C -parameter and energy correlators. This has been done within a paradigm where recoil from gluon insertion is linear in the gluon momentum [35,36]. While the paradigm remains to be fully established, our results are sufficiently striking as to strongly motivate further study of its theoretical foundations. The most surprising finding is that, despite the complicated structure of the soft-gluon cloud from which a nonperturbative gluon is emitted, the all-order result is a straightforward exponentiation, Eq. (11), of the first-order result [46]. This remains true even with a finite gluon mass, as long as one uses the D -scheme. Furthermore, this analytic approach reproduces the Q dependence and event-shape dependence seen in PYTHIA, to within higher-order and higher-power effects that are larger than might have been expected from earlier studies. We look forward to future phenomenological applications of these insights.

Note added. Recently, Ref. [62] appeared, carrying out an operator-based calculation for the anomalous scaling of the EEC hadronization correction in the D -scheme. Its result is consistent with ours, providing further support for the validity of the underlying picture.

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Data availability. The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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