

Chiral enhancement in soft-photon bremsstrahlung and charge asymmetry in low-energy lepton-proton scattering

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We perform a Lorentz gauge evaluation of the single soft-photon bremsstrahlung radiative corrections to the unpolarized elastic lepton-proton scattering cross section at low energies, using the framework of heavy baryon chiral perturbation theory (HB χ PT). We systematically incorporate all next-to-leading-order [i.e., $\mathcal{O}(\alpha^3/M)$] contributions to the cross section arising from radiative and proton's low-energy recoil effects. An important component of the soft-photon bremsstrahlung corrections arises from the interference between lepton and proton bremsstrahlung processes, which are characterized as charge-odd. We identify a class of Feynman diagrams involving the final-state radiating proton that induces a chiral enhancement, thereby modifying the naive amplitude hierarchy based on the standard chiral power-counting scheme. The recently derived HB χ PT result for the analytically computed exact two-photon exchange counterpart at next-to-leading order is combined with our charge-odd soft-photon bremsstrahlung contribution to yield the complete charge-odd radiative correction. By incorporating the leading hadronic correction to the one-photon exchange process within the charge-even radiative correction, we obtain a prediction for the charge asymmetry observable, which can be compared with forthcoming MUon Scattering Experiment data on elastic lepton-antilepton scattering off the proton. In all our calculations, we use finite lepton masses.

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I. INTRODUCTION

The higher-order quantum electrodynamic (QED) radiative corrections to the one-photon exchange (OPE) are crucial for the precision determination of the unpolarized elastic (anti)lepton-proton ($\ell^\pm p$) scattering cross sections. In experiments, the undetected *soft* bremsstrahlung photons play a vital role in both elastic and inelastic processes, namely, $\ell^\pm p \rightarrow \ell^\pm p$ and $\ell^\pm p \rightarrow \ell^\pm p \gamma^*$, respectively. By a soft-photon we mean that the energy of the emitted photon is tiny,

i.e., $E_\gamma \rightarrow 0$,¹ rendering both elastic and inelastic processes physically indistinguishable.² The bremsstrahlung contributions to the cross section are further categorized as being “charge-even” and “charge-odd.” This approach is based on the type of interference matrix elements considered between products of Feynman diagrams with the external lepton and proton legs radiating the photon. For the $\ell^\pm p$ scatterings, the charge-even cross section is the sum of the ℓ^- and ℓ^+ contributions, while the charge-odd counterpart is their difference. Furthermore, in this paper, the individual squared amplitudes and the lepton-lepton or proton-proton

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¹The distinction between soft-photon and hard-photon bremsstrahlung can be defined in terms of an experimental resolution threshold Δ_γ , which sets the minimum photon energy required for an event to be detectable. This threshold, determined by the detector's acceptance and design, renders the hard-photon events observable only above Δ_γ while soft-photon emissions fall below this limit.

²In this work, we will use the symbol γ^* to denote a real (on-shell) bremsstrahlung photon, in contrast to a virtual (off-shell) photon, which will be denoted by γ .

interference bremsstrahlung matrix elements contributing to the charge-even cross section $\sigma_{\gamma\gamma^*}^{(\text{even})}$ will be referred to as the “direct-interference” terms. In contrast, the lepton-proton interference bremsstrahlung matrix elements contributing to the charge-odd cross section $\sigma_{\gamma\gamma^*}^{(\text{odd})}$ shall be referred to as the “exchange-interference” terms. Among the one-loop virtual photon-loop corrections in QED, only the two-photon exchange (TPE) radiative correction contributes to the charge-odd cross section $\sigma_{\gamma\gamma}^{(\text{odd})}$ (for recent TPE analyses, see, e.g., Refs. [1–20], including the review of Ref. [21]). The remaining virtual photon-loop corrections—such as the vertex, self-energy, and vacuum polarization corrections—contribute to the charge-even cross section $\sigma_{\gamma\gamma}^{(\text{even})}$. The total charge-even and charge-odd contributions can therefore be summarized as

$$\sigma_{\text{tot}}^{(\text{even})} = \sigma_{\gamma\gamma}^{(\text{even})} + \sigma_{\gamma\gamma^*}^{(\text{even})}, \quad \text{and} \quad (1)$$

$$\sigma_{\text{tot}}^{(\text{odd})} = \sigma_{\gamma\gamma}^{(\text{odd})} + \sigma_{\gamma\gamma^*}^{(\text{odd})}, \quad (2)$$

respectively. It should be noted, however, that neither the virtual photon loops nor the soft-photon bremsstrahlung cross sections are physical observables. Each of them is affected by infrared (IR) divergence—an artifact that precludes direct comparison with measured cross section data. To facilitate a meaningful comparison, the unphysical divergences are eliminated via the concomitant inclusion of a degenerate phase-space of inelastic real soft-photon emissions, constituting an unavoidable background to the basic elastic scattering process [22–25]. The IR divergences stemming from the virtual photon-loop integrals are systematically canceled order by order in perturbation theory, by the corresponding divergences from the soft-photon bremsstrahlung phase-space integrals when the incoherent sum of the virtual photon loops and soft bremsstrahlung cross sections is considered. Consequently, the virtual corrections to the elastic cross section receive finite modifications from the soft-photon bremsstrahlung.

One can experimentally measure the difference between the lepton and antilepton proton scattering cross sections via the charge asymmetry observable $\mathcal{A}_{\ell^\pm}$, defined by the following ratio of the differential cross sections³:

$$\mathcal{A}_{\ell^\pm} = \frac{d\sigma_{\text{tot}}^{(\text{odd})}}{d\sigma_{\text{tot}}^{(\text{even})}} = \frac{d\sigma_{\text{el}}^{(-)} - d\sigma_{\text{el}}^{(+)}}{d\sigma_{\text{el}}^{(-)} + d\sigma_{\text{el}}^{(+)}}. \quad (3)$$

³Note that our sign convention in the definition of the charge asymmetry is opposite to the one in Refs. [9,26], namely,

$$\widetilde{\mathcal{A}}_{\ell^\pm} = -\mathcal{A}_{\ell^\pm} = \frac{d\sigma_{\text{el}}^{(+)} - d\sigma_{\text{el}}^{(-)}}{d\sigma_{\text{el}}^{(+)} + d\sigma_{\text{el}}^{(-)}},$$

for reasons elucidated later in the text (see Sec. II E).

In particular, the measurement of $\mathcal{A}_{\ell^\pm}$ provides a rather “clean” procedure to estimate the charge-odd TPE effects, with various systematic errors contributing to the radiative corrections dropping out in the ratio. Several ongoing experimental efforts plan to measure this asymmetry. The MUon Scattering Experiment (MUSE) at the Paul Scherrer Institute (PSI) [27–29] is an ongoing effort designed to simultaneously scatter leptons and antileptons off a proton target at unprecedented low four-momentum transfers ($|Q^2| \lesssim 0.08 \text{ GeV}^2/c$ for a maximum beam three-momentum $p_{\text{beam}} \lesssim 210 \text{ MeV}/c$). A major goal of MUSE is to pin down the low-energy TPE corrections with a systematic uncertainty of 0.1%, thereby offering potential insights into resolving the long-standing discrepancies in the proton charge radius [30–35]—an issue that remains unsettled in experimental analyses of low-energy electron-proton scattering (see, e.g., Ref. [36] for a review of recent experimental efforts).

To facilitate the confrontation of theoretical predictions with the future high-precision MUSE data on ℓ -p elastic cross sections, an accurate estimation of the low-energy TPE contribution, along with its exchange-interference soft-photon bremsstrahlung counterpart, is needed. Furthermore, given that these two corrections are rather poorly constrained in comparison with the other radiative corrections, a careful assessment of the role of low-energy nonperturbative effects due to the proton’s structure, as well as the chiral recoil-radiative corrections [14], is deemed essential. While a substantial body of recent literature focuses on TPE effects [1–20], relatively few studies address the estimation of charge-odd soft-photon bremsstrahlung corrections and their impact on charge asymmetry at low energies [9,14,26,37]. For recent developments in this direction within the framework of QED-inspired hadronic models, we refer the reader to the works presented in Refs. [9,26,37]. However, a key limitation of model-dependent approaches lies in their potential to obscure subtle off-shell, nonperturbative features of the proton’s low-energy structure through the use of *ad hoc* on-shell phenomenological form factors. To address this limitation, a robust, model-independent analysis of soft-photon bremsstrahlung corrections at low energies is essential—one that systematically accounts for uncertainties arising from the perturbative truncation of higher-order effects. This objective can be effectively pursued within the framework of low-energy effective field theory (EFT).

In this paper, we present an EFT evaluation of the leading-order (LO) and next-to-leading-order (NLO) bremsstrahlung corrections within the well-known framework of the SU(2) formulation of heavy baryon chiral perturbation theory (HB χ PT) [38–44]. Such an EFT is intended to capture the essential nonperturbative features of low-energy QCD manifested through the chiral dynamics of pions and nucleons in response to gauge-invariant hadronic and electroweak external currents below a

predetermined cutoff momentum scale, $\Lambda_\chi \simeq 4\pi f_\pi \simeq 1 \text{ GeV}/c$, where f_π is the pion decay constant. Low-energy symmetries, such as chiral symmetry (and its breakings), play a crucial role in shaping the spectrum and underlying dynamics. The short-distance/ultraviolet physics beyond the scale Λ_χ is tacitly subsumed into a well-defined set of low-energy constants (LECs). These LECs are universally determined either phenomenologically or via lattice-QCD calculations. The hadrons are effectively treated as elementary particles in this framework, and the formalism incorporates a simultaneous low-energy expansion in inverse powers of the nucleon mass M , i.e., Q/M , along with the standard chiral expansion in powers of $Q/\Lambda_\chi \sim Q/M$, where Q is a generic momentum scale near the pion mass m_π . Such an EFT power-counting scheme ensures a systematic perturbative improvement of the theoretical predictions to an intended level of accuracy.

In the work of Talukdar *et al.* [14], the soft-photon bremsstrahlung radiative corrections to the elastic cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] were evaluated in HB χ PT at low energies relevant to MUSE kinematics. The evaluations were performed by adopting the Coulomb gauge. In this work, we will adopt the gauge and demonstrate that there exists a certain class of chirally enhanced bremsstrahlung diagrams from subleading orders, which effectively contribute to the cross section at LO and NLO. These enhanced contributions are generated from the final-state proton-radiating amplitudes having insertions of $\mathcal{O}(M^0)$ and $\mathcal{O}(1/M)$ final-state proton propagator components. As we will demonstrate in the next section, this calls for the promotion of some of these amplitudes to lower orders, which are formally next-to-next-to-leading order (NNLO) and next-to-next-to-next-to-leading order (N³LO) by the chiral power counting. Moreover, this also implies that the “effectively promoted” LO and NLO corrections to the bremsstrahlung cross section are now likely to receive additional contributions from renormalized pion loops. This possibility is somewhat unexpected, standing in contrast to the conventional scenario in HB χ PT, where hadronic structure-dependent radiative corrections from the proton are anticipated to contribute only at NNLO in the cross section. In light of the recent *exact* analytical evaluation of the TPE up to and including NLO in HB χ PT by Choudhary *et al.* [19] performed in the Lorentz gauge, it is appropriate to adopt the same gauge for the charge-odd bremsstrahlung calculation to maintain consistency in the cancellation of IR divergences. The same applies to the charge-even bremsstrahlung corrections, wherein the IR-divergent terms must be systematically canceled with those stemming from the charge-even virtual corrections.

The purpose of this paper is to present a consistent analytical evaluation of the soft-photon bremsstrahlung contributions to the elastic cross section up to and including NLO. Together with the virtual photon-loop corrections, obtained earlier in Refs. [14,19], we determine the full

IR-finite radiative corrections to the elastic cross section of $\mathcal{O}(\alpha^3/M)$. In particular, by including the exact TPE results of Choudhary *et al.*, the accuracy of our radiative corrections is expected to supersede the previous predictions by Talukdar *et al.*, who used the soft-photon approximation (SPA) to analytically evaluate the intricate 4-point scalar TPE loop integrals [9,11,12,14,45–49]. The radiative correction results are subsequently used to evaluate the charge asymmetry $\mathcal{A}_{\ell^\pm}$, which in turn assesses the extent to which the TPE contributions could dominate the observable over the ostensibly residual charge-odd bremsstrahlung contributions. Thus, an accurate measurement of the asymmetry is crucial to the precision with which the TPE effects can be extracted in future measurements.

Our paper is organized as follows: In Sec. II, we present the basic ingredients of the HB χ PT framework that allows a consistent evaluation of the LO and NLO soft-photon bremsstrahlung corrections to the elastic lepton-proton cross section. We incorporate the low-energy recoil-radiative effects in a perturbative approach. In addition, we also consider the leading hadronic structure-dependent corrections to the LO Born or OPE cross section at NNLO accuracy. Section III deals with our numerical results and their discussion. Here we combine our bremsstrahlung and hadronic OPE correction results with the charge-even virtual photon-loop corrections obtained by Talukdar *et al.* [14] and the charge-odd exact TPE results of Choudhary *et al.* [19]. Our predictions pertain to the full radiative corrections and charge asymmetry, in addition to estimating the soft-photon bremsstrahlung corrections at NLO accuracy. Finally, we summarize and conclude in Sec. IV with an outlook for subsequent future work. Appendixes A and B contain a detailed collection of relevant Feynman amplitudes and soft-photon phase-space integrals, as well as a brief description of the so-called S-frame transformations needed for evaluating the bremsstrahlung contributions.

II. THEORETICAL FRAMEWORK OF HB χ PT

We deploy the EFT framework of HB χ PT [38–44], where the “heavy” proton is treated nonrelativistically, while leptons, pions, and the photon are treated relativistically. The radiative corrections are evaluated up to $\mathcal{O}(\alpha/M)$, which is formally NLO in the power counting. In the following, we will explicitly present only the relevant LO, NLO, and NNLO components of the pion-nucleon effective chiral Lagrangian $\mathcal{L}_{\pi N}$ that are required for our calculations.⁴ This includes parts of the chiral Lagrangian

⁴While our analysis accounts for nonvanishing contributions from N³LO Feynman amplitudes, all insertions from the N³LO chiral Lagrangian $\mathcal{L}_{\pi N}^{(3)}$ [see Eq. (A37) in Appendix A] either yield vanishing contributions to the bremsstrahlung cross section or contribute only at orders beyond the target $\mathcal{O}(\alpha^3/M)$ NLO accuracy. Consequently, $\mathcal{L}_{\pi N}^{(3)}$ will not be explicitly required in the analysis.

involving purely pionic as well as the standard leptonic interactions in QED. For details, we refer to the exposition presented in Refs. [14,41–44]. Thus, to NNLO accuracy in this work,

$$\mathcal{L}_{\pi N} = \mathcal{L}_{\pi N}^{(0)} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \dots, \quad (4)$$

where the complete lowest chiral order parts of the πN Lagrangian are given as

$$\mathcal{L}_{\pi N}^{(0)} = \bar{N}_v (i v \cdot \mathcal{D} + g_A S \cdot u) N_v; \quad (5)$$

$$\begin{aligned} \mathcal{D}_\mu = \partial_\mu + \frac{1}{2} \left[u^\dagger \left(\mathbb{I} \partial_\mu + i \tau^3 \frac{e}{2} A_\mu \right) u \right. \\ \left. + u \left(\mathbb{I} \partial_\mu + i \tau^3 \frac{e}{2} A_\mu \right) u^\dagger \right] + i \mathbb{I} \frac{e}{2} A_\mu, \end{aligned} \quad (6)$$

where $\mathcal{D}_\mu$ represents the chiral covariant derivative, including the terms with the photon vector field A_μ . Furthermore, $\mathbb{I}$ denotes the 2×2 identity matrix, and τ^3 is the third component of the Pauli isospin matrix. The next chiral order Lagrangian is

$$\begin{aligned} \mathcal{L}_{\pi N}^{(1)} = \bar{N}_v \left[\frac{1}{2M} (v \cdot \mathcal{D})^2 - \frac{1}{2M} \mathcal{D} \cdot \mathcal{D} - \frac{ig_A}{2M} \{ S \cdot \mathcal{D}, v \cdot u \} \right. \\ + c_1 \text{Tr}(\chi_+) + \left(c_2 - \frac{g_A^2}{8M} \right) (v \cdot u)^2 + c_3 u \cdot u \\ + \left(c_4 + \frac{1}{4M} \right) [S^\mu, S^\nu] u_\mu u_\nu + c_5 \text{Tr}(\tilde{\chi}_+) \\ \left. - \frac{i}{4M} [S^\mu, S^\nu] \left\{ (1 + c_6) f_{\mu\nu}^+ + c_7 \text{Tr}(f_{\mu\nu}^+) \right\} \right] N_v. \end{aligned} \quad (7)$$

In the above expressions, $N_v = (\mathbf{p}_v, \mathbf{n}_v)^T$ denotes the heavy nucleon isodoublet spinor field, and $u = \sqrt{U}$ is the SU(2) isotriplet field, where the nonlinear pion field $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is defined, e.g., via the sigma gauge parametrization, i.e.,

$$U = u^2 = \sqrt{1 - \frac{\boldsymbol{\pi}^2}{f_\pi^2}} + \frac{i}{f_\pi} \boldsymbol{\tau} \cdot \boldsymbol{\pi} \quad \text{and} \quad u_\mu = i u^\dagger \partial_\mu U u^\dagger. \quad (8)$$

The $\boldsymbol{\tau}$ denotes the Pauli isospin matrices, and $g_A = 1.267$ is the nucleon axial coupling constant. The four-vectors v^μ and S^μ denote the velocity and spin of the nucleon, respectively, satisfying $S \cdot v = 0$. We find it convenient to choose $v^\mu = (1, \mathbf{0})$ and $S^\mu = (0, \boldsymbol{\sigma}/2)$. In addition, $c_{1,\dots,7}$ are the seven LECs with well-known finite values which have been determined in phenomenological analyses, e.g., Ref. [50]. In particular, the values $c_6 = \kappa_v$ and $c_7 = (\kappa_s - \kappa_v)/2$ are determined via the nucleon isovector and isoscalar anomalous magnetic moments, $\kappa_v = 3.71$ and $\kappa_s = -0.12$, respectively [51]. The remaining terms in the

NLO Lagrangian are as given below:

$$\begin{aligned} \chi_+ = u^\dagger \chi u + u \chi^\dagger u, \quad \tilde{\chi}_+ = \chi_+ - \frac{\mathbb{I}}{2} \text{Tr}(\chi_+), \\ f_{\mu\nu}^+ = e(\partial_\mu A_\nu - \partial_\nu A_\mu)(u \hat{Q} u^\dagger + u^\dagger \hat{Q} u), \quad \text{where} \\ \hat{Q} = \frac{1}{2}(\mathbb{I} + \tau^3) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (9)$$

In our analysis, we have ignored isospin violation except for considering the physical mass of the proton. Hence, we only consider the limit $m_d = m_u$, i.e., $\tilde{\chi}_+ \rightarrow 0$, meaning that the term proportional to c_5 vanishes.

Finally, we display the relevant portions of the NNLO πN Lagrangian which are needed in our analysis; the ellipsis denotes the excluded parts of the complete NNLO Lagrangian, and $\mathcal{L}_{\pi N}^{(2;\text{counter})}$ [51] includes a string of isoscalar and isovector counterterms, used for renormalizing the UV-divergent pion loops but not needed explicitly in this work (also see, e.g., Refs. [42,43] for more details):

$$\begin{aligned} \mathcal{L}_{\pi N}^{(2)} = \bar{N}_v \left[-\frac{i}{4M^2} (v \cdot \mathcal{D})^3 + \frac{i}{8M^2} \left\{ \mathcal{D}^2 (v \cdot \mathcal{D}) \right. \right. \\ \left. \left. - (v \cdot \mathcal{D}^\dagger) \mathcal{D}^{\dagger 2} \right\} + \dots \right] N_v + \mathcal{L}_{\pi N}^{(2;\text{counter})}. \end{aligned} \quad (10)$$

Before we proceed with our analysis of the radiative corrections, we first discuss the kinematics associated with the unpolarized elastic lepton-proton scattering. In addition, we include the kinematics of the corresponding inelastic single soft-photon emission processes with an energy $E_{\gamma^*} \lesssim \Delta_{\gamma^*}$, which is lower than any other energy scale in the problem of interest and defines the limit of an experimentally detectable photon. Our analysis is performed in the laboratory frame (lab frame) where the initial target proton is at rest. In this frame, we adopt the following four-momentum assignments:

$$\begin{aligned} \ell^\pm(p) + p(P_p) \longrightarrow \ell^\pm(p') + p(P'_p) \quad \text{and} \\ \ell^\pm(p) + p(P_p) \longrightarrow \ell^\pm(p') + p(P'_p) + \gamma^*(k); \quad \ell \equiv e, \mu. \end{aligned}$$

Here $p^\mu = (E, \mathbf{p})$ and $P_p^\mu = (M, \mathbf{0})$ denote the initial (anti) lepton and proton four-momenta, respectively, and $p'^\mu = (E', \mathbf{p}')$ and $P'_p^\mu = (E'_p, \mathbf{P}'_p)$ their corresponding outgoing four-momenta. Hence, $Q^\mu = (p - p')^\mu = (P'_p - P_p)^\mu$ is the four-momentum transferred between the (anti)lepton and proton in the above elastic process.

Regarding the inelastic soft-photon bremsstrahlung process, $k^\mu = (E_{\gamma^*}, \mathbf{k})$ is the four-momentum of the emitted soft-photon. In this situation, we either have the four-momentum transfer Q^μ or $q'^\mu = (Q - k)^\mu$, depending on whether the photon is radiated by the (anti)lepton or proton, respectively. Since we are only concerned with the undetectable soft-photon emission, the real photon emission

amplitudes are evaluated in the limit $k \rightarrow 0$. One crucial assumption is that the soft-photon emission does not alter the elastic kinematics. This implies that the four-momentum transfer for the soft-photon bremsstrahlung process q^μ is practically indistinguishable from its elastic counterpart Q^μ , e.g., Refs. [23,45–49].

It is notable that in HB χ PT, by virtue of reparametrization invariance, the proton's initial- and final-state four-momenta can be expressed as $P_p^\mu = Mv^\mu + p_p^\mu$ and $P_p'^\mu = Mv^\mu + p_p'^\mu$, where p_p^μ and $p_p'^\mu$ are the corresponding residual four-momenta. In the lab frame, we therefore have $p_p = (0, \mathbf{0})$, which implies that $v \cdot p_p = 0$ and $v \cdot p_p' = v \cdot Q = E - E' = -Q^2/(2M) > 0$. (For a t -channel process, we always have $Q^2 < 0$.) Finally, we mention that the standard form of the heavy proton propagator used in HB χ PT, accurate up to chiral order $\nu = 2$ [i.e., $\mathcal{O}(1/M^2)$], is given by

$$iS_{\text{full}}^{(p)}(l) = iS^{(\nu=0)}(l) + iS^{(\nu=1)}(l) + iS^{(\nu=2)}(l),$$

where

$$\begin{aligned} iS^{(0)}(l) &= \frac{i}{v \cdot l + i0}, \\ iS^{(1)}(l) &= \frac{i}{2M} \left[1 - \frac{l^2}{(v \cdot l + i0)^2} \right], \quad \text{and} \\ iS^{(2)}(l) &= \frac{i}{4M^2} \left[\frac{(v \cdot l)^3 - l^2(v \cdot l)}{(v \cdot l + i0)^2} \right] + \mathcal{O}\left(\frac{1}{M^3}\right). \end{aligned} \quad (11)$$

Here, l^μ denotes the generic residual off-shell proton momentum. As already pointed out in the Introduction, an important subtlety arises with the choice of the proton's off-shell momentum as $l^\mu \rightarrow Q^\mu$. Introducing the term $v \cdot Q \sim \mathcal{O}(1/M)$ in the propagator denominators results in $\mathcal{O}(M)$ chiral enhancements for both the LO and NLO components. As shown in the following analysis, the use of the proton propagator of Eq. (11) leads to a modified ordering of a certain class of Feynman amplitudes as

prescribed by the standard HB χ PT power-counting scheme. For an alternative heavy-baryon propagator description, we refer the reader to, e.g., Ref. [52].

A. The Born or OPE process

The lab-frame unpolarized differential cross section for the elastic OPE/Born (anti)lepton-proton scattering process is given by

$$[d\sigma_{el}]_\gamma = \frac{(2\pi)^4 \delta^4(p + p_p - p' - p_p')}{4M|\mathbf{p}|} \times \frac{d^3\mathbf{p}'}{(2\pi)^3 2E'} \frac{d^3\mathbf{p}_p'}{(2\pi)^3 2E_p'} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_\gamma|^2, \quad (12)$$

where the total OPE amplitude can be expressed as the following sum of Born amplitudes of different chiral powers, written as

$$\mathcal{M}_\gamma = \mathcal{M}_\gamma^{(0)} + \mathcal{M}_\gamma^{(1)} + \mathcal{M}_\gamma^{(2)} + \dots, \quad (13)$$

where the superscripts denote the different chiral orders. In particular, the amplitude $\mathcal{M}_\gamma^{(2)}$ includes the leading hadronic corrections arising from the proton's low-energy structure at NNLO in the chiral power counting. The OPE amplitude arising from the LO chiral Lagrangian is given by

$$\mathcal{M}_\gamma^{(0)} = -\frac{e^2}{Q^2} \left[\bar{u}(p') \gamma^\mu u(p) \right] \left[\chi^\dagger(p_p') v_\mu \chi(p_p) \right], \quad (14)$$

where $\chi(p_p)$ and $\chi^\dagger(p_p')$ denote the initial- and final-state two-component Pauli spinors for the nonrelativistic proton, and $u(p)$ and $\bar{u}(p')$ are the corresponding Dirac spinors for the relativistic leptons. The OPE amplitudes due to insertions of NLO and NNLO interactions from $\mathcal{L}_{\pi N}^{(1,2)}$ are given by

$$\mathcal{M}_\gamma^{(1)} = -\frac{e^2}{2MQ^2} \left[\bar{u}(p') \gamma^\mu u(p) \right] \left[\chi^\dagger(p_p') \left\{ (p_p + p_p')_\mu - v_\mu v \cdot (p_p + p_p') + (2 + \kappa_s + \kappa_v)[S_\mu, S \cdot Q] \right\} \chi(p_p) \right], \quad (15)$$

and

$$\mathcal{M}_\gamma^{(2)} = -\frac{e^2}{8M^2Q^2} \left[\bar{u}(p') \gamma^\mu u(p) \right] \left[\chi^\dagger(p_p') \left\{ (2(v \cdot Q)^2 - Q^2)v_\mu - (v \cdot Q)Q_\mu \right\} \chi(p_p) \right] - \frac{e^2}{Q^2} \left[\bar{u}(p') \gamma^\mu u(p) \right] \left[\chi^\dagger(p_p') \mathcal{V}_\mu^{(2)} \chi(p_p) \right], \quad (16)$$

respectively. In particular, the amplitude Eq. (16) is an “effective” Born amplitude obtained after renormalizing the genuine NNLO Born amplitude (first line) by incorporating the proton's hadronic structure in the proton-photon NNLO vertex (second line) via the amputated three-point Green's function [14]:

$$\mathcal{V}_\mu^{(2)} = (F_1^p - 1)v_\mu + \frac{1}{2M} \left\{ (F_1^p - 1)Q_\mu + 2(F_1^p + F_2^p - 1 - \kappa_p)[S_\mu, S \cdot Q] \right\} + \frac{Q^2}{8M^2} (F_1^p + 2F_2^p - 1)v_\mu + \mathcal{O}\left(\frac{1}{M^3}\right). \quad (17)$$

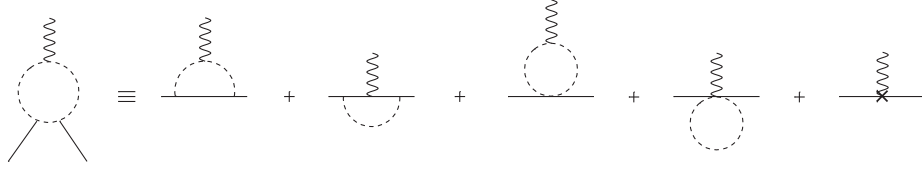


FIG. 1. Pion loops and counterterms at NNLO in HB χ PT with LO interactions at the proton's vertices contributing to its form factors. The solid, dashed, and wiggly lines denote the proton, pion, and photon propagators, respectively. The $\times$ denotes relevant counterterms stemming from $\mathcal{L}_{\pi N}^{\text{counter}}$ dependence, which absorbs the UV divergences arising from the pion loops (see Ref. [51] for more details).

The Green's function tacitly incorporates the contributions from UV-divergent pion loops and counterterms, as shown in Fig. 1, parametrizing the proton's hadronic structure via the Dirac and Pauli form factors $F_{1,2}^p$. At very low- Q^2 values, the form factors allow a Taylor expansion in terms of the mean-squared radii, $\langle r_{1,2}^2 \rangle \sim \mathcal{O}(1/\Lambda_\chi^2) \sim \mathcal{O}(1/M^2)$, and the anomalous magnetic moment of the proton, $\kappa_p = (\kappa_v + \kappa_s)/2 = 1.795$ [51]:

$$\begin{aligned} F_1^p(Q^2) &= 1 + \frac{Q^2}{6} \langle r_1^2 \rangle + \mathcal{O}(M^{-3}), \quad \text{and} \\ F_2^p(Q^2) &= \kappa_p + \frac{Q^2}{6} \langle r_2^2 \rangle + \mathcal{O}(M^{-3}). \end{aligned} \quad (18)$$

However, to $\mathcal{O}(1/M^2)$ we only need the contribution from the mean-squared $\langle r_1^2 \rangle$, which in turn could be related to the mean-squared charge or electric radius of the proton up to terms of $\mathcal{O}(1/M^3)$, where the rms radius is given by

$$r_p \equiv \langle r_E^2 \rangle^{1/2} \approx \sqrt{\langle r_1^2 \rangle + \frac{3\kappa_p}{2M^2}}. \quad (19)$$

Thus, the full chirally corrected elastic Born differential cross section up to and including NNLO can be expressed in the following form:

$$\left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_\gamma = \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \left\{ 1 + \delta_\chi^{(2)}(Q^2, r_p) \right\}, \quad (20)$$

where the LO Born contribution is given by [14]

$$\left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 = \frac{\alpha^2 |\mathbf{p}'|}{Q^2 |\mathbf{p}|} \left(1 - \frac{Q^2}{4M^2} \right) \left[1 + \frac{4EE'}{Q^2} \right]. \quad (21)$$

Here the fractional contribution $\delta_\chi^{(2)}$ not only includes the dynamical hadronic contributions from the NNLO effective interaction vertex, Eq. (17), but also $\mathcal{O}(1/M^2)$ kinematical recoil corrections. The latter terms arise from the contributions of the amplitudes $\mathcal{M}_\gamma^{(0,1)}$, which do not receive contributions from the pion-loop effects, *vis-a-vis*, the proton's finite-size structure. Consequently, we have the following charge-even result for the hadronic vertex corrections, as borrowed from the work of Talukdar *et al.* [14]:

$$\delta_\chi^{(2)}(Q^2, r_p) = \frac{Q^2}{3} \left[r_p^2 - \frac{3\kappa_p}{2M^2} \right] + \frac{Q^2}{4M^2} \left[1 + 2\kappa_p + (1 + \kappa_p)^2 \left(\frac{Q^2 + 4m_l^2 - 4E^2}{Q^2 + 4E^2} \right) \right] + \mathcal{O}\left(\frac{1}{M^3}\right). \quad (22)$$

For HB χ PT Feynman rules and other calculational details, we refer the reader to Refs. [41–44,51].

B. The soft-photon bremsstrahlung process

The lab-frame differential cross section for the single photon emission bremsstrahlung process is given by

$$[d\sigma_{br}]_{\gamma\gamma^*} = \frac{(2\pi)^4 \delta^4(p + p_p - p' - p'_p - k)}{4M|\mathbf{p}|} \frac{d^3\mathbf{p}'}{(2\pi)^3 2E'} \frac{d^3\mathbf{p}'_p}{(2\pi)^3 2E'_p} \frac{d^3\mathbf{k}'}{(2\pi)^3 2E_{\gamma^*}} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{\gamma\gamma^*}|^2, \quad (23)$$

where $\mathcal{M}_{\gamma\gamma^*}$ denotes the sum of all possible chiral order amplitudes in the Lorentz gauge, which can *potentially* contribute to the single photon emission cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. These include amplitudes up to N⁴LO, whose general, frame-independent expressions—presented without approximation—are compiled in Appendix A. However, in this section, we will present only the subset

of the nonvanishing amplitudes specific to the lab frame that pertains to a single soft-photon (γ_{soft}^*) emission with energy within the range $0 \leq E_{\gamma^*} \lesssim \Delta_{\gamma^*}$, which we will refer to as the soft-photon limit (SPL) of the bremsstrahlung amplitudes. It is important to emphasize that the SPL must not be confused with the soft-photon approximation (SPA), which is commonly employed in TPE calculations—see, for example, our recent TPE analysis in Ref. [20].

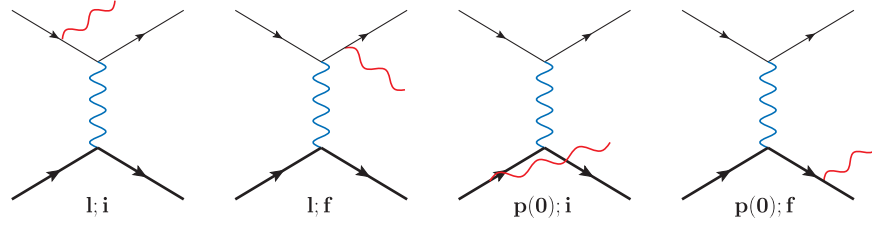


FIG. 2. Photon bremsstrahlung diagrams at LO [i.e., $\mathcal{O}(e^3)$], contributing to the radiative corrections of the ℓ -p elastic scattering cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The thin, thick, and wiggly lines denote the propagators for the lepton, proton, and photon (red for the soft bremsstrahlung photon and blue for the exchanged off-shell photon), respectively. The nomenclature of the individual diagrams is based on whether the radiated photon originates from the initial-state “i” or final-state “f” lepton “l” or proton “p”, where the label “(0)” indicates a LO proton-radiating vertex.

1. LO SPL bremsstrahlung amplitudes

The four LO [i.e., $\mathcal{O}(e^3)$] amplitudes [cf. Eqs. (A1)–(A4)] for the soft-photon bremsstrahlung processes, as displayed in Fig. 2, are given by

$$\mathcal{M}_{\gamma\gamma^*}^{[l;i]} \xrightarrow{\gamma_{\text{soft}}^*} e\mathcal{M}_\gamma^{(0)} \left(\frac{p \cdot \varepsilon^*}{p \cdot k} \right), \quad (24)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[l:f]} \xrightarrow{\gamma_{\text{soft}}^*} -e\mathcal{M}_\gamma^{(0)} \left(\frac{p' \cdot \varepsilon^*}{p' \cdot k} \right), \quad (25)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} \xrightarrow{\gamma_{\text{soft}}^*} -e\mathcal{M}_\gamma^{(0)} \left(\frac{v \cdot \varepsilon^*}{v \cdot k} \right), \quad (26)$$

and

$$\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} \xrightarrow{\gamma_{\text{soft}}^*} e\mathcal{M}_\gamma^{(0)} \left(\frac{v \cdot \varepsilon^*}{v \cdot Q} \right), \quad (27)$$

where the polarization vector of the emitted photon is ε_μ^* .

Here, we use the notation $\xrightarrow{\gamma_{\text{soft}}^*}$ to denote that these expressions are valid only in SPL. The contractions of the photon momentum k^μ with the external momenta p^μ and p'^μ in the numerators of the amplitudes are effectively set to zero, i.e., $k \cdot p \sim k \cdot p' \sim 0$. The resulting amplitudes are consistent with the well-known Low’s soft-photon theorem [53], which asserts that in the soft limit, the bremsstrahlung amplitudes are proportional to the LO Born amplitude $\mathcal{M}_\gamma^{(0)}$, Eq. (14). The amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[l;i/f]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[p(0);i/f]}$, respectively, represent the initial-state/final-state radiating lepton and proton, with LO proton-photon vertices. There are two key points to be emphasized here:

- (1) In contrast to the Coulomb gauge used by Talukdar *et al.* [14], the Lorentz gauge yields nonvanishing LO bremsstrahlung amplitudes proportional to $v \cdot \varepsilon^*$ —specifically $\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}$ —in which the bremsstrahlung photon is emitted by the proton. These amplitudes, therefore, contribute to the NLO [i.e., $\mathcal{O}(\alpha^3/M)$] cross section.

- (2) Each of the two LO amplitudes, $\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}$, involves a $\mathcal{O}(M^0)$ proton propagator ($iS^{(0)}$) insertion characterized by the denominators $-v \cdot k + i0$ and $v \cdot Q + i0$, respectively [see Eq. (11)]. As $v \cdot Q = -Q^2/(2M)$, the latter amplitude corresponds to the final-state proton radiation, and since the proton propagator scales as $\mathcal{O}(M)$, it implies that the amplitude is chirally enhanced. In contrast, the former amplitude with the initial-state radiating proton scales as expected, i.e., $\mathcal{O}(M^0)$. This modifies the naive hierarchy of ordering of Feynman diagrams as mandated by using chiral counting, enhancing the class of all diagrams with the final-state radiating proton by $\mathcal{O}(M)$, as compared to those with the initial-state radiating proton. A similar enhancement is also manifested in the subleading diagrams with an $\mathcal{O}(1/M)$ propagator ($iS^{(1)}$) insertion in the final-state (however, not pertinent to $iS^{(\nu \geq 2)}$ insertions), as we subsequently demonstrate.

2. NLO SPL bremsstrahlung amplitudes

There are nine NLO [i.e., $\mathcal{O}(e^3/M)$] bremsstrahlung amplitudes [see Eqs. (A5)–(A13)], as displayed in Fig. 3. In SPL, all but one decompose to yield either the LO Born amplitude $\mathcal{M}_\gamma^{(0)}$, Eq. (14), or the amplitude

$$\mathcal{N}_\gamma^{(1)} = -\frac{e^2}{2MQ^2} \left[\bar{u}(p') \gamma^\mu u(p) \right] \left[\chi(p_p')^\dagger Q_\mu \chi(p_p) \right], \quad (28)$$

which, in essence, forms part of the full NLO Born amplitude, Eq. (15). The resulting amplitudes are given by

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[l;i]} \xrightarrow{\gamma_{\text{soft}}^*} e\mathcal{N}_\gamma^{(1)} \left(\frac{p \cdot \varepsilon^*}{p \cdot k} \right) \\ + \frac{e}{2M} \mathcal{M}_\gamma^{(0)} \left(\frac{p \cdot \varepsilon^*}{p \cdot k} \right) (v \cdot k - v \cdot Q), \end{aligned} \quad (29)$$

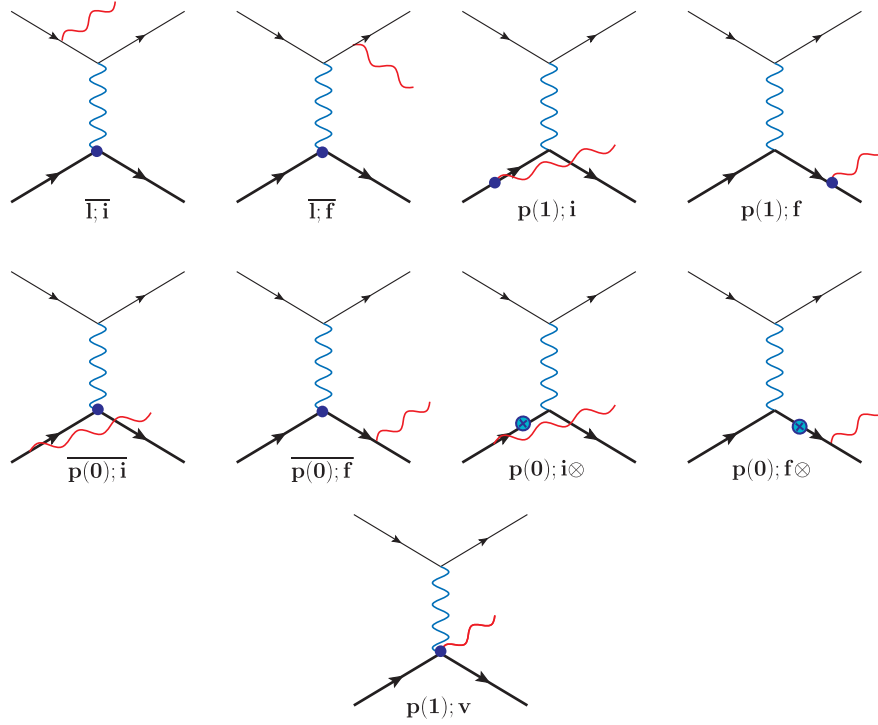


FIG. 3. Photon bremsstrahlung diagrams at NLO [i.e., $\mathcal{O}(e^3/M)$] with NLO proton-photon interaction vertex (filled blob $\bullet$) or $\mathcal{O}(1/M)$ proton propagator (crossed circle $\otimes$) insertions, contributing to the radiative corrections for the ℓ -p elastic scattering cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The thin, thick, and wiggly lines denote the propagators for the lepton, proton, and photon (red for the soft bremsstrahlung photon and blue for the exchanged off-shell photon), respectively. The nomenclature of the individual diagrams is based on whether the radiated photon originates from the initial-state “i” or final-state “f” lepton “l” or proton “p.” The labels “(0)” and “(1)” indicate LO and NLO proton-radiating vertices, respectively, while the overline signifies an NLO exchange-photon proton vertex insertion.

$$\mathcal{M}_{\gamma\gamma^*}^{[\overline{l};f]} \xrightarrow{\gamma_{\text{soft}}^*} -e\mathcal{N}_\gamma^{(1)} \left(\frac{p' \cdot \varepsilon^*}{p' \cdot k} \right) - \frac{e}{2M} \mathcal{M}_\gamma^{(0)} \left(\frac{p' \cdot \varepsilon^*}{p' \cdot k} \right) (v \cdot k - v \cdot Q), \quad (30)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[p(1);i]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{-e}{2M} \mathcal{M}_\gamma^{(0)} \left[v \cdot \varepsilon^* - \frac{k \cdot \varepsilon^*}{v \cdot k} \right], \quad (31)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[p(1);f]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{e}{2M} \frac{\mathcal{M}_\gamma^{(0)}}{v \cdot Q} \left[(2Q - k) \cdot \varepsilon^* - [v \cdot (2Q - k)](v \cdot \varepsilon^*) \right], \quad (32)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0)};i]} \xrightarrow{\gamma_{\text{soft}}^*} -e\mathcal{N}_\gamma^{(1)} \left(\frac{v \cdot \varepsilon^*}{v \cdot k} \right) + \frac{e}{M} \mathcal{M}_\gamma^{(0)} \left[\frac{(v \cdot Q)}{2(v \cdot k)} - 1 \right] (v \cdot \varepsilon^*), \quad (33)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0)};f]} \xrightarrow{\gamma_{\text{soft}}^*} e\mathcal{N}_\gamma^{(1)} \left(\frac{v \cdot \varepsilon^*}{v \cdot Q} \right) - \frac{e}{2M} \mathcal{M}_\gamma^{(0)} (v \cdot \varepsilon^*), \quad (34)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[p(0);i \otimes]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{e}{2M} \mathcal{M}_\gamma^{(0)} (v \cdot \varepsilon^*), \quad (35)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[p(0);f \otimes]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{e}{2M} \mathcal{M}_\gamma^{(0)} \left[1 - \frac{Q^2}{(v \cdot Q)^2} \right] (v \cdot \varepsilon^*), \quad (36)$$

and

$$\mathcal{M}_{\gamma\gamma^*}^{[p(1);v]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{e}{M} \mathcal{M}_\gamma^{(0)} (v \cdot \varepsilon^*) + \frac{e^3}{MQ^2} \left[\bar{u}(p') \gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) \varepsilon_\mu^* \chi(p_p) \right]. \quad (37)$$

A notable feature of the Born-like amplitude $\mathcal{N}_\gamma^{(1)}$ is that its interference with the LO Born amplitude $\mathcal{M}_\gamma^{(0)}$ vanishes identically, i.e.,

$$\sum_{\text{spins}} \left[\mathcal{M}_\gamma^{(0)*} \mathcal{N}_\gamma^{(1)} \right] = 0, \quad (38)$$

leading to a considerable simplification in evaluating the NLO contributions to the cross sections. As shown in Fig. 3, the amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[\overline{l};f]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0)};i/f]}$, respectively, represent the initial-state/final-state radiating lepton and proton processes, each having an insertion of the NLO exchange-photon proton vertex. The amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[p(1);i/f]}$

and $\mathcal{M}_{\gamma\gamma^*}^{[p(0);i/f\otimes]}$, respectively, represent the initial-state/final-state radiating proton processes with an insertion of an NLO proton-photon emission vertex or an $\mathcal{O}(1/M)$ component of the proton's propagator ($iS^{(1)}$), as in Eq. (11). The ninth amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(1);v]}$ is the bremsstrahlung amplitude containing an insertion of the NLO seagull proton-photon vertex, which does not factorize into one of the Born amplitudes [14]. As previously noted, unlike the Coulomb gauge evaluation in Ref. [14], all terms in the amplitudes proportional to $v \cdot \varepsilon \neq 0$ remain nonvanishing in our Lorentz gauge analysis. Furthermore, as stated before, the presence of the term $v \cdot Q$ in the denominators of the LO and NLO components of the final-state proton propagators ($iS^{(0,1)}$) in the three scattering amplitudes— $\mathcal{M}_{\gamma\gamma^*}^{[p(1);f]}$, $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}$, and $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]}$ —modifies their naive ordering, kinematically enhancing them so that they effectively contribute as LO [i.e., $\mathcal{O}(M^0)$] or even higher [i.e., $\mathcal{O}(M)$]. It is precisely these final-state propagator terms that motivate us to include contributions from diagrams with higher chiral order components—formally appearing at NNLO and N³LO—which could potentially contribute to the $\mathcal{O}(\alpha^3/M)$ NLO cross section. Such higher-order amplitudes are discussed next.

3. NNLO SPL bremsstrahlung amplitudes

At NNLO, two types of dynamical contributions constitute the bremsstrahlung matrix elements: (i) the

$\mathcal{O}(e^3/M^2)$ recoil-radiative corrections and (ii) the $\mathcal{O}(e^3/\Lambda_\chi^2)$ hadronic corrections. The former contributions arise from insertions of subleading-order proton-photon interaction vertices and proton propagator component insertions, while the latter arise from pion loops and counterterms that determine the proton rms charge radius contributions (cf. Fig. 1). Below, we explore each of the two distinct types of contributions.

- (i) *Recoil-radiative corrections*: As detailed in Appendix A, these consist primarily of 15 bremsstrahlung amplitudes that are formally expected to contribute at NNLO [see Eqs. (A14)–(A28)]. However, as shown in Fig. 4, only eight of these amplitudes, Eqs. (A14)–(A21), survive in SPL and contribute to the bremsstrahlung cross section at our intended NLO accuracy, namely, $\mathcal{O}(\alpha^3/M)$. The rest of the amplitudes, Eqs. (A22)–(A28), either vanish or are neglected as they generate higher-order terms. In SPL, these eight amplitudes either decompose to yield the LO Born amplitude $\mathcal{M}_\gamma^{(0)}$, Eq. (14), or the Born-like amplitude $\mathcal{N}_\gamma^{(1)}$, Eq. (28):

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[i;i]} \xrightarrow{\gamma_{\text{soft}}^*} & \frac{-e}{4M} \mathcal{N}_\gamma^{(1)} \left(\frac{p \cdot \varepsilon^*}{p \cdot k} \right) [v \cdot (Q - k)] \\ & + \frac{e}{8M^2} \mathcal{M}_\gamma^{(0)} \left(\frac{p \cdot \varepsilon^*}{p \cdot k} \right) \\ & \times \left[2[v \cdot (Q - k)]^2 - (Q - k)^2 \right], \quad (39) \end{aligned}$$

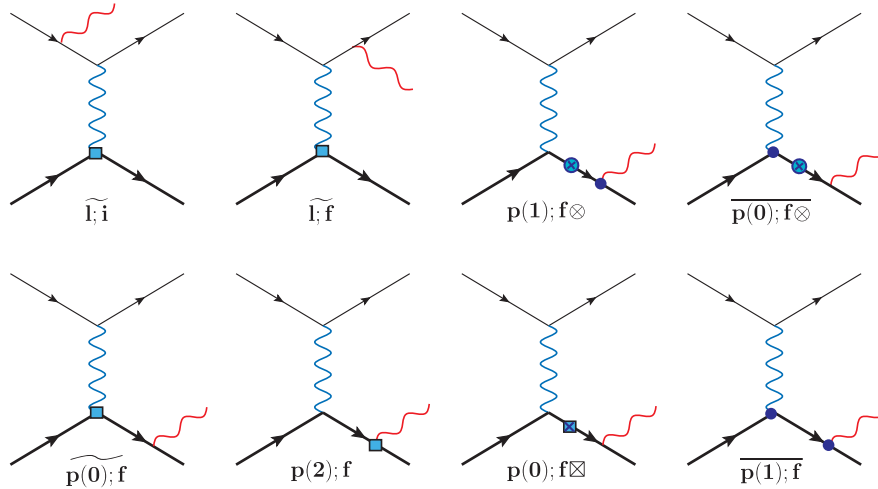


FIG. 4. Chirally enhanced photon bremsstrahlung diagrams of NNLO [i.e., $\mathcal{O}(e^3/M^2)$] with insertions of NLO (filled blob ●) and NNLO (filled box ◻) proton-photon interaction vertices, and $\mathcal{O}(1/M)$ (crossed circle ⊗) and $\mathcal{O}(1/M^2)$ (crossed box ⊞) proton propagator components, contributing to the radiative corrections for the ℓ -p elastic scattering cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The thin, thick, and wiggly lines denote the propagators for the lepton, proton, and photon (red for the soft bremsstrahlung photon and blue for the exchanged off-shell photon), respectively. The nomenclature of the individual diagrams is based on whether the radiated photon originates from the initial-state “i” or final-state “f” lepton “(l)” or proton “p.” The labels “(0),” “(1),” and “(2)” denote LO, NLO, and NNLO proton-radiating vertices, respectively, while the overline and tilde signify NLO and NNLO exchange-photon proton vertex insertions, respectively. Topologies involving initial-state proton radiation are not chirally enhanced and hence contribute beyond NLO in the cross section; they are therefore deferred to Fig. 13 in Appendix A.

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[l:f]} \xrightarrow{\gamma_{\text{soft}}^*} & \frac{e}{4M} \mathcal{N}_\gamma^{(1)} \left(\frac{p' \cdot \varepsilon^*}{p' \cdot k} \right) [v \cdot (Q - k)] \\ & - \frac{e}{8M^2} \mathcal{M}_\gamma^{(0)} \left(\frac{p' \cdot \varepsilon^*}{p' \cdot k} \right) \\ & \times [2[v \cdot (Q - k)]^2 - (Q - k)^2], \quad (40) \end{aligned}$$

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[p(1);f]} \xrightarrow{\gamma_{\text{soft}}^*} & \frac{e}{2M} \left(\frac{\mathcal{N}_\gamma^{(1)}}{v \cdot Q} - \frac{\mathcal{M}_\gamma^{(0)}}{2M} \right) [(2Q - k) \cdot \varepsilon^* \\ & - (v \cdot \varepsilon^*)[v \cdot (2Q - k)]], \quad (41) \end{aligned}$$

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} \xrightarrow{\gamma_{\text{soft}}^*} & \frac{-e}{4M} \mathcal{N}_\gamma^{(1)} (v \cdot \varepsilon^*) \\ & - \frac{eQ^2}{8M^2} \mathcal{M}_\gamma^{(0)} \left(\frac{v \cdot \varepsilon^*}{v \cdot Q} \right) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (42) \end{aligned}$$

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[p(2);f]} \xrightarrow{\gamma_{\text{soft}}^*} & \frac{e}{8M^2} \frac{\mathcal{M}_\gamma^{(0)}}{v \cdot Q} [2(v \cdot k)^2 (v \cdot \varepsilon^*) \\ & - (v \cdot k)(\varepsilon^* \cdot k)], \quad (43) \end{aligned}$$

$$\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\boxtimes]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{-eQ^2}{4M^2} \mathcal{M}_\gamma^{(0)} \left(\frac{v \cdot \varepsilon^*}{v \cdot Q} \right) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (44)$$

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[p(1);f\boxtimes]} \xrightarrow{\gamma_{\text{soft}}^*} & \frac{e}{4M^2} \mathcal{M}_\gamma^{(0)} \left(1 - \frac{Q^2}{(v \cdot Q)^2} \right) \\ & \times [(2Q - k) \cdot \varepsilon^* - (v \cdot \varepsilon^*)[v \cdot (2Q - k)]], \quad (45) \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\boxtimes]} \xrightarrow{\gamma_{\text{soft}}^*} & \frac{e}{2M} \left(\mathcal{N}_\gamma^{(1)} - \frac{v \cdot Q}{2M} \mathcal{M}_\gamma^{(0)} \right) \\ & \times \left(1 - \frac{Q^2}{(v \cdot Q)^2} \right) (v \cdot \varepsilon^*). \quad (46) \end{aligned}$$

Above, we include all Feynman diagrams with proton-photon interaction vertices and propagator insertions up to $\mathcal{O}(1/M^2)$, capturing all possible topologies necessary to retain amplitude terms of the same order, such that they will contribute to the NLO cross section of $\mathcal{O}(\alpha^3/M)$. As shown in Fig. 4, the

amplitude $\mathcal{M}_{\gamma\gamma^*}^{[l;i/f]}$ represents the initial-state/final-state radiating lepton process with the insertion of an NNLO proton-photon vertex. The amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[p(2);f]}$ represent diagrams with the LO final-state proton propagator with NNLO proton-photon vertices, while the amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\boxtimes]}$ has a $\mathcal{O}(1/M^2)$ final-state proton propagator ($iS^{(2)}$) insertion. Finally, the amplitudes

$\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\boxtimes]}$, $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\boxtimes]}$, and $\mathcal{M}_{\gamma\gamma^*}^{[p(1);f]}$ represent topologies with final-state radiating protons, having up to two insertions of NLO components. It is again important to recognize that diagrams involving the final-state proton propagators, although classified as higher-order contributions [i.e., $\mathcal{O}(1/M^2)$], are chirally enhanced by the factor $(v \cdot Q)^{-1} = -2M/Q^2$ appearing in the proton's propagator denominator. As indicated earlier, this enhancement modifies their naive chiral ordering. Furthermore, it is notable that

the genuine $\mathcal{O}(1/M^2)$ amplitudes, such as $\mathcal{M}_{\gamma\gamma^*}^{[l;i/f]}$, are also needed here so that their interference with the chirally enhanced LO amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} \sim \mathcal{O}(M)$ yields a nonvanishing contribution to the $\mathcal{O}(\alpha^3/M)$ NLO cross section. In contrast, the seven other commensurate $\mathcal{O}(1/M^2)$ amplitudes, involving either the initial-state proton propagators [see Eqs. (A22)–(A27)] or the NNLO seagull vertex, Eq. (A28) (cf. Fig. 13 in Appendix A), yield no contribution to the NLO cross section; they vanish upon integrating the soft-photon bremsstrahlung energies over the range $0 \leq E_\gamma \leq \Delta_{\gamma^*}$.

- (ii) *Hadronic corrections:* A total of six possible NNLO bremsstrahlung amplitudes, Eqs. (A29)–(A34), are given in Appendix A. They effectively renormalize the proton's NNLO recoil correction vertices via the hadronic interaction $\mathcal{V}_\mu^{(2)}$ [see Eqs. (16) and (17)]. These terms entail UV-divergent pion loops [cf. Fig. 1] as well as their counterterms, and they contribute to the proton's form factors $F_{1,2}^p$. In the works of Refs. [41,51], the form factors were evaluated to NNLO accuracy in HB χ PT, which in turn yield the proton's rms charge radius r_p that scales as $\mathcal{O}(1/M^2)$, Eq. (19). In SPL, only the first four amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[\pi(a),\dots,\pi(d)]}$, Eqs. (A29)–(A32), contribute to nonvanishing terms for the NLO bremsstrahlung cross section. The corresponding Feynman diagrams are displayed in Fig. 5. In the same vein as some of the genuine NNLO recoil-radiative amplitudes, the pion-loop amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[\pi(g),\pi(h)]}$, Eqs. (A33) and (A34) (cf. Fig. 13), do not yield any contributions to the NLO cross section. The resulting SPL amplitudes read as follows:

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[\pi(a)]} \xrightarrow{\gamma_{\text{soft}}^*} & e \left[\mathcal{M}_\gamma^{(0)} \mathcal{G}_{12}^p(Q^2) + 2\mathcal{N}_\gamma^{(1)} \{F_1^p(Q^2) - 1\} \right] \\ & \times \left(\frac{p \cdot \varepsilon^*}{p \cdot k} \right) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (47) \end{aligned}$$

$$\begin{aligned} \mathcal{M}_{\gamma\gamma^*}^{[\pi(b)]} \xrightarrow{\gamma_{\text{soft}}^*} & -e \left[\mathcal{M}_\gamma^{(0)} \mathcal{G}_{12}^p(Q^2) + 2\mathcal{N}_\gamma^{(1)} \{F_1^p(Q^2) - 1\} \right] \\ & \times \left(\frac{p' \cdot \varepsilon^*}{p' \cdot k} \right) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (48) \end{aligned}$$

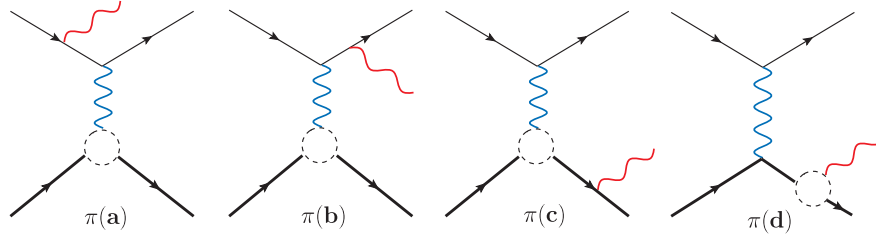


FIG. 5. Photon bremsstrahlung diagrams at NNLO [i.e., $\mathcal{O}(e^3/\Lambda_\chi^2)$], with $\Lambda_\chi \sim M \approx 1$ GeV/c], contributing to the radiative corrections for the ℓ -p elastic scattering cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The thin, thick, and wiggly lines denote the propagators for the lepton, proton, and photon (red for the real soft bremsstrahlung photon and blue for the exchanged off-shell photon), respectively. The dashed circular blob denotes the insertion of proton form factors (rms radius), incorporated via renormalized pion loops shown in Fig. 1. The diagrams $\pi(c)$ and $\pi(d)$ are chirally enhanced compared to the first two diagrams. Topologies involving initial-state proton radiation are not enhanced and hence contribute beyond NLO in the cross section; they are therefore deferred to Fig. 13 in Appendix A.

$$\mathcal{M}_{\gamma\gamma^*}^{[\pi(c)]} \xrightarrow{\gamma_{\text{soft}}^*} -e \left[\mathcal{M}_\gamma^{(0)} \mathcal{G}_{12}^p(Q^2) + 2\mathcal{N}_\gamma^{(1)} \{F_1^p(Q^2) - 1\} \right] \times \left(\frac{v \cdot \varepsilon^*}{v \cdot Q} \right) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad \text{and} \quad (49)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[\pi(d)]} \xrightarrow{\gamma_{\text{soft}}^*} -e \mathcal{M}_\gamma^{(0)} \left(\frac{v \cdot \varepsilon^*}{v \cdot Q} \right) \mathcal{G}_{12}^p(Q^2) - \frac{e}{M} \mathcal{M}_\gamma^{(0)} \left(\frac{Q \cdot \varepsilon^*}{v \cdot Q} \right) \{F_1^p(Q^2) - 1\} + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (50)$$

where

$$\mathcal{G}_{12}^p(Q^2) = F_1^p(Q^2) - 1 + \frac{3Q^2}{8M^2} \left[F_1^p(Q^2) - 1 + \frac{2}{3} F_2^p(Q^2) \right], \quad (51)$$

is a function of the proton's form factors $F_{1,2}^p$. As is evident from the Taylor expansion of the form factors, Eq. (18), $\mathcal{G}_{12}^p \sim \mathcal{O}(1/M^2)$. Thus, we again find that the amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[\pi(c), \pi(d)]}$ are enhanced relative to $\mathcal{M}_{\gamma\gamma^*}^{[\pi(a), \pi(b)]}$ by $\mathcal{O}(M)$, effectively rendering them NLO by the modified power counting.

4. N³LO SPL bremsstrahlung amplitudes

At N³LO, there are nine possible recoil-radiative amplitudes of $\mathcal{O}(e^3/M^3)$, Eqs. (A38)–(A43), involving NLO, NNLO and N³LO proton-photon interaction vertices, and $\mathcal{O}(1/M)$ and $\mathcal{O}(1/M^2)$ proton propagator ($iS^{(1,2)}$) insertions. In addition, we have five pion-loop amplitudes of $\mathcal{O}(e^3/M\Lambda_\chi^2)$, Eqs. (A47)–(A51), involving the proton's form factors. However, in SPL, only three of the former and two of the latter amplitudes survive to contribute to the $\mathcal{O}(\alpha^3/M)$ NLO cross section (cf. Fig. 6). They read as follows:

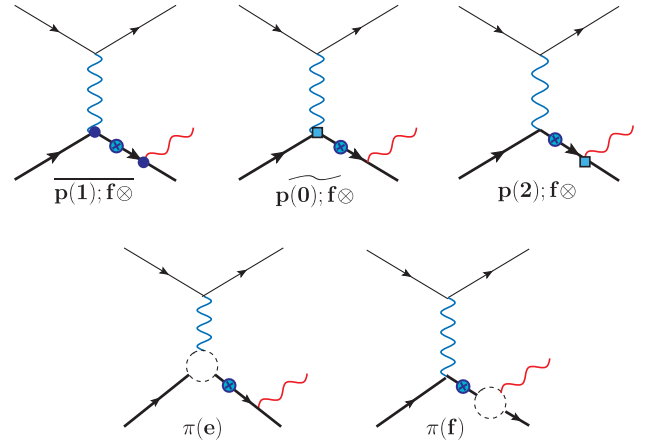


FIG. 6. Chirally enhanced photon bremsstrahlung diagrams at N³LO [i.e., $\mathcal{O}(e^3/M^3) \sim \mathcal{O}(e^3/(M\Lambda_\chi^2))$]. The insertions are as follows: NLO (filled blob $\bullet$) and NNLO (filled box $\square$) proton-photon interaction vertices, the proton form factors (dashed circular blob, cf. Fig. 1), and the $\mathcal{O}(1/M)$ proton propagator component (crossed circle $\otimes$), contributing to the radiative corrections for the ℓ -p elastic scattering cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The thin, thick, and wiggly lines denote propagators for the lepton, proton, and photon (red for the soft bremsstrahlung photon and blue for the exchanged off-shell photon), respectively. The nomenclature of the individual diagrams is based on whether the radiated photon originates from the initial-state “(i)” or final-state “(f)” proton “(p).” Diagrams with form factors have the symbol “ π ,” namely, “ $\pi(e)$ ” and “ $\pi(f)$.” Topologies involving the insertions of final-state proton propagator components and the $\nu = 3$ chiral-order vertex, which do not contribute to the NLO cross section, are deferred to Fig. 14 in Appendix A. Topologies involving initial-state proton radiation lack enhancement, contributing only at higher orders, and are thus omitted.

$$\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{e}{4M^2} \left(\mathcal{N}_\gamma^{(1)} - \frac{v \cdot Q}{2M} \mathcal{M}_\gamma^{(0)} \right) \times \left(1 - \frac{Q^2}{(v \cdot Q)^2} \right) \left[\varepsilon^* \cdot (2Q - k) - (v \cdot \varepsilon^*) [v \cdot (2Q - k)] \right], \quad (52)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{-e}{8M^2} \left(\mathcal{N}_\gamma^{(1)} (v \cdot Q) - \frac{1}{2M} \mathcal{M}_\gamma^{(0)} \right) \times [2(v \cdot Q)^2 - Q^2] \times \left(1 - \frac{Q^2}{(v \cdot Q)^2} \right) (v \cdot \varepsilon^*), \quad (53)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[p(2);f\otimes]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{-e}{16M^3} \mathcal{M}_\gamma^{(0)} \frac{Q^2}{(v \cdot Q)^2} \left[2(v \cdot k)^2 (v \cdot \varepsilon^*) - (v \cdot k)(k \cdot \varepsilon^*) \right] + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (54)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[\pi(e)]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{-e}{M} \left(\mathcal{N}_\gamma^{(1)} [F_1^p(Q^2) - 1] + \frac{1}{2} \mathcal{M}_\gamma^{(0)} \mathcal{G}_{12}^p(Q^2) \right) (v \cdot \varepsilon^*) \times \left(1 - \frac{Q^2}{(v \cdot Q)^2} \right) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (55)$$

$$\mathcal{M}_{\gamma\gamma^*}^{[\pi(f)]} \xrightarrow{\gamma_{\text{soft}}^*} \frac{-e}{2M} \mathcal{M}_\gamma^{(0)} \left[[F_1^p(Q^2) - 1] (Q \cdot \varepsilon^*) + \mathcal{G}_{12}^p(Q^2) (v \cdot \varepsilon^*) \right] \times \left(1 - \frac{Q^2}{(v \cdot Q)^2} \right) + \mathcal{O}\left(\frac{1}{M^3}\right), \quad (56)$$

where the expression for $\mathcal{G}_{12}^p$ incorporating the proton's form factors was already given in Eq. (51). Here, the amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]}$ has two NLO proton-photon vertices and one $\mathcal{O}(1/M)$ proton propagator ($iS^{(1)}$) insertion, while the amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[p(2);f\otimes]}$ have one NNLO proton-photon vertex and $\mathcal{O}(1/M)$ propagator ($iS^{(1)}$) insertions each. All the remaining chirally enhanced N³LO amplitudes, except for $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}$, as shown in Fig 14 of Appendix A, vanish upon integrating over the soft-photon bremsstrahlung energies in our NLO cross

section calculation. As for the amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}$, the $\mathcal{O}(M)$ enhancement by the final-state proton propagator is equally compensated by the $v \cdot Q \sim \mathcal{O}(1/M)$ term stemming from the N³LO vertex, thereby restoring it to the N³LO chiral counting scaling, i.e., $\mathcal{O}(1/M^3)$. Thus, the amplitude contributes to the cross section only at $\mathcal{O}(1/M^2)$, i.e., beyond NLO. Moreover, it is worth noting that all

commensurate amplitudes with insertions of $\mathcal{O}(1/M^\nu)$ ($\nu \geq 2$) final-state proton propagator components effectively scale as $iS^{(\nu)}(v \cdot Q) \sim \mathcal{O}(1/M^{\nu-1})$ and therefore do not lead to any type of enhancement.

5. N⁴LO SPL bremsstrahlung amplitudes

At N⁴LO, there are two chirally enhanced amplitudes, $\mathcal{M}_{\gamma\gamma^*}^{[p(3);f\otimes]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]}$, presented in Eqs. (A54) and (A55), respectively (cf. Fig. 15 in Appendix A). These amplitudes appear to scale as $\mathcal{O}(1/M^2)$, suggesting that they can potentially contribute to the $\mathcal{O}(\alpha^3/M)$ NLO cross section. However, akin to the N³LO diagrams shown in Fig 14, the amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(3);f\otimes]}$ vanishes when computing the cross section. As for the amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]}$, the expected $\mathcal{O}(1/M^4)$ scaling reemerges, rendering it irrelevant at our intended $\mathcal{O}(\alpha^3/M)$ accuracy of the NLO cross section.

6. Sum of SPL bremsstrahlung amplitudes

Before evaluating the soft-photon bremsstrahlung cross section, we summarize the complete set of relevant amplitudes required to achieve accuracy up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$]. To this end, if $\widetilde{\mathcal{M}}_{\gamma\gamma^*} \subset \mathcal{M}_{\gamma\gamma^*}$ denotes the corresponding sum of the relevant subset of bremsstrahlung amplitudes up to $\mathcal{O}(e^3/M^3)$, which is needed at our working accuracy to evaluate the NLO cross section [cf. Eq. (23)], then the contributing amplitudes LO through N³LO are consolidated as

$$\widetilde{\mathcal{M}}_{\gamma\gamma^*} = \mathcal{M}_{\text{LO}} + \mathcal{M}_{\text{NLO}} + \widetilde{\mathcal{M}}_{\text{NNLO}} + \widetilde{\mathcal{M}}_{\text{N}^3\text{LO}}, \quad (57)$$

where the tildes indicate only those parts considered in this section that contribute nonvanishingly to our lab frame $\mathcal{O}(\alpha^3/M)$ cross section, as summarized below:

$$\mathcal{M}_{\text{LO}} = \mathcal{M}_{\gamma\gamma^*}^{[t;i]} + \mathcal{M}_{\gamma\gamma^*}^{[t;f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}, \quad (58)$$

$$\begin{aligned} \mathcal{M}_{\text{NLO}} = & \mathcal{M}_{\gamma\gamma^*}^{[\widetilde{t};i]} + \mathcal{M}_{\gamma\gamma^*}^{[\widetilde{t};f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);f]} \\ & + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} \\ & + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);v]}, \end{aligned} \quad (59)$$

$$\begin{aligned} \widetilde{\mathcal{M}}_{\text{NNLO}} = & \mathcal{M}_{\gamma\gamma^*}^{[\widetilde{t};i]} + \mathcal{M}_{\gamma\gamma^*}^{[\widetilde{t};f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} \\ & + \mathcal{M}_{\gamma\gamma^*}^{[p(2);f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]} \\ & + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(a)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(b)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(c)]} \\ & + \mathcal{M}_{\gamma\gamma^*}^{[\pi(d)]}, \quad \text{and} \end{aligned} \quad (60)$$

$$\begin{aligned}\widetilde{\mathcal{M}}_{\text{N}^3\text{LO}} = & \mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(2);f\otimes]} \\ & + \mathcal{M}_{\gamma\gamma^*}^{[\pi(e)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(f)]}.\end{aligned}\quad (61)$$

Finally, we note that the total squared matrix element for the bremsstrahlung process ($\mathcal{S}_{br}$) could be decomposed into distinct charge-even ($\mathcal{S}_{\text{even}}^{(\nu)}$) and charge-odd ($\mathcal{S}_{\text{odd}}^{(\nu)}$) chiral order (ν) components based on lepton charge symmetry of the products of bremsstrahlung amplitudes, i.e.,

$$\mathcal{S}_{br} \equiv \sum_{\text{spins}} |\mathcal{M}_{\gamma\gamma^*}|^2 = \sum_{\nu=0}^3 \mathcal{S}_{\text{even}}^{(\nu)} + \sum_{\nu=0}^3 \mathcal{S}_{\text{odd}}^{(\nu)}. \quad (62)$$

As we subsequently demonstrate, at the desired level of accuracy, the matrix elements of chiral order components $\mathcal{S}_{\text{even}}^{(\nu \geq 3)}$ and $\mathcal{S}_{\text{odd}}^{(\nu \geq 4)}$, arising from the interference of the aforementioned amplitudes, do not contribute to the NLO cross section. Next, we separately determine these charge-even and charge-odd components.

C. The charge-even bremsstrahlung component

The component of the cross section for the inelastic charge-even (anti)lepton-proton bremsstrahlung process is obtained from the direct-interference (see Introduction) product of Feynman diagrams in which the bremsstrahlung photon is emitted from either the (anti)lepton or the proton legs of both graphs. As a result, the cross section remains symmetric under conjugation of the lepton's charge. In particular, we evaluate the charge-even differential cross section, Eq. (23), corresponding to the single soft-photon emission process contributing to the lowest-order QED corrections to the elastic $\ell^\pm$ -p scattering up to and including NLO accuracy [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The only nonvanishing contributions to the charge-even results stem from the LO and NLO diagrams (cf. Figs. 2 and 3). Apart from a few chirally enhanced diagrams beyond NLO, which contribute higher-order terms [i.e., $\mathcal{O}(\alpha^3/M^2)$] to the charge-even cross section, most yield vanishing contributions when integrated over the soft-photon emission energies within the range $0 \leq E_\gamma \leq \Delta_{\gamma^*}$. Their noncontribution relies on the fact that, in dimensional regularization (DR), scaleless loop integrals vanish identically, i.e.,

$$\int \frac{d^d k}{(2\pi)^d} \frac{(k^2)^m}{(v \cdot k + i0)^n} \xrightarrow{\text{DR}} 0. \quad (63)$$

1. $\nu=0$ charge-even contribution

The charge-even cross section component for soft-photon bremsstrahlung at LO [i.e., $\mathcal{O}(\alpha^3)$] is obtained from the following charge-even components of the squared

matrix elements, stemming from the LO bremsstrahlung amplitudes (cf. Fig. 2):

$$\begin{aligned}\mathcal{S}_{\text{even}}^{(\nu=0)} = & \sum_{\text{spins}} |\mathcal{M}_{\text{LO}}|_{\text{even}}^2 \\ = & \sum_{\text{spins}} \left[|\mathcal{M}_{\gamma\gamma^*}^{[l;i]}|^2 + |\mathcal{M}_{\gamma\gamma^*}^{[l;f]}|^2 + |\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]}|^2 \right. \\ & + |\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}|^2 + 2\text{Re} \left(\mathcal{M}_{\gamma\gamma^*}^{[l;i]\dagger} \mathcal{M}_{\gamma\gamma^*}^{[l;f]} \right. \\ & \left. \left. + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i]\dagger} \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} \right) \right].\end{aligned}\quad (64)$$

After integrating over the phase-space, the LO differential cross section for charge-even bremsstrahlung corrections in SPL can be expressed as [14]

$$\begin{aligned}\left[\frac{d\sigma_{br}^{(\text{LO};\text{even})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} = & -\frac{\alpha}{2\pi^2} \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \\ & \times [L_{ii} + L_{ff} - L_{if}],\end{aligned}\quad (65)$$

where $\left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0$ denotes the LO Born/OPE differential cross section for the elastic lepton-proton scattering process, Eq. (21). Here, L_{ii} , L_{ff} , and L_{if} are IR-divergent integrals which result from the integration over the soft-photon energy $0 \leq E_{\gamma^*} \leq \Delta_{\gamma^*}$. Following the methodology adopted in Refs. [11,45–49], we boost to the so-called S-frame (the center-of-mass frame of the recoiling proton and emitted soft photon), where such phase-space integrations are conveniently evaluated. This approach facilitates the extraction of the corresponding IR-finite contributions, $\tilde{L}_{ii}$, $\tilde{L}_{ff}$, and $\tilde{L}_{if}$, using DR. However, here we quote the integration results borrowed from Ref. [14], as expressed in lab-frame coordinates (see Appendix B for details of the evaluations in the S-frame):

$$\begin{aligned}L_{ii} = & \pi \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] + 2\pi\tilde{L}_{ii}; \\ \tilde{L}_{ii} = & \frac{1}{2} \ln \left(\frac{4\Delta_{\gamma^*}^2 E^2}{m_l^2 E'^2} \right) - \frac{1}{\beta} \ln \sqrt{\frac{1+\beta}{1-\beta}},\end{aligned}\quad (66)$$

$$\begin{aligned}L_{ff} = & \pi \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] + 2\pi\tilde{L}_{ff}; \\ \tilde{L}_{ff} = & \frac{1}{2} \ln \left(\frac{4\Delta_{\gamma^*}^2 E^2}{m_l^2 E'^2} \right) - \frac{1}{\beta'} \ln \sqrt{\frac{1+\beta'}{1-\beta'}},\end{aligned}\quad (67)$$

and

$$\begin{aligned}
L_{\text{if}} &= \pi \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] \left(\frac{\nu_l^2 + 1}{\nu_l} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] + 2\pi \tilde{L}_{\text{if}}; \\
\tilde{L}_{\text{if}} &= \frac{\nu_l^2 + 1}{2\nu_l} \left[\ln \left(\frac{4\Delta_{\gamma^*}^2 E^2}{m_l^2 E'^2} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] + \ln^2 \sqrt{\frac{1+\beta'}{1-\beta'}} - \ln^2 \sqrt{\frac{1+\beta}{1-\beta}} + \text{Li}_2 \left(1 - \frac{\lambda_\nu E' - E}{(1-\beta')E'\xi_\nu} \right) \right. \\
&\quad \left. + \text{Li}_2 \left(1 - \frac{\lambda_\nu E' - E}{(1+\beta')E'\xi_\nu} \right) - \text{Li}_2 \left(1 - \frac{\lambda_\nu E' - E}{(1-\beta)E\lambda_\nu \xi_\nu} \right) - \text{Li}_2 \left(1 - \frac{\lambda_\nu E' - E}{(1+\beta)E\lambda_\nu \xi_\nu} \right) \right], \quad (68)
\end{aligned}$$

where $\nu_l(Q^2) = \sqrt{1 - \frac{4m_l^2}{Q^2}}$, $\xi_\nu(Q^2) = \frac{2\nu_l}{(\nu_l+1)(\nu_l-1)}$, and $\lambda_\nu(Q^2) = \frac{\nu_l-1}{\nu_l+1}$.

Here, $\beta = |\mathbf{p}|/E$ and $\beta' = |\mathbf{p}'|/E'$ denote the initial and final lepton velocities, $\gamma_E = 0.577216\dots$ is the Euler-Mascheroni constant, and $\varepsilon_{\text{IR}} = (4-D)/2 < 0$. Furthermore, the well-known dilogarithm or Spence function that appears in Eq. (68) is defined by

$$\text{Li}_2(z) = - \int_0^z dt \frac{\ln(1-t)}{t}, \quad \forall z \in \mathbb{C}. \quad (69)$$

We note that in Eq. (64), none of the terms in the squared matrix elements, $|\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}|^2$, contribute to the cross section as they lead to scaleless integrals vanishing according to the condition given in Eq. (63). Thus, at LO, there is no contribution from amplitudes where the proton radiates a single soft photon to the charge-even bremsstrahlung cross section. The corresponding fractional contribution to the charge-even differential cross section at LO is expressed as follows⁵:

$$\begin{aligned}
\delta_{\gamma\gamma^*}^{(\text{LO;even})}(Q^2, \Delta_{\gamma^*}) &= \left[\frac{d\sigma_{br}^{(\text{LO;even})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} / \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 = \mathbf{IR}_{\gamma\gamma^*}^{(0)}(Q^2) - \frac{\alpha}{\pi} (\tilde{L}_{\text{ii}} + \tilde{L}_{\text{ff}} - \tilde{L}_{\text{if}}) \\
&= \mathbf{IR}_{\gamma\gamma^*}^{(0)}(Q^2) + \frac{\alpha}{\pi} \left[\left(\frac{\nu_l^2 + 1}{2\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - 1 \right) \ln \left(\frac{4\Delta_{\gamma^*}^2}{m_l^2} \right) + \frac{2}{\beta} \ln \sqrt{\frac{1+\beta}{1-\beta}} \right. \\
&\quad \left. + \left(\frac{\nu_l^2 + 1}{2\nu_l} \right) \left\{ \text{Li}_2 \left(1 - \frac{\lambda_\nu - 1}{(1-\beta)\xi_\nu} \right) + \text{Li}_2 \left(1 - \frac{\lambda_\nu - 1}{(1+\beta)\xi_\nu} \right) - \text{Li}_2 \left(1 - \frac{\lambda_\nu - 1}{(1-\beta)\lambda_\nu \xi_\nu} \right) \right. \right. \\
&\quad \left. \left. - \text{Li}_2 \left(1 - \frac{\lambda_\nu - 1}{(1+\beta)\lambda_\nu \xi_\nu} \right) \right\} \right] + \delta_{\gamma\gamma^*}^{(\text{LO-1/M;even})}(Q^2, \Delta_{\gamma^*}), \quad (70)
\end{aligned}$$

where the $\mathcal{O}(1/M)$ component of the charge-even LO result is given by

$$\begin{aligned}
\delta_{\gamma\gamma^*}^{(\text{LO-1/M;even})}(Q^2, \Delta_{\gamma^*}) &= \frac{\alpha Q^2}{2\pi M E \beta^2} \left[1 - 2\beta^2 \left(\frac{\nu_l^2 + 1}{2\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - 1 \right) - \left\{ \frac{1}{\beta} - \beta \left(\frac{\nu_l^2 + \nu_l + 1}{\nu_l} \right) \right\} \ln \sqrt{\frac{1+\beta}{1-\beta}} \right. \\
&\quad \left. + \beta \left(\frac{\nu_l^2 + 1}{2\nu_l} \right) \left\{ \frac{\lambda_\nu(1+\beta) - 1}{\xi_\nu(1-\beta) - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1-\beta)\xi_\nu} \right) - \frac{\lambda_\nu(1-\beta) - 1}{\xi_\nu(1+\beta) - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1+\beta)\xi_\nu} \right) \right. \right. \\
&\quad \left. \left. - \frac{\lambda_\nu \beta}{(1+\beta)\lambda_\nu \xi_\nu - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1+\beta)\xi_\nu \lambda_\nu} \right) - \frac{\lambda_\nu \beta}{(1-\beta)\lambda_\nu \xi_\nu - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1-\beta)\xi_\nu \lambda_\nu} \right) \right\} \right] + \mathcal{O}\left(\frac{1}{M^2}\right), \quad (71)
\end{aligned}$$

and the LO IR-divergent part of the charge-even soft-photon bremsstrahlung component is given as

$$\mathbf{IR}_{\gamma\gamma^*}^{(0)}(Q^2) = \frac{\alpha}{\pi} \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] \left(\frac{\nu_l^2 + 1}{2\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - 1 \right). \quad (72)$$

⁵Regarding our expression in Eq. (70), a few typographical errors were identified in the corresponding expression, Eq. (67) in Ref. [14]. These errors have been rectified in the recent SPA-based TPE work of Goswami *et al.* [20].

The same charge-even result also applies to antilepton scattering off the proton. Note that the above expression is not genuinely LO in the strict sense of chiral power counting, as it includes the $\mathcal{O}(1/M)$ kinematically suppressed terms from quantities such as E' and β' , where $E' = E - Q^2/(2M)$ and

$$\beta' = \beta - \frac{Q^2}{2ME} \left(\beta - \frac{1}{\beta} \right) + \mathcal{O}\left(\frac{1}{M^2}\right).$$

Such higher-order terms must be systematically accounted for to be consistent with the NLO contributions, which we subsequently examine. We further note that the above LO

Lorentz gauge charge-even bremsstrahlung result is identical to the one obtained previously by Talukdar *et al.* [14] adopting the Coulomb gauge. The IR divergence of Eq. (72) is specifically canceled when added to the virtual photon-loop corrections (attributed to the one-loop self-energy, vertex, and vacuum polarization corrections) to the elastic cross section in HB χ PT. These virtual QED radiative corrections $\delta_{\gamma\gamma}^{(\text{LO;even})}$ were analytically computed in several earlier works (see, e.g., Refs. [11,14,46]). Thus, we have the following corrections to the charge-even elastic cross section arising from all the LO amplitudes involving the virtual photon loops:

$$\begin{aligned} \delta_{\gamma\gamma}^{(\text{LO;even})}(Q^2) &= \left[\frac{d\sigma_{\text{virt}}^{(\text{LO;even})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma} / \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \\ &= \mathbf{IR}_{\gamma\gamma}^{(0)}(Q^2) + \frac{\alpha}{\pi} \left[\frac{\nu_l^2 + 1}{4\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] \ln \left[\frac{\nu_l^2 - 1}{4\nu_l^2} \right] + \frac{2\nu_l^2 + 1}{2\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - 2 \right. \\ &\quad + \frac{\nu_l^2 + 1}{2\nu_l} \left\{ \text{Li}_2 \left(\frac{\nu_l + 1}{2\nu_l} \right) - \text{Li}_2 \left(\frac{\nu_l - 1}{2\nu_l} \right) \right\} + \frac{1}{\nu_l} \left(\frac{2m_l^2}{Q^2 + 4E^2} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] \\ &\quad + \sum_{f=e,\mu,\tau} \left\{ \frac{2}{3} \left(\nu_f^2 - \frac{8}{3} \right) + \nu_f \left(\frac{3 - \nu_f^2}{3} \right) \ln \left[\frac{\nu_f + 1}{\nu_f - 1} \right] \right\} - \frac{2}{3} \left(\nu_\pi^2 + \frac{1}{3} \right) + \frac{\nu_\pi^3}{3} \ln \left[\frac{\nu_\pi + 1}{\nu_\pi - 1} \right] \Bigg] \\ &\quad - \frac{\alpha Q^2 E}{\pi M} \left(\frac{2m_l}{Q^2 + 4E^2} \right)^2 \frac{1}{\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] + \mathcal{O}\left(\frac{1}{M^2}\right), \end{aligned} \quad (73)$$

where $\nu_{f,\pi} = \sqrt{1 - 4m_{f,\pi}^2/Q^2}$. We find that the IR-divergent component is exactly equal in magnitude and opposite in sign to that arising from the charge-even bremsstrahlung correction:

$$\mathbf{IR}_{\gamma\gamma}^{(0)}(Q^2) = -\mathbf{IR}_{\gamma\gamma^*}^{(0)}(Q^2) = -\frac{\alpha}{\pi} \left[\frac{1}{|\epsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] \left(\frac{\nu^2 + 1}{2\nu} \ln \left[\frac{\nu + 1}{\nu - 1} \right] - 1 \right). \quad (74)$$

Following the consistent subtraction of infrared singularities, we obtain the total one-loop IR-finite fractional radiative correction to the charge-even (anti)lepton-proton elastic differential cross section stemming from all LO amplitudes [14]:

$$\begin{aligned} \delta_{2\gamma}^{(\text{LO;even})}(Q^2, \Delta_{\gamma^*}) &= \left[\frac{d\sigma_{el}^{(\text{LO;even})}(Q^2)}{d\Omega'_l} \right]_{2\gamma} / \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 = \delta_{\gamma\gamma}^{(\text{LO;even})}(Q^2) + \delta_{\gamma\gamma^*}^{(\text{LO;even})}(Q^2, \Delta_{\gamma^*}) \\ &= \frac{\alpha}{\pi} \left[\frac{\nu_l^2 + 1}{4\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] \ln \left[\frac{\nu_l^2 - 1}{4\nu_l^2} \right] - 2 + \frac{2\nu_l^2 + 1}{2\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - \frac{\nu_l^2 + 1}{2\nu_l} \left\{ \text{Li}_2 \left(\frac{\nu_l - 1}{2\nu_l} \right) - \text{Li}_2 \left(\frac{\nu_l + 1}{2\nu_l} \right) \right\} \right. \\ &\quad + \frac{1}{\nu_l} \left(\frac{2m_l^2}{Q^2 + 4E^2} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] + \sum_{f=e,\mu,\tau} \left\{ \frac{2}{3} \left(\nu_f^2 - \frac{8}{3} \right) + \nu_f \left(\frac{3 - \nu_f^2}{3} \right) \ln \left[\frac{\nu_f + 1}{\nu_f - 1} \right] \right\} - \frac{2}{3} \left(\nu_\pi^2 + \frac{1}{3} \right) \\ &\quad + \frac{\nu_\pi^2}{3} \ln \left[\frac{\nu_\pi + 1}{\nu_\pi - 1} \right] + \left\{ \left(\frac{\nu_l^2 + 1}{2\nu_l} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - 1 \right\} \ln \left(\frac{4\Delta_{\gamma^*}^2}{m_l^2} \right) + \frac{2}{\beta} \ln \sqrt{\frac{1 + \beta}{1 - \beta}} + \left(\frac{\nu_l^2 + 1}{2\nu_l} \right) \\ &\quad \times \left\{ \text{Li}_2 \left(1 + \frac{1 - \lambda_\nu}{(1 - \beta)\xi_\nu} \right) + \text{Li}_2 \left(1 + \frac{1 - \lambda_\nu}{(1 + \beta)\xi_\nu} \right) - \text{Li}_2 \left(1 + \frac{1 - \lambda_\nu}{(1 - \beta)\lambda_\nu \xi_\nu} \right) - \text{Li}_2 \left(1 + \frac{1 - \lambda_\nu}{(1 + \beta)\lambda_\nu \xi_\nu} \right) \right\} \Bigg] \end{aligned}$$

$$\begin{aligned}
& -\frac{\alpha Q^2}{2\pi M E \beta^2} \left[-1 - 2\beta^2 + \frac{\beta^2}{\nu_l} \left\{ 1 + \nu_l^2 + 2 \left(\frac{2m_l E \beta}{Q^2 + 4E^2} \right)^2 \right\} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] + \left\{ \frac{1}{\beta} - \beta \left(\frac{\nu_l^2 + \nu_l + 1}{\nu_l} \right) \right\} \ln \sqrt{\frac{1+\beta}{1-\beta}} \right. \\
& - \beta \left(\frac{\nu_l^2 + 1}{2\nu_l} \right) \left\{ \frac{\lambda_\nu(1+\beta) - 1}{\xi_\nu(1-\beta) - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1-\beta)\xi_\nu} \right) - \frac{\lambda_\nu(1-\beta) - 1}{\xi_\nu(1+\beta) - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1+\beta)\xi_\nu} \right) \right. \\
& \left. \left. - \frac{\lambda_\nu \beta}{(1+\beta)\lambda_\nu \xi_\nu - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1+\beta)\xi_\nu \lambda_\nu} \right) - \frac{\lambda_\nu \beta}{(1-\beta)\lambda_\nu \xi_\nu - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1-\beta)\xi_\nu \lambda_\nu} \right) \right\} \right] + \mathcal{O}\left(\frac{1}{M^2}\right), \quad (75)
\end{aligned}$$

where $\delta_{\gamma\gamma^*}^{(\text{LO};\text{even})}$ is obtained from Eqs. (70) and (71).

2. $\nu = 1$ charge-even contribution

Here, we compute the genuinely dynamical NLO charge-even contributions by involving the NLO soft-photon bremsstrahlung amplitudes. This requires the following charge-even component of the interference of the LO and NLO matrix elements to be determined (cf. Figs. 2 and 3):

$$\begin{aligned}
\mathcal{S}_{\text{even}}^{(\nu=1)} &= \sum_{\text{spins}} \left[\mathcal{M}_{\text{LO}}^\dagger \mathcal{M}_{\text{NLO}} + \text{H.c.} \right]_{\text{even}} \\
&= 2\mathcal{R}e \sum_{\text{spins}} \left[\left(\mathcal{M}_{\gamma\gamma^*}^{[i;i]} + \mathcal{M}_{\gamma\gamma^*}^{[i;f]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[\bar{i};i]} + \mathcal{M}_{\gamma\gamma^*}^{[\bar{i};f]} \right) \right. \\
&\quad + \left(\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[p(1);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);f]} \right) \\
&\quad + \mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0);i}]} + \mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0);f}]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} \\
&\quad \left. + \mathcal{M}_{\gamma\gamma^*}^{[p(1);v]} \right]. \quad (76)
\end{aligned}$$

After integration over the bremsstrahlung phase-space, we obtain the following expression for the NLO charge-even differential cross section in SPL:

$$\begin{aligned}
& \left[\frac{d\sigma_{br}^{(\text{NLO};\text{even})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} \\
&= -\frac{\alpha}{2\pi^2 M} \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 [LL_{ii} + LL_{ff} - LL_{if}], \quad (77)
\end{aligned}$$

where LL_{ii} , LL_{ff} , and LL_{if} are IR-finite integrals, conveniently evaluated in the S-frame (see Appendix B for details). However, here we require their expressions in the lab frame, which are given as follows:

$$LL_{ii} = LL_{ff} = \frac{2\pi\Delta_{\gamma^*} E}{E'}, \quad \text{and} \quad (78)$$

$$LL_{if} = \frac{4\pi\Delta_{\gamma^*} E}{E'} \left(\frac{1 + \nu_l^2}{2\nu_l} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right]. \quad (79)$$

As in the LO case, we find that all NLO matrix elements corresponding to the proton emitting a single soft photon in Eq. (76) lead to scaleless integrals that vanish in DR. The corresponding NLO correction to the charge-even differential cross section is finite and given by

$$\begin{aligned}
& \delta_{\gamma\gamma^*}^{(\text{NLO};\text{even})}(Q^2, \Delta_{\gamma^*}) \\
&= \left[\frac{d\sigma_{br}^{(\text{NLO};\text{even})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} / \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \\
&= \frac{2\alpha\Delta_{\gamma^*}}{\pi M} \left[\left(\frac{1 + \nu_l^2}{2\nu_l} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - 1 \right] + \mathcal{O}\left(\frac{1}{M^2}\right). \quad (80)
\end{aligned}$$

Notably, unlike at LO, our charge-even result involving the NLO amplitudes differs from the calculation of Talukdar *et al.* [14] where, by adopting the Coulomb gauge condition, $v \cdot k = 0$, it yielded higher-order terms in the cross section, namely, $\delta_{\gamma\gamma^*}^{(\text{NLO};\text{even})} \sim \mathcal{O}(1/M^2)$. Beyond these NLO matrix elements, no further charge-even bremsstrahlung contributions to the NLO cross sections arise. All possible interference matrix elements involving the products of chirally enhanced NNLO and N³LO bremsstrahlung amplitudes, namely, those contributing to $\mathcal{S}_{\text{even}}^{(\nu=2)}$ and $\mathcal{S}_{\text{even}}^{(\nu=3)}$, either yield cross section terms of $\mathcal{O}(1/M^2)$ or vanish upon integrating over the bremsstrahlung phase-space. Thus, we obtain the total corrections to the charge-even (anti)lepton-proton differential cross section due to the soft-photon bremsstrahlung calculated up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] below:

$$\begin{aligned}
& \delta_{\gamma\gamma^*}^{(\text{even})}(Q^2, \Delta_{\gamma^*}) \\
&= \left[\frac{d\sigma_{br}^{(\text{even})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} / \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \\
&= \delta_{\gamma\gamma^*}^{(\text{LO};\text{even})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma^*}^{(\text{NLO};\text{even})}(Q^2, \Delta_{\gamma^*}) \\
&\equiv \mathbf{IR}_{\gamma\gamma^*}^{(0)}(Q^2) + \delta_{\gamma\gamma^*}^{(0;\text{even})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma^*}^{(1;\text{even})}(Q^2, \Delta_{\gamma^*}) \\
&\quad + \mathcal{O}\left(\frac{1}{M^2}\right), \quad (81)
\end{aligned}$$

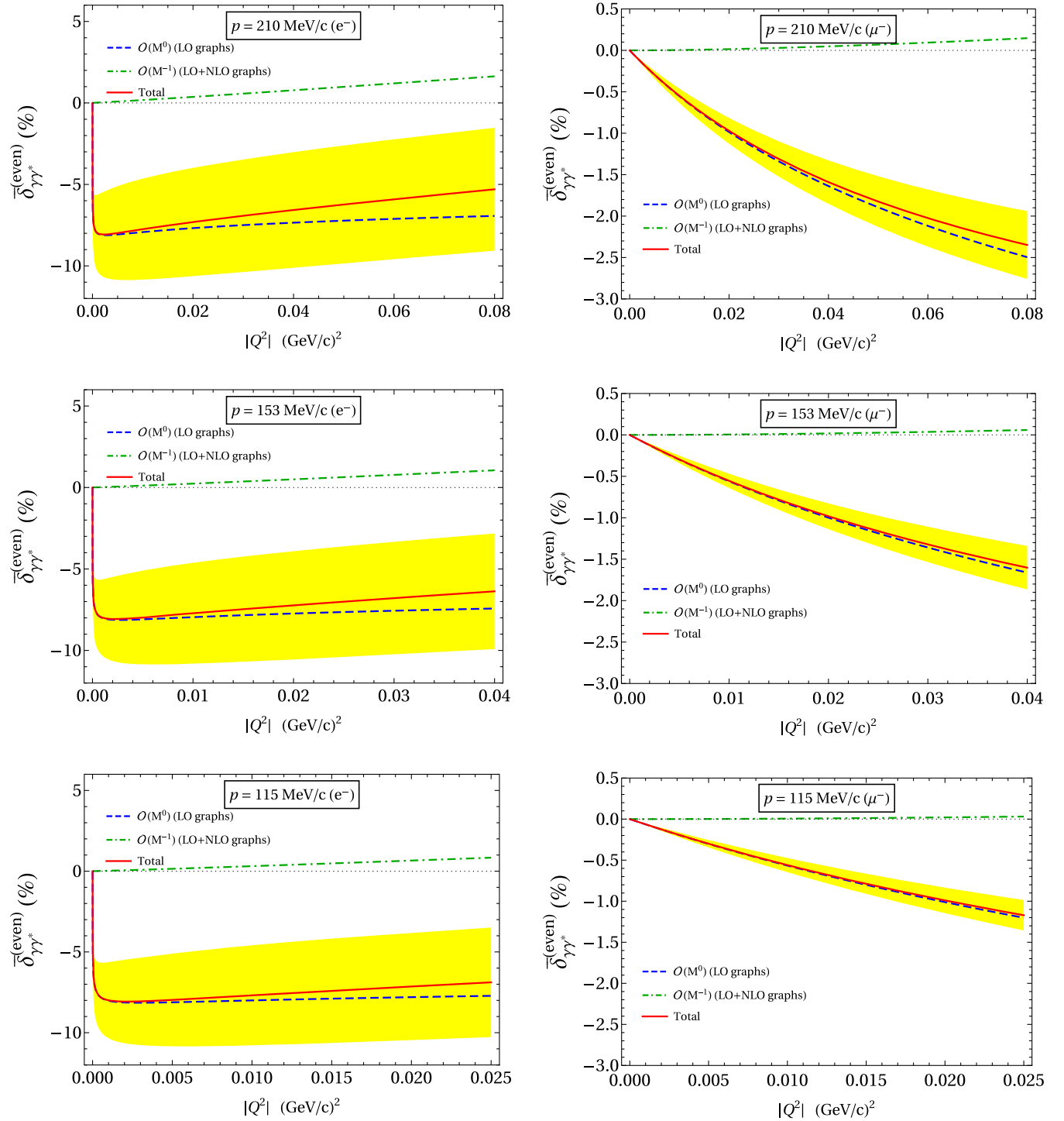


FIG. 7. Finite fractional charge-even soft-photon bremsstrahlung corrections, $\bar{\delta}_{\gamma\gamma^*}^{(\text{even})} = \delta_{\gamma\gamma^*}^{(\text{even})} - \mathbf{IR}_{\gamma\gamma^*}^{(0)}$ [expressed as a percentage relative to the LO Born/OPE differential cross section, Eq. (21)], to the $e^\pm$ -p (left panel) and $\mu^\pm$ -p (right panel) unpolarized elastic scatterings presented as a function of $|Q^2|$ in HB χ PT. The results correspond to three fixed values of the incident lepton beam momenta proposed by the MUSE experiment: 210 MeV/c, 153 MeV/c, and 115 MeV/c. The blue (green) dashed (dashed-dot) curve represents the sum of all genuine chiral LO [NLO], i.e., $\mathcal{O}(M^0)$ [$\mathcal{O}(M^{-1})$] terms $\delta_{\gamma\gamma^*}^{(0;\text{even})}$ [$\delta_{\gamma\gamma^*}^{(1;\text{even})}$], given in Eq. (82) [Eq. (83)], and the red solid curve is their sum (LO + NLO) $\delta_{\gamma\gamma^*}^{(0;\text{even})} + \delta_{\gamma\gamma^*}^{(1;\text{even})}$ (labeled “Total”). All curves are evaluated at a nominal detector cutoff of $\Delta_{\gamma^*} = 1\%$ of the incident lepton beam energy E . The yellow shaded band indicates the sensitivity of the full result to the acceptance in the range $0.5\%E < \Delta_{\gamma^*} < 2\%E$.

where $\delta_{\gamma\gamma^*}^{(\text{LO};\text{even})}$ was already given by Eq. (70). For convenience of displaying our numerical results (cf. Fig. 7), we have equivalently expressed the above contributions by isolating the genuine LO ($\nu = 0$) and NLO ($\nu = 1$) chiral

components of the finite soft-photon bremsstrahlung correction, as given by $\delta_{\gamma\gamma^*}^{(0;\text{even})} \equiv \delta_{\gamma\gamma^*}^{(\text{LO};\text{even})} - \delta_{\gamma\gamma^*}^{(\text{LO}-1/M;\text{even})}$ and $\delta_{\gamma\gamma^*}^{(1;\text{even})} \equiv \delta_{\gamma\gamma^*}^{(\text{LO}-1/M;\text{even})} + \delta_{\gamma\gamma^*}^{(\text{NLO};\text{even})}$, respectively, namely,

$$\delta_{\gamma\gamma^*}^{(0;\text{even})}(Q^2, \Delta_{\gamma^*}) = \frac{\alpha}{\pi} \left[\left(\frac{\nu_l^2 + 1}{2\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - 1 \right) \ln \left(\frac{4\Delta_{\gamma^*}^2}{m_l^2} \right) + \frac{2}{\beta} \ln \sqrt{\frac{1+\beta}{1-\beta}} + \left(\frac{\nu_l^2 + 1}{2\nu_l} \right) \right. \\ \left. \times \left\{ \text{Li}_2 \left(1 - \frac{\lambda_\nu - 1}{(1-\beta)\xi_\nu} \right) + \text{Li}_2 \left(1 - \frac{\lambda_\nu - 1}{(1+\beta)\xi_\nu} \right) - \text{Li}_2 \left(1 - \frac{\lambda_\nu - 1}{(1-\beta)\lambda_\nu \xi_\nu} \right) - \text{Li}_2 \left(1 - \frac{\lambda_\nu - 1}{(1+\beta)\lambda_\nu \xi_\nu} \right) \right\} \right], \quad (82)$$

and

$$\delta_{\gamma\gamma^*}^{(1;\text{even})}(Q^2, \Delta_{\gamma^*}) = \frac{\alpha Q^2}{2\pi M E \beta^2} \left[1 - 2\beta^2 \left(\frac{\nu_l^2 + 1}{2\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - 1 \right) - \left\{ \frac{1}{\beta} - \beta \left(\frac{\nu_l^2 + \nu_l + 1}{\nu_l} \right) \right\} \ln \sqrt{\frac{1+\beta}{1-\beta}} \right. \\ \left. + \beta \left(\frac{\nu_l^2 + 1}{2\nu_l} \right) \left\{ \frac{\lambda_\nu(1+\beta) - 1}{\xi_\nu(1-\beta) - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1-\beta)\xi_\nu} \right) - \frac{\lambda_\nu(1-\beta) - 1}{\xi_\nu(1+\beta) - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1+\beta)\xi_\nu} \right) \right. \right. \\ \left. \left. - \frac{\lambda_\nu \beta}{(1+\beta)\lambda_\nu \xi_\nu - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1+\beta)\xi_\nu \lambda_\nu} \right) - \frac{\lambda_\nu \beta}{(1-\beta)\lambda_\nu \xi_\nu - \lambda_\nu + 1} \ln \left(\frac{\lambda_\nu - 1}{(1-\beta)\xi_\nu \lambda_\nu} \right) \right\} \right. \\ \left. + \frac{4\Delta_{\gamma^*} E \beta^2}{Q^2} \left[\left(\frac{1+\nu_l^2}{2\nu_l} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - 1 \right] \right]. \quad (83)$$

Similarly, it is interesting to note that there are no genuine NLO amplitudes involving the virtual photon loops that contribute to charge-even corrections to the NLO cross section, i.e., $\delta_{\gamma\gamma}^{(\text{NLO};\text{even})} = 0$, as previously reported in Ref. [14]. Hence, the total one-loop IR-finite radiative correction to the charge-even (anti)lepton-proton elastic differential cross section calculated up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] is given as

$$\delta_{2\gamma}^{(\text{even})}(Q^2, \Delta_{\gamma^*}) = \left[\frac{d\sigma_{el}^{(\text{even})}(Q^2)}{d\Omega_l'} \right]_{2\gamma} \bigg/ \left[\frac{d\sigma_{el}(Q^2)}{d\Omega_l'} \right]_0 \\ = \delta_{2\gamma}^{(\text{LO};\text{even})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma^*}^{(\text{NLO};\text{even})}(Q^2, \Delta_{\gamma^*}), \quad (84)$$

where $\delta_{2\gamma}^{(\text{LO};\text{even})}$ was already given by Eq. (75). Figure 8 displays the numerical estimates of $\delta_{2\gamma}^{(\text{even})}$ for $e^\pm$ -p and $\mu^\pm$ -p scatterings.

D. The charge-odd bremsstrahlung component

The cross section for the charge-odd (anti)lepton-proton bremsstrahlung process is obtained from the exchange-interference (see Introduction) products of amplitudes, where the photon is emitted by the (anti)lepton in one and the proton

in the other. Consequently, the cross section changes sign when the lepton's charge polarity is reversed. Below, we evaluate the charge-odd differential cross section in Eq. (23), corresponding to the single soft-photon emission process, which contributes to the lowest-order QED corrections of the $\ell^\pm$ -p cross section, up to and including NLO accuracy [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. In contrast to the charge-even bremsstrahlung cross section, we will now obtain nonvanishing contributions from chirally enhanced NNLO and N³LO amplitudes with a single soft-photon radiated from the final-state proton.

1. $\nu=0$ charge-odd contribution

To compute the charge-odd cross section for soft-photon bremsstrahlung at LO, it is necessary to determine the following charge-odd component of the squared LO matrix elements (cf. Fig. 2):

$$\mathcal{S}_{\text{odd}}^{(\nu=0)} = \sum_{\text{spins}} |\mathcal{M}_{\text{LO}}|_{\text{odd}}^2 \\ = 2\text{Re} \sum_{\text{spins}} \left[\left(\mathcal{M}_{\gamma\gamma^*}^{[l;i]} + \mathcal{M}_{\gamma\gamma^*}^{[l:f]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} \right) \right]. \quad (85)$$

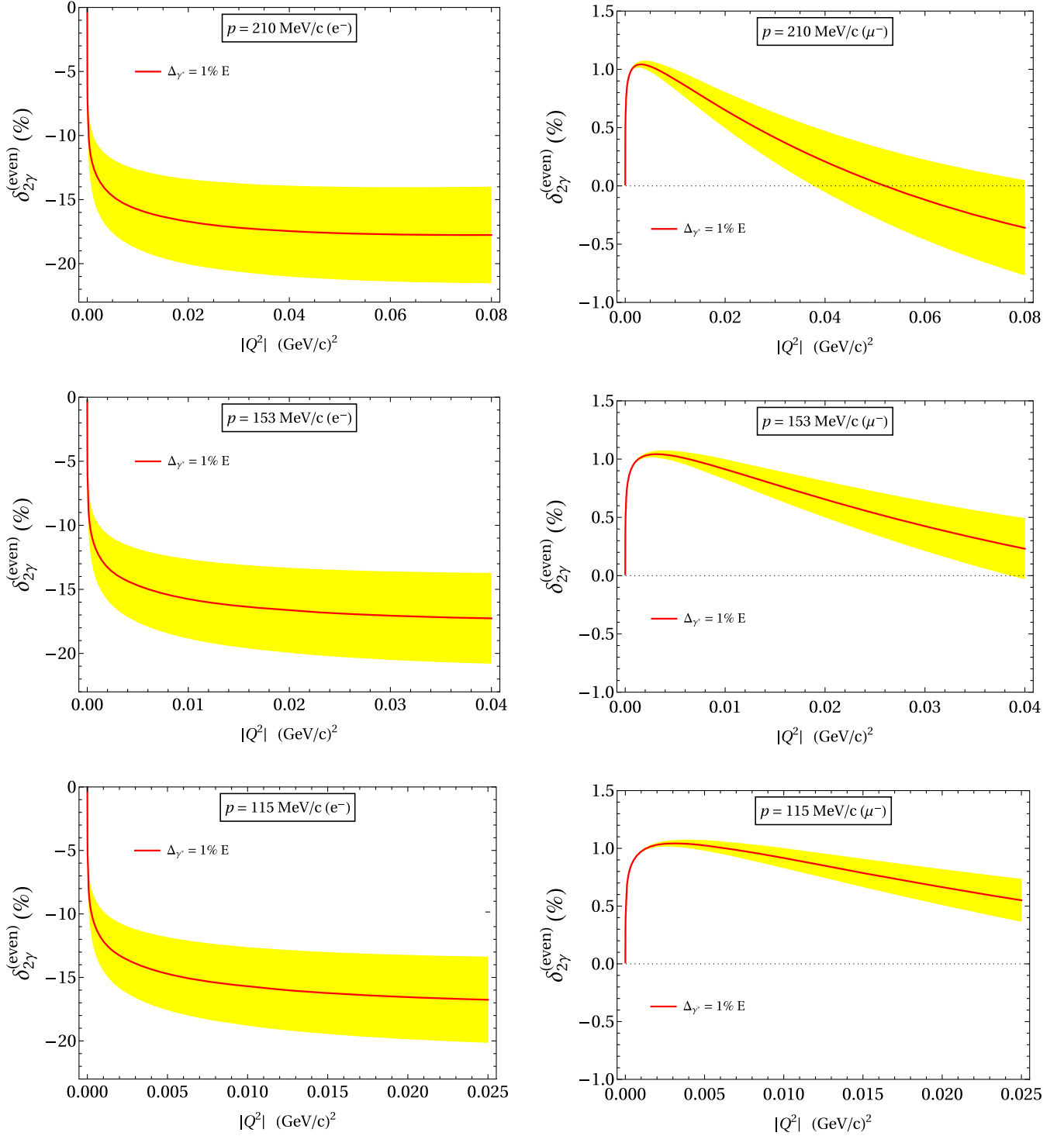


FIG. 8. Finite fractional charge-even radiative corrections [expressed as a percentage relative to the LO Born/OPE differential cross section, Eq. (21)] to the $e^\pm$ -p (left panel) and $\mu^\pm$ -p (right panel) unpolarized elastic scatterings presented as a function of $|Q^2|$ in HB χ PT. The results correspond to three fixed values of the incident lepton beam momenta proposed by the MUSE experiment: 210 MeV/c, 153 MeV/c, and 115 MeV/c. The red solid curve represents the total charge-even radiative corrections, $\delta_{2\gamma}^{(\text{even})}$, Eq. (84), evaluated at the nominal detector cutoff $\Delta_{\gamma^*} = 1\%$ of the incident lepton beam energy E . The yellow shaded band indicates the sensitivity of our result to the detector acceptance in the range $0.5\%E < \Delta_{\gamma^*} < 2\%E$.

After phase-space integration, we obtain the LO charge-odd lepton-proton differential cross section in SPL as

$$\left[\frac{d\sigma_{\gamma\gamma^*}^{(\text{LO;odd})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} = \frac{\alpha M}{\pi^2 Q^2} \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \left[EI_1 - E'I_2 + \frac{Q^2}{2M}(EI_3 - E'I_4) \right], \quad (86)$$

where $I_{1,2}$ are IR-finite and $I_{3,4}$ are IR-divergent integrals. The integrals are most conveniently evaluated in the S-frame [cf. Appendix B]. However, here we need their corresponding expressions in the lab frame, namely,

$$I_1 = \frac{4\pi\Delta_{\gamma^*}}{\beta E'} \ln \sqrt{\frac{1+\beta}{1-\beta}}, \quad (87)$$

$$I_2 = \frac{4\pi\Delta_{\gamma^*}E}{\beta' E'^2} \ln \sqrt{\frac{1+\beta'}{1-\beta'}}, \quad (88)$$

$$I_3 = \frac{2\pi}{\beta E} \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] \ln \sqrt{\frac{1+\beta}{1-\beta}} + 2\pi\tilde{I}_3; \quad (89)$$

$$\tilde{I}_3 = \frac{1}{\beta E} \left[\ln \left(\frac{4\Delta_{\gamma^*}^2 E^2}{m_l^2 E'^2} \right) \ln \sqrt{\frac{1+\beta}{1-\beta}} + \frac{1}{2} \text{Li}_2 \left(\frac{2\beta}{\beta-1} \right) - \frac{1}{2} \text{Li}_2 \left(\frac{2\beta}{\beta+1} \right) \right], \quad \text{and}$$

$$I_4 = \frac{2\pi}{\beta' E'} \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] \ln \sqrt{\frac{1+\beta'}{1-\beta'}} + 2\pi\tilde{I}_4; \quad (90)$$

$$\tilde{I}_4 = \frac{1}{\beta' E'} \left[\ln \left(\frac{4\Delta_{\gamma^*}^2 E^2}{m_l^2 E'^2} \right) \ln \sqrt{\frac{1+\beta'}{1-\beta'}} + \frac{1}{2} \text{Li}_2 \left(\frac{2\beta'}{\beta'-1} \right) - \frac{1}{2} \text{Li}_2 \left(\frac{2\beta'}{\beta'+1} \right) \right].$$

The corresponding fractional contribution to the charge-odd lepton-proton differential cross section arising solely from the LO soft-photon bremsstrahlung amplitudes is given as

$$\begin{aligned} \delta_{\gamma\gamma^*}^{(\text{LO;odd})}(Q^2, \Delta_{\gamma^*}) &= \left[\frac{d\sigma_{br}^{(\text{LO;odd})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} \bigg/ \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \\ &= \mathbf{IR}_{\gamma\gamma^*}^{(1)}(Q^2) + \frac{\alpha M}{\pi^2 Q^2} \left[EI_1 - E'I_2 + \frac{\pi Q^2}{M}(EI_3 - E'\tilde{I}_4) \right] \\ &= \mathbf{IR}_{\gamma\gamma^*}^{(1)}(Q^2) - \frac{2\alpha\Delta_{\gamma^*}}{\pi E\beta^2} \left[1 + \left(\beta - \frac{1}{\beta} \right) \ln \sqrt{\frac{1+\beta}{1-\beta}} \right] + \delta_{\gamma\gamma^*}^{(\text{LO-1}/M;\text{odd})}(Q^2, \Delta_{\gamma^*}), \end{aligned} \quad (91)$$

where

$$\begin{aligned} \delta_{\gamma\gamma^*}^{(\text{LO-1}/M;\text{odd})}(Q^2, \Delta_{\gamma^*}) &= \frac{-\alpha Q^2}{2\pi M E \beta^2} \left[\ln \left(\frac{4\Delta_{\gamma^*}^2}{m_l^2} \right) \left\{ 1 + \left(\beta - \frac{1}{\beta} \right) \ln \sqrt{\frac{1+\beta}{1-\beta}} \right\} - \frac{3\Delta_{\gamma^*}}{E\beta^2} + \frac{1}{2} \left(\beta - \frac{1}{\beta} \right) \left\{ \text{Li}_2 \left(\frac{2\beta}{\beta-1} \right) - \text{Li}_2 \left(\frac{2\beta}{\beta+1} \right) \right\} \right. \\ &\quad \left. - \frac{2}{\beta} \left\{ 1 + \frac{\Delta_{\gamma^*}\beta}{E} \left(1 + \frac{3}{2\beta^2} \right) \left(\beta - \frac{1}{\beta} \right) \right\} \ln \sqrt{\frac{1+\beta}{1-\beta}} \right] + \mathcal{O} \left(\frac{1}{M^2} \right), \end{aligned} \quad (92)$$

with the IR-divergent part given by

$$\begin{aligned} \mathbf{IR}_{\gamma\gamma^*}^{(1)}(Q^2) &= \frac{\alpha}{\pi} \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] \left[\frac{1}{\beta} \ln \sqrt{\frac{1+\beta}{1-\beta}} - \frac{1}{\beta'} \ln \sqrt{\frac{1+\beta'}{1-\beta'}} \right] \\ &= -\frac{\alpha Q^2}{2\pi M E \beta^2} \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] \left[1 + \left(\beta - \frac{1}{\beta} \right) \ln \sqrt{\frac{1+\beta}{1-\beta}} \right] + \mathcal{O} \left(\frac{1}{M^2} \right). \end{aligned} \quad (93)$$

The following two features of the charge-odd soft-photon bremsstrahlung result merit attention:

- (i) First, akin to the charge-even contributions, we again observe that the above expression is not genuinely LO since it also includes $\mathcal{O}(1/M)$ terms which are commensurate with the NLO corrections. Moreover, the LO contribution receives additional modifications from chirally enhanced graphs of $\mathcal{O}(M)$ and $\mathcal{O}(M^2)$, which, under standard power counting, would be classified as NLO and NNLO terms.
- (ii) Second, in our charge-odd bremsstrahlung cross section up to and including NLO, the IR divergence arises solely from the LO amplitudes. However, again at the genuine LO, i.e., $\mathcal{O}(M^0)$ correction to the cross section exhibits complete cancellation of the IR-divergent terms, leaving a residual $\mathcal{O}(1/M)$ IR divergence that effectively contributes to the NLO cross section.⁶ As revealed in our subsequent

analysis, no additional charge-odd IR divergences arise from the genuine NLO corrections to the soft-photon bremsstrahlung cross section.

To cancel the residual $\mathcal{O}(1/M)$ IR singularity, it is necessary to include the TPE counterpart, such that the total charge-odd radiative correction to the elastic cross section is obtained as a finite physical quantity. Recently, the low-energy TPE corrections to the elastic cross section were analytically evaluated exactly up to and including NLO in HB χ PT by Choudhary *et al.* [19]. It turns out that in the corrections to the TPE cross section up to $\mathcal{O}(1/M)$, the IR divergences arise solely from the LO TPE box amplitudes with no additional singularities arising from the genuine NLO TPE amplitudes. The two LO IR-divergent parts of the TPE exactly cancel, leaving only a residual IR divergence of $\mathcal{O}(1/M)$ that contributes to the NLO TPE cross section [19]. From Eq. (33) of Ref. [19], we find the following analytical expression for the IR-divergent part of the TPE cross section up to and including NLO in HB χ PT:

$$\begin{aligned} \mathbf{IR}_{\text{TPE}}^{(1)}(Q^2) &= -\mathbf{IR}_{\gamma\gamma^*}^{(1)}(Q^2) = -\frac{\alpha}{\pi} \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln\left(\frac{4\pi\mu^2}{m_l^2}\right) \right] \left[\frac{1}{\beta} \ln \sqrt{\frac{1+\beta}{1-\beta}} - \frac{1}{\beta'} \ln \sqrt{\frac{1+\beta'}{1-\beta'}} \right] \\ &= \frac{\alpha Q^2}{2\pi M E \beta^2} \left[\frac{1}{|\varepsilon_{\text{IR}}|} + \gamma_E - \ln\left(\frac{4\pi\mu^2}{m_l^2}\right) \right] \left[1 + \left(\beta - \frac{1}{\beta} \right) \ln \sqrt{\frac{1+\beta}{1-\beta}} \right] + \mathcal{O}\left(\frac{1}{M^2}\right). \end{aligned} \quad (94)$$

With equal and opposite charge-odd IR-divergent terms arising from the TPE and exchange-interference (see Introduction) soft-photon bremsstrahlung counterparts, the net charge-odd radiative correction to the elastic cross section up to and including NLO is therefore finite. The work of Choudhary *et al.* [19] used analytical techniques, such as integration by parts and partial fractions, to evaluate the TPE box diagrams, which, from the LO amplitudes, yielded the following form of the result:

$$\begin{aligned} \delta_{\gamma\gamma}^{(\text{LO;odd})}(Q^2) &= \frac{\left[\frac{d\sigma_{\text{virt}}^{(\text{LO;odd})}(Q^2)}{d\Omega_l'} \right]_{\gamma\gamma}}{\left[\frac{d\sigma_{\text{el}}(Q^2)}{d\Omega_l'} \right]_0} \\ &= \mathbf{IR}_{\text{TPE}}^{(1)}(Q^2) + \pi\alpha \frac{\sqrt{-Q^2}}{2E} \left[\frac{1}{1 + \frac{Q^2}{4E^2}} \right] \\ &\quad + \delta_{\gamma\gamma}^{(ab;1/M)}(Q^2). \end{aligned} \quad (95)$$

Here, the second term is our HB χ PT version of the McKinley-Feshbach term [54], and the IR-finite term

$\delta_{\gamma\gamma}^{(ab;1/M)} \sim \mathcal{O}(1/M)$ is taken from Choudhary *et al.* [19], wherein one can find the detailed expression. In particular, the latter $\mathcal{O}(1/M)$ correction effectively modifies the contributions from the seven other genuine NLO TPE diagrams [19]. Hence, the total one-loop finite fractional radiative correction to the charge-odd elastic differential cross section is

$$\begin{aligned} \delta_{2\gamma}^{(\text{LO;odd})}(Q^2, \Delta_{\gamma^*}) &= \frac{\left[\frac{d\sigma_{\text{el}}^{(\text{LO;odd})}(Q^2)}{d\Omega_l'} \right]_{2\gamma}}{\left[\frac{d\sigma_{\text{el}}(Q^2)}{d\Omega_l'} \right]_0} \\ &= \delta_{\gamma\gamma^*}^{(\text{LO;odd})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma}^{(\text{LO;odd})}(Q^2), \end{aligned} \quad (96)$$

where $\delta_{\gamma\gamma^*}^{(\text{LO;odd})}$ is given by Eqs. (91) and (92), and $\delta_{\gamma\gamma}^{(\text{LO;odd})}$ is given by Eq. (95).

2. $\nu=1$ charge-odd contribution

Next, we compute the genuinely dynamical NLO charge-odd contributions to the cross section by involving the NLO soft-photon bremsstrahlung amplitudes. The following charge-odd component of the interference of the LO and NLO matrix elements must be determined in this case (cf. Figs. 2 and 3):

⁶This feature is reminiscent of the analogous singularity structure of the LO TPE cross section, as discussed in the work of Choudhary *et al.* [19], constituting the charge-odd virtual correction counterpart of the exchange-interference (see Introduction) bremsstrahlung cross section at LO.

$$\begin{aligned}
S_{\text{odd}}^{(\nu=1)} &= \sum_{\text{spins}} \left[\mathcal{M}_{\text{LO}}^\dagger \mathcal{M}_{\text{NLO}} + \text{H.c.} \right]_{\text{odd}} \\
&= 2\mathcal{R}e \sum_{\text{spins}} \left[\left(\mathcal{M}_{\gamma\gamma^*}^{[l;i]} + \mathcal{M}_{\gamma\gamma^*}^{[l;f]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[p(1);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);f]} + \mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0);i}]} + \mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0);f}]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i\otimes]} \right. \right. \\
&\quad \left. \left. + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);v]} \right) + \left(\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[l;i]} + \mathcal{M}_{\gamma\gamma^*}^{[l;f]} \right) \right]. \tag{97}
\end{aligned}$$

After the integration over the bremsstrahlung phase-space, the corresponding charge-odd lepton-proton differential cross section in SPL can be expressed as

$$\begin{aligned}
\left[\frac{d\sigma_{br}^{(\text{NLO;odd})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} &= \frac{\alpha M}{\pi^2 Q^2} \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \left[EI_1 - E'I_2 + \frac{Q^2}{2M} \left\{ I_1 + I_2 + \frac{2}{Q^2} (EI_5 - E'I_6) \right\} \right. \\
&\quad \left. + \frac{Q^2}{M^2} \left\{ EI_1 - E'I_2 + \frac{4m_l^2(E+E')}{Q^2 + 4EE'} (I_1 - I_2) \right\} \right] + \mathcal{O}\left(\frac{1}{M^2}\right), \tag{98}
\end{aligned}$$

where $I_{1,2,5,6}$ are all IR-finite integrals. These integrals are most conveniently evaluated in the S-frame whose expressions are presented in Appendix B. Translating those results to the lab frame, the corresponding expressions for $I_{1,2}$ have already been presented in Eqs. (87) and (88), while those for I_5 and I_6 are, respectively, given as

$$I_5 = \frac{2\pi\Delta_{\gamma^*}^2 E^2}{\beta EE'^2} \ln \sqrt{\frac{1+\beta}{1-\beta}}, \quad \text{and} \tag{99}$$

$$I_6 = \frac{2\pi\Delta_{\gamma^*}^2 E^2}{\beta' E'^3} \ln \sqrt{\frac{1+\beta'}{1-\beta'}}. \tag{100}$$

We then obtain the fractional contribution to the charge-odd lepton-proton differential cross section at NLO [i.e., $\mathcal{O}(\alpha^3/M)$] arising from the LO and NLO bremsstrahlung amplitudes. As mentioned earlier, the NLO correction is IR finite and given by the following expression:

$$\begin{aligned}
\delta_{\gamma\gamma^*}^{(\text{NLO;odd})}(Q^2, \Delta_{\gamma^*}) &= \left[\frac{d\sigma_{br}^{(\text{NLO;odd})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} \bigg/ \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \\
&= -\frac{2\alpha\Delta_{\gamma^*}}{\pi E\beta^2} \left[1 - \left(\beta + \frac{1}{\beta} \right) \ln \sqrt{\frac{1+\beta}{1-\beta}} \right] + \delta_{\gamma\gamma^*}^{(\text{NLO-1/M;odd})}(Q^2, \Delta_{\gamma^*}) \quad \text{where} \\
\delta_{\gamma\gamma^*}^{(\text{NLO-1/M;odd})}(Q^2, \Delta_{\gamma^*}) &= -\frac{\alpha Q^2}{2\pi M E \beta^2} \left[\frac{2\Delta_{\gamma^*}^2}{Q^2} - \frac{3\Delta_{\gamma^*}}{E\beta^2} \left(1 + \frac{2\beta^2}{3} \right) + \left\{ \frac{2\Delta_{\gamma^*}^2}{Q^2} \left(\beta - \frac{1}{\beta} \right) \right. \right. \\
&\quad \left. \left. + \frac{\Delta_{\gamma^*}}{E} \left(\frac{3}{\beta^3} + \frac{1}{\beta} + 2\beta \right) \right\} \ln \sqrt{\frac{1+\beta}{1-\beta}} \right] + \mathcal{O}\left(\frac{1}{M^2}\right). \tag{101}
\end{aligned}$$

We find that the above expression is not genuinely NLO [i.e., $\mathcal{O}(1/M)$]. Interestingly, the first term contains a LO [i.e., $\mathcal{O}(M^0)$] term due to the chiral enhancement. This contribution originates from the NLO amplitudes involving the final-state proton radiation, each containing a proton propagator that scales as $\mathcal{O}(M)$, namely, $\mathcal{M}_{\gamma\gamma^*}^{[p(1);f]}$, $\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0);f}]}$, and $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]}$, which yield the above LO component. It is important to recognize that the $\mathcal{O}(1/M)$ terms from the previously considered LO charge-odd bremsstrahlung correction, namely, $\delta_{\gamma\gamma^*}^{(\text{LO-1/M;odd})}$ in Eq. (92), must be retained alongside $\delta_{\gamma\gamma^*}^{(\text{NLO;odd})}$ to properly account for the complete NLO contribution. In addition, one must also include the potential contributions to the

NLO cross section arising from the $\mathcal{O}(M)$ chirally enhanced NNLO and N³LO bremsstrahlung amplitudes, which we examine in the following analysis.

3. $\nu=2$ charge-odd contribution

The computation of the charge-odd component of the cross section for soft-photon bremsstrahlung up to and including NLO entails the inclusion of additional exchange-interference matrix elements involving products of LO and NLO amplitudes with chirally enhanced N²LO amplitudes (cf. Figs. 2–4):

$$\begin{aligned} \mathcal{S}_{\text{odd}}^{(\nu=2)} &= \sum_{\text{spins}} |\mathcal{M}_{\text{NLO}}|^2_{\text{odd}} + \sum_{\text{spins}} \left[\mathcal{M}_{\text{LO}}^\dagger \widetilde{\mathcal{M}}_{\text{NNLO}} + \text{H.c.} \right]_{\text{odd}} \\ &= 2\mathcal{R}e \sum_{\text{spins}} \left[\left(\mathcal{M}_{\gamma\gamma^*}^{[i;\bar{i}]} + \mathcal{M}_{\gamma\gamma^*}^{[i;\bar{f}]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[p(1);f]} + \mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0);f}]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} \right) + \left(\mathcal{M}_{\gamma\gamma^*}^{[i;\bar{i}]} + \mathcal{M}_{\gamma\gamma^*}^{[i;\bar{f}]} \right)^\dagger \right. \\ &\quad \times \left(\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(1);f}]} + \mathcal{M}_{\gamma\gamma^*}^{[\widetilde{p(0);f}]} + \mathcal{M}_{\gamma\gamma^*}^{[p(2);f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0);f\otimes}]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(c)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(d)]} \right) \\ &\quad \left. + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]^\dagger} \left(\mathcal{M}_{\gamma\gamma^*}^{[i;\bar{i}]} + \mathcal{M}_{\gamma\gamma^*}^{[i;\bar{f}]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(a)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(b)]} \right) \right] + \mathcal{O}\left(\frac{1}{M^2}\right). \end{aligned} \quad (102)$$

Here, the $\mathcal{O}(1/M^2)$ terms correspond to neglected products of amplitudes lacking chiral enhancement, resulting in higher-order contributions to the charge-odd bremsstrahlung cross section. We have relegated all such $\mathcal{O}(1/M^2)$ terms $\in \mathcal{S}_{\text{odd}}^{(\nu=2)}$ to Appendix A [see Eq. (A36)].

After the phase-space integration, the terms displayed in Eq. (102) yield the following additional correction terms to the charge-odd lepton-proton differential cross section up to and including NLO in SPL:

$$\begin{aligned} \left[\frac{d\sigma_{br}^{(\text{NNLO;odd})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} &= \frac{\alpha}{2\pi^2 Q^2} \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \left[Q^2(I_1 + I_2) + 2(EI_5 - E'I_6) + \frac{Q^2}{2M} \left\{ 2\left(\frac{r_p^2 M^2}{3} + 1\right)(EI_1 - E'I_2) \right. \right. \\ &\quad \left. \left. + (I_5 + I_6) + \frac{3}{Q^2}(EI_7 - E'I_8) + \frac{1}{Q^2}(EI_9 - E'I_{10}) \right\} \right] + \mathcal{O}\left(\frac{1}{M^2}\right) \\ &= \frac{\alpha}{2\pi^2 Q^2} \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \left[Q^2(I_1 + I_2) + 2(EI_5 - E'I_6) + \frac{Q^2}{2M}(I_5 + I_6) \right] + \mathcal{O}\left(\frac{1}{M^2}\right), \end{aligned} \quad (103)$$

where the finite phase-space integrals $I_{1,2,5,6}$ have already been presented earlier. The integrals $I_{7,8,9,10}$ were evaluated by boosting to the S-frame [cf. Appendix B] and subsequently translated back into the lab frame after their evaluation. They are given below:

$$I_7 = \frac{4\pi\Delta_{\gamma^*}^3 E^2}{3\beta E'^3} \ln \sqrt{\frac{1+\beta}{1-\beta}}, \quad (104)$$

$$I_8 = \frac{4\pi\Delta_{\gamma^*}^3 E^3}{3\beta' E'^4} \ln \sqrt{\frac{1+\beta'}{1-\beta'}}, \quad (105)$$

$$I_9 = \frac{4\pi\Delta_{\gamma^*}^2 E}{2\beta E'} \left[\beta' - \left(1 - \frac{\beta'}{\beta}\right) \ln \sqrt{\frac{1+\beta}{1-\beta}} \right], \quad \text{and} \quad (106)$$

$$I_{10} = -\frac{4\pi\Delta_{\gamma^*}^2 E^3}{2\beta' E'^3} \left[\beta - \left(1 - \frac{\beta}{\beta'}\right) \ln \sqrt{\frac{1+\beta'}{1-\beta'}} \right]. \quad (107)$$

Here, we reiterate that in our analysis, the proton's rms charge radius r_p is determined as a phenomenological fixed LEC. Since it implicitly accounts for all possible pion-loop contributions responsible for the proton's hadronic structure, it must therefore scale as $r_p^2 \sim \mathcal{O}(1/M^2)$. Thus, all terms enclosed within the curly brackets “{...}” in Eq. (103) appear to scale as $\mathcal{O}(1)$. However, the subtlety here is that all the following combinations of integrals scale as $(EI_1 - E'I_2) \sim (EI_7 - E'I_8) \sim (EI_9 - E'I_{10}) \sim \mathcal{O}(1/M)$;

namely, they are all kinematically suppressed with $\mathcal{O}(1)$ terms canceling out. Multiplied by the factor $Q^2/(2M)$, all the terms within the curly brackets, except for $I_5 \sim I_6 \sim \mathcal{O}(1)$, scale as $\mathcal{O}(1/M^2)$ and hence are ultimately discarded. Accordingly, the only additional finite fractional contribution to the charge-odd lepton-proton differential cross section up to and including NLO, involving the chirally enhanced NNLO amplitudes, is given as

$$\begin{aligned} \delta_{\gamma\gamma^*}^{(\text{NNLO;odd})}(Q^2, \Delta_{\gamma^*}) &= \left[\frac{d\sigma_{br}^{(\text{NNLO;odd})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} \bigg/ \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \\ &= \frac{4\alpha\Delta_{\gamma^*}}{\pi E\beta} \ln \sqrt{\frac{1+\beta}{1-\beta}} + \delta_{\gamma\gamma^*}^{(\text{NNLO}-1/M;\text{odd})}(Q^2, \Delta_{\gamma^*}) \quad \text{where} \\ \delta_{\gamma\gamma^*}^{(\text{NNLO}-1/M;\text{odd})}(Q^2, \Delta_{\gamma^*}) &= -\frac{\alpha Q^2}{2\pi M E \beta^2} \left[\frac{2\Delta_{\gamma^*}^2}{Q^2} \left(1 - \frac{\beta^2 E^2}{Q^2} \right) - \frac{2\Delta_{\gamma^*}}{E} - \left\{ \frac{2\Delta_{\gamma^*}^2}{Q^2\beta} - \frac{2\Delta_{\gamma^*}}{E} \left(2\beta + \frac{1}{\beta} \right) \right\} \ln \sqrt{\frac{1+\beta}{1-\beta}} \right] + \mathcal{O}\left(\frac{1}{M^2}\right). \end{aligned} \quad (108)$$

It is indeed interesting that even a LO [i.e., $\mathcal{O}(M^0)$] contribution to the charge-odd bremsstrahlung cross section transcends into the otherwise naively counted NNLO correction as obtained above. The origin of such a term is attributed to the NNLO final-state radiating proton amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]}$ [cf. Eq. (40)] interfering with the two LO radiating lepton amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[l;i/f]}$, as seen in Eq. (102). Moreover, we find that the hadronic structure-dependent terms—those proportional to the proton charge radius r_p —arising from the NNLO bremsstrahlung amplitudes with

renormalized pion loops ultimately cancel out and do not contribute to the $\mathcal{O}(\alpha^3/M)$ NLO cross section.

4. $\nu=3$ charge-odd contribution

Finally, we account for the contribution from the chirally enhanced $N^3\text{LO}$ amplitudes (cf. Fig. 6) in the evaluation of the charge-odd NLO cross section for soft-photon bremsstrahlung. This involves computing the following interference among the LO, NLO, and the chirally enhanced NNLO and $N^3\text{LO}$ amplitudes:

$$\begin{aligned} \mathcal{S}_{\text{odd}}^{(\nu=3)} &= \sum_{\text{spins}} \left[\mathcal{M}_{\text{NLO}}^\dagger \widetilde{\mathcal{M}}_{\text{NNLO}} + \mathcal{M}_{\text{LO}}^\dagger \widetilde{\mathcal{M}}_{\text{N}^3\text{LO}} + \text{H.c.} \right]_{\text{odd}} \\ &= 2\text{Re} \left[\left(\mathcal{M}_{\gamma\gamma^*}^{[l;i]} + \mathcal{M}_{\gamma\gamma^*}^{[l:f]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} \right) + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]\dagger} \left(\mathcal{M}_{\gamma\gamma^*}^{[l;i]} + \mathcal{M}_{\gamma\gamma^*}^{[l:f]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(a)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(b)]} \right) \right. \\ &\quad \left. + \left(\mathcal{M}_{\gamma\gamma^*}^{[l;i]} + \mathcal{M}_{\gamma\gamma^*}^{[l:f]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(2);f\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(e)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(f)]} \right) \right] + \mathcal{O}\left(\frac{1}{M^2}\right), \end{aligned} \quad (109)$$

where $\mathcal{O}(1/M^2)$ terms again denote neglected higher-order contributions to the cross section, arising from the interference between the NLO amplitudes $\in \mathcal{M}_{\text{NLO}}$ and the renormalized pion-loop NNLO amplitudes $\in \widetilde{\mathcal{M}}_{\text{NNLO}}$. Such interference terms $\in \mathcal{S}_{\text{odd}}^{(\nu=3)}$ are not chirally enhanced and hence are excluded in the above expression, relegating them to Appendix A [see Eq. (A53)].

Upon integrating over the bremsstrahlung phase-space, all contributions to the cross section from the interference terms $\sum_{\text{spins}} [\mathcal{M}_{\text{LO}}^\dagger \widetilde{\mathcal{M}}_{\text{N}^3\text{LO}} + \text{H.c.}]$ either correspond to

higher-order terms of $\mathcal{O}(1/M^2)$ or lead to scaleless integrals that vanish in dimensional regularization, as per the condition given in Eq. (63). Consequently, none of the chirally enhanced $N^3\text{LO}$ amplitudes ultimately contribute to the $\mathcal{O}(\alpha^3/M)$ NLO cross section (cf. Fig. 6). The remaining interference terms between $\mathcal{M}_{\text{NLO}}$ and the enhanced $\widetilde{\mathcal{M}}_{\text{NNLO}}$ amplitudes grouped in $\mathcal{S}_{\text{odd}}^{(\nu=3)}$ solely account for the following correction to the NNLO counterpart, i.e., Eq. (103), contributing to the charge-odd soft-photon bremsstrahlung differential cross section in the lab frame:

$$\begin{aligned}
\Delta \left[\frac{d\sigma_{br}^{(\text{NNLO;odd})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*} &= \frac{\alpha}{4\pi^2 M} \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \left[\left(\frac{2r_p^2 M^2}{3} - 1 \right) (EI_1 - E'I_2) + (I_5 + I_6) \right. \\
&\quad \left. + \frac{3}{Q^2} (EI_7 - E'I_8) + \frac{1}{Q^2} (EI_9 - E'I_{10}) \right] + \mathcal{O}\left(\frac{1}{M^2}\right) \\
&= \frac{\alpha}{4\pi^2 M} \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 (I_5 + I_6) + \mathcal{O}\left(\frac{1}{M^2}\right).
\end{aligned} \tag{110}$$

Here, we again find that the proton's hadronic structure-dependent terms, arising from the NNLO bremsstrahlung amplitudes with renormalized pion loops, ultimately cancel out and do not contribute to the $\mathcal{O}(\alpha^3/M)$ NLO cross section. Accordingly, the additional finite fractional correction to Eq. (108), arising from the interference between the NLO and chirally enhanced NNLO amplitudes and contributing to the charge-odd lepton-proton differential cross section at NLO [i.e., $\mathcal{O}(\alpha^3/M)$], is given as

$$\begin{aligned}
\tilde{\delta}_{\gamma\gamma^*}^{(\text{NNLO;odd})}(Q^2, \Delta_{\gamma^*}) &= \Delta \left[\frac{d\sigma_{br}^{(\text{NNLO;odd})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} \bigg/ \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \\
&= -\frac{\alpha Q^2}{2\pi M E \beta^2} \left[-\frac{4\Delta_{\gamma^*}^2 E^2 \beta^2}{Q^4} - \frac{2\Delta_{\gamma^*}^2 \beta}{Q^2} \ln \sqrt{\frac{1+\beta}{1-\beta}} \right] + \mathcal{O}\left(\frac{1}{M^2}\right),
\end{aligned} \tag{111}$$

where, as stated above, $\tilde{\delta}_{\gamma\gamma^*}^{(\text{NNLO;odd})} \sim \mathcal{O}(\alpha/M)$ originates from the first two terms in Eq. (110) which are generated by enhanced NNLO terms, while the genuine N³LO terms of $\mathcal{O}(1/M^2)$ have been dropped. Hence, the total fractional corrections to the charge-odd (anti)lepton-proton differential cross section due to the soft-photon bremsstrahlung, calculated up to and including NLO in HB χ PT, can be consolidated by the sum of contributions that we obtained in Eqs. (91), (92), (101), (108), and (111) (cf. Fig. 9 for numerical estimates for e^- -p and μ^- -p scatterings):

$$\begin{aligned}
\delta_{\gamma\gamma^*}^{(\text{odd})}(Q^2, \Delta_{\gamma^*}) &= \left[\frac{d\sigma_{br}^{(\text{odd})}(Q^2)}{d\Omega'_l} \right]_{\gamma\gamma^*}^{(E_{\gamma^*} \leq \Delta_{\gamma^*})} \bigg/ \left[\frac{d\sigma_{el}(Q^2)}{d\Omega'_l} \right]_0 \\
&= \delta_{\gamma\gamma^*}^{(\text{LO;odd})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma^*}^{(\text{NLO;odd})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma^*}^{(\text{NNLO;odd})}(Q^2, \Delta_{\gamma^*}) + \tilde{\delta}_{\gamma\gamma^*}^{(\text{NNLO;odd})}(Q^2, \Delta_{\gamma^*}) \\
&\equiv \mathbf{IR}_{\gamma\gamma^*}^{(1)}(Q^2) + \delta_{\gamma\gamma^*}^{(0;\text{odd})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma^*}^{(1;\text{odd})}(Q^2, \Delta_{\gamma^*}) + \mathcal{O}\left(\frac{1}{M^2}\right),
\end{aligned} \tag{112}$$

where the genuine chiral LO ($\nu = 0$) and NLO ($\nu = 1$) IR-finite soft-photon bremsstrahlung corrections are given as

$$\begin{aligned}
\delta_{\gamma\gamma^*}^{(0;\text{odd})}(\Delta_{\gamma^*}) &= \left[\delta_{\gamma\gamma^*}^{(\text{LO;odd})}(Q^2, \Delta_{\gamma^*}) - \delta_{\gamma\gamma^*}^{(\text{LO}-1/M;\text{odd})}(Q^2, \Delta_{\gamma^*}) \right] + \left[\delta_{\gamma\gamma^*}^{(\text{NLO;odd})}(Q^2, \Delta_{\gamma^*}) - \delta_{\gamma\gamma^*}^{(\text{NLO}-1/M;\text{odd})}(Q^2, \Delta_{\gamma^*}) \right] \\
&\quad + \left[\delta_{\gamma\gamma^*}^{(\text{NNLO;odd})}(Q^2, \Delta_{\gamma^*}) - \delta_{\gamma\gamma^*}^{(\text{NNLO}-1/M;\text{odd})}(Q^2, \Delta_{\gamma^*}) \right] \\
&= -\frac{4\alpha\Delta_{\gamma^*}}{\pi E \beta^2} \left[1 - \left(\beta + \frac{1}{\beta} \right) \ln \sqrt{\frac{1+\beta}{1-\beta}} \right],
\end{aligned} \tag{113}$$

which is interestingly a Q^2 -independent result, and

$$\begin{aligned}
\delta_{\gamma\gamma^*}^{(1;\text{odd})}(Q^2, \Delta_{\gamma^*}) &= \delta_{\gamma\gamma^*}^{(\text{LO}-1/M;\text{odd})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma^*}^{(\text{NLO}-1/M;\text{odd})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma^*}^{(\text{NNLO}-1/M;\text{odd})}(Q^2, \Delta_{\gamma^*}) + \tilde{\delta}_{\gamma\gamma^*}^{(\text{NNLO;odd})}(Q^2, \Delta_{\gamma^*}) \\
&= -\frac{\alpha Q^2}{2\pi M E \beta^2} \left[\ln \left(\frac{4\Delta_{\gamma^*}^2}{m_l^2} \right) \left\{ 1 + \left(\beta - \frac{1}{\beta} \right) \ln \sqrt{\frac{1+\beta}{1-\beta}} \right\} + \frac{1}{2} \left(\beta - \frac{1}{\beta} \right) \left\{ \text{Li}_2 \left(\frac{2\beta}{\beta-1} \right) - \text{Li}_2 \left(\frac{2\beta}{1+\beta} \right) \right\} \right. \\
&\quad \left. + \frac{4\Delta_{\gamma^*}^2}{Q^2} \left(1 - \frac{3\beta^2 E^2}{2Q^2} \right) - \frac{4\Delta_{\gamma^*}}{E} \left(1 + \frac{3}{2\beta^2} \right) - \left\{ \frac{4\Delta_{\gamma^*}^2}{Q^2 \beta} - \frac{2\Delta_{\gamma^*}}{E} \left(2\beta + \frac{1}{\beta} + \frac{3}{\beta^3} \right) + \frac{2}{\beta} \right\} \ln \sqrt{\frac{1+\beta}{1-\beta}} \right],
\end{aligned} \tag{114}$$

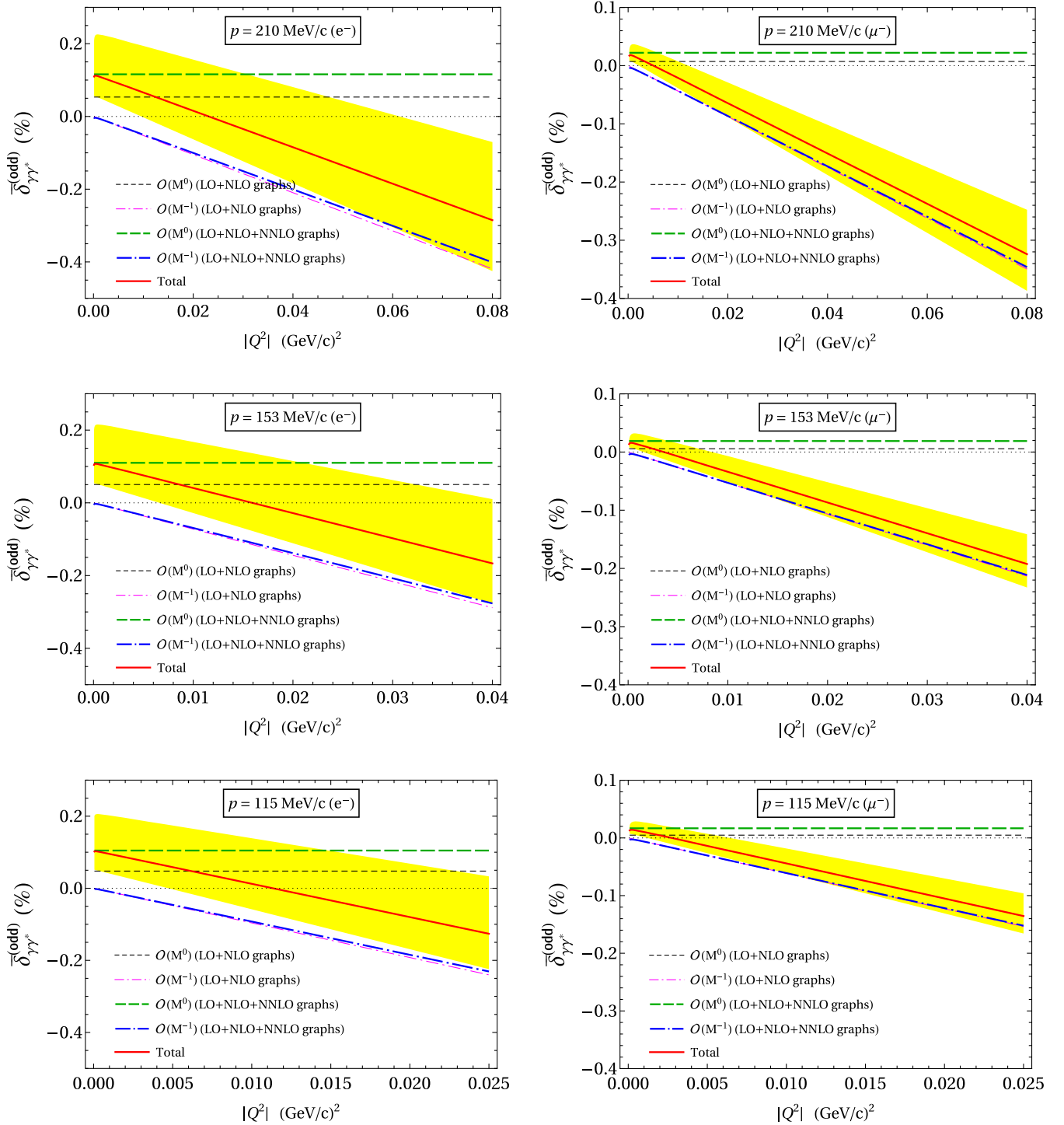


FIG. 9. Finite fractional charge-odd soft-photon bremsstrahlung corrections $\bar{\delta}_{\gamma\gamma^*}^{(\text{odd})} = \delta_{\gamma\gamma^*}^{(\text{odd})} - \mathbf{IR}_{\gamma\gamma^*}^{(1)}$ [expressed as a percentage relative to the LO Born/OPE differential cross section, Eq. (21)] to the e^- -p (left panel) and μ^- -p (right panel) unpolarized elastic scatterings presented as a function of $|Q^2|$ in HB χ PT. The results correspond to three fixed values of the incident lepton beam momenta proposed by the MUSE experiment: 210 MeV/c, 153 MeV/c, and 115 MeV/c. The green (blue) long dashed (long dashed-dot) curve represents the sum of all genuine chiral LO [NLO], i.e., $\mathcal{O}(M^0)$ [$\mathcal{O}(M^{-1})$] terms $\delta_{\gamma\gamma^*}^{(0;\text{odd})}$ [$\delta_{\gamma\gamma^*}^{(1;\text{odd})}$], given in Eq. (113) [Eq. (114)], which includes the chirally enhanced contributions from NNLO bremsstrahlung graphs with final-state proton propagator components. The black (magenta) short dashed (short dashed-dot) curve represents the corresponding results without the enhanced contributions from the NNLO graphs. The red solid curve finally represents the complete chirally enhanced sum (LO + NLO), namely, $\delta_{\gamma\gamma^*}^{(0;\text{odd})} + \delta_{\gamma\gamma^*}^{(1;\text{odd})}$ (labeled as “Total”). All curves are evaluated at a nominal detector cutoff $\Delta_{\gamma^*} = 1\%$ of the incident lepton beam energy E . The yellow shaded band indicates the sensitivity of our full result to the detector acceptance in the range $0.5\%E < \Delta_{\gamma^*} < 2\%E$.

respectively. The above result is consistent with the standard expectation in HB χ PT analysis that up to corrections of $\mathcal{O}(1/M)$ (i.e., chiral order $\nu = 1$) in the radiative corrections, the proton's hadronic finite-size structure effectively does not play any role. As previously noted, the residual $\mathcal{O}(1/M)$ IR divergence $\mathbf{IR}_{\gamma\gamma^*}^{(1)}$, Eq. (93), is exactly canceled by the charge-odd counterpart stemming from the residual $\mathcal{O}(1/M)$ IR divergences of the LO TPE box (a) and crossed-box (b) amplitudes (see Refs. [12,14,19] for detailed discussions of the TPE diagrams in HB χ PT), i.e., $\mathbf{IR}_{\text{TPE}}^{(1)} = -\mathbf{IR}_{\gamma\gamma^*}^{(1)}$. The net contribution to the charge-odd radiative correction to the elastic cross section is therefore a finite physical quantity. In the recent work by Choudhary *et al.* [19], the contributions to the elastic cross section from the sum of seven genuine NLO TPE amplitudes—specifically, the box diagrams (c), (e), and (g); the crossed-box diagrams (d), (f), and (h); and the seagull diagram (i), which are not shown in this paper—were evaluated exactly using analytical methods. In this work, we will denote these contributions collectively as

$$\begin{aligned} \delta_{\gamma\gamma}^{(\text{NLO;odd})}(Q^2) &\equiv \delta_{\text{box}}^{(c)}(Q^2) + \delta_{\text{xbox}}^{(d)}(Q^2) + \delta_{\text{box}}^{(e)}(Q^2) \\ &\quad + \delta_{\text{xbox}}^{(f)}(Q^2) + \delta_{\text{box}}^{(g)}(Q^2) + \delta_{\text{xbox}}^{(h)}(Q^2) \\ &\quad + \delta_{\text{seagull}}^{(i)}(Q^2). \end{aligned} \quad (115)$$

Details of the analytical expressions for the individual TPE amplitudes are provided in Ref. [19]. The complete analytical expression of $\mathcal{O}(1/M)$ is too lengthy to be reproduced here. However, for the sake of brevity, we only provide a symbolic expression for the total fractional contribution to the charge-odd lepton-proton differential cross section arising from virtual corrections up to and including NLO in HB χ PT, which effectively encapsulates the complete impact of the virtual TPE corrections:

$$\begin{aligned} \delta_{\gamma\gamma}^{(\text{odd})}(Q^2) &= \frac{\left[\frac{d\sigma_{\text{virt}}^{(\text{odd})}(Q^2)}{d\Omega_\ell'} \right]_{\gamma\gamma}}{\left[\frac{d\sigma_{\text{el}}(Q^2)}{d\Omega_\ell'} \right]_0} \\ &= \delta_{\gamma\gamma}^{(\text{LO;odd})}(Q^2) + \delta_{\gamma\gamma}^{(\text{NLO;odd})}(Q^2) \\ &= \mathbf{IR}_{\text{TPE}}^{(1)}(Q^2) + \delta_{\gamma\gamma}^{(0;\text{odd})}(Q^2) + \delta_{\gamma\gamma}^{(1;\text{odd})}(Q^2) \\ &\quad + \mathcal{O}\left(\frac{1}{M^2}\right), \end{aligned} \quad (116)$$

where the genuine chiral LO ($\nu = 0$) and NLO ($\nu = 1$) IR-finite TPE corrections are given, respectively, as [19]

$$\delta_{\gamma\gamma}^{(0;\text{odd})}(Q^2) = \pi\alpha \frac{\sqrt{-Q^2}}{2E} \left[\frac{1}{1 + \frac{Q^2}{4E^2}} \right] \quad (117)$$

and

$$\delta_{\gamma\gamma}^{(1;\text{odd})}(Q^2) = \delta_{\gamma\gamma}^{(ab;1/M)}(Q^2) + \delta_{\gamma\gamma}^{(\text{NLO;odd})}(Q^2). \quad (118)$$

Hence, it follows that the total one-loop IR-finite fractional radiative correction to the charge-odd lepton-proton elastic differential cross section calculated up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT is given as (cf. Fig. 10 for numerical estimates for e^- -p and μ^- -p scatterings):

$$\begin{aligned} \delta_{2\gamma}^{(\text{odd})}(Q^2, \Delta_{\gamma^*}) &= \frac{\left[\frac{d\sigma_{\text{el}}^{(\text{LO;odd})}(Q^2)}{d\Omega_\ell'} \right]_{2\gamma}}{\left[\frac{d\sigma_{\text{el}}(Q^2)}{d\Omega_\ell'} \right]_0} \\ &= \delta_{\gamma\gamma^*}^{(\text{odd})}(Q^2, \Delta_{\gamma^*}) + \delta_{\gamma\gamma}^{(\text{odd})}(Q^2), \end{aligned} \quad (119)$$

where $\delta_{\gamma\gamma^*}^{(\text{odd})}$ is the total charge-odd bremsstrahlung contribution up to and including NLO, Eq. (112), and $\delta_{\gamma\gamma}^{(\text{odd})}$ is the corresponding total TPE contribution, Eq. (116).

In the interest of evaluating the total differential cross section for elastic (anti)lepton-proton scattering, we incorporate all QED and recoil-radiative corrections of $\mathcal{O}(\alpha/M)$ accurately up to and including NLO, as well as the hadronic corrections of $\mathcal{O}(1/\Lambda_\chi^2)$ arising at NNLO in HB χ PT, by summing the charge-odd and charge-even fractional contributions with respect to the LO Born contribution, i.e.,

$$\begin{aligned} \left[\frac{d\sigma_{\text{el}}(Q^2)}{d\Omega_\ell'} \right]^{(\mp)} &= \left[\frac{d\sigma_{\text{el}}(Q^2)}{d\Omega_\ell'} \right]_0 \\ &\quad \times [1 + \delta_{\text{el}}^{(\text{even})}(Q^2, r_p) \pm \delta_{2\gamma}^{(\text{odd})}(Q^2, \Delta_{\gamma^*})], \end{aligned} \quad (120)$$

where the charge-odd contributions, $+\delta_{\gamma\gamma^*}^{(\text{odd})}$ and $-\delta_{\gamma\gamma^*}^{(\text{odd})}$, correspond to the case of lepton-proton (ℓ^- -p) and antilepton-proton (ℓ^+ -p) scattering, respectively. In particular, the total charge-even contribution to the elastic cross section $\delta_{\text{el}}^{(\text{even})}$ includes not only the charge-even real and virtual radiative correction components but also the hadronic corrections to the LO OPE arising at NNLO in chiral expansion, $\delta_\chi^{(2)} \sim \mathcal{O}(1/\Lambda_\chi^2)$, as given in Eq. (22). Thus, the total charge-even fractional corrections (with respect to the LO Born cross section) can be quantified by the following sum:

$$\delta_{\text{el}}^{(\text{even})}(Q^2, \Delta_{\gamma^*}, r_p) = \delta_\chi^{(2)}(Q^2, r_p) + \delta_{2\gamma}^{(\text{even})}(Q^2, \Delta_{\gamma^*}), \quad (121)$$

where $\delta_{2\gamma}^{(\text{even})}$ represents our total charge-even radiative correction result, as obtained in Eq. (84) [cf. Fig. 8]. In Fig. 11, we summarize our numerical estimates for the total radiative corrections calculated up to and including NLO

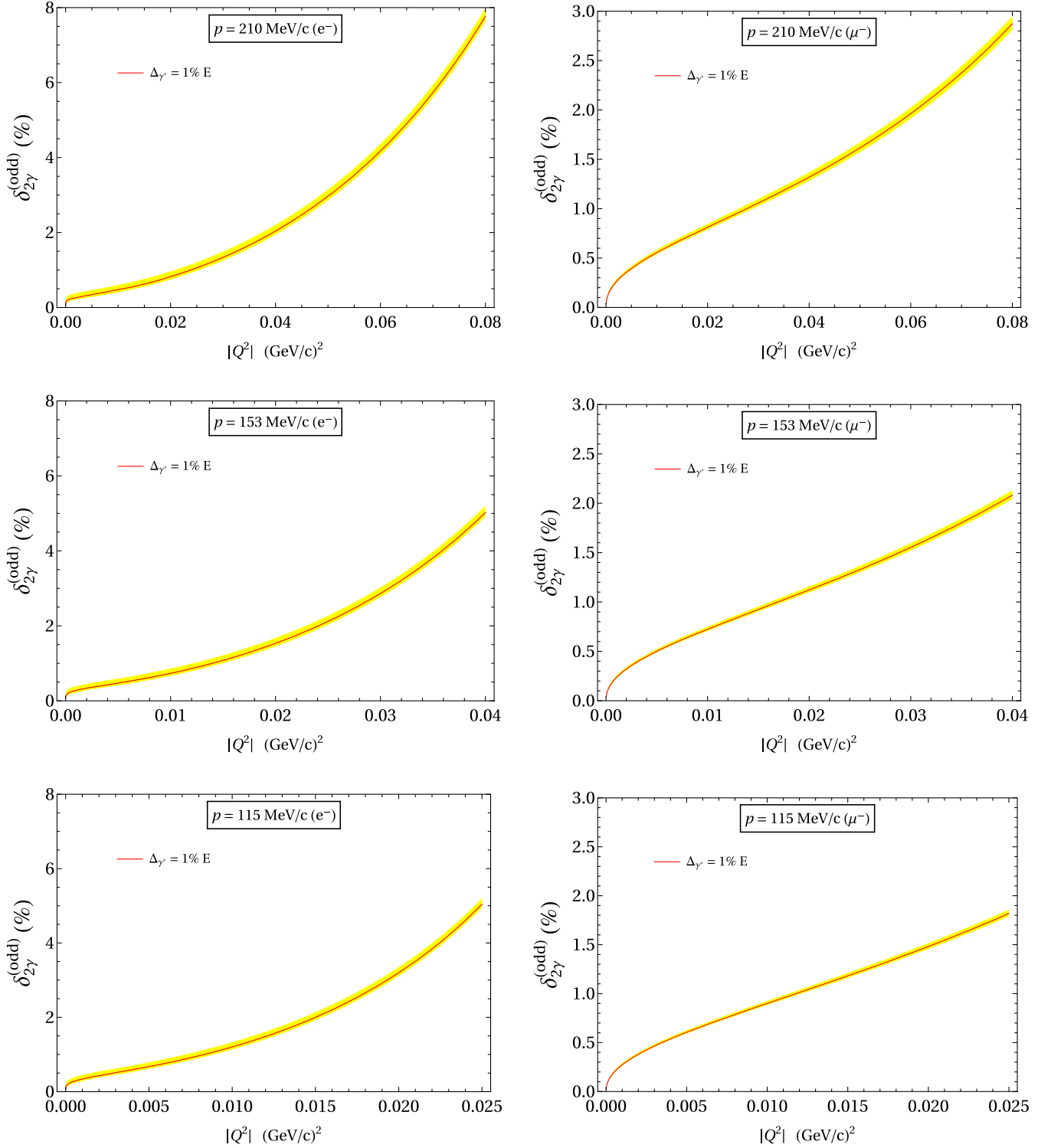


FIG. 10. Finite fractional charge-odd radiative corrections [expressed as a percentage relative to the LO Born/OPE differential cross section, Eq. (21)] to the e^-p (left panel) and μ^-p (right panel) unpolarized elastic scatterings presented as a function of $|Q^2|$ in HB χ PT. The results correspond to three fixed values of the incident lepton beam momenta proposed by the MUSE experiment: 210 MeV/c, 153 MeV/c, and 115 MeV/c. The red solid curve represents the total charge-odd radiative corrections $\delta_{2\gamma}^{(\text{odd})}$ given in Eq. (119), evaluated at a nominal detector cutoff $\Delta_{\gamma^*} = 1\%$ of the incident lepton beam energy E . The shaded yellow band indicates the sensitivity of our result to the detector acceptance in the range $0.5\%E < \Delta_{\gamma^*} < 2\%E$.

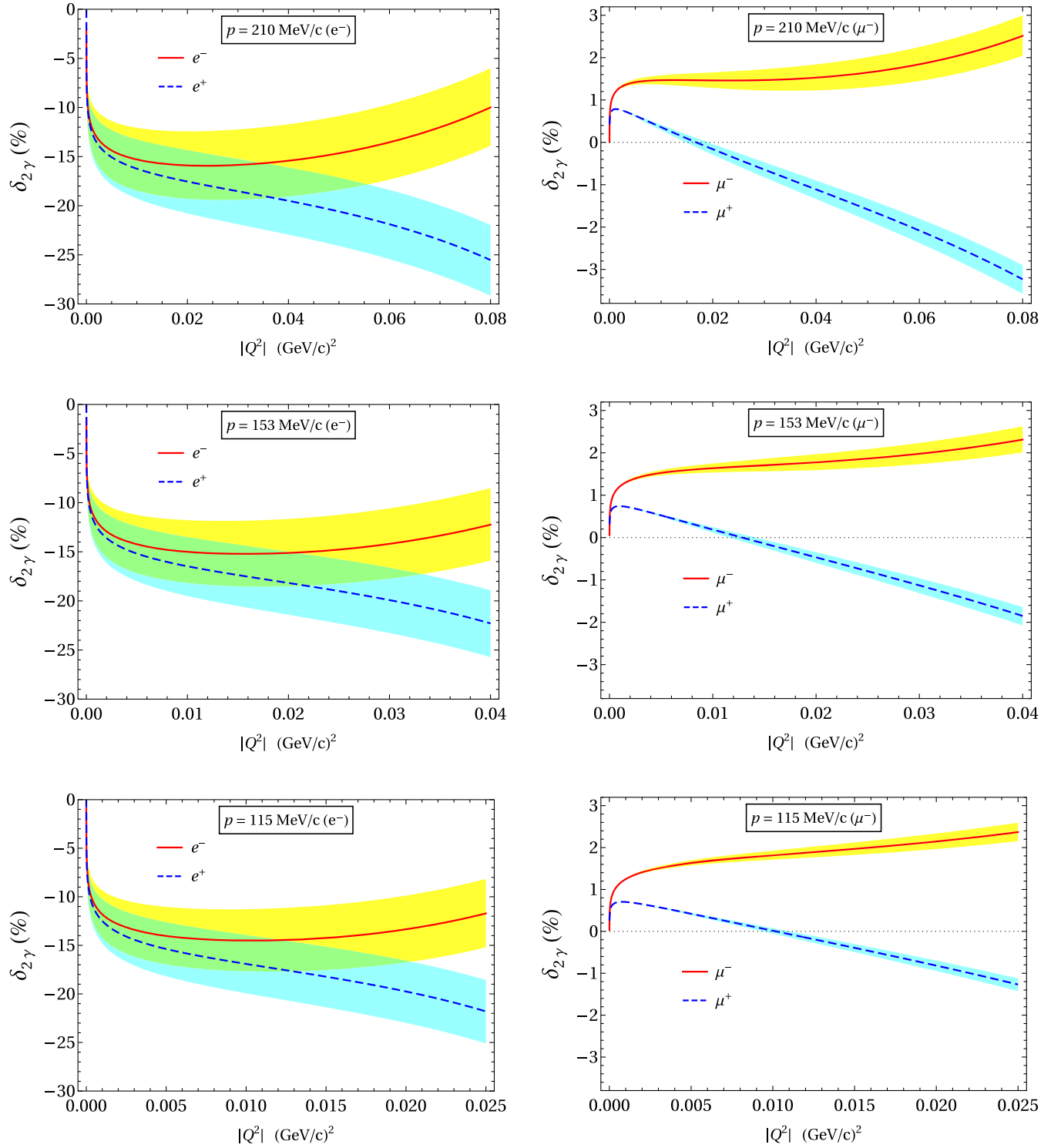


FIG. 11. Using Eq. (122), the total finite fractional radiative corrections for the ℓ^- -p (solid red line) and ℓ^+ -p (blue dashed line) unpolarized elastic scatterings [expressed as a percentage relative to the LO OPE or Born differential cross section, Eq. (21)] up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] presented as a function of $|Q^2|$ in HB χ PT. The $e^\pm$ -p results are on the left panel, and $\mu^\pm$ -p is on the right panel. The results correspond to the three fixed values of the incident lepton beam momenta proposed by the MUSE experiment: 210 MeV/c, 153 MeV/c, and 115 MeV/c. All curves are evaluated at a nominal detector cutoff $\Delta_{\gamma^*} = 1\%$ of the incident (anti)lepton beam energy E . The yellow (cyan) shaded band indicates the sensitivity of the lepton (antilepton) scattering results with a detector acceptance in the range $0.5\%E < \Delta_{\gamma^*} < 2\%E$.

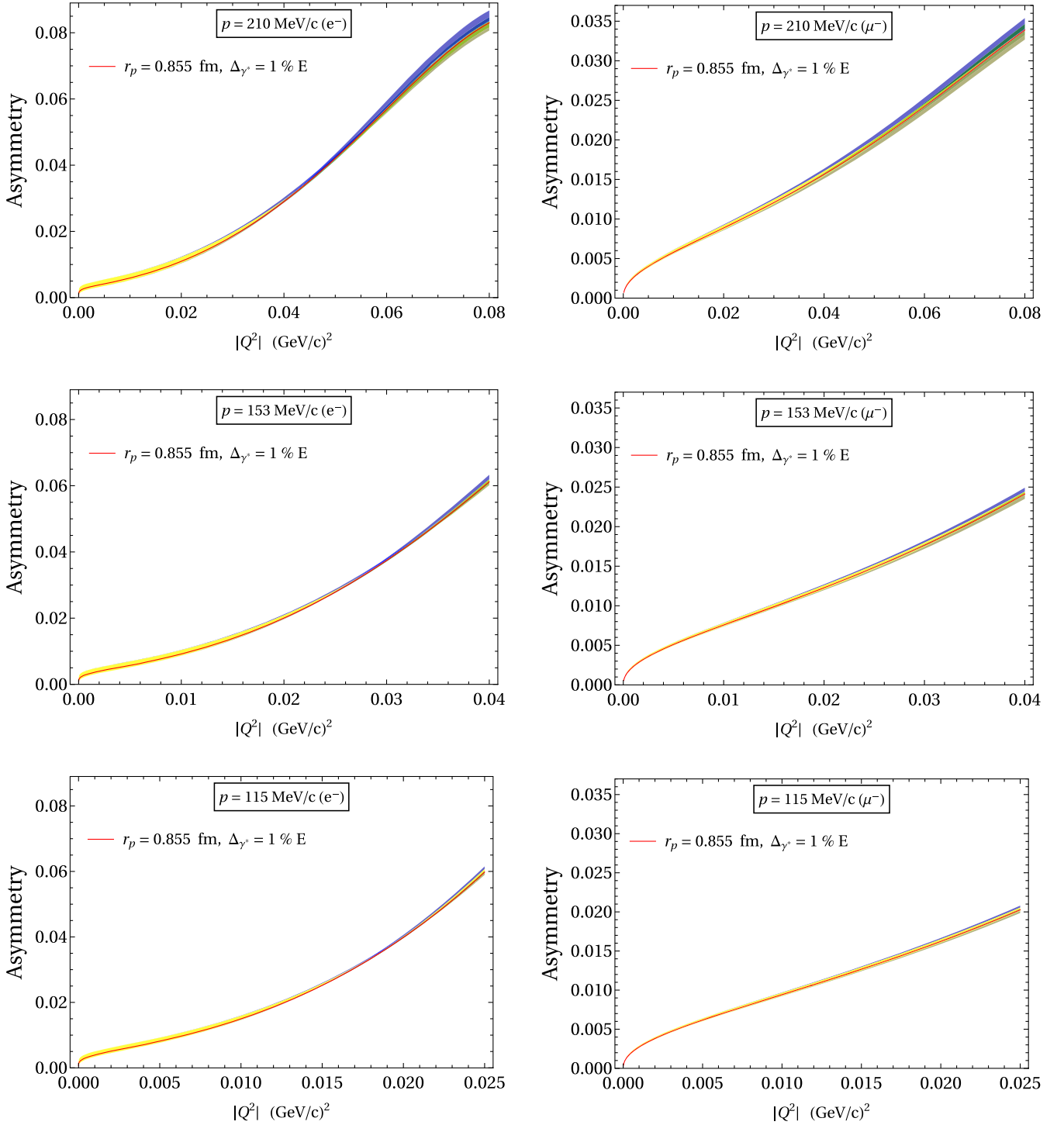


FIG. 12. Charge asymmetry ($\mathcal{A}_{\ell^\pm}$), Eq. (123), for $e^\pm$ -p (left panels) and $\mu^\pm$ -p (right panels) unpolarized elastic scattering cross sections up to and including NLO in HB χ PT presented as a function of $|Q^2|$. The results correspond to three fixed values of the incident lepton beam momenta proposed by the MUSE experiment: 210 MeV/c, 153 MeV/c, and 115 MeV/c. The yellow shaded band indicates the sensitivity of our results with a detector acceptance in the range $0.5\% < \Delta_{\gamma^*} < 2\%$ of the incident (anti)lepton beam energy E , while the blue shaded band reflects the uncertainty for a wide range of experimentally extracted values of the proton's rms charge radius (r_p) between the CREMA [56,57] [$r_p = 0.84087(39)$ fm] and MAMI ISR [58] [$r_p = 0.87 \pm (0.014)_{\text{stat}} \pm (0.024)_{\text{syst}} \pm (0.003)_{\text{mod}}$ fm] measurements. The solid red curve corresponds to our prototype result, evaluated at a nominal detector cutoff $\Delta_{\gamma^*} = 1\%E$ and an rms radius $r_p = 0.855$ fm, averaged over the given extracted range.

[i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT for lepton proton (ℓ^- -p) and antilepton proton (ℓ^+ -p) scatterings, namely,

$$\delta_{2\gamma}^{(\ell^\mp p)}(Q^2, \Delta_{\gamma^*}) = \delta_{2\gamma}^{(\text{even})}(Q^2, \Delta_{\gamma^*}) \pm \delta_{2\gamma}^{(\text{odd})}(Q^2, \Delta_{\gamma^*}); \quad \ell \equiv e, \mu, \quad (122)$$

where $\delta_{2\gamma}^{(\text{odd})}$ was obtained in Eq. (119).

E. The lepton-antilepton charge asymmetry

As inferred in the Introduction, one major goal of our paper is to predict the Q^2 dependence of the charge asymmetry $\mathcal{A}_{\ell^\pm}$. It yields a measure of the difference between higher-order quantum corrections affecting the respective cross sections of lepton and antilepton elastic scatterings with the proton. Such an asymmetry arises primarily due to the interference between the OPE and TPE amplitudes and is expected to be small in magnitude, yet non-negligible. Its determination provides one of the most effective means of isolating TPE effects from the corresponding scattering cross sections. We expect that the MUSE Collaboration will publish their results in this context in the near future, with an expected systematic uncertainty of 0.1%. To achieve comparable accuracy, theoretical predictions within low-energy EFT must account for contributions at least up to NLO [i.e., $\mathcal{O}(\alpha^3/M)$], if not beyond. To that end, as per the definition, Eq. (3), based on our aforementioned HB χ PT analysis, we provide the following NLO determination of the asymmetry observable:

$$\begin{aligned} \mathcal{A}_{\ell^\pm}(Q^2, \Delta_{\gamma^*}, r_p) &= \frac{d\sigma_{\text{tot}}^{(\text{odd})}}{d\sigma_{\text{tot}}^{(\text{even})}} = \frac{d\sigma_{\text{el}}^{(-)} - d\sigma_{\text{el}}^{(+)}}{d\sigma_{\text{el}}^{(-)} + d\sigma_{\text{el}}^{(+)}} \\ &= \frac{\delta_{2\gamma}^{(\text{odd})}(Q^2, \Delta_{\gamma^*})}{1 + \delta_{\text{el}}^{(\text{even})}(Q^2, \Delta_{\gamma^*}, r_p)}. \end{aligned} \quad (123)$$

Our definition of the charge asymmetry employs an opposite sign compared to Refs. [9,26], chosen solely for convenience and for aesthetic reasons of presenting our results. This choice has no bearing on the underlying physics. Our choice is motivated by our exact one-loop TPE results presented in Fig. 12, indicating that in the low-energy kinematic region relevant to MUSE, the radiative corrected lepton-proton cross section exceeds the corresponding antilepton-proton cross section (cf. Fig. 11), leading to a net positive contribution to the charge asymmetries of either lepton species. It is worth noting that a recent updated analysis of radiative corrections by Bytev *et al.* [55], based on the scattering data from the OLYMPUS experiment at DESY, ostensibly indicates that $d\sigma_{\text{el}}^{(+)} < d\sigma_{\text{el}}^{(-)}$ at small momentum transfers. However, the current paucity of available very-low-energy scattering data precludes a definitive conclusion in this regard.

The scenario is expected to change with the availability of high-precision MUSE data. Our results, as shown in Fig. 12, arise from the fact that the charge-odd components, namely, the one-loop TPE and the soft-photon lepton-proton exchange-interference bremsstrahlung, enter with positive signs for leptons and negative signs for antileptons. Earlier QED-based analyses employing the SPA [9,26]—an approximation originally introduced by Tsai [47], in which one of the two photon propagators is placed on shell—predicted a negative TPE correction at low energies, as this approximation neglects contributions from the hard kinematic region of the one-loop TPE graphs. Consequently, the SPA result contradicts the classic McKinley-Feshbach expression [54], which yields a positive correction for e -p scattering. Our recent HB χ PT evaluation of the TPE within the same SPA framework [20] exhibited a similar behavior. Thus, under our sign convention, $\mathcal{A}_{\ell^\pm}$ acquires a negative value for lepton scattering in all SPA-based analyses. This feature inherent to the SPA methodology was the primary reason that its use, particularly at low energies, was discouraged in Ref. [20]. In contrast, the exact analytical one-loop TPE evaluation in HB χ PT by Choudhary *et al.* [19], with its corrections, yields a positive sign. Thus, in the present work, the adoption of the exact one-loop TPE result naturally leads to a positive sign of the charge asymmetry.

III. RESULTS AND DISCUSSION

In this section, we present our numerical results pertaining to the elastic (anti)lepton-proton scattering cross section computed within the framework of HB χ PT, up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] corrections. Our numerical results for the soft-photon bremsstrahlung calculations are obtained using SPL, where the emitted real soft-photon four-momentum $k \rightarrow 0$. This corresponds to photon energies below a certain detector threshold (Δ_{γ^*}). In all our subsequent numerical estimates, Δ_{γ^*} will be treated as a free parameter to assess the sensitivity of our EFT results to external factors, such as variations arising from specific experimental design choices.

Before outlining our numerical results in the different figures, we briefly discuss some key aspects of our bremsstrahlung amplitudes that underpin our cross section calculations.

- (i) As presented in Sec. II, one aspect is the emergence of chiral enhancement of certain amplitudes which involve the insertion of final-state proton propagator components (iS^ν) containing the factor $(v \cdot Q)^{-1} = -2M/Q^2$ in their denominators. This highlights the significance of contributions that go beyond the naive diagrammatical hierarchy implied by chiral power-counting expectations. Specifically, radiative processes involving the final-state photon emission necessitate the inclusion of chirally enhanced

amplitudes that could formally arise from contributions up to $N^4\text{LO}$ ($\nu = 4$ chiral order) since they could *per se* yield a non-negligible contribution to the intended $\mathcal{O}(\alpha^3/M)$ accuracy of the NLO differential cross section. Consequently, to maintain consistency in evaluating the cross section up to and including NLO accuracy, it is essential to account for all relevant amplitudes—starting from the lowest-order contribution at $\mathcal{O}(e^3M)$, stemming from the LO chirally enhanced amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}$ [see Eq. (27)] and extending up to the highest contributing order at $\mathcal{O}(e^3/M^2)$. The latter can originate from the chirally enhanced amplitudes, potentially including those as high as $N^4\text{LO}$, which effectively contribute at $\mathcal{O}(1/M^2)$ —specifically, $\mathcal{M}_{\gamma\gamma^*}^{[p(3);f\otimes]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[\widehat{p(0);f\otimes}]}$ [see Eqs. (A54) and (A55) in Appendix A]. In this process, we only retain the relevant amplitudes that (i) do not vanish in the lab frame using SPL, (ii) yield nonvanishing contributions either to the charge-even or charge-odd bremsstrahlung cross section upon phase-space integration using DR, i.e., subject to the condition Eq. (63), and (iii) could potentially yield cross section terms of $\mathcal{O}(\alpha^3/M)$ accuracy in the SPL. Figures 2–6 display all such graphs that satisfy these three criteria. For completeness, Appendix A provides the general frame-independent expressions for all contributing amplitudes, including the numerous other $\mathcal{O}(e^3/M^2)$ amplitudes that do not enter our NLO soft-bremsstrahlung cross section in the lab frame.

- (ii) It is only upon performing the bremsstrahlung phase-space integrations that two additional subtle features underlying our analytical result for the $\mathcal{O}(\alpha^3/M)$ NLO bremsstrahlung differential cross section become apparent—features that are not readily apparent before integration. First, all $N^3\text{LO}$ amplitudes shown in Fig. 6 drop out for the charge-even case due to the DR condition, Eq. (63), while for the charge-odd case, some contributions similarly vanish in DR, while others result in $\mathcal{O}(\alpha^3/M^2)$ expressions [cf. Eqs. (110) and (111)]. Second, all NNLO hadronic amplitudes (those with renormalized pion loops) shown in Fig. 5 drop out as they lead to $\mathcal{O}(1/M^2)$ cross section [see Eqs. (103) and (108)]. Consequently, the bremsstrahlung cross section up to and including NLO receives no contribution from the proton’s hadronic finite-size structure effects. In other words, at this level of accuracy, the proton effectively behaves as a pointlike particle.

We now turn to the numerical results shown in Figs. 7–12, which correspond to the proposed low-energy kinematics

relevant to the ongoing MUSE experiment [27–29]. First, we consider our results for the finite part of the fractional charge-even bremsstrahlung correction to the unpolarized elastic (anti)lepton-proton scattering cross sections, defined by

$$\bar{\delta}_{\gamma\gamma^*}^{(\text{even})}(Q^2, \Delta_{\gamma^*}) = \delta_{\gamma\gamma^*}^{(\text{even})}(Q^2, \Delta_{\gamma^*}) - \mathbf{IR}_{\gamma\gamma^*}^{(0)}(Q^2). \quad (124)$$

As displayed in Fig. 7, the red solid line represents the full result up to and including NLO [i.e., $\mathcal{O}(\alpha/M)$] in HB χ PT. The following salient characteristics emerge from the plots:

- (1) The corrections for both electron and muon scatterings are predominantly insensitive to the variation of the incident lepton beam momentum in the MUSE experimental kinematic range $p = |\mathbf{p}| \in [115, 210]$ MeV/c.
- (2) The genuine LO [i.e., $\mathcal{O}(M^0)$] correction $\delta_{\gamma\gamma^*}^{(0,\text{even})}$, Eq. (82), without subleading terms, is almost a negative constant and relatively large, $\sim 7\%$, for the entire range of squared four-momentum transfers $-Q^2 > 0$ for electron scattering. The $\mathcal{O}(1/M)$ corrections $\delta_{\gamma\gamma^*}^{(1,\text{even})}$, Eq. (83), on the other hand, are quite small but positive, monotonically increasing very gradually in the same range.
- (3) For the muon scattering, the corresponding genuine LO correction has, in contrast, a sharp negative slope with a maximum of about 2.5%, while for the $\mathcal{O}(1/M)$ corrections, the behavior is analogous, albeit a factor of 10 smaller than the electron case.
- (4) The full charge-even bremsstrahlung correction for electron scattering is much more sensitive to the variation with the detector threshold, $\Delta_{\gamma^*} \in [0.5\%, 2\%]$, than in muon scattering, as evident from the contrasting widths of the yellow shaded band. This indicates the suppression of charge-even bremsstrahlung effects for heavier leptons.
- (5) It is important to emphasize that our calculated charge-even bremsstrahlung correction is smaller than what was reported in the work of Talukdar *et al.* [14]. This is primarily due to the adoption of a different subtraction scheme for eliminating the IR divergences. In our analysis, the IR-divergent contribution is a constant. This contrasts with the $|Q^2|$ -dependent subtraction adopted by Talukdar *et al.*

Second, we consider our results displayed in Fig. 8, where we plot the total fractional charge-even radiative corrections $\delta_{2\gamma}^{(\text{even})}$, Eq. (84), calculated up to and including NLO [i.e., $\mathcal{O}(\alpha/M)$] to the unpolarized elastic (anti)lepton-proton scattering cross sections. The results include the analytically evaluated charge-even virtual photon-loop correction (i.e., self-energy, vertex, and vacuum polarization one-loop effects) counterparts taken from Talukdar *et al.* [14]. The key features observed in the plots are summarized below:

- (1) There is little sensitivity of both electron and muon scattering results on the variation of the incident lepton beam momentum ($p = |\mathbf{p}|$) in the MUSE kinematic range.
- (2) For electron scattering, the total correction remains substantially negative throughout the range of $|Q^2|$ values considered at MUSE, exhibiting a very gradual increase in magnitude with rising $|Q^2|$.
- (3) In contrast to the electron scattering, the result for muon scattering exhibits a change in sign at the low-momentum transfer region $-Q^2 \sim 0.05 \text{ GeV}^2/c^2$. Consequently, at the lowest MUSE beam momentum of $p = 115 \text{ MeV}/c$, the correction stays positive across the entire kinematic range below $|Q^2| \simeq 0.025 \text{ GeV}^2/c^2$, while for the largest beam momentum of $p = 210 \text{ MeV}/c$, the correction is positive only within the range $|Q^2| \lesssim 0.05 \text{ GeV}^2/c^2$. Beyond that, it turns negative, with magnitude increasing monotonically with a negative slope. However, the overall magnitude remains strongly suppressed, staying within the 1% level throughout the kinematic interval.
- (4) For the same reason discussed in point 4 of Fig. 7, the magnitude of the correction in our analysis is smaller than that reported by Talukdar *et al.* [14].

Third, we consider our results for the finite part of the fractional charge-odd bremsstrahlung correction to the unpolarized elastic electron-proton and muon-proton scattering cross sections, defined by

$$\bar{\delta}_{\gamma\gamma^*}^{(\text{odd})}(Q^2, \Delta_{\gamma^*}) = \delta_{\gamma\gamma^*}^{(\text{odd})}(Q^2, \Delta_{\gamma^*}) - \mathbf{IR}_{\gamma\gamma^*}^{(1)}(Q^2). \quad (125)$$

Given the charge-odd character, the corresponding correction for antilepton scattering is obtained by inverting the overall sign. As displayed in Fig. 9, the red solid line represents our full lepton scattering result up to and including NLO [i.e., $\mathcal{O}(\alpha/M)$], obtained from the sum of Eqs. (113) and (114). The salient features of the plots are summarized as follows:

- (1) A comparison with the charge-even bremsstrahlung correction displayed in Fig. 7 reveals a striking contrast—the charge-odd bremsstrahlung correction is smaller by more than an order of magnitude.
- (2) Similar to the charge-even case, there is little sensitivity of both electron and muon scattering results on the variation of the incident lepton beam momentum.
- (3) Unlike the charge-even bremsstrahlung correction, the magnitudes of the full (including enhancement) charge-odd bremsstrahlung correction for the electron and muon scatterings are similar (they are slightly larger for the electron case). They become more negative with increasing $|Q^2|$ and have comparable slopes.

- (4) The genuine LO [i.e., $\mathcal{O}(M^0)$] correction $\delta_{\gamma\gamma^*}^{(0,\text{odd})}$, Eq. (113), is a positive constant, independent of $|Q^2|$ for both electron and muon scatterings, whereas the $\mathcal{O}(1/M)$ correction $\delta_{\gamma\gamma^*}^{(1,\text{odd})}$, Eq. (114), yields a negative slope in each case.
- (5) The observed difference between the “enhanced” (green long dashed and blue long dashed-dot curves) and “unenanced” (black short dashed and magenta short dashed-dot curves) results arises from the inclusion of additional $\mathcal{O}(M^0)$ and $\mathcal{O}(1/M)$ correction terms in the NLO cross section—specifically, the contributions $\delta_{\gamma\gamma^*}^{(\text{NNLO};\text{odd})}$, Eq. (108), and $\tilde{\delta}_{\gamma\gamma^*}^{(\text{NNLO};\text{odd})}$, Eq. (111), which stem from $\nu = 2, 3$ chiral order matrix elements $\mathcal{S}_{\text{odd}}^{\nu=2,3}$ involving chirally enhanced NNLO amplitudes (the contribution from the N^3LO amplitudes drops out, as we discussed). We find that the enhanced contribution significantly modifies the $\mathcal{O}(M^0)$ results—almost doubling them for electron scattering and increasing them even further for muon scattering—whereas the $\mathcal{O}(1/M)$ results exhibit only a minimal change that is scarcely discernible.
- (6) A comparison of the yellow shaded band widths reveals that the full result exhibits greater sensitivity to Δ_{γ^*} variation in electron scattering than in muon scattering.

Fourth, we consider our results in Fig. 10 for the total fractional charge-odd radiative corrections $\delta_{2\gamma}^{(\text{odd})}$, Eq. (119), to the unpolarized elastic electron-proton and muon-proton scattering cross sections, up to and including NLO in $\text{HB}\chi\text{PT}$. The results include the analytically evaluated exact TPE virtual correction counterpart $\delta_{\gamma\gamma}^{(\text{odd})}$, as given in Eq. (116), taken from Fig. 5 of Choudhary *et al.* [19].⁷ The following are the key features observed in the plots:

- (1) As demonstrated in the TPE analysis by Choudhary *et al.*, unlike other types of radiative corrections, the TPE contributions for both electron and muon scatterings exhibit a pronounced sensitivity to the variation in the incident lepton beam momentum in the MUSE kinematic range $p = |\mathbf{p}| \in [115, 210] \text{ MeV}/c$. Consequently, the dominance of the finite part of the TPE contribution, i.e., $\bar{\delta}_{\gamma\gamma}^{(\text{odd})} = \delta_{\gamma\gamma}^{(\text{odd})} - \mathbf{IR}_{\text{TPE}}^{(1)}$, is likewise manifested in the results for the charge-odd radiative correction.

⁷The present analysis considers an updated numerical estimate of the exact TPE result, incorporating a correction for an error identified in our Mathematica-based code used to generate Fig. 5 of Ref. [19]. The revised plots yield a substantial increase (up to nearly 3% for electron scattering and 0.5% for muon scattering) in the magnitude of the $\mathcal{O}(1/M)$ corrections to the elastic cross section arising from the LO TPE diagrams (a) and (b). An erratum to Ref. [19] for our corrections has been accepted by EPJA for publication.

- (2) For both electron and muon scatterings, the corrections stay positive and exhibit a monotonic increase with increasing $|Q^2|$, indicating a net enhancement to the cross section. Here, the TPE corrections $\delta_{\gamma\gamma}^{(\text{odd})}$ dominate over the charge-odd bremsstrahlung corrections given in Eq. (112). Indeed, the negligible impact of the bremsstrahlung correction suggests that the charge-odd radiative correction shown in Fig. 10 could serve as a reliable proxy for the “physical TPE” contribution, making the latter amenable to extraction from asymmetry measurements.
- (3) The magnitude of $\delta_{2\gamma}^{(\text{odd})}$ is more than a factor of 2 larger for electron scattering than for muon scattering at the same beam momenta. This discrepancy primarily arises from the greater sensitivity of electrons to TPE effects, which are significantly suppressed for the heavier muons.
- (4) The rather narrow yellow shaded bands—particularly in the case of the muon scattering—indicate that the TPE corrections are largely insensitive to variations in the detector threshold Δ_{γ^*} .

Fifth, we consider Fig. 11 where we display the total fractional radiative correction $\delta_{2\gamma}^{(\ell^{\mp}\text{p})}$, Eq. (122), to the unpolarized elastic (anti)lepton-proton cross section of

$e^{\pm}\text{-p}$ and $\mu^{\pm}\text{-p}$ scattering processes, up to and including NLO in HB χ PT. An analysis of the plots reveals the following key features:

- (1) The corrections are sizable and negative across the entire range of low-energy MUSE kinematics for both electron and positron scatterings, with the latter exhibiting a larger magnitude. This difference becomes increasingly pronounced with rising $|Q^2|$, driven by the enhanced charge-odd contributions.
- (2) In contrast to electron and positron scatterings, the radiative corrections for muon and antimuon scatterings are about a magnitude smaller. For muon scattering, the correction is positive and increases gradually with rising $|Q^2|$. Interestingly, however, for antimuon scattering, the correction is positive only for a limited range of momentum transfers, below $|Q^2| \lesssim 0.02 \text{ GeV}^2/c^2$, beyond which it turns negative for all $|Q^2|$ values pertinent to MUSE.
- (3) In general, the dependence on the incident lepton beam momentum is weak.
- (4) The variation with Δ_{γ^*} , as indicated by the yellow and cyan shaded bands for lepton and antilepton scattering, points toward a notably higher sensitivity in the electron(positron) scattering process relative to the muon(antimuon) counterpart.

TABLE I. Study of the charge asymmetry ($\mathcal{A}_{e^{\pm}}$) in unpolarized elastic $e^{\pm}\text{-p}$ and $\mu^{\pm}\text{-p}$ scattering cross sections, calculated up-to-and-including NLO in HB χ PT. The analysis is performed for a median value of the proton’s rms charge radius, $r_p \approx 0.855 \text{ fm}$, between the CREMA [56,57] [$r_p = 0.84087(39) \text{ fm}$] and MAMI ISR [58] [$r_p = 0.87 \pm (0.014)\text{stat} \pm (0.024)\text{syst} \pm (0.003)_{\text{mod}} \text{ fm}$] measurement results, while varying the detector cutoff Δ_{γ^*} within the range $[0.5\%, 2\%]$ of the incident lepton beam energy E . Numerical results are presented at the maximum allowed four-momentum transfer, $|Q^2| = |Q_{\text{max}}^2|$, corresponding to each of the three MUSE kinematical domains, defined by the incident lepton beam momenta, $p = 115, 153$, and $210 \text{ MeV}/c$.

$ Q^2 \text{ (GeV}^2/c^2\text{)}$	$p \text{ (MeV}/c\text{)}$	$\Delta_{\gamma^*} \text{ (in \% of } E\text{)}$	$\mathcal{A}_{e^{\pm}} \text{ (error)}$	$\mathcal{A}_{\mu^{\pm}} \text{ (error)}$
0.025	115	0.5	0.061(1)	0.020(1)
		1	0.059(1)	0.020(1)
		2	0.059(1)	0.021(1)
	153	0.5	0.028(1)	0.014(1)
		1	0.028(1)	0.015(1)
		2	0.028(1)	0.015(1)
	210	0.5	0.014(1)	0.010(1)
		1	0.014(1)	0.010(1)
		2	0.015(1)	0.011(1)
0.04	153	0.5	0.062(1)	0.024(1)
		1	0.061(1)	0.024(1)
		2	0.060(1)	0.024(1)
	210	0.5	0.029(1)	0.015(1)
		1	0.029(1)	0.016(1)
		2	0.030(1)	0.016(1)
	210	0.5	0.084(1)	0.033(1)
		1	0.082(1)	0.034(1)
		2	0.081(1)	0.034(1)

TABLE II. Study of the charge asymmetry ($\mathcal{A}_{e^\pm}$) in unpolarized elastic $e^\pm$ -p and $\mu^\pm$ -p scattering cross sections, calculated up to and including NLO in HB χ PT. The analysis is performed for a fixed benchmark value of the detector cutoff, $\Delta_{\gamma^*} = 1\%$ of the incoming lepton beam energy E , while varying the proton's rms charge radius (r_p) [which scales as $\mathcal{O}(1/M^2)$] between the CREMA [56,57] [$r_p = 0.84087(39)$ fm] and MAMI ISR [58] [$r_p = 0.87 \pm (0.014)\text{stat} \pm (0.024)\text{syst} \pm (0.003)_{\text{mod}}$ fm] measurement values. Numerical results are presented at the maximum allowed four-momentum transfer, $|Q^2| = |Q_{\text{max}}^2|$, corresponding to each of the three MUSE kinematical domains, defined by incident lepton beam momenta, $p = 115, 153$, and 210 MeV/ c .

$ Q^2 $ (GeV ² / c^2)	p (MeV/ c)	r_p (fm) $\mathcal{O}(1/M^2)$	$\mathcal{A}_{e^\pm}$ (error)	$\mathcal{A}_{\mu^\pm}$ (error)
0.025	115	0.84087 (CREMA)	0.059(1)	0.020(1)
		0.87 (MAMI ISR)	0.060(1)	0.020(1)
	153	0.84087 (CREMA)	0.027(1)	0.015(1)
		0.87 (MAMI ISR)	0.028(1)	0.015(1)
	210	0.84087 (CREMA)	0.014(1)	0.010(1)
		0.87 (MAMI ISR)	0.014(1)	0.010(1)
0.04	153	0.84087 (CREMA)	0.060(1)	0.024(1)
		0.87 (MAMI ISR)	0.061(1)	0.024(1)
	210	0.84087 (CREMA)	0.028(1)	0.015(1)
		0.87 (MAMI ISR)	0.029(1)	0.016(1)
0.08	210	0.84087 (CREMA)	0.081(1)	0.033(1)
		0.87 (MAMI ISR)	0.084(1)	0.034(1)

Finally, we consider our result for the charge asymmetry observable $\mathcal{A}_{e^\pm}$, Eq. (123), which incorporates the total charge-odd and charge-even radiative corrections up to and including NLO in HB χ PT. These are as illustrated in Fig. 12 (see also Tables I and II). The former correction $\delta_{2\gamma}^{(\text{odd})}$ arises from the exchange-interference bremsstrahlung (see Introduction) and TPE effects, while the latter $\delta_{el}^{(\text{even})}$ arises from the direct-interference bremsstrahlung, QED one-loop virtual corrections (other than the charge-odd TPE), and the leading hadronic corrections $\delta_\chi^{(2)}$ to the OPE process of $\mathcal{O}(1/\Lambda_\chi^2) \sim \mathcal{O}(1/M^2)$. Consequently, our asymmetry results rely on two free parameters, namely, the detector cutoff Δ_{γ^*} , which enters via the bremsstrahlung corrections, and the proton's rms charge radius r_p , which enters into the hadronic OPE correction given in Eq. (22). To quantify the sensitivity of our prediction to these two inputs, we examine the variation of our results across a reasonable range of parameter values, namely, $\Delta_{\gamma^*} \in [0.5\%, 2\%]$ of the incident lepton beam energy E , and $r_p = 0.87 \pm (0.014)\text{stat} \pm (0.024)\text{syst} \pm (0.003)_{\text{mod}}$ fm to $0.84087(39)$ fm. The former r_p value corresponds to the recent precision extractions from the electron-proton scattering ISR measurements by the A1 Collaboration at MAMI [58], and the latter corresponds to the well-acclaimed high-precision muonic hydrogen spectroscopy measurements by the CREMA Collaboration at PSI [56,57] about a decade ago. These choices facilitate an assessment

of the extent to which the current phenomenological discrepancy in the proton radius contributes to the theoretical error in our asymmetry predictions. The following key features are evident from the plots in Fig. 12 and data presented in Tables I and II:

- (1) The asymmetry remains positive and exhibits a monotonic increase with $|Q^2|$ across all incident lepton beam momenta and species.⁸ For electron (muon) scattering, the asymmetry increases beyond $\gtrsim 0.08$ ($\lesssim 0.035$) for the highest momentum transfers at MUSE, $|Q^2| \approx 0.08$ GeV²/ c^2 .
- (2) A clear, increasing trend with a narrow width is observed with increasing lepton beam momentum for both electrons and muons.
- (3) The asymmetry exhibits a rather small sensitivity to either parameter variation of Δ_{γ^*} and r_p , given their rapid increase with $|Q^2|$ for both electron and muon scatterings.

⁸Notably, in our recent SPA-based HB χ PT analysis of the TPE correction [20], we compared the charge asymmetry obtained using the TPE corrections evaluated within the SPA framework to that derived from our earlier exact analytical TPE results of Ref. [19]. In that study, the resulting asymmetry differs considerably from the one shown in Fig. 12, in particular, for the electron case, yielding a negative value. This discrepancy is attributed to the same reason as discussed previously in footnote 7.

TABLE III. Summary of our EFT error estimates (as a percentage, relative to the Born cross section at MUSE kinematical settings) in the finite part of the NLO charge-even ($\Delta\delta_{\gamma\gamma^*}^{(\text{even})}$) and charge-odd ($\Delta\delta_{\gamma\gamma^*}^{(\text{odd})}$) bremsstrahlung, NLO TPE ($\Delta\delta_{\gamma\gamma}^{(\text{odd})}$) and charge-even virtual/photon-loop ($\Delta\delta_{\gamma\gamma}^{(\text{even})}$), and the NNLO hadronic OPE ($\Delta\delta_\chi^{(2)}$) corrections for (anti)lepton-proton scatterings. Hence, applying the quadrature rule for combining uncorrelated uncertainties, the corresponding errors in the charge-even radiative ($\Delta\delta_{2\gamma}^{(\text{even})}$), charge-even total ($\Delta\delta_{el}^{(\text{even})}$), charge-odd radiative ($\Delta\delta_{2\gamma}^{(\text{odd})}$) corrections, and the asymmetry observable ($\Delta\mathcal{A}_{\ell^\pm}$) are determined.

$\ell^\pm$	$\Delta\delta_{\gamma\gamma^*}^{(\text{even})}$ (%)	$\Delta\delta_{\gamma\gamma^*}^{(\text{odd})}$ (%)	$\Delta\delta_{\gamma\gamma}^{(\text{even})}$ (%)	$\Delta\delta_{\gamma\gamma}^{(\text{odd})}$ (%)	$\Delta\delta_\chi^{(2)}$ (%)	$\Delta\delta_{2\gamma}^{(\text{even})}$ (%)	$\Delta\delta_{el}^{(\text{even})}$ (%)	$\Delta\delta_{2\gamma}^{(\text{odd})}$ (%)	$\Delta\mathcal{A}_{\ell^\pm}$ (%)
$e^\pm$	0.2	0.02	0.00023	0.4	2.3	0.2	2.31	0.4	0.5
$\mu^\pm$	0.01	0.01	0.88	0.2	2.3	0.88	2.46	0.2	0.3

- (4) At low values of $|Q^2|$, the results show greater sensitivity to the variations in the detector threshold Δ_{γ^*} , but this sensitivity diminishes as $|Q^2|$ increases.
- (5) In contrast, the sensitivity to changes in the proton's rms radius [where $r_p \sim \mathcal{O}(1/M^2)$] and, to some extent, Δ_{γ^*} becomes more significant at higher $|Q^2|$.
- (6) The dependence of the electron-positron asymmetry on the detector threshold Δ_{γ^*} is, however, quite subtle as evidenced by both Fig. 12 and Table I. The asymmetry first decreases and then increases again with increasing Δ_{γ^*} . Notably, at small $|Q^2|$ values, the yellow band exhibits a visible width, which constricts at intermediate $|Q^2|$ and then becomes prominent again at higher $|Q^2|$ values. For muon-antimuon asymmetry, the behavior is more or less straightforward, with the width appearing to increase monotonically with increasing Δ_{γ^*} across all MUSE kinematical settings.
- (7) Table II reveals that, for fixed $|Q^2|$ and incident lepton beam momenta, the asymmetry increases monotonically with the proton's rms charge radius across all MUSE kinematical settings and for both lepton species.

Prior to concluding this section, we address the theoretical uncertainties associated with our analysis. Besides the parametric sensitivities to Δ_{γ^*} and r_p , our results are subject to another principal source of systematic theoretical uncertainty associated with higher-order EFT contributions. Specifically, the charge-odd bremsstrahlung correction $\delta_{\gamma\gamma^*}^{(\text{odd})}$ in electron scattering and the total charge-even radiative correction $\delta_{2\gamma}^{(\text{even})}$ in muon scattering both exhibit a zero crossing in the small $|Q^2|$ domain. This trend underscores a kinematic regime where the approximation within the EFT framework breaks down, rendering NNLO contributions essential for a more robust estimation of the corrections. While a complete NNLO calculation of all radiative corrections lies beyond the scope of this work, a naive estimate of the associated uncertainty can still be made. This is done by retaining all kinematically suppressed terms in the $\mathcal{O}(1/M)$ cross section that depend on the energy E' and velocity β' of the outgoing lepton—i.e., without making the approximations $E' \rightarrow E$ and $\beta' \rightarrow \beta$. In

this way, we may obtain an estimate of the percentage error associated with the different charge-odd and charge-even radiative corrections, including that of the TPE, as displayed in Table III. In particular, the table suggests that the NNLO [i.e., $\mathcal{O}(1/M^2)$] hadronic and higher-order radiative effects induce deviations (relative to the LO Born cross sections at MUSE kinematical settings) of up to

$$\begin{aligned} \Delta\delta_{el}^{(\text{even})} &= \sqrt{(\Delta\delta_{\gamma\gamma^*}^{(\text{even})})^2 + (\Delta\delta_{\gamma\gamma}^{(\text{even})})^2 + (\Delta\delta_\chi^{(2)})^2} \\ &= \begin{cases} 2.31\% & \text{for } \ell = e, \\ 2.46\% & \text{for } \ell = \mu, \end{cases} \end{aligned} \quad (126)$$

$$\begin{aligned} \Delta\delta_{2\gamma}^{(\text{even})} &= \sqrt{(\Delta\delta_{\gamma\gamma^*}^{(\text{even})})^2 + (\Delta\delta_{\gamma\gamma}^{(\text{even})})^2} \\ &= \begin{cases} 0.2\% & \text{for } \ell = e, \\ 0.88\% & \text{for } \ell = \mu, \end{cases} \end{aligned} \quad (127)$$

and

$$\begin{aligned} \Delta\delta_{2\gamma}^{(\text{odd})} &= \sqrt{(\Delta\delta_{\gamma\gamma^*}^{(\text{odd})})^2 + (\Delta\delta_{\gamma\gamma}^{(\text{odd})})^2} \\ &= \begin{cases} 0.4\% & \text{for } \ell = e, \\ 0.2\% & \text{for } \ell = \mu, \end{cases} \end{aligned} \quad (128)$$

where the uncertainty in our NNLO hadronic correction result, stemming from the N³LO corrections, is based on a naive dimensional assessment, namely, $\Delta\delta_\chi^{(2)} \sim (\sqrt{-Q^2}/M)^3$. Consequently, our charge asymmetry predictions are subject to deviations of up to

$$\begin{aligned} \Delta\mathcal{A}_{\ell^\pm} &= \mathcal{A}_{\ell^\pm} \sqrt{\left(\frac{\Delta\delta_{el}^{(\text{even})}}{1 + \delta_{el}^{(\text{even})}}\right)^2 + \left(\frac{\Delta\delta_{2\gamma}^{(\text{odd})}}{\delta_{2\gamma}^{(\text{odd})}}\right)^2} \\ &= \begin{cases} 0.5\% & \text{for } \ell = e, \\ 0.3\% & \text{for } \ell = \mu. \end{cases} \end{aligned} \quad (129)$$

It is also worth noting that while the proton's rms charge radius does not affect the $\mathcal{O}(1/M)$ cross section, its influence nevertheless reappears in the higher-order uncertainty estimates, which now have been incorporated.

IV. SUMMARY AND CONCLUSION

In this work, we have carried out a comprehensive Lorentz gauge analysis of the soft-photon bremsstrahlung radiative corrections to the unpolarized elastic lepton-proton scattering within the framework of HB χ PT. The study includes all contributions to the cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$], systematically incorporating both charge-even and charge-odd bremsstrahlung corrections. Subsequently, the charge asymmetry observable $A_{\ell^\pm}$ is predicted by incorporating the NLO virtual photon radiative corrections—both charge-even (self-energy, vacuum polarization, and vertex) and the charge-odd TPE—alongside the NNLO hadronic corrections to the LO Born/OPE process. Our result thereby represents a model-independent prediction of the charge asymmetry in (anti)lepton-proton scattering, specifically in scenarios where finite lepton mass effects cannot be neglected.

In our evaluation of the bremsstrahlung contributions from the undetected soft photons—i.e., photons with energies below the detector threshold Δ_{γ^*} —we employed the so-called SPL methodology, wherein the emitted photon four-momentum is taken to be vanishingly small ($k \rightarrow 0$). This approximation, a well-established technique in QED, enables a systematic and transparent evaluation of the IR divergences [23,45–49]. In this work, we identified a class of chirally enhanced amplitudes involving the final-state radiating proton, notably those containing $\mathcal{O}(M^0)$ and $\mathcal{O}(1/M)$ components of the proton's propagator with denominators of the form

$$\frac{1}{v \cdot Q + i\epsilon} = \frac{1}{-Q^2/2M + i\epsilon} \sim \mathcal{O}(M).$$

As a result, certain kinematically enhanced NLO ($\nu = 1$) and NNLO ($\nu = 2$) amplitudes were promoted to lower effective orders for consistency and thereby contributed to the $\mathcal{O}(\alpha^3/M)$ NLO cross section. This departure from the standard chiral ordering of Feynman diagrams in HB χ PT constitutes a manifestation of chiral enhancement within the chiral power-counting scheme. This carries important theoretical implications. In particular, they challenge the conventional notion that the NNLO insertions and pion-loop diagrams do not contribute to low-energy cross sections below NNLO order [i.e., below $\mathcal{O}(1/M^2)$]. Such enhanced NNLO amplitudes have a negligible effect on the charge-even bremsstrahlung corrections at our working precision. In contrast, the bremsstrahlung charge-odd correction results get substantially modified, introducing nearly 100%(200%) corrections of the genuine LO [i.e., $\mathcal{O}(\alpha^3 M^0)$] for electron (muon) scattering within the MUSE kinematic regime (cf. Fig. 9). However, the NLO [i.e., $\mathcal{O}(\alpha^3/M)$] bremsstrahlung corrections exhibit only a very nominal change, and the overall numerical impact of these enhancements is not spectacular since the charge-odd corrections are dominated by the one-loop TPE contribution.

Our general finding for both lepton scattering processes is that the finite part of the charge-even bremsstrahlung correction $\bar{\delta}_{\gamma\gamma^*}^{(\text{even})}$ exceeds the charge-odd counterpart $\bar{\delta}_{\gamma\gamma^*}^{(\text{odd})}$ by more than an order of magnitude (compare Figs. 7 and 9). The finite charge-odd bremsstrahlung correction $\bar{\delta}_{\gamma\gamma^*}^{(\text{odd})}$ in the MUSE regime was found to reach values of only about 0.3% for both e^-p and μ^-p scattering, assuming a benchmark value of the detector cutoff parameter, $\Delta_{\gamma^*} = 1\%$, of the incident lepton beam energy. For electron scattering, in particular, the observed zero crossing of the correction within the relevant energy range suggests that the NNLO [i.e., $\mathcal{O}(\alpha^3/M^2)$] terms may become important, pointing to the potential influence of hadronic structure effects not captured at NLO (recalling that all dependence on the proton's rms charge radius r_p drops out of the NLO charge-odd cross section). Therefore, a more comprehensive evaluation, systematically incorporating all NNLO contributions to the cross section, is left for future work.

As for the charge-even bremsstrahlung correction $\bar{\delta}_{\gamma\gamma^*}^{(\text{even})}$, our results with the benchmark detector cutoff, $\Delta_{\gamma^*} = 1\%$, indicate that the total correction can be as high as 8% for $e^\pm p$ scattering while remaining below 2.5% for the $\mu^\pm p$ case under identical detector conditions (cf. Fig. 7). These values are smaller than those reported in a prior radiative correction work of Talukdar *et al.* [14], owing to the adoption of the specific $|Q^2|$ -dependent IR subtraction scheme in that analysis, as opposed to the momentum-independent subtraction scheme adopted in this work, as elucidated in Sec. III.

We next combined our bremsstrahlung results, $\bar{\delta}_{\gamma\gamma^*}^{(\text{even})}$ and $\bar{\delta}_{\gamma\gamma^*}^{(\text{odd})}$, with the NLO charge-even virtual correction $\delta_{\gamma\gamma}^{(\text{even})}$ from Talukdar *et al.* [14] and the exact NLO charge-odd TPE correction $\delta_{\gamma\gamma}^{(\text{odd})}$ from Choudhary *et al.* [19] to determine the full charge-even $\delta_{2\gamma}^{(\text{even})}$ and charge-odd $\delta_{2\gamma}^{(\text{odd})}$ radiative correction components of the elastic cross section (cf. Figs. 8 and 10). Because our charge-odd bremsstrahlung results are relatively small, the latter correction is overwhelmingly dominated by the much larger TPE contribution—exceeding it by nearly 2 orders of magnitude. This ensures that a precise extraction of the TPE remains feasible in the near future, as it should ostensibly dominate asymmetry measurements.

Furthermore, these corrections enabled us to determine the corresponding total radiative corrections, $\delta_{2\gamma}^{(\ell^+p)}$ for the leptons and $\delta_{2\gamma}^{(\ell^-p)}$ for the antileptons, using Eq. (122). The results reveal that within the MUSE regime, the total radiative correction up to and including NLO could reach up to about 10% for e^-p scattering and 25% for e^+p scattering, whereas for μ^-p scattering, they go up to 2.5% and for μ^+p scattering up to 3% (cf. Fig. 11). Compared to the earlier results of Talukdar *et al.* (cf. the unresummed results, as represented in Fig. 13 of Ref. [14]),

our findings show notable qualitative and quantitative differences—reaching up to 15% for e^-p and 4% for μ^-p scattering.

The final part of our analysis dealt with the prediction of the charge asymmetry between lepton and antilepton scatterings with the proton. As indicated in Eq. (123), the asymmetry result is obtained from combining our NLO (even and odd) radiative corrections with the NNLO hadronic correction $\delta_\chi^{(2)}$ to the elastic Born/OPE cross section. The latter result, which contributes to the total charge-even component $\delta_{el}^{(\text{even})}$, has also been adopted from the work of Talukdar *et al.* [14]. The asymmetry result, displayed in Fig. 12, reveals that it increases with $|Q^2|$ for both electron and muon scatterings. Our prediction indicates that the asymmetry could reach levels of approximately 8% for $e^\pm p$ and 3% for $\mu^\pm p$ scattering processes—magnitudes that should be accessible to experimental verification by MUSE in the near term. Looking forward, we plan to improve upon Tsai’s soft-photon bremsstrahlung framework [47] by dispensing with approximations such as the SPL and the use of the artificial cutoff Δ_γ^* . Our goal is to develop a more robust method capable of extracting soft-photon contributions directly from the elastic end point of the radiative-tail differential spectrum, allowing a more precise theoretical prediction of the charge asymmetry. From the experimental standpoint, the forthcoming MUSE results are highly anticipated, as they will offer a crucial test of our predicted asymmetry.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: BREMSSTRAHLUNG IN HB χ PT

In this work, our primary goal is to evaluate the single soft-photon bremsstrahlung cross section for the (anti) lepton-proton inelastic scattering process in the lab frame, up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The analysis entails evaluating a wide range of amplitudes up to $\mathcal{O}(e^3/M^2)$, which may originate from as high as N⁴LO, i.e., $\nu = 4$ chirally enhanced amplitudes. To that end, in this Appendix we collect all such relevant amplitudes without approximation, in a manifestly covariant (frame-independent) manner, adopting the Lorentz gauge. These amplitudes are organized according to their standard chiral orderings, beginning with the LO, which naively scale as $\mathcal{O}(e^3 M^0)$, and extending up to N⁴LO, with a naive scaling of $\mathcal{O}(e^3/M^4)$.

1. LO [i.e., $\mathcal{O}(e^3 M^0)$] amplitudes

At this order, four relevant amplitudes contribute to the bremsstrahlung cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. They are given as follows (cf. Fig. 2):

$$i\mathcal{M}_{\gamma\gamma^*}^{[i;i]} = \frac{ie^3}{(Q-k)^2} \left[\bar{u}(p')\gamma^\mu \frac{(\not{p}-\not{k}+m_l)}{(p-k)^2-m_l^2} \not{\epsilon}^* u(p) \right] \left[\chi^\dagger(p'_p) v_\mu \chi(p_p) \right], \quad (\text{A1})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[i;f]} = \frac{ie^3}{(Q-k)^2} \left[\bar{u}(p')\not{\epsilon}^* \frac{(\not{p}'+\not{k}+m_l)}{(p'+k)^2-m_l^2} \gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) v_\mu \chi(p_p) \right], \quad (\text{A2})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} = \frac{-ie^3}{Q^2[v \cdot (p_p - k) + i0]} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) v_\mu (v \cdot \epsilon^*) \chi(p_p) \right], \quad (\text{A3})$$

and

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} = \frac{-ie^3}{Q^2[v \cdot (p'_p + k) + i0]} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) (v \cdot \epsilon^*) v_\mu \chi(p_p) \right]. \quad (\text{A4})$$

In particular, in the lab frame, the amplitude $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}$ involving the LO [i.e., $\mathcal{O}(M^0)$] final-state proton propagator component is chirally enhanced under the SPL.

2. NLO [i.e., $\mathcal{O}(e^3/M)$] amplitudes

At this order, nine relevant amplitudes can contribute to the bremsstrahlung cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. They are given as follows (cf. Fig. 3):

$$i\mathcal{M}_{\gamma\gamma^*}^{[i;\bar{i}]} = \frac{ie^3}{2M(Q-k)^2} \left[\bar{u}(p')\gamma^\mu \frac{(\not{p}-\not{k}+m_l)}{(p-k)^2-m_l^2} \epsilon^* u(p) \right] \left[\chi^\dagger(p'_p) \left\{ (p_p+p'_p)_\mu - v_\mu[v \cdot (p_p+p'_p)] \right\} \chi(p_p) \right], \quad (\text{A5})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[i;\bar{f}]} = \frac{ie^3}{2M(Q-k)^2} \left[\bar{u}(p')\epsilon^* \frac{(\not{p}'+\not{k}+m_l)}{(p'+k)^2-m_l^2} \gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) \left\{ (p_p+p'_p)_\mu - v_\mu[v \cdot (p_p+p'_p)] \right\} \chi(p_p) \right], \quad (\text{A6})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(1);\bar{i}]} = \frac{-ie^3}{2MQ^2[v \cdot (p_p-k) + i0]} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) v_\mu \left\{ (2p_p-k) \cdot \epsilon^* - (v \cdot \epsilon^*)[v \cdot (2p_p-k)] \right\} \chi(p_p) \right], \quad (\text{A7})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(1);\bar{f}]} = \frac{-ie^3}{2MQ^2[v \cdot (p'_p+k) + i0]} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) \left\{ (2p'_p+k) \cdot \epsilon^* - (v \cdot \epsilon^*)[v \cdot (2p'_p+k)] \right\} v_\mu \chi(p_p) \right], \quad (\text{A8})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0)};\bar{i}]} = \frac{-ie^3}{2MQ^2[v \cdot (p_p-k) + i0]} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) \left\{ (p_p+p'_p-k)_\mu - v_\mu[v \cdot (p_p+p'_p-k)] \right\} (v \cdot \epsilon^*) \chi(p_p) \right], \quad (\text{A9})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0)};\bar{f}]} = \frac{-ie^3}{2MQ^2[v \cdot (p'_p+k) + i0]} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) (v \cdot \epsilon^*) \left\{ (p_p+p'_p+k)_\mu - v_\mu[v \cdot (p_p+p'_p+k)] \right\} \chi(p_p) \right], \quad (\text{A10})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(1);\bar{v}]} = \frac{-ie^3}{MQ^2} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) \left\{ \epsilon_\mu^* - v_\mu(v \cdot \epsilon^*) \right\} \chi(p_p) \right], \quad (\text{A11})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(0);\bar{i}\otimes]} = \frac{-ie^3}{2MQ^2} \left(1 - \frac{(p_p-k)^2}{[v \cdot (p_p-k) + i0]^2} \right) \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) v_\mu (v \cdot \epsilon^*) \chi(p_p) \right], \quad (\text{A12})$$

and

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(0);\bar{f}\otimes]} = \frac{-ie^3}{2MQ^2} \left(1 - \frac{(p'_p+k)^2}{[v \cdot (p'_p+k) + i0]^2} \right) \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p) (v \cdot \epsilon^*) v_\mu \chi(p_p) \right]. \quad (\text{A13})$$

In particular, in the lab frame, the amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[p(1);\bar{f}]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0)};\bar{f}]}$, involving the LO [i.e., $\mathcal{O}(M^0)$], and $\mathcal{M}_{\gamma\gamma^*}^{[p(0);\bar{f}\otimes]}$, involving the NLO [i.e., $\mathcal{O}(1/M)$] components of the final-state proton propagator, are all chirally enhanced in SPL.

3. NNLO [i.e., $\mathcal{O}(e^3/M^2)$] amplitudes

At this order, 15 relevant amplitudes can contribute to the bremsstrahlung cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. They are given as follows (cf. Fig. 4 and the first two rows in Fig. 13):

$$i\mathcal{M}_{\gamma\gamma^*}^{[l;\bar{i}]} = \frac{ie^3}{8M^2(Q-k)^2} \left[\bar{u}(p')\gamma^\mu \frac{(\not{p}-\not{k}+m_l)}{(p-k)^2-m_l^2} \epsilon^* u(p) \right] \times \left[\chi^\dagger(p'_p) \left\{ 2[v \cdot (Q-k)]^2 v_\mu - [v \cdot (Q-k)](Q-k)_\mu - (Q-k)^2 v_\mu \right\} \chi(p_p) \right], \quad (\text{A14})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[l;\bar{f}]} = \frac{ie^3}{8M^2(Q-k)^2} \left[\bar{u}(p')\epsilon^* \frac{(\not{p}'+\not{k}+m_l)}{(p'+k)^2-m_l^2} \gamma^\mu u(p) \right] \times \left[\chi^\dagger(p'_p) \left\{ 2[v \cdot (Q-k)]^2 v_\mu - [v \cdot (Q-k)](Q-k)_\mu - (Q-k)^2 v_\mu \right\} \chi(p_p) \right], \quad (\text{A15})$$

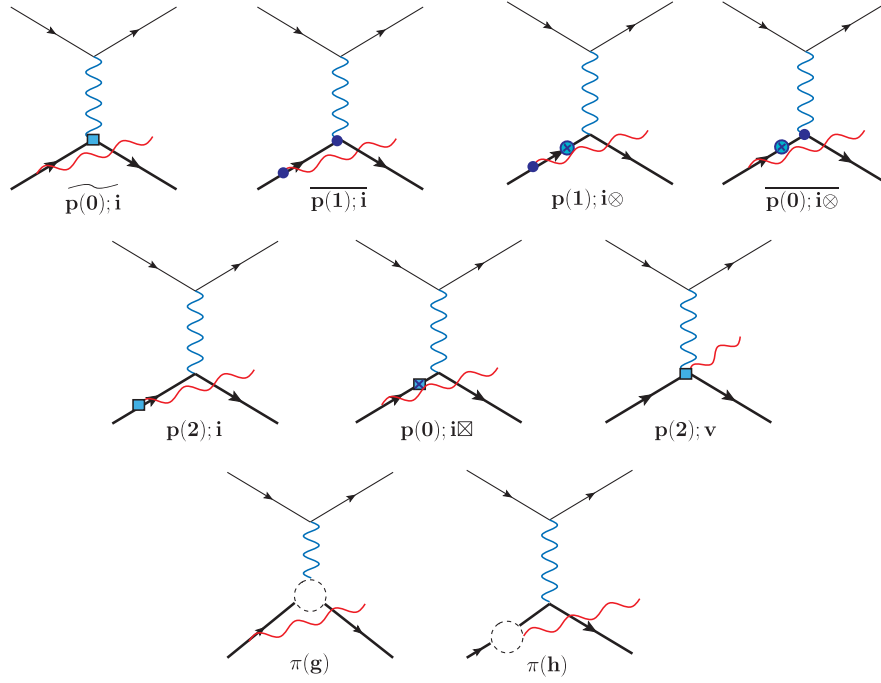


FIG. 13. Other photon bremsstrahlung diagrams of NNLO [i.e., $\mathcal{O}(e^3/M^2) \sim \mathcal{O}(e^3/\Lambda_\chi^2)$] with insertions of NLO (filled blob $\bullet$) and NNLO (filled box $\square$) proton-photon interaction vertices, proton's form factors (dashed circular blob, cf. Fig. 1), and $\mathcal{O}(1/M)$ (crossed circle $\otimes$) and $\mathcal{O}(1/M^2)$ (crossed box $\boxtimes$) proton propagator components not contributing to our ℓ -p elastic scattering cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The thin, thick, and wiggly lines denote the propagators for the lepton, proton, and photon (red for the soft bremsstrahlung photon and blue for the exchanged off-shell photon), respectively. The nomenclature of the individual diagrams is based on whether the radiated photon originates from the initial-state “i” or final-state proton “p.” The labels “(0),” “(1),” and “(2)” denote LO, NLO, and NNLO proton-radiating vertices, respectively, while the overline and tilde signify NLO and NNLO exchange-photon proton vertex insertions, respectively.

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]} = \frac{-ie^3}{4M^2Q^2} \left(1 - \frac{(p'_p + k)^2}{[v \cdot (p'_p + k) + i0]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \times \left[\chi^\dagger(p'_p) \left\{ (2p'_p + k) \cdot \epsilon^* - (v \cdot \epsilon^*) [v \cdot (2p'_p + k)] \right\} v_\mu \chi(p_p) \right], \quad (\text{A16})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0);f\otimes}]} = \frac{-ie^3}{4M^2Q^2} \left(1 - \frac{(p'_p + k)^2}{[v \cdot (p'_p + k) + i0]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \times \left[\chi^\dagger(p'_p) (v \cdot \epsilon^*) \left\{ (p_p + p'_p + k)_\mu - v_\mu [v \cdot (p_p + p'_p + k)] \right\} \chi(p_p) \right], \quad (\text{A17})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\widetilde{p(0);f}]} = \frac{-ie^3}{8M^2Q^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) (v \cdot \epsilon^*) \left\{ 2(v \cdot Q)^2 v_\mu - (v \cdot Q) Q_\mu - Q^2 v_\mu \right\} \chi(p_p) \right], \quad (\text{A18})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(2);f]} = \frac{-ie^3}{8M^2Q^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) \left\{ 2(v \cdot k)^2 (v \cdot \epsilon^*) - (v \cdot k)(k \cdot \epsilon^*) - k^2(v \cdot \epsilon^*) \right\} v_\mu \chi(p_p) \right], \quad (\text{A19})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\boxtimes]} = \frac{-ie^3}{4M^2Q^2} \left(\frac{[v \cdot (p'_p + k)]^3 - [v \cdot (p'_p + k)](p'_p + k)^2}{[v \cdot (p'_p + k) + i0]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) (v \cdot \epsilon^*) v_\mu \chi(p_p) \right], \quad (\text{A20})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(1);f}]} = \frac{-ie^3}{4M^2Q^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \times \left[\chi^\dagger(p'_p) \left\{ (2p'_p + k) \cdot \epsilon^* - (v \cdot \epsilon^*) [v \cdot (2p'_p + k)] \right\} \left\{ (p_p + p'_p + k)_\mu - v_\mu [v \cdot (p_p + p'_p + k)] \right\} \chi(p_p) \right], \quad (\text{A21})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} = \frac{ie^3}{8M^2Q^2[v \cdot (p_p - k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) \left\{ 2(v \cdot Q)^2 v_\mu - (v \cdot Q)Q_\mu - Q^2 v_\mu \right\} (v \cdot \varepsilon^*) \chi(p_p) \right], \quad (\text{A22})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(1);i]} = \frac{ie^3}{4M^2Q^2[v \cdot (p_p - k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \times \left[\chi^\dagger(p'_p) \left\{ (p_p + p'_p - k)_\mu - v_\mu [v \cdot (p_p + p'_p - k)] \right\} \left\{ (2p_p - k) \cdot \varepsilon^* - (v \cdot \varepsilon^*) [v \cdot (2p_p - k)] \right\} \chi(p_p) \right], \quad (\text{A23})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(1);i\otimes]} = \frac{-ie^3}{4M^2Q^2} \left(1 - \frac{(p_p - k)^2}{[v \cdot (p_p - k) + i0]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \times \left[\chi^\dagger(p'_p) v_\mu \left\{ (2p_p - k) \cdot \varepsilon^* - (v \cdot \varepsilon^*) [v \cdot (2p_p - k)] \right\} \chi(p_p) \right], \quad (\text{A24})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(0);i\otimes]} = \frac{-ie^3}{4M^2Q^2} \left(1 - \frac{(p_p - k)^2}{[v \cdot (p_p - k) + i0]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \times \left[\chi^\dagger(p'_p) \left\{ (p_p + p'_p - k)_\mu - v_\mu [v \cdot (p_p + p'_p - k)] \right\} (v \cdot \varepsilon^*) \chi(p_p) \right], \quad (\text{A25})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(2);i]} = \frac{ie^3}{8M^2Q^2[v \cdot (p_p - k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \times \left[\chi^\dagger(p'_p) v_\mu \left\{ 2(v \cdot k)^2 (v \cdot \varepsilon^*) - (v \cdot k)(k \cdot \varepsilon^*) - k^2 (v \cdot \varepsilon^*) \right\} \chi(p_p) \right], \quad (\text{A26})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(0);i\boxtimes]} = \frac{-ie^3}{4M^2Q^2} \left(\frac{[v \cdot (p_p - k)]^3 - [v \cdot (p_p - k)](p_p - k)^2}{[v \cdot (p_p - k) + i0]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) v_\mu (v \cdot \varepsilon^*) \chi(p_p) \right], \quad \text{and} \quad (\text{A27})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(2);v]} = \frac{-ie^3}{4M^2Q^2} [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) \left\{ 3[v \cdot (Q - k)] v_\mu (v \cdot \varepsilon^*) - (Q - k)_\mu (v \cdot \varepsilon^*) - v_\mu (Q - k) \cdot \varepsilon^* - [v \cdot (Q - k)] \varepsilon_\mu^* \right\} \chi(p_p) \right]. \quad (\text{A28})$$

Of the 15 recoil-radiative amplitudes listed above, only the first eight amplitudes (cf. Fig. 4), namely, $\mathcal{M}_{\gamma\gamma^*}^{[L;i]}$, $\mathcal{M}_{\gamma\gamma^*}^{[L;f]}$, $\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]}$, $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]}$, $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f]}$, $\mathcal{M}_{\gamma\gamma^*}^{[p(2);f]}$, $\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\boxtimes]}$, and $\mathcal{M}_{\gamma\gamma^*}^{[p(1);f]}$, Eqs. (A14)–(A21), contribute to the soft-photon bremsstrahlung cross sections in the lab frame at our intended $\mathcal{O}(1/M)$ accuracy. Apart from $\mathcal{M}_{\gamma\gamma^*}^{[L;i]}$ and $\mathcal{M}_{\gamma\gamma^*}^{[L;f]}$, the rest of these contributing amplitudes in the SPL, with either LO or NLO components of the final-state proton propagator, are chirally enhanced in the lab frame.

In addition to the recoil-radiative amplitudes, six pion-loop amplitudes (cf. Fig. 5 and the last row in Fig. 13) arise from insertions of the proton's Dirac (F_1) and Pauli (F_2) form factors, which can also contribute to the $\mathcal{O}(\alpha^3/M)$ NLO cross section. The following are their expressions:

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(a)]} = \frac{ie^3}{(Q - k)^2} \left[\bar{u}(p')\gamma^\mu \frac{(\not{p} - k + m_l)}{(p - k)^2 - m_l^2} \varepsilon^* u(p) \right] \left[\chi^\dagger(p'_p) \mathcal{V}_\mu^{(2)} \chi(p_p) \right], \quad (\text{A29})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(b)]} = \frac{ie^3}{(Q - k)^2} \left[\bar{u}(p')\gamma^\mu \frac{(\not{p} - k + m_l)}{(p - k)^2 - m_l^2} \varepsilon^* u(p) \right] \left[\chi^\dagger(p'_p) \mathcal{V}_\mu^{(2)} \chi(p_p) \right], \quad (\text{A30})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(c)]} = \frac{-ie^3}{Q^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) \mathcal{V}_\mu^{(2)} (v \cdot \varepsilon^*) \chi(p_p) \right], \quad (\text{A31})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(d)]} = \frac{-ie^3}{Q^2[v \cdot (p'_p + k) + i0]} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p)(\mathcal{V}^{(2)} \cdot \varepsilon^*)v_\mu \chi(p_p) \right], \quad (\text{A32})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(g)]} = \frac{-ie^3}{Q^2[v \cdot (p_p - k) + i0]} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p)\mathcal{V}_\mu^{(2)}(v \cdot \varepsilon^*)\chi(p_p) \right], \quad \text{and} \quad (\text{A33})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(h)]} = \frac{-ie^3}{Q^2[v \cdot (p_p - k) + i0]} \left[\bar{u}(p')\gamma^\mu u(p) \right] \left[\chi^\dagger(p'_p)v_\mu(\mathcal{V}_\mu^{(2)} \cdot \varepsilon^*)\chi(p_p) \right], \quad (\text{A34})$$

where $\mathcal{V}_\mu^{(2)}$ is the effective hadronic correction vertex, given in Eq. (17) in the main text, which renormalizes the LO proton-photon vertex. Of the six pion-loop amplitudes listed above, only the first four amplitudes, $\mathcal{M}_{\gamma\gamma^*}^{[\pi(a), \dots, \pi(d)]}$ (cf. Fig. 5), Eqs. (A29)–(A32), contribute to the soft-photon bremsstrahlung cross sections in the lab frame at our intended $\mathcal{O}(1/M)$ accuracy. In particular, the amplitudes $\mathcal{M}_{\gamma\gamma^*}^{[\pi(c), \pi(d)]}$, Eqs. (A29) and (A30), involving the LO component of the final-state proton propagator are chirally enhanced in the lab frame in the SPL.

The remaining nine amplitudes, Eqs. (A22)–(A28), (A33), and (A34), shown in Fig. 13, are discarded in our main text analysis, as they do not contribute to the soft-photon bremsstrahlung cross section in the lab frame. A detailed investigation reveals that for the computation of the cross sections, they all either lead to scaleless phase-space integrals vanishing under dimensional regularization (DR) (charge-even case) or yield terms of $\mathcal{O}(1/M^2)$, which are beyond our working precision (charge-odd case).

Consequently, in the analysis of the charge-even soft-photon bremsstrahlung cross section at NLO, all relevant contributions arising from the $\nu \geq 2$ chiral order matrix elements scale as $\mathcal{S}_{\text{even}}^{(\nu \geq 2)} \sim \mathcal{O}(1/M^2)$ and are hence discarded. In contrast, the analysis presented in Sec. II addresses the nontrivial contribution of chirally enhanced NNLO amplitudes to the charge-odd soft-photon bremsstrahlung cross section at NLO. This necessitates the evaluation of the interference terms in $\mathcal{S}_{\text{odd}}^{(\nu=2)}$, Eq. (102), namely,

$$\mathcal{S}_{\text{odd}}^{(\nu=2)} = \sum_{\text{spins}} |\mathcal{M}_{\text{NLO}}|^2_{\text{odd}} + \sum_{\text{spins}} \left[\mathcal{M}_{\text{LO}}^\dagger \widetilde{\mathcal{M}}_{\text{NNLO}} + \text{H.c.} \right]_{\text{odd}}, \quad (\text{A35})$$

where $\mathcal{M}_{\text{LO}}$, $\mathcal{M}_{\text{NLO}}$, and $\widetilde{\mathcal{M}}_{\text{NNLO}}$ are the sum of the soft-photon bremsstrahlung amplitudes in the lab frame, as given in Eqs. (58)–(60), in the main text. However, the general structure of some of the amplitudes contributing to charge-odd $\nu = 2$ chiral order matrix elements, namely,

$$\begin{aligned} \mathcal{S}_{\text{odd}}^{(\nu=2)} &\supset 2\mathcal{R}e \sum_{\text{spins}} \left[\left(\mathcal{M}_{\gamma\gamma^*}^{[i;i]} + \mathcal{M}_{\gamma\gamma^*}^{[i;\bar{f}]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[p(1);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} \right) \right. \\ &\quad \left. + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i;\otimes]} + \mathcal{M}_{\gamma\gamma^*}^{[p(1);v]} \right) \\ &\quad \left. + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i]^\dagger} \left(\mathcal{M}_{\gamma\gamma^*}^{[i;i]} + \mathcal{M}_{\gamma\gamma^*}^{[i;\bar{f}]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(a)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(b)]} \right) \right] \\ &\sim \mathcal{O}\left(\frac{1}{M^2}\right), \end{aligned} \quad (\text{A36})$$

makes it clear that they yield $\mathcal{O}(1/M^2)$ terms in the cross section which lie beyond the accuracy of our present analysis. Accordingly, the above combination of amplitudes is excluded from the subsequent treatment of the charge-odd component.

4. N³LO [i.e., $\mathcal{O}(e^3/M^3)$] amplitudes

At this order, it is necessary to include Feynman amplitudes having the contributions from the N³LO proton-photon interactions stemming from the chiral Lagrangian of $\nu = 3$ order. The relevant $\mathcal{O}(1/M^3)$ interaction terms are

$$\mathcal{L}_{\pi N}^{(3)} = \bar{N}_v \left[\frac{1}{8M^3} \left\{ (v \cdot \mathcal{D}) \mathcal{D}_\mu \mathcal{D}^\mu (v \cdot \mathcal{D}) - (v \cdot \mathcal{D})^4 \right\} + \dots \right] N_v,$$

with

$$\begin{aligned} \mathcal{D}_\mu &= \partial_\mu + \frac{1}{2} \left[u^\dagger \left(\mathbb{I} \partial_\mu + i\tau^3 \frac{e}{2} A_\mu \right) u + u \left(\mathbb{I} \partial_\mu + i\tau^3 \frac{e}{2} A_\mu \right) u^\dagger \right] \\ &\quad + i\mathbb{I} \frac{e}{2} A_\mu. \end{aligned} \quad (\text{A37})$$

The above ellipsis indicates additional operators in the chiral Lagrangian not relevant for the present analysis. In this case, nine relevant amplitudes can contribute to the bremsstrahlung cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. In particular, in the lab frame, all such amplitudes in SPL are chirally enhanced with LO and NNLO components of the final-state proton propagator. They are given as follows (cf. the first row in Fig. 6 and the first two rows in Fig. 14):

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]} = \frac{-ie^3}{8M^3Q^2} \left(1 - \frac{(p'_p + k)^2}{[v \cdot (p'_p + k) + i0]^2}\right) [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) \left\{ (2p'_p + k) \cdot \varepsilon^* - (v \cdot \varepsilon^*)[v \cdot (2p'_p + k)] \right\} \right. \\ \left. \times \left\{ (p_p + p'_p + k)_\mu - v_\mu[v \cdot (p_p + p'_p + k)] \right\} \chi(p_p) \right], \quad (\text{A38})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(0);f\otimes]} = \frac{-ie^3}{16M^3Q^2} \left(1 - \frac{(p'_p + k)^2}{[v \cdot (p'_p + k) + i0]^2}\right) [\bar{u}(p')\gamma^\mu u(p)] \\ \times \left[\chi^\dagger(p'_p)(v \cdot \varepsilon^*) \left\{ 2(v \cdot Q)^2 v_\mu - (v \cdot Q)Q_\mu - Q^2 v_\mu \right\} \chi(p_p) \right], \quad (\text{A39})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(2);f\otimes]} = \frac{-ie^3}{16M^3Q^2} \left(1 - \frac{(p'_p + k)^2}{[v \cdot (p'_p + k) + i0]^2}\right) [\bar{u}(p')\gamma^\mu u(p)] \\ \times \left[\chi^\dagger(p'_p) \left\{ 2(v \cdot k)^2(v \cdot \varepsilon^*) - (v \cdot k)(k \cdot \varepsilon^*) - k^2(v \cdot \varepsilon^*) \right\} v_\mu \chi(p_p) \right], \quad (\text{A40})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(2);f}]} = \frac{-ie^3}{16M^3Q^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \\ \times \left[\chi^\dagger(p'_p) \left\{ 2(v \cdot k)^2(v \cdot \varepsilon^*) - (v \cdot k)(k \cdot \varepsilon^*) - k^2(v \cdot \varepsilon^*) \right\} \left\{ (p_p + p'_p + k)_\mu - v_\mu[v \cdot (p_p + p'_p + k)] \right\} \chi(p_p) \right], \quad (\text{A41})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\widetilde{p(1);f}]} = \frac{-ie^3}{16M^3Q^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \\ \times \left[\chi^\dagger(p'_p) \left\{ (2p'_p + k) \cdot \varepsilon^* - (v \cdot \varepsilon^*)[v \cdot (2p'_p + k)] \right\} \left\{ 2(v \cdot Q)^2 v_\mu - (v \cdot Q)Q_\mu - Q^2 v_\mu \right\} \chi(p_p) \right], \quad (\text{A42})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\overline{p(0);f\otimes}]} = \frac{-ie^3}{8M^3Q^2} \left(\frac{[v \cdot (p'_p + k)]^3 - [v \cdot (p'_p + k)](p'_p + k)^2}{[v \cdot (p'_p + k) + i0]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \\ \times \left[\chi^\dagger(p'_p)(v \cdot \varepsilon^*) \left\{ (p_p + p'_p + k)_\mu - v_\mu[v \cdot (p_p + p'_p + k)] \right\} \chi(p_p) \right], \quad (\text{A43})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]} = \frac{-ie^3}{8M^3Q^2} \left(\frac{[v \cdot (p'_p + k)]^3 - [v \cdot (p'_p + k)](p'_p + k)^2}{[v \cdot (p'_p + k) + i0]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \\ \times \left[\chi^\dagger(p'_p) \left\{ (2p'_p + k) \cdot \varepsilon^* - (v \cdot \varepsilon^*)[v \cdot (2p'_p + k)] \right\} v_\mu \chi(p_p) \right], \quad (\text{A44})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(3);f]} = \frac{ie^3}{8M^3Q^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) \left\{ (v \cdot k)k^2 - (v \cdot k)^3 \right\} (v \cdot \varepsilon^*) v_\mu \chi(p_p) \right], \quad (\text{A45})$$

and

$$i\mathcal{M}_{\gamma\gamma^*}^{[\widetilde{p(0);f}]} = \frac{ie^3}{8M^3Q^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p)(v \cdot \varepsilon^*) \left\{ (v \cdot Q)Q^2 - (v \cdot Q)^3 \right\} v_\mu \chi(p_p) \right]. \quad (\text{A46})$$

In particular, the amplitudes incorporating the N³LO vertex structure are presented in Eqs. (A45) and (A46). Of the nine amplitudes listed above, only the first three enhanced amplitudes (cf. first row in Fig. 6)—namely, $\mathcal{M}_{\gamma\gamma^*}^{[p(1);f\otimes]}$, $\mathcal{M}_{\gamma\gamma^*}^{[\widetilde{p(0);f\otimes}]}$, and $\mathcal{M}_{\gamma\gamma^*}^{[p(2);f\otimes]}$, Eqs. (A38)–(A40), involving the NLO component of the final-state proton propagator—contribute to the soft-photon bremsstrahlung cross sections in the lab frame at our intended $\mathcal{O}(1/M)$ accuracy.

In addition to the recoil-radiative amplitudes, there are five pion-loop amplitudes (cf. the last rows in Figs. 6 and 14) arising from the insertions of the proton's form factors ($F_{1,2}^p$), which can potentially contribute to the $\mathcal{O}(\alpha^3/M)$ NLO bremsstrahlung cross section. The following are their expressions:

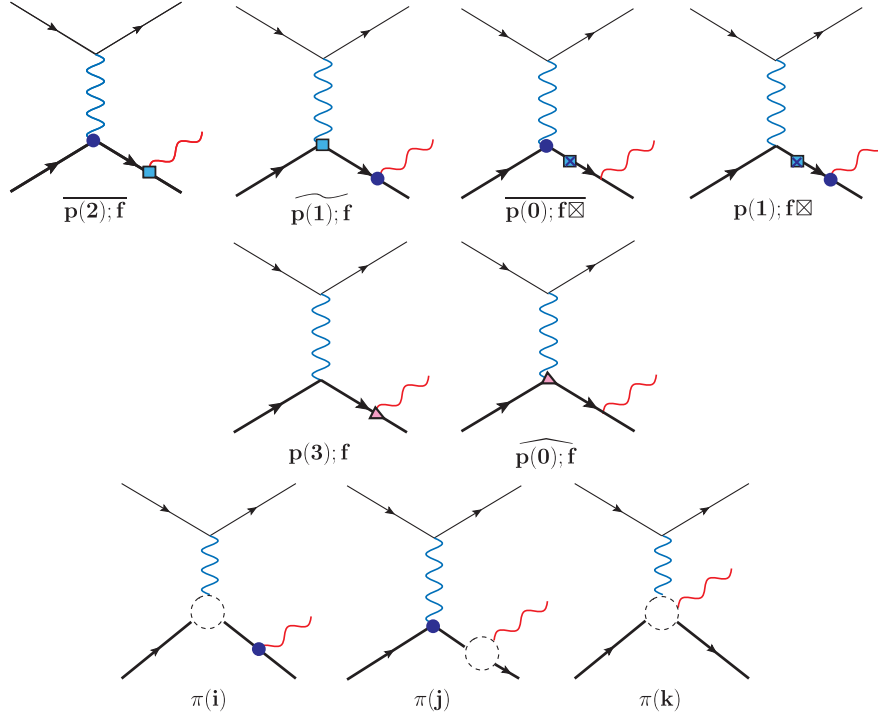


FIG. 14. Photon bremsstrahlung diagrams of $N^3\text{LO}$ [i.e., $\mathcal{O}(e^3/M^3) \sim \mathcal{O}(e^3/(M\Lambda_\chi^2))$] with insertions of NLO (filled blob $\bullet$), NNLO (filled box $\square$), and $N^3\text{LO}$ (triangle $\triangle$) proton-photon interaction vertices, proton's form factors (dashed circular blob, cf. Fig. 1), and $\mathcal{O}(1/M^2)$ (crossed box $\boxtimes$) proton propagator components, not contributing to our ℓ -p elastic scattering cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The thin, thick, and wiggly lines denote the propagators for the lepton, proton, and photon (red for the soft bremsstrahlung photon and blue for the exchanged off-shell photon), respectively. The nomenclature of the individual diagrams is based on the radiated photon originating from the final-state “f” proton “p.” The labels “(0),” “(1),” “(2),” and “(3)” denote LO, NLO, NNLO, and $N^3\text{LO}$ proton-radiating vertices, respectively, while the overline, tilde, and wedge signify NLO, NNLO, and $N^3\text{LO}$ exchange-photon proton vertices, respectively. All of the above—except for the last one involving the two-photon pion-loop—correspond to chirally enhanced amplitudes.

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(e)]} = \frac{-ie^3}{2MQ^2} \left(1 - \frac{(p'_p + k)^2}{[v \cdot (p'_p + k + i0)]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] [\chi^\dagger(p'_p)(\mathcal{V}^{(2)} \cdot \epsilon^*)v_\mu \chi(p_p)], \quad (\text{A47})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(f)]} = \frac{-ie^3}{2MQ^2} \left(1 - \frac{(p'_p + k)^2}{[v \cdot (p'_p + k + i0)]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] [\chi^\dagger(p'_p)(v \cdot \epsilon^*)\mathcal{V}_\mu^{(2)}\chi(p_p)], \quad (\text{A48})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(i)]} = \frac{-ie^3}{2MQ^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] [\chi^\dagger(p'_p)(\mathcal{V}^{(2)} \cdot \epsilon^*)\{(p_p + p'_p + k)_\mu - v_\mu[v \cdot (p_p + p'_p + k)]\}\chi(p_p)], \quad (\text{A49})$$

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(j)]} = \frac{-ie^3}{2MQ^2[v \cdot (p'_p + k) + i0]} [\bar{u}(p')\gamma^\mu u(p)] [\chi^\dagger(p'_p)\{(2p'_p + k) \cdot \epsilon^* - (v \cdot \epsilon^*)[v \cdot (2p'_p + k)]\}\mathcal{V}_\mu^{(2)}\chi(p_p)], \quad (\text{A50})$$

and

$$i\mathcal{M}_{\gamma\gamma^*}^{[\pi(k)]} = \frac{-ie^3}{MQ^2} [\bar{u}(p')\gamma^\mu u(p)] [\chi^\dagger(p'_p)\{\Gamma_1(Q^2)\epsilon_\mu^* - \Gamma_2(Q^2)v_\mu(v \cdot \epsilon^*)\}\chi(p_p)]. \quad (\text{A51})$$

Here, $\Gamma_1 \sim \mathcal{O}(1/\Lambda_\chi^2)$ and $\Gamma_2 \sim \mathcal{O}(1/\Lambda_\chi^2)$ are two other generic Q^2 -dependent form factor functions that could be generated from evaluating the pion-loop diagrams that renormalize the NLO two-photon seagull vertex (cf. the last graph in Fig. 3). Except for the pion-loop seagull amplitude $\mathcal{M}_{\gamma\gamma^*}^{[\pi(k)]}$, Eq. (A51), the other four listed amplitudes involve LO and NLO components of the final-state proton propagator. Hence, in SPL, these amplitudes are expected to be chirally enhanced in the

lab frame. However, only the first two amplitudes, $\mathcal{M}_{\gamma\gamma^*}^{[\pi(e),\pi(f)]}$ (cf. Fig. 6), Eqs. (A47) and (A48), contribute to the soft-photon bremsstrahlung cross sections in the lab frame at our intended $\mathcal{O}(1/M)$ accuracy.

The remaining nine amplitudes, Eqs. (A41)–(A46) and Eqs. (A49)–(A51), shown in Fig. 14, are discarded in the main text analysis since they do not contribute to the soft-photon bremsstrahlung cross section in the lab frame. They either lead to scaleless phase-space integrals vanishing under DR, subject to the condition Eq. (63) (charge-even case), or yield cross section terms of $\mathcal{O}(1/M^2)$ which are beyond our working precision (charge-odd case).

In our analysis presented in Sec. II, we examine the contribution of additional chirally enhanced $N^3\text{LO}$ amplitudes to the charge-odd soft-photon bremsstrahlung cross

section at NLO. This requires us to evaluate the interference matrix elements of $\nu = 3$ chiral order in $\mathcal{S}_{\text{odd}}^{(\nu=3)}$, Eq. (109), namely,

$$\mathcal{S}_{\text{odd}}^{(\nu=3)} = \sum_{\text{spins}} \left[\mathcal{M}_{\text{NLO}}^\dagger \widetilde{\mathcal{M}}_{\text{NNLO}} + \mathcal{M}_{\text{LO}}^\dagger \widetilde{\mathcal{M}}_{\text{N}^3\text{LO}} + \text{H.c.} \right]_{\text{odd}}, \quad (\text{A52})$$

where $\mathcal{M}_{\text{LO}}$, $\mathcal{M}_{\text{NLO}}$, $\widetilde{\mathcal{M}}_{\text{NNLO}}$, and $\widetilde{\mathcal{M}}_{\text{N}^3\text{LO}}$ are the sum of the soft-photon bremsstrahlung amplitudes in the lab frame, as given in Eqs. (58)–(60), in the main text. However, the general structure of some of the amplitudes contributing to the first interference term of the amplitudes (along with their Hermitian conjugates), namely,

$$\begin{aligned} \sum_{\text{spins}} \left[\mathcal{M}_{\text{NLO}}^\dagger \widetilde{\mathcal{M}}_{\text{NNLO}} + \text{H.c.} \right]_{\text{odd}} &\supset 2\mathcal{R}e \left[\left(\mathcal{M}_{\gamma\gamma^*}^{[\bar{l}i]} + \mathcal{M}_{\gamma\gamma^*}^{[\bar{l}f]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[p(1);f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(2);f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f\boxtimes]} \right. \right. \\ &\quad \left. \left. + \mathcal{M}_{\gamma\gamma^*}^{[\pi(c)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(d)]} \right) + \left(\mathcal{M}_{\gamma\gamma^*}^{[p(1);f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);f]} + \mathcal{M}_{\gamma\gamma^*}^{[p(0);i\boxtimes]} \right. \right. \\ &\quad \left. \left. + \mathcal{M}_{\gamma\gamma^*}^{[p(1);v]} \right)^\dagger \left(\mathcal{M}_{\gamma\gamma^*}^{[\bar{l}i]} + \mathcal{M}_{\gamma\gamma^*}^{[\bar{l}f]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(a)]} + \mathcal{M}_{\gamma\gamma^*}^{[\pi(b)]} \right) \right] \sim \mathcal{O}\left(\frac{1}{M^2}\right), \quad (\text{A53}) \end{aligned}$$

makes it evident that they yield contributions at $\mathcal{O}(1/M^2)$, which are beyond the required accuracy of our NLO cross section. Thus, the above combination of amplitudes is excluded from our treatment of the charge-odd component.

5. $N^4\text{LO}$ [i.e. $\mathcal{O}(e^3/M^4)$] amplitudes

Finally, there are only two possible amplitudes at this order, which can contribute to the radiative recoil corrections to the cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. They are given as follows (cf. Fig. 15):

$$i\mathcal{M}_{\gamma\gamma^*}^{[p(3);f\boxtimes]} = \frac{ie^3}{16M^4Q^2} \left(1 - \frac{(p'_p + k)^2}{[v \cdot (p'_p + k + i0)]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) \{ (v \cdot k)k^2 - (v \cdot k)^3 \} (v \cdot \varepsilon^*) v_\mu \chi(p_p) \right], \quad (\text{A54})$$

and

$$i\mathcal{M}_{\gamma\gamma^*}^{[\widehat{p(0);f\boxtimes}]} = \frac{ie^3}{16M^4Q^2} \left(1 - \frac{(p'_p + k)^2}{[v \cdot (p'_p + k + i0)]^2} \right) [\bar{u}(p')\gamma^\mu u(p)] \left[\chi^\dagger(p'_p) (v \cdot \varepsilon^*) \{ (v \cdot Q)Q^2 - (v \cdot Q)^3 \} v_\mu \chi(p_p) \right]. \quad (\text{A55})$$

Both the above amplitudes involve the NLO component of the final-state proton propagator and are therefore chirally enhanced in the lab frame in SPL. However, the first amplitude leads to a scaleless phase-space integral which vanishes under DR, while the second amplitude yields cross section terms of $\mathcal{O}(1/M^2)$ which lie beyond our working precision. Hence, both amplitudes are discarded from our analysis in the main text.

APPENDIX B: BREMSSTRAHLUNG PHASE-SPACE INTEGRATION

1. S-frame kinematics

The determination of the lab frame soft-photon bremsstrahlung cross section entails the phase-space integration over the so-called radiative tail differential distribution, namely,

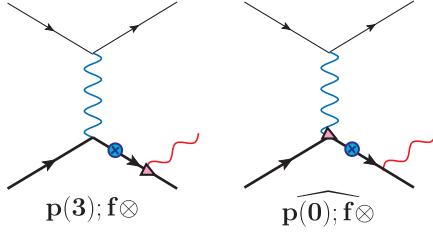


FIG. 15. Chirally enhanced photon bremsstrahlung diagrams of N⁴LO [i.e., $\mathcal{O}(e^3/M^4)$] with insertions of the N³LO proton-photon interaction vertex (triangle $\triangle$) and the $\mathcal{O}(1/M)$ proton propagator component (crossed box $\otimes$), not contributing to our ℓ -p elastic scattering cross section up to and including NLO [i.e., $\mathcal{O}(\alpha^3/M)$] in HB χ PT. The thin, thick, and wiggly lines denote the propagators for the lepton, proton, and photon (red for the soft bremsstrahlung photon and blue for the exchanged off-shell photon), respectively. The nomenclature of the individual diagrams is based on the radiated photon originating from the final-state “f” proton “p.” The labels “(0)” and “(3)” denote LO and N³LO proton-radiating vertices, respectively, while the wedge signifies an N³LO exchange-photon proton vertex insertion.

$$[d\tilde{\sigma}_{br}]_{\gamma\gamma^*} = \frac{(2\pi)^4 \delta_k^{\text{lab}}}{8ME'_p |\mathbf{p}|} \frac{d^3 \mathbf{p}'}{(2\pi)^3 2E'} \frac{d^3 \mathbf{k}'}{(2\pi)^3 2E_{\gamma^*}} \times \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{\gamma\gamma^*}|^2,$$

with the energy-conserving δ -function given by

$$\delta_k^{\text{lab}} \equiv \delta(E + M - E' - \sqrt{(\mathbf{Q} - \mathbf{k})^2 + M^2} - E_{\gamma^*}), \quad (\text{B1})$$

as obtained after partially integrating the final-state proton three-momentum. The resulting integration is complicated by the dependence of the δ -function on the photon emission angles, which renders the radiation spectrum anisotropic. This confines the photon-energy integration domain to an ellipsoidal region that is analytically challenging to evaluate. An alternative practical method for evaluating the phase-space integrals employs a technique originally introduced by Tsai in Ref. [47], later formalized in Refs. [48,49], and subsequently applied in various studies of radiative corrections [11,14,45,46]. This technique of evaluation of the integrals first entails boosting from the laboratory to the S-frame, namely, the center-of-mass frame of the recoiling proton and the emitted soft-photon, i.e.,

$$\mathbf{p}'_p + \mathbf{k} = \mathbf{Q} \xrightarrow{\text{S-frame}} \mathbf{Q}^S = 0, \quad (\text{B2})$$

where $\mathbf{p}'_p$ and $\mathbf{k}$ are the respective lab frame three-momenta of the recoil proton and the emitted soft photon, and $\mathbf{Q}$ is the three-momentum transferred in the ℓ -p elastic scattering process. In this way, the integrals reduce to standard spherical forms, as the δ -function above becomes independent of the photon angles in SPL. This effectively transforms

the lab-frame kinematics for the bremsstrahlung process, $\ell + p \rightarrow \ell + p + \gamma^*$, into the kinematics for the “reverse bremsstrahlung” process, namely, $\ell + p + \gamma^* \rightarrow \ell + p$, in the S-frame with $(k^S)^\mu \equiv (E_{\gamma^*}^S, \mathbf{k}^S) \rightarrow 0$, subject to the constraint

$$\lim_{k \rightarrow 0} \delta_k^{\text{lab}} \xrightarrow{\text{S-frame}} \delta^S \equiv \delta(E^S + E_p^S - E'^S - E_{\gamma^*}^S). \quad (\text{B3})$$

With the lab-frame detector resolution defined by the parameter Δ_{γ^*} , the following relationships can be justified:

$$\begin{aligned} & \text{(i) } E_p'^S \approx M, \quad \text{(ii) } E^S \approx E' = \frac{E}{\eta}, \quad \text{(iii) } E'^S \approx E, \\ & \text{(iv) } E_p^S \approx E'_p, \quad \text{(v) } \cos \theta_S \approx \cos \theta, \quad \text{(vi) } \Delta_S \approx \eta \Delta_{\gamma^*}, \end{aligned} \quad (\text{B4})$$

where θ and θ_S denote the lab-frame and S-frame scattering angles for the outgoing lepton, respectively. It is also notable that Δ_S/η is the upper limit of the soft-photon energy integral corresponding to the detector resolution Δ_S in the S-frame. A comprehensive treatment of the S-frame kinematics is provided in the Appendix of the work by Talukdar *et al.* [14].

2. Integrations of the charge-even component

The charge-even soft-photon bremsstrahlung differential cross section up to and including NLO in HB χ PT is obtained by integrating over the elastic radiative tail distribution. This involves the following integrals, expressed in terms of lab-frame variables:

$$\begin{aligned} & \int \frac{d^3 \mathbf{k}}{k} \frac{1}{(p \cdot k)^2} \delta_k^{\text{lab}}, & \int \frac{d^3 \mathbf{k}}{k} \frac{1}{(p' \cdot k)^2} \delta_k^{\text{lab}}, \\ & \int \frac{d^3 \mathbf{k}}{k} \frac{(p \cdot p')}{(p \cdot k)(p' \cdot k)} \delta_k^{\text{lab}}, & \int \frac{d^3 \mathbf{k}}{k} \frac{(v \cdot k)}{(p \cdot k)^2} \delta_k^{\text{lab}}, \\ & \int \frac{d^3 \mathbf{k}}{k} \frac{(v \cdot k)}{(p' \cdot k)^2} \delta_k^{\text{lab}}, & \int \frac{d^3 \mathbf{k}}{k} \frac{(p \cdot p')(v \cdot k)}{(p \cdot k)(p' \cdot k)} \delta_k^{\text{lab}}, \end{aligned}$$

where $E_{\gamma^*}^* = k = |\mathbf{k}| \rightarrow 0$ is the soft-photon energy or three-momentum. To simplify the treatment of the “inelastic” δ_k^{lab} -functions, Eq. (B1), Tsai’s technique [47] is employed, which involves boosting to the S-frame. The transformation renders δ_k^{lab} effectively “elastic,” resulting in the simplified δ^S -function of Eq. (B2). With the δ -function becoming independent of the photon angles in the SPL, the integrals are readily evaluated in the S-frame. As discussed in the main text, some of these integrals exhibit IR divergences, which are isolated using DR by analytical continuation to $D - 1 = 3 - 2\varepsilon_{\text{IR}}$ spatial dimensions, with $\varepsilon_{\text{IR}} < 0$. The resulting S-frame integrals, reexpressed in terms of lab-frame kinematic variables via the kinematic transformations (B4), are presented below:

$$L_{ii}^{(S)} \xrightarrow{\text{DR}} \frac{m_l^2}{2} \int \frac{d^{D-1}k^S}{k^S} \frac{1}{(p^S \cdot k^S)^2} \\ = \pi \left[\frac{1}{|\epsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] + 2\pi\tilde{L}_{ii}^{(S)},$$

where

$$\tilde{L}_{ii}^{(S)} = \frac{1}{2} \ln \left(\frac{4\Delta_{\gamma^*}^2}{m_l^2} \right) - \frac{1}{\beta'} \ln \sqrt{\frac{1+\beta'}{1-\beta'}}, \quad (\text{B5})$$

$$L_{ff}^{(S)} \xrightarrow{\text{DR}} \frac{m_l^2}{2} \int \frac{d^{D-1}k^S}{k^S} \frac{1}{(p'^S \cdot k^S)^2} \\ = \pi \left[\frac{1}{|\epsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] + 2\pi\tilde{L}_{ff}^{(S)},$$

where

$$\tilde{L}_{ff}^{(S)} = \frac{1}{2} \ln \left(\frac{4\Delta_{\gamma^*}^2}{m_l^2} \right) - \frac{1}{\beta} \ln \sqrt{\frac{1+\beta}{1-\beta}}, \quad (\text{B6})$$

$$L_{if}^{(S)} \xrightarrow{\text{DR}} \int \frac{d^{D-1}k^S}{k^S} \frac{(p^S \cdot p'^S)}{(p^S \cdot k^S)(p'^S \cdot k^S)} \\ = \pi \left[\frac{1}{|\epsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] \frac{\nu_l^2 + 1}{\nu_l} \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] \\ + 2\pi\tilde{L}_{if}^{(S)},$$

where

$$\tilde{L}_{if}^{(S)} = \frac{\nu_l^2 + 1}{2\nu_l} \left[\ln \left(\frac{4\Delta_{\gamma^*}^2}{m_l^2} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right] - \ln^2 \sqrt{\frac{1+\beta'}{1-\beta'}} \right. \\ \left. + \ln^2 \sqrt{\frac{1+\beta}{1-\beta}} + \text{Li}_2 \left(1 - \frac{\lambda_\nu E - E'}{(1-\beta)E\xi_\nu} \right) \right. \\ \left. + \text{Li}_2 \left(1 - \frac{\lambda_\nu E - E'}{(1+\beta)E\xi_\nu} \right) \right. \\ \left. - \text{Li}_2 \left(1 - \frac{\lambda_\nu E - E'}{(1-\beta')E'\lambda_\nu\xi_\nu} \right) \right. \\ \left. - \text{Li}_2 \left(1 - \frac{\lambda_\nu E - E'}{(1+\beta')E'\lambda_\nu\xi_\nu} \right) \right], \quad (\text{B7})$$

$$LL_{ii}^{(S)} \xrightarrow{\text{DR}} \int \frac{d^{D-1}k^S}{k^S} \frac{(v \cdot k^S)}{(p^S \cdot k^S)^2} = 2\pi\Delta_{\gamma^*}, \quad (\text{B8})$$

$$LL_{ff}^{(S)} \xrightarrow{\text{DR}} \int \frac{d^{D-1}k^S}{k^S} \frac{(v \cdot k^S)}{(p'^S \cdot k^S)^2} = 2\pi\Delta_{\gamma^*}, \quad (\text{B9})$$

$$LL_{if}^{(S)} \xrightarrow{\text{DR}} \int \frac{d^{D-1}k^S}{k^S} \frac{(p^S \cdot p'^S)(v \cdot k^S)}{(p^S \cdot k^S)(p'^S \cdot k^S)} \\ = 4\pi\Delta_{\gamma^*} \left(\frac{1+\nu_l^2}{2\nu_l} \right) \ln \left[\frac{\nu_l + 1}{\nu_l - 1} \right]. \quad (\text{B10})$$

Here, μ denotes the renormalization scale, and $\gamma_E = 0.577216\dots$ is the Euler-Mascheroni constant. After the phase-space integrations are carried out, the resulting expressions are boosted back into the lab frame using the same transformation relations (B4). The corresponding lab-frame expressions for these integrals have been presented in the main text, as given in Eqs. (66)–(68), (78), and (79).

3. Integrations of the charge-odd component

For the evaluation of our charge-odd soft-photon bremsstrahlung differential cross section up to and including NLO in HB χ PT, we encounter the following types of phase-space integrals, as expressed in terms of the lab-frame kinematical variables:

$$\begin{aligned} & \int \frac{d^3\mathbf{k}}{k} \frac{1}{(p \cdot k)} \delta_k^{\text{lab}}, & \int \frac{d^3\mathbf{k}}{k} \frac{1}{(p' \cdot k)} \delta_k^{\text{lab}}, \\ & \int \frac{d^3\mathbf{k}}{k} \frac{1}{(p \cdot k)(v \cdot k)} \delta_k^{\text{lab}}, & \int \frac{d^3\mathbf{k}}{k} \frac{1}{(p' \cdot k)(v \cdot k)} \delta_k^{\text{lab}}, \\ & \int \frac{d^3\mathbf{k}}{k} \frac{(v \cdot k)}{(p \cdot k)} \delta_k^{\text{lab}}, & \int \frac{d^3\mathbf{k}}{k} \frac{(v \cdot k)}{(p' \cdot k)} \delta_k^{\text{lab}}, \\ & \int \frac{d^3\mathbf{k}}{k} \frac{(v \cdot k)^2}{(p \cdot k)} \delta_k^{\text{lab}}, & \int \frac{d^3\mathbf{k}}{k} \frac{(v \cdot k)^2}{(p' \cdot k)} \delta_k^{\text{lab}}, \\ & \int \frac{d^3\mathbf{k}}{k} \frac{(k \cdot Q)}{(p \cdot k)} \delta_k^{\text{lab}}, & \int \frac{d^3\mathbf{k}}{k} \frac{(k \cdot Q)}{(p' \cdot k)} \delta_k^{\text{lab}}. \end{aligned}$$

The integrals are initially evaluated in the S-frame and then expressed in terms of lab-frame kinematic variables using the transformation rules (B4) outlined earlier. The following are their expressions:

$$I_1^{(S)} = \int \frac{d^3k^S}{k^S} \frac{1}{(p^S \cdot k^S)} = \frac{4\pi\Delta_{\gamma^*}}{\beta'E'} \ln \sqrt{\frac{1+\beta'}{1-\beta'}}, \quad (\text{B11})$$

$$I_2^{(S)} = \int \frac{d^3k^S}{k^S} \frac{1}{(p'^S \cdot k^S)} = \frac{4\pi\Delta_{\gamma^*}}{\beta E} \ln \sqrt{\frac{1+\beta}{1-\beta}}, \quad (\text{B12})$$

$$I_3^{(S)} \xrightarrow{\text{DR}} \int \frac{d^{D-1}k^S}{k^S} \frac{1}{(p^S \cdot k^S)(v \cdot k^S)} \\ = \frac{2\pi}{\beta'E'} \left[\frac{1}{|\epsilon_{\text{IR}}|} + \gamma_E - \ln \left(\frac{4\pi\mu^2}{m_l^2} \right) \right] \ln \sqrt{\frac{1+\beta'}{1-\beta'}} \\ + 2\pi\tilde{I}_3^{(S)}, \quad \text{where}$$

$$\tilde{I}_3^{(S)} = \frac{1}{\beta'E'} \left[\ln \left(\frac{4\Delta_{\gamma^*}^2}{m_l^2} \right) \ln \sqrt{\frac{1+\beta'}{1-\beta'}} \right. \\ \left. + \frac{1}{2} \text{Li}_2 \left(\frac{2\beta'}{\beta' - 1} \right) - \frac{1}{2} \text{Li}_2 \left(\frac{2\beta'}{\beta' + 1} \right) \right], \quad (\text{B13})$$

$$\begin{aligned}
I_4^{(S)} &\stackrel{\text{DR}}{\mapsto} \int \frac{d^{D-1}k^S}{k^S} \frac{1}{(p'^S \cdot k^S)(v \cdot k^S)} \\
&= \frac{2\pi}{\beta E} \left[\frac{1}{|\epsilon_{\text{IR}}|} + \gamma_E - \ln\left(\frac{4\pi\mu^2}{m_l^2}\right) \right] \ln \sqrt{\frac{1+\beta}{1-\beta}} \\
&\quad + 2\pi \tilde{I}_4^{(S)},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{I}_4^{(S)} &= \frac{1}{\beta E} \left[\ln\left(\frac{4\Delta_{\gamma^*}^2}{m_l^2}\right) \ln \sqrt{\frac{1+\beta}{1-\beta}} \right. \\
&\quad \left. + \frac{1}{2} \text{Li}_2\left(\frac{2\beta}{\beta-1}\right) - \frac{1}{2} \text{Li}_2\left(\frac{2\beta}{\beta+1}\right) \right], \quad (\text{B14})
\end{aligned}$$

$$I_5^{(S)} = \int \frac{d^3k^S}{k^S} \frac{(v \cdot k^S)}{(p^S \cdot k^S)} = \frac{2\pi\Delta_{\gamma^*}^2}{\beta' E'} \ln \sqrt{\frac{1+\beta'}{1-\beta'}}, \quad (\text{B15})$$

$$I_6^{(S)} = \int \frac{d^3k^S}{k^S} \frac{(v \cdot k^S)}{(p'^S \cdot k^S)} = \frac{2\pi\Delta_{\gamma^*}^2}{\beta E} \ln \sqrt{\frac{1+\beta}{1-\beta}}, \quad (\text{B16})$$

$$I_7^{(S)} = \int \frac{d^3k^S}{k^S} \frac{(v \cdot k^S)^2}{(p^S \cdot k^S)} = \frac{4\pi\Delta_{\gamma^*}^3}{3\beta' E'} \ln \sqrt{\frac{1+\beta'}{1-\beta'}}, \quad (\text{B17})$$

$$I_8^{(S)} = \int \frac{d^3k^S}{k^S} \frac{(v \cdot k^S)^2}{(p'^S \cdot k^S)} = \frac{4\pi\Delta_{\gamma^*}^3}{3\beta E} \ln \sqrt{\frac{1+\beta}{1-\beta}}, \quad (\text{B18})$$

$$\begin{aligned}
I_9^{(S)} &= \int \frac{d^3k^S}{k^S} \frac{(k^S \cdot Q^S)}{(p^S \cdot k^S)} \\
&= \frac{-2\pi E \Delta_{\gamma^*}^2}{E' \beta'} \left[\left(1 - \frac{\beta}{\beta'}\right) \ln \sqrt{\frac{1+\beta'}{1-\beta'}} - \beta \right], \quad (\text{B19})
\end{aligned}$$

$$\begin{aligned}
I_{10}^{(S)} &= \int \frac{d^3k^S}{k^S} \frac{(k^S \cdot Q^S)}{(p'^S \cdot k^S)} \\
&= \frac{2\pi E' \Delta_{\gamma^*}^2}{E \beta} \left[\left(1 - \frac{\beta'}{\beta}\right) \ln \sqrt{\frac{1+\beta}{1-\beta}} - \beta' \right]. \quad (\text{B20})
\end{aligned}$$

The corresponding expressions for these integrals, after being boosted back to the lab frame, have been presented in the main text, as given in Eqs. (87)–(90), (99), (100), and (104)–(107).

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