

Notes on off-shell conformal integrals and correlation functions at five points

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We study five-point off-shell conformal integrals and the associated half-Bogomol'nyi-Prasad-Sommerfield (BPS) correlation functions at two loops in the 't Hooft coupling expansion of maximally supersymmetric Yang-Mills theory. We construct a basis of uniform-transcendental (UT) pure integrals spanning six distinct topologies by diagonalizing leading singularities subject to conformal invariance. By fixing conformal frames, this basis can be mapped to known two-loop four-mass integral families. We then compute the integrated results by combining canonical differential equations with integration-by-parts reduction. As an application, we present symbol-level integrated results for the two-loop five-point half-BPS correlators, including both maximal and nonmaximal sectors.

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I. INTRODUCTION

In recent years, significant progress has been made in multiple aspects of research within the framework of maximally supersymmetric Yang-Mills ($\mathcal{N} = 4$ SYM) theory, as an ideal theoretical model for quantum field theory (QFT) and the most famous example of AdS/CFT correspondence [1]. Owing to its remarkable physical properties, such as the Yangian symmetry and various dualities between Wilson-loops, scattering amplitudes and correlation functions [2–8], this theory has become a vital laboratory for understanding scattering amplitudes in QFT [9,10]. Simultaneously, studies of its correlation functions have emerged as a key playground for exploring observables in conformal field theories.

As one of the most important classes of operators in this theory, scalar single-trace half-Bogomol'nyi-Prasad-Sommerfield (BPS) operators and their correlation functions play a central role in this field (see Ref. [11] for a review). In the strong coupling regime, they correspond to supergravity amplitudes of type IIB string theory on $\text{AdS}_5 \times S^5$, providing an important window into quantum gravity in curved spacetime [12–17]. These physical quantities, especially their “large-charge” limits and polygon correlators [18–25], are also ideal objects for integrability studies, being well described by approaches such as hexagonalization [26–29]. On the other hand, at the

integrand level in the weak coupling regime, the discovery of hidden permutation symmetries in four-point correlation functions [30,31] has enabled us to determine astonishing results for the integrand of this physical quantity up to twelve loops in perturbation theory, through graph-theoretic bootstrap methods called *f-graph* bootstrap [32–35]. Moreover, the *positive geometry* framework [36,37], which has achieved remarkable success in the realm of scattering amplitudes [38–41], has also been applied to this physical quantity. The discovery of the *Correlahedron* [42] and its genuinely new interpretation via *Chambers* [43,44], reveal profound mathematical structures underlying this observable. Finally, by introducing superpartners of half-BPS operators [7,8,45], twistor-space reformulation proves to be powerful in determining integrand result at higher points and higher loops [46,47].

On the other hand, in contrast to integrand level, research on integrated correlators in the weak-coupling expansion has so far been very limited. Currently, fully computed cases are limited to four-point correlators up to three loops, and to general numbers of operators only at one loop [48–52] (see also the previous studies on conformal integrals [53–57]). Several factors prevent calculations at higher points and higher loops: First, although conformally invariant, correlators always involve off-shell dynamical inputs, resulting in $4N - 15$ kinematic variables for $N \geq 5$ operators. Moreover, the integrand for this observable is generally nonplanar in the dual space. Precisely due to this nature, even when restricted to multiple polylogarithms (MPL) cases, the complexity of functions and singularities grows rapidly with the number of points and loops, soon exceeding the reach of tools developed from scattering amplitudes and Feynman integrals studies. Finally, in the already computed integrands, cuts beyond multiple polylogarithms are ubiquitous [44,46,47], indicating that more

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complicated structures such as iterative elliptic integrals will likely appear frequently in yet-unknown results.

In this paper, we present the first integrated results for five-point correlators at two loops. To this end, we construct a basis of uniform-transcendental pure conformal integrals [58] by diagonalizing leading singularities subject to conformal invariance, which is motivated by the geometric background of correlators at four points [44]. Generally speaking, this basis can also express any conformal integrand at two-loop order in the 't Hooft coupling expansion and with five off-shell external legs. Finally, thanks to conformal invariance, we can map these integrals to known four-mass families [59] by frame fixing and compute them via canonical differential equations (CDE) and the master integrals (MI) [60,61]. As a practical example, we expand both maximal and nonmaximal sectors onto our basis, and provide their integrated results at symbol level [62,63].

II. 'T HOOFT COUPLING EXPANSION OF HALF-BPS CORRELATORS

In this work, correlation functions of half-BPS operators $O_2(x_i, y_i) = \text{tr}[(y_i \cdot \Phi(x_i))^2]$ will be the main focus, where y_i^I are six-dimensional $SO(6)$ harmonic variables [30,45], and O_2 comes from contraction of y_i^I , with the operator $O_{20'}^{IJ} = \text{tr}(\Phi^I \Phi^J) - \frac{1}{6} \delta^{IJ} \text{tr}(\Phi^K \Phi^K)$, belonging to the $20'$ representation of the R-symmetry group. We consider the 't Hooft coupling expansion of the correlation functions

$$\langle O_2(x_1, y_1), \dots, O_2(x_n, y_n) \rangle = \sum_{l=0}^{\infty} a^l G_n^{(L)}, \quad (1)$$

where $a = g^2 N_c / (4\pi^2)$ and we will specifically focus on the case when $n = 5$. In variables $x_{i,j}^2 := (x_i - x_j)^2$ and $d_{i,j}^2 = -y_{i,j}^2 / x_{i,j}^2$, the correlation function $G_n^{(L)}$ can always be formally written as

$$G_n^{(L)} = C \sum_{b_{i,j}} \left(\prod_{1 \leq i < j \leq n} (d_{i,j})^{b_{i,j}} f_{b_{i,j}}^{(L)}(x_{i,j}^2) \right), \quad (2)$$

where C is an overall constant depending on the N_c factor, and $b_{i,j}$ are summed over all possible integer partitions, such that for any $1 \leq i \leq n$, $\sum_{j=1, j \neq i}^n b_{i,j} = 2$ holds. Furthermore, the above structure captures the dependence of G_n on $y_{i,j}^2$, and $f_{b_{i,j}}$ only depend on $x_{i,j}^2$. Especially for the five-point case, the correlator has the form

$$G_5^{(L)} = \langle (d_{12}^2)^2 d_{34}^2 d_{45}^2 d_{53}^2 \times f_{23}^{(L)} + d_{12}^2 d_{23}^2 d_{34}^2 d_{45}^2 d_{51}^2 \times f_5^{(L)} \rangle_{S_5}, \quad (3)$$

with $f_{23}^{(L)}$ and $f_5^{(L)}$ being conformal invariant transcendental functions of $x_{i,j}^2$.

In [45], a supersymmetric version of correlation function was proposed in the superspace (x_i, θ_i) by considering the correlation functions of n stress-tensor supermultiplets

$$\langle \mathcal{T}(x_1, \theta_1), \dots, \mathcal{T}(x_n, \theta_n) \rangle. \quad (4)$$

The original $20'$ operator serves as the bosonic component of the multiplet $\mathcal{T}(x_i, \theta_i)$; hence the bosonic correlator (1) can always be extracted by taking the component with Grassmann degree zero. Furthermore, $\mathcal{N} = 4$ superconformal symmetry implies the decomposition

$$\langle \mathcal{T}(x_1, \theta_1), \dots, \mathcal{T}(x_n, \theta_n) \rangle = \mathcal{G}_{n,0} + \dots + \mathcal{G}_{n,n-4}, \quad (5)$$

where p in $\mathcal{G}_{n,p}$ denotes the Grassmann degree. At $n = 5$, besides the $\mathcal{G}_{5,0}^{(L)} \propto G_5^{(L)}$, there is one more nontrivial component $\mathcal{G}_{5,1}$, whose form reads generally

$$\mathcal{G}_{5,1}^{(L)} = \langle \mathcal{R}_5 \times f_{\max}^{(L)} \rangle_{S_5}, \quad (6)$$

where $\mathcal{R}_5$ is a universal nilpotent prefactor [31,45].

In this work, we focus on the case at two loops, i.e., $f_{\max}^{(2)}$, $f_{23}^{(2)}$, and $f_5^{(2)}$. According to Grassman degree, we will also call $\mathcal{G}_{5,1}$ the *maximal sector*, and $\mathcal{G}_{5,0}$ the *next-to-maximal sector* or nonmaximal sector. They are conformal invariant functions depending on *five* cross-ratios of $x_{i,j}^2$. In the literature, the integrand level of the two cases was calculated via hidden symmetry and *f-graph* [30], or twistor-space reformulation [46], which serve as the starting point of our calculation.

III. CONSTRUCTING CONFORMAL UT INTEGRAL BASIS

In [46], a local integral representation for nonmaximal correlators was constructed in terms of conformally invariant integrals, including kissing boxes, double boxes, pentaboxes, and double pentagons [64]. While this representation correctly reproduces the correlator integrand, the choice of integral basis can be improved in several respects. In particular, the individual integrals do not exhibit the uniform transcendentality (UT). Moreover, the physical leading singularities are not manifest at the level of individual integrals. Spurious leading singularities can arise in the individual integrals, which cancel only through the combinations among other integrals in the basis.

Guided by the geometric description of correlators [43,44] and the idea of the unitarity method in scattering amplitudes, as a first step, in this section we systematically construct a basis of UT, pure integrals [58,60], for the half-BPS five-point correlators at two loops. Importantly, our integrals serve as integral basis for observables with five off-shell external legs, applied more generally to a wide class of observables in the future.

Topologies of the five-point integrand. Generally speaking, given a set of external kinematics at fixed loop order, the construction of a local integral basis in four dimensions starts from enumerating all nontrivial maximal four-dimensional cuts and introducing a sufficient set of integrals to reproduce them. For observables with conformal symmetry, this further restricts the integral basis to be composed of $SO(2,4)$ -invariant projective integrals. As pointed out in [65], for any loop momentum, integrals involving more than five propagators are redundant in four dimensions. This follows from the vanishing of the Gram determinant. In four dimensions, the numerator of an integrand with six propagators can always be reduced to a linear combination of propagators, thereby expressing it in terms of pentagon integrals.

Following this logic, at five points there are eight distinct candidate integral topologies. Among them, six are independent and form the basis shown in Fig. 1. The remaining two topologies, displayed in Fig. 2, are excluded from our basis. The first is removed to eliminate redundancies arising from Gram relations, while the second does not admit a maximal residue: upon sequentially imposing the cut conditions, a double pole is encountered before the final cut can be taken. As a consequence, this topology integrates to functions of lower transcendental weight and therefore is not expected to contribute to the correlator integrand.

Diagonalizing the leading singularities. Having identified the relevant topologies, the next step is to construct proper numerators so that the resulting integrals are UT and carry unit residue. In this way, physical leading singularities for a certain observable will be directly visible in the expanding coefficients, and the basis exhibits uniform weight at an integrated level. The strategy is to choose numerators such that residues on all cut configurations either vanish or normalize to rational numbers, while any

double poles are eliminated. A related discussion for this point can be found also in [44].

Specified at two loops, when diagonalizing the leading singularities and getting the UT basis, all topologies always fall into the following three classes:

1. *Scalar integrals.* For this class of integrals, each loop momentum is associated with exactly four propagators. Therefore, no nontrivial numerator depending on the loop momenta is needed for these cases, and a UT basis can be constructed by simply normalizing the scalar integrals by its maximal residue. Specified to the maximal cut topologies in Fig. 1, the double box and kissing box belong to this case. As an example, consider the following double-box integral and its maximal cut [66], we obtain maximal residue as

$$2 \begin{array}{c} \text{3} \quad \text{4} \\ \text{---} \quad \text{---} \\ \text{a} \quad \text{b} \\ \text{---} \quad \text{---} \\ \text{1} \end{array} 5 = \frac{1}{\lambda_{1,23,45}} \quad (7)$$

where the cut propagators are indicated by double lines, and $\lambda_{1,23,45}$ is defined in Appendix A, Eq. (A3). Dividing this residue value, the corresponding unit-residue integral we will identify as basis is [67]

$$B_{1,23,45} := \int \frac{d^4 x_a d^4 x_b \lambda_{1,23,45}}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,b}^2 x_{b,1}^2 x_{b,4}^2 x_{b,5}^2}. \quad (8)$$

Throughout, the two loop variables a, b are symmetrized, though we often suppress the explicit ($a \leftrightarrow b$). Especially, when considering the special case $B_{1,23,34}$, the corresponding integral reduces to four-point two-loop ladder integral [68], and the basis is

$$h_{12,34} := \int \frac{d^4 x_a d^4 x_b x_{1,2}^2 \Delta_5}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,b}^2 x_{b,1}^2 x_{b,2}^2 x_{b,4}^2}, \quad (9)$$

with Δ_5 defined in Appendix A, Eq. (A2).

The kissing boxes in Fig. 1 are treated analogously, and we define the integrals,

$$F_{4 \times 5} := \int \frac{d^4 x_a \Delta_4}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,5}^2} \times \frac{d^4 x_b \Delta_5}{x_{b,1}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2},$$

$$F_{5 \times 5} := \frac{1}{2} \int \frac{d^4 x_a \Delta_5}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,4}^2} \times \frac{d^4 x_b \Delta_5}{x_{b,1}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2}. \quad (10)$$

2. *Pentabox integral.* For a pentabox integral with eight propagators in Fig. 1, it contains one maximal pentabox cut and four double-box cuts, and these five cuts can be used to determine the numerators of the integral uniquely.

Let us explicitly construct the basis corresponding to the pentabox integral. Since x_b is adjacent to five dual points $x_2, \dots, x_5$ and x_a , an extra numerator $\mathcal{N} \propto x_{b,1}^2$ should be

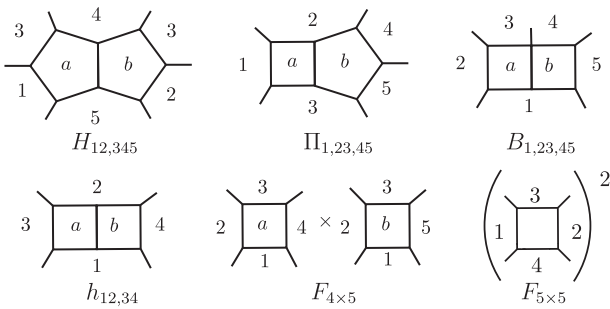


FIG. 1. Two-loop five-point relevant topologies.

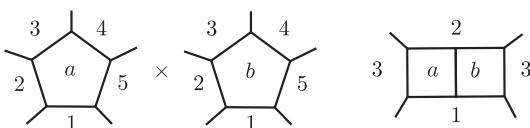


FIG. 2. Excluded two-loop five-point topologies.

considered. Consequently, the maximal cut fixes the numerator to be $\Delta_1 x_{b,1}^2 x_{2,3}^2$,

$$x_{b,1}^2 \times 1 \times \text{diagram} = \frac{1}{\Delta_1 x_{2,3}^2}. \quad (11)$$

Besides, we should also consider the four double-box subtopology cuts for the pentabox,

$$\begin{aligned} \text{(i)} \quad & \text{diagram} \quad \text{(ii)} \quad \text{diagram} \\ \text{(iii)} \quad & \text{diagram} \quad \text{(iv)} \quad \text{diagram} \end{aligned} \quad (12)$$

We obtain nontrivial residues at (i) and (ii), while we find vanishing residues for (iii) and (iv). Therefore, we propose the UT basis for this cut as

$$\Pi_{1,23,45} := \int \frac{d^4 x_a d^4 x_b \Delta_1 (x_{b,1}^2 x_{2,3}^2 - x_{b,2}^2 x_{1,3}^2 - x_{b,3}^2 x_{1,2}^2)}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,b}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2 x_{b,5}^2}. \quad (13)$$

3. Double-pentagon integral. Finally, we should consider the double-pentagon integral basis with nine propagators. The number of the propagators is more than the degree of freedom for the loop momenta. Unlike the previous cases, the overall normalization cannot be fixed by simply requiring unit residue from a maximal cut. We therefore introduce an unfixed coefficient N at the first step

$$\frac{N \times x_{3,5}^2 x_{3,4}^2 x_{4,5}^2 x_{a,1}^2 x_{b,2}^2}{x_{a,1}^2 x_{a,3}^2 x_{a,4}^2 x_{a,5}^2 x_{a,b}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2 x_{b,5}^2}. \quad (14)$$

N is a conformal invariant that depends only on conformal cross ratios of $x_{i,j}^2$.

To fix the normalization factor as well as construct the subsequent terms in the numerator, we consider all pentabox cuts and double-box cuts of the integrand (14). Elegantly, residues on any of these subtopology cuts always decompose into a common rational term and a distinct square-root term. For example, on the double-box cut

$$N \times x_{3,5}^2 x_{3,4}^2 x_{4,5}^2 x_{a,1}^2 x_{b,2}^2 \times \text{diagram} \quad (15)$$

the residue of the double-pentagon is

$$N \left(\frac{1}{2} - \frac{x_{2,3}^2 x_{4,5}^2 - x_{2,4}^2 x_{3,5}^2 - x_{2,5}^2 x_{3,4}^2}{2\Delta_1} \right). \quad (16)$$

This fact suggests fixing $N = 1$ and adding terms corresponding to different subtopologies in numerators to cancel the square-root term, such that finally on any of these cuts it only has rational numbers as maximal residues. In addition to that, spurious double poles must be removed. Imposing these conditions, we uniquely determine the numerator, and the final integral is

$$H_{12,345} := \int \frac{d^4 x_a d^4 x_b (\mathcal{G}_{\{b,2,3,4,5\}}^{a,1,3,4,5} + \sum_{i,j} (-1)^{i+j} T_{1345,2345}^{i,j})}{x_{a,1}^2 x_{a,3}^2 x_{a,4}^2 x_{a,5}^2 x_{a,b}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2 x_{b,5}^2}, \quad (17)$$

where the Gram determinants $\mathcal{G}_B^A$ and the numerators $T_{1345,2345}^{i,j}$ are defined in Appendix A, Eqs. (A4) and (A5).

Result of two-loop integrands. By now we propose six types of UT integrals, which are sufficient for any observables with five off-shell legs at two-loop order. Finally, we expand integrands of five-point correlation functions [30,46] onto our basis. Note that since all basis integrals are pure and of uniform weight, all prefactors calculated in the following expansion will be essential to the final results of the correlation function. The expansion is obtained by a systematic algebraic reduction onto the UT basis. Although the procedure is algorithmic, carrying it out in practice is nontrivial due to the size of the expressions and requires extensive bookkeeping and cross-checks.

For the maximal sector,

$$\begin{aligned} f_{\max}^{(2)} = & \left\langle \frac{x_{1,3}^2 x_{1,5}^2 x_{2,4}^2 + x_{1,3}^2 x_{1,4}^2 x_{2,5}^2 + x_{1,2}^2 x_{1,5}^2 x_{3,4}^2 + x_{1,2}^2 x_{1,4}^2 x_{3,5}^2}{4\lambda_{1,23,45}} B_{1,23,45} - \frac{c_5}{12\mathcal{G}_{\{1,2,3,4,5\}}} H_{12,345} + \frac{x_{1,2}^2 x_{3,4}^2}{4\Delta_5} h_{12,34} \right. \\ & \left. + \frac{x_{1,2}^2 x_{3,4}^2 + x_{1,4}^2 x_{2,3}^2}{4\Delta_5} \Pi_{5,13,24} + \frac{c_5}{12\mathcal{G}_{\{1,2,3,4,5\}}} F_{5 \times 5} + \frac{x_{1,4}^2 x_{1,5}^2 x_{2,3}^2 + x_{2,4}^2 x_{2,5}^2 x_{1,3}^2 + x_{3,4}^2 x_{3,5}^2 x_{1,2}^2}{12\Delta_4 \Delta_5} F_{4 \times 5} - \frac{c_5 \mathcal{G}_{\{1,2,3,5\}}^{\{1,2,3,5\}}}{12\mathcal{G}_{\{1,2,3,4,5\}}} F_{4 \times 5} \right\rangle_{S_5}, \end{aligned} \quad (18)$$

where $c_5 := x_{1,2}^2 x_{2,3}^2 x_{3,4}^2 x_{4,5}^2 x_{1,5}^2 + 11$ perms, which is permutation invariant constant.

For the next-to-maximal sector,

$$f_{23}^{(2)} = \left\langle \frac{2x_{1,2}^2 x_{3,5}^2 x_{4,5}^2}{\lambda_{5,14,23}} B_{5,14,23} - \frac{H_{34,125}}{4} + \frac{x_{1,2}^2 x_{3,4}^2}{2\Delta_5} (h_{12,34} - h_{34,12} + h_{14,23} + h_{23,14} + h_{13,24} + h_{24,13} + 2\Pi_{5,34,12}) \right. \\ \left. + \frac{\mathcal{G}_{\{3,4\}}^{\{1,2\}}}{4\Delta_5} (-h_{14,23} - h_{23,14} + h_{13,24} + h_{24,13} - 4\Pi_{5,13,24}) + \frac{\mathcal{G}_{\{1,2,3,5\}}^{\{1,2,3,4\}} + x_{1,2}^2 \mathcal{G}_{\{2,3,5\}}^{\{1,3,4\}} + x_{1,2}^2 \mathcal{G}_{\{1,3,5\}}^{\{2,3,4\}} - 4x_{1,2}^4 \mathcal{G}_{\{3,5\}}^{\{3,4\}}}{4\Delta_4 \Delta_5} F_{4 \times 5} \right\rangle_{K_{23}}, \quad (19)$$

$$f_5^{(2)} = \left\langle -\frac{4x_{1,2}^2 x_{1,5}^2 x_{3,4}^2}{\lambda_{1,24,35}} B_{1,24,35} + \frac{2\mathcal{G}_{\{1,3,4\}}^{\{1,2,5\}}}{\lambda_{1,25,34}} B_{1,25,34} + \frac{H_{12,345}}{2} + \frac{x_{1,2}^2 x_{3,4}^2}{2\Delta_5} (-5h_{12,34} - 5h_{34,12} + 2h_{14,23} - 2h_{23,14} + h_{13,24} \right. \\ \left. + h_{24,13} + 3\Pi_{5,12,34} + 3\Pi_{5,34,12} - 4\Pi_{5,14,23} - 3\Pi_{5,13,24} - 3\Pi_{5,24,13}) \right. \\ \left. + \frac{\mathcal{G}_{\{3,4\}}^{\{1,2\}}}{2\Delta_5} (h_{12,34} + h_{34,12} - 2h_{14,23} + 2h_{23,14} + 3h_{13,24} + 3h_{24,13} \right. \\ \left. + \Pi_{5,12,34} + \Pi_{5,34,12} + 4\Pi_{5,14,23} - \Pi_{5,13,24} - \Pi_{5,24,13}) \right. \\ \left. + \frac{-3\mathcal{G}_{\{1,2,3,5\}}^{\{1,2,3,4\}} + 4x_{1,3}^2 \mathcal{G}_{\{1,2,5\}}^{\{2,3,4\}} + 2x_{1,2}^2 x_{2,3}^2 \mathcal{G}_{\{1,5\}}^{\{3,4\}} - 6x_{1,2}^2 x_{1,5}^2 x_{2,3}^2 x_{3,4}^2}{2\Delta_4 \Delta_5} F_{4 \times 5} \right\rangle_{D_5}. \quad (20)$$

In the expressions, we use notation $K_{23} = S_2 \times S_3$, S_n the n -point permutation group and D_n the n -point dihedral group. We have checked the correctness by comparing our results with the integrand proposed in [30,46], and recorded them in the Appendix A.

IV. INTEGRATED-LEVEL RESULTS OF THE CONFORMAL INTEGRAL BASIS

After constructing conformal integrals with unit leading singularities and UT property, we compute their results at integrated level.

Matching MIs by fixing conformal frame. Due to the complexity of the external kinematics, a direct integration for our conformal MIs from Feynman parametrization will be out of reach, and we will adopt the CDE and integration-by-parts (IBP) method [60,61] to solve the problem. To begin with, it is very important to notice that all the basis integrals we proposed in the last section actually belong to the two-loop integral family in [59] with four massive external legs, after we fix the conformal frame by sending one of the external dual points to ∞ .

Let us offer some explanations of this procedure based on Fig. 3. In this example, we set $x_1 \rightarrow \infty$ and correspondingly treat all $x_{i,1}^2 \rightarrow \infty$ as well. Consequently, the kinematics goes to a four-point fully massive kinematics without conformal property, which depends on variables $s = (P_1 + P_2)^2$, $t = (P_2 + P_3)^2$ and $m_i^2 = P_i^2$. Conformal cross ratios of x_i , as independent kinematic variables, turn

to ratios of $\{s, t, m_i^2\}$. For instance,

$$\frac{x_{3,4}^2 x_{1,5}^2}{x_{4,5}^2 x_{1,3}^2} \rightarrow \frac{x_{3,4}^2}{x_{4,5}^2} := \frac{m_3^2}{m_4^2} \quad (21)$$

where in the last step we identify $P_i = x_{i+1} - x_i$ for $i = 2, \dots, 4$, $P_1 := x_2 - x_5$ and then find the matching between conformal cross ratios and nonconformal kinematics variables. By inverting the monomial relations, four-point massive integrals can be used to compute five-point conformal integrals. This matching is our beginning point for IBP calculation.

As an example, according to our definition, $B_{1,23,45}$ under this limit is

$$B_{1,23,45} \rightarrow \int d^4 x_a d^4 x_b \frac{r_6}{x_{a,2}^2 x_{a,3}^2 x_{b,4}^2 x_{b,5}^2 x_{a,b}^2}. \quad (22)$$

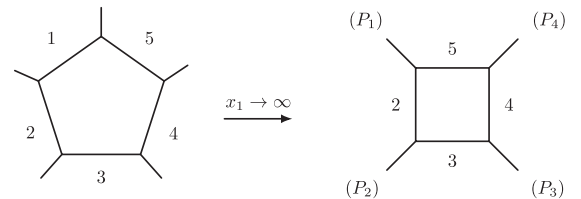


FIG. 3. Fixing frame of conformal kinematics to get general kinematics.

Furthermore, by translation $x_{a,2} \rightarrow \ell_1 - P_1$, $x_{b,5} \rightarrow \ell_2$, we finally we have

$$B_{1,23,45} \rightarrow \int d^4\ell_1 d^4\ell_2 \frac{r_6}{D_2 D_3 D_5 D_6 D_7}, \quad (23)$$

where r_6 and D_i are defined in Appendix B, Eqs. (B4) and (B2), respectively. This is exactly the MI $\mathcal{I}_{17}$ from [59].

IBP reduction for the conformal MIs. Based on this procedure, conformal MIs can always be identified with integrals that belong to the two-loop double-box integral family in [59] for four massive particles scattering processes. This suggests that five-point two-loop correlators are always Chen-iterative integrals [69] with only d log kernels [70].

The next step is to perform the IBP reduction and obtain the integrated results. In [59], results of all MIs at symbol level were calculated based on the CDE. We include the necessary review of that work in Appendix B, and only mention here that there are 74 UT MIs in this integral family. We use Kira [71] to perform the reduction. Two points are worth noting here. First, following the restriction of conformal invariance, numerators of the integral basis are simple enough, such that most IBP reduction can be completed by simply expanding the numerators onto irreducible scalar products algebraically. Second, although the IBP calculations are performed in dimensional regularization in $D = 4 - 2\epsilon$ dimensions, due to the favorable properties of the well-designed basis, the coefficients obtained after reducing to MIs are always constant rational numbers. Consequently, these finite integrals will be UT functions even at higher ϵ order.

By adopting this procedure, we find the following elegant correspondence between our basis and the UT basis in [59] as

$$\begin{aligned} B_{1,23,45} & \xrightarrow{\{x_{5,2}, x_{2,3}, x_{3,4}, x_{4,5}\}} \mathcal{I}_{17}, \\ h_{12,34} & \xrightarrow{\{x_{1,4}, x_{4,2}, x_{2,3}, x_{3,1}\}} \mathcal{I}_1, \\ \Pi_{1,23,45} & \xrightarrow{\{x_{2,4}, x_{4,3}, x_{3,1}, x_{1,2}\}} \mathcal{I}_3, \\ F_{4 \times 5} & \xrightarrow{\{x_{3,5}, x_{5,2}, x_{2,4}, x_{4,3}\}} \mathcal{I}_9, \\ H_{12,345} & \xrightarrow{\{x_{3,1}, x_{1,4}, x_{4,2}, x_{2,3}\}} -2\mathcal{I}_4 - 3\mathcal{I}_{73}, \end{aligned} \quad (24)$$

where $\{x_{i,j}, x_{j,k}, x_{k,l}, x_{l,i}\}$ stands for the identification between the difference of dual points from conformal kinematics and momenta $\{P_i\}_{i=1,\dots,4}$ from four-point massive kinematics. Especially $F_{5 \times 5}$ can be obtained by identifying $x_5 \rightarrow x_4$ in $F_{4 \times 5}$ furthermore. We obtain the results of all basis integrals at five points and two loops then.

Symbol result. By adding up the basis, we can get the final integrated result of five-point correlators in both the maximal and next-to-maximal sector. In [59], although the

CDE system was worked out, the MIs were only expressed by Chen iterated integrals over d log forms. We will therefore focus on their symbol results in this work. The two results both contain **106** symbol letters with **20** different square roots. Among all these symbol letters, $11 + 20 = \mathbf{31}$ letters already appeared in one-loop differential equations, containing five one-loop four-mass box square roots Δ_i , while other $15 \times 5 = \mathbf{75}$ symbol letters are from integral $B_{1,23,45}$ and its permutations, containing 15 square roots, which are $\lambda_{1,23,45}$ and their permutations. We provide the symbol results for all conformal integral basis elements and the full result in the ancillary data files [72].

V. CONCLUSION

This work discussed a UT and pure basis for observables with five off-shell external legs and at two-loop order, consisting of six conformal integrals. Especially, we simplify and compute, for the first time the correlation functions of five half-BPS operators at two-loop order in $\mathcal{N} = 4$ SYM theory at symbol level, both for maximal and nonmaximal sectors.

Our work also raises several important open questions. First, the integrand for two-loop higher-point nonmaximal sector correlators has also been provided in [46,47], and its result involves elliptic integrals. It would be highly significant if these results could be computed using existing elliptic symbol techniques similar to two-loop conformal integral basis for amplitudes [73,74]. Second, building on the success of the four-point Correlahedron, whether the Correlahedron framework can be extended to higher-point nonmaximal sectors remains a crucial problem. Third, the hidden $10D$ symmetry discovered in the four-point case connects correlation functions of different Kaluza-Klein modes [75–78]; it would be profoundly important to generalize this structure to higher points, which has already been observed in strong coupling expansion [17,79]. Finally, the half-BPS correlation function itself is related to numerous other physical quantities, such as energy correlation functions of scalar flow in $\mathcal{N} = 4$ SYM theory [80–82]. We expect our results to facilitate future studies in these directions.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [72].

APPENDIX A: FIVE-POINT CONFORMAL INTEGRALS

For completeness, we collect the definitions of the conformal integrals that appear in the five-point analysis and describe their properties in more detail. Note that in this appendix, the loop momenta x_a and x_b in each integral are symmetrized at the integrand level, ensuring a well-defined correlator integrand. The three main topologies are

$$\begin{aligned} H_{12,345} &:= \int d^4x_a d^4x_b \frac{\mathcal{G}_{\{b,2,3,4,5\}}^{\{a,1,3,4,5\}} + \sum_{i,j} (-1)^{i+j} T_{1345,2345}^{i,j}}{x_{a,1}^2 x_{a,3}^2 x_{a,4}^2 x_{a,5}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2 x_{b,5}^2}, \\ \Pi_{1,23,45} &:= \int d^4x_a d^4x_b \frac{\Delta_1 (x_{a,1}^2 x_{2,3}^2 - x_{a,2}^2 x_{1,3}^2 - x_{a,3}^2 x_{1,2}^2)}{x_{a,2}^2 x_{a,3}^2 x_{a,4}^2 x_{a,5}^2 x_{b,1}^2 x_{b,2}^2 x_{b,3}^2}, \\ B_{1,23,45} &:= \int d^4x_a d^4x_b \frac{\lambda_{1,23,45}}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,b}^2 x_{b,1}^2 x_{b,4}^2 x_{b,5}^2}. \quad (\text{A1}) \end{aligned}$$

Here the square roots are defined as

$$\Delta_5 := \sqrt{(x_{1,3}^2 x_{2,4}^2 - x_{1,2}^2 x_{3,4}^2 - x_{1,4}^2 x_{2,3}^2)^2 - 4x_{1,2}^2 x_{3,4}^2 x_{1,4}^2 x_{2,3}^2} \quad (\text{A2})$$

and

$$\lambda_{1,23,45} := \sqrt{(x_{1,3}^2 x_{1,5}^2 x_{2,4}^2 - x_{1,3}^2 x_{1,4}^2 x_{2,5}^2 - x_{1,2}^2 x_{1,5}^2 x_{3,4}^2 + x_{1,2}^2 x_{1,4}^2 x_{3,5}^2)^2 - 4x_{1,2}^2 x_{1,3}^2 x_{1,4}^2 x_{2,3}^2 x_{1,5}^2 x_{4,5}^2}. \quad (\text{A3})$$

Cyclic images of Δ_5 with $x_j \rightarrow x_{j+i}$ are denoted as Δ_i , and other $\lambda_{m,ij,kl}$ are permutations of $\lambda_{1,23,45}$ according to the indices. The square root $\lambda_{m,ij,kl}$ first appears at five points and two loops, which is invariant under the exchanges

$$X_i \leftrightarrow X_j, \quad X_k \leftrightarrow X_l, \quad (X_i, X_j) \leftrightarrow (X_k, X_l).$$

The Gram determinants are defined by

$$\mathcal{G}_B^A := \det(x_{a,b}^2)|_{a \in A, b \in B}, \quad \mathcal{G}_A := \mathcal{G}_A^A. \quad (\text{A4})$$

The numerator $\mathcal{T}_{1345,2345}^{i,j}$ is obtained by deleting the i th and j th columns from the sets (1345, 2345), with the remaining entries forming a 3×3 Gram determinant. For example,

$$\mathcal{T}_{1345,2345}^{2,4} = x_{a,3}^2 x_{b,5}^2 \mathcal{G}_{\{2,3,4\}}^{\{1,4,5\}}. \quad (\text{A5})$$

The remaining integrals are either degenerate limits of these three topologies or products of lower-loop integrals:

Degenerate limit. In the degenerate limit $k = l$ of $B_{i,jk,lm}$, we obtain

$$h_{12,34} := \int d^4x_a d^4x_b \frac{x_{1,2}^2 \Delta_5}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,b}^2 x_{b,1}^2 x_{b,2}^2 x_{b,4}^2}, \quad (\text{A6})$$

where the square root $\lambda_{m,ij,kl}$ reduces to Δ_i .

Products of one-loop boxes. Finally, we record the products of one-loop box integrals,

$$\begin{aligned} F_{4 \times 5} &:= \int d^4x_a d^4x_b \frac{\Delta_4}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,5}^2} \times \frac{\Delta_5}{x_{b,1}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2}, \\ F_{5 \times 5} &:= \frac{1}{2} \int d^4x_a d^4x_b \frac{\Delta_5}{x_{a,1}^2 x_{a,2}^2 x_{a,3}^2 x_{a,4}^2} \times \frac{\Delta_5}{x_{b,1}^2 x_{b,2}^2 x_{b,3}^2 x_{b,4}^2}. \quad (\text{A7}) \end{aligned}$$

APPENDIX B: TWO-LOOP FOUR-EXTERNAL-MASS FEYNMAN INTEGRAL FAMILY

In the main text, we reduced conformal invariant basis integrals onto two-loop four-external-mass master integrals [59] by the IBP method. In this appendix, we review necessary facts that are used in that calculation.

The two-loop Feynman integral family with four different external masses can be generally written as

$$G_{\alpha_1, \dots, \alpha_n} \equiv \int \prod_{i=1}^L \frac{d^D l_i}{i\pi^{D/2}} \frac{D_8^{-n_8} D_9^{-n_9}}{D_1^{\alpha_1} \dots D_7^{\alpha_7}}. \quad (\text{B1})$$

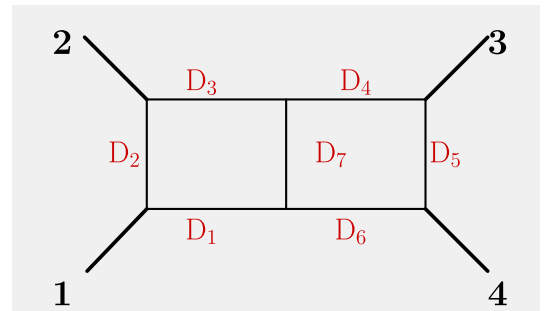


FIG. 4. Two-loop diagram with four external massive legs.

Propagators in this expression are

$$\begin{aligned} D_1 &= l_1^2, & D_2 &= (l_1 - p_1)^2, & D_3 &= (l_1 - p_1 - p_2)^2, \\ D_4 &= (l_2 + p_1 + p_2)^2, & D_5 &= (l_2 + p_1 + p_2 + p_3)^2, & D_6 &= l_2^2, \\ D_7 &= (l_1 + l_2)^2, & D_8 &= (l_1 - p_1 - p_2 - p_3)^2, & D_9 &= (l_2 + p_1)^2. \end{aligned} \quad (\text{B2})$$

D_8 and D_9 serve as irreducible scalar products of this family. $G_{1,1,1,1,1,1,1,0,0}$ from this family is the two-loop ladder diagram shown in Fig. 4, whose result is conformal invariant and was calculated in [68].

IBP reduction shows that this family contains 74 master integrals. Explicit result for these master integrals can be found in [59]. Here we list the UT master integrals that are used in the main text.

$$\begin{aligned} \mathcal{I}_1 &= sr_1 G_{1,1,1,1,1,1,1,0,0}, & \mathcal{I}_3 &= r_2(-m_3^2 G_{0,1,1,1,1,1,1,0,0} - m_4^2 G_{1,1,0,1,1,1,1,0,0} + s G_{1,1,1,1,1,1,1,-1,0}), \\ \mathcal{I}_4 &= s G_{1,1,1,1,1,1,1,-1,-1} + \frac{1}{2} s(s - m_1^2 - m_2^2) G_{1,1,1,1,1,1,1,-1,0} + \frac{1}{2} s(s - m_3^2 - m_4^2) G_{1,1,1,1,1,1,1,0,-1} + (\text{lower-sector integrals}), \\ \mathcal{I}_9 &= r_2 r_4 G_{1,1,1,1,1,1,0,0,0}, & \mathcal{I}_{17} &= r_6 G_{0,1,1,0,1,1,1,0,0}, & \mathcal{I}_{73} &= \frac{s G_{0,0,2,0,0,2,1,0,0}}{\epsilon^2} \end{aligned} \quad (\text{B3})$$

where kinematics variables are defined as $p_i^2 = m_i^2$ and $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$, and the roots r_i defined as

$$\begin{aligned} r_1^2 &= s^2 t^2 - 2 s t m_1^2 m_3^2 + m_1^4 m_3^4 - 2 s t m_2^2 m_4^2 - 2 m_1^2 m_2^2 m_3^2 m_4^2 + m_2^4 m_4^4, \\ r_2^2 &= s^2 - 2 s m_1^2 + m_1^4 - 2 s m_2^2 - 2 m_1^2 m_2^2 + m_2^4, \\ r_4^2 &= s^2 - 2 s m_3^2 + m_3^4 - 2 s m_4^2 - 2 m_3^2 m_4^2 + m_4^4, \\ r_6^2 &= s^2 + 2 s t + t^2 - 2 s m_1^2 - 2 t m_1^2 + m_1^4 - 2 s m_3^2 - 2 t m_3^2 + 2 m_1^2 m_3^2 + m_3^4 - 4 m_2^2 m_4^2. \end{aligned} \quad (\text{B4})$$

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